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Spécialité: Analyse mathématiques et applications

Sur la conception d'observateurs d'état des systèmes non linéaires et applications

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List of publications

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Introduction

State estimation is in general important for real-world applications, such as chemical engineering systems, robotics, neural networks, and population dynamic models. On other hand, estimation is sometimes necessary and essential for specific applications like autonomous vehicles. Estimation is important because the system state variables are used in control design schemes, diagnosis, monitoring, and other modern applications like self-synchronization and consensus achievement for multi-agent systems. Estimation is also essential due to unavailability of the system states in real-time. Indeed, full state variables cannot be measured or are too expensive to measure in many applications such as vehicle dynamics [1], due to unavailability of sensors at any cost. To this end, several methods have been developed in the literature for nonlinear systems, namely extended Kalman filter; moving horizon estimation; and state observer techniques [2–8]. Observer design technique is the most popular one used for state estimation in the recent literature. Several observer design approaches have been established, namely high-gain observer [9], sliding mode observer, and LMIs-based observer [10–12].

The design of high-gain observers was essentially motivated by its simplicity to implement due to the use of only one single tuning parameter. However, there are three limitations that make high-gain observer weak and difficult to be used in sensitive industrial applications. The first limitation is related to numerical issues concerning large systems as high values of the observer gain are required. The second limitation is the sensitivity to measurement noise because high values of the observer gains amplify the noise [13]. The third and last limitation is the peaking phenomenon characterized by large amplitudes of the estimated states in the transient phase.

To overcome these restrictions, several solutions have been proposed in the literature, a new high-gain observer has been proposed in [14]. Their contribution consists in limiting the power of the tuning parameter to 2. However, the dimension of the observer is equal to $2(n - 1)$ where n is the dimension of the original system, and the power n is only distributed between different additional state variables injected in the observer. Then this power n reappears in the bound of the estimation error when the system is subject to measurement noise. This particular design

has been reconsidered in [15, 16] by including saturations to avoid the peaking phenomenon. Another recent high-gain observer with the same dimension as the original system and where the observer's gain power is limited to 1 was proposed in [17] for the same class of systems considered in [15]. The authors in [18] proposed a technique called "HG/LMI observer" which follows the standard high-gain methodology with the same state observer structure of dimension n by exploiting the LPV/LMI technique developed in [19]. They introduced an index j_i , with $0 \leq j_0 \leq n$. Hence, the power of the proposed high-gain is limited to j_0 , but we need to solve 2^{j_0} LMIs instead of one with the standard high-gain observer.

Thesis Contributions

1. In Chapter 2, we present new high-gain type observer design methods for nonlinear systems [20], this new designs provide lower gains compared to the high-gain observer. We combine the improved high-gain methodology with the LMI-based observer design technique to build a more general observer that allows us to exploit the benefits of both approaches.
2. Chapter 3 is devoted to the analysis of the feasibility of LMI conditions in the framework of the design of observers for a class of nonlinear systems [21]. Two significant contributions were presented. First, we propose a general observer synthesis method which allows to overcome the conservatism of existing LMI techniques in the literature. Secondly, through non-linear transformations, LMI conditions with guaranteed feasibility have been established. Thus the LMIs obtained admit solutions for any finite value of the Lipschitz constant characterizing the non-linearity of the system.
3. In Chapter 4, we deal with the problem of exact finite-time estimation for a class of switched discrete-time linear systems with delayed-output measurements [22]. Two estimation algorithms are proposed. The first one is based on explicit computation of the system state by using a combination of delayed inputs/outputs, while the second algorithm uses two asymptotic observers linked by a mathematical relation. Numerical examples are given to illustrate the validity and effectiveness of the proposed algorithms.

Thesis Organization

Chapter 1 is organized as follows. Sections 1.1 recalls the mathematical definitions of the concept of observability and some preliminaries on the stability of systems in the sense of Lyapunov. Section gives a state of the art on the high-gain type observers. This chapter contains the necessary tools for understanding and developing the contributions and the rest of the manuscript and allows to situate the contributions of this work in relation to existing work in the literature.

Chapter 2 is organized as follows. Section 2.1 presents a motivating example then the HG/LMI observer based on [8]. Section 2.2 presents a more general HG/LMI observer which is the main contribution of this chapter. Section 2.2.3 provides a numerical example and an application to a tumor growth model in order to show the validity and effectiveness of the proposed design technique.

Chapter 3 is organized as follows. Section 3.1 is deduced to an introduction and some useful preliminaries. Section 3.2 presents a general LMI Design Method. Section 3.3 provides some

LMI Feasibility Improvements by introducing linear and nonlinear transformations.

Chapter 4 is organized as follows. Section 4.2 is devoted to the preliminary results on exact and finite-time estimation for linear systems with delayed outputs. The main generalization to switched systems and output feedback stabilization through a set of LMI conditions is presented in Section 4.3. Further extensions to a class of switched singular systems are developed in Section 4.4. Section 4.2.6 provides some illustrative examples to show the validity and effectiveness of the estimation algorithms and their use in output feedback stabilization loops. Finally, this chapter is ended by a conclusion and some prospects given in Section 4.5.

Preliminaries

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1.1 Notions of Stability and Lyapunov Functions

Let the following differential equation (autonomous system because f does not explicitly depend on time t)

$$\dot{x} = f(x), \quad (1.1)$$

where $x \in D \subset \mathbb{R}^n$, D a domain containing the origin. We assume that f is Piece-wise continuous with respect to t ; and Locally Lipschitz with respect to x , *i.e.*: satisfying

$$\|f(x_1) - f(x_2)\| \leq k\|x_1 - x_2\|, \quad \text{with } k > 0 \quad (1.2)$$

$f(x_0)$ bounded, where x_0 represents the initial condition at $t = 0$.

Definition 1.1

- x^* is an equilibrium point of (1.1) if

$$x(0) = x^* \Rightarrow x(t) = x^*, \forall t \geq 0$$

- Equivalent condition :

$$x^* \text{ an equilibrium point} \Leftrightarrow f(x^*) = 0, \forall t \geq 0$$

Without loss of generality, we'll study stability of the origin for the autonomous system (1.1).

Definition 1.2

1. **Stability:** the origin is a stable equilibrium point for (1.1) if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that:

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0.$$

Otherwise, the origin is said to be unstable.

2. **Asymptotic stability:** the origin is asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} \|x(t)\| = 0.$$

3. **Exponential stability:** the origin is a locally exponentially stable equilibrium point for (1.1), if there exist constants $\alpha \geq 1$ and $\beta > 0$ such that:

$$\|x(t)\| \leq \alpha \|x(0)\| \exp(-\beta t), \forall t \geq 0, \forall x(0) \in \mathcal{B}_r.$$

When $\mathcal{B}_r = \mathbb{R}^n$, we say that the origin is globally exponentially stable.

Basically, an equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.

Note that applying the previous definitions to prove stability requires an explicit computation of the solution corresponding to a given initial condition, which is a complicated or rather an impossible task. So, new tools are necessary to characterize stability, namely Lyapunov methods. Next, two main theorems of Lyapunov stability are presented.

1.1.1 Lyapunov's indirect method

Stability of the origin for system (1.1) is related to the stability of the linearization of the system (1.1) around the equilibrium point.

Theorem 1.1

Let $x = 0$ be an equilibrium point for the nonlinear system (1.3), $f : D \rightarrow \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$\dot{x} = f(x) \Rightarrow \underbrace{\dot{x} = Ax}_{\text{linear dynamics}}, \text{ where } A = \left. \frac{\partial f}{\partial x} \right|_{x=0}.$$

Then

1. The origin is locally asymptotically stable if $Re(\lambda_i) < 0$ for all eigenvalues of A .
2. The origin is unstable if $Re(\lambda_i) > 0$ for at least one of the eigenvalues of A .
3. If $Re(\lambda_i) = 0$ for some i , then
 - If $Re(\lambda_i) \leq 0$ for all i , the origin is marginally stable.
 - Else the origin is unstable.

1.1.2 Lyapunov's direct method**Philosophy of the method : Energy and dissipation functions**

The philosophy of this method is based on a fundamental observation of physics: if the total energy of a system is dissipated continuously, then the system will reach an equilibrium point. We can therefore conclude that a system is stable by examining the variation of a scalar function V , representing the total energy.

Theorem 1.2: Lyapunov function and Lyapunov stability

Let $x = 0$ be an equilibrium point for the nonlinear system (1.3) and let D a neighborhood of the origin. Consider a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $V(x) = 0$ if and only if $x = 0$;
- $V(x) > 0$ if and only if $x \neq 0$;
- $\dot{V}(x(t)) = \frac{d}{dt}V(x(t)) = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x) = \nabla V(x) \cdot f(x) \leq 0, \forall x \neq 0$.

Then $V(x)$ is called a Lyapunov function and the system is stable.

- Asymptotic stability: $\dot{V}(x) < 0$ for $x \neq 0$ is required.
- Global asymptotic stability: An additional condition $\|x\| \rightarrow +\infty \Rightarrow V(x) \rightarrow +\infty$, called "radial unboundedness" is required to conclude.

1.2 Observer and Observability

The observability of a process is a very important concept in the field of state estimation. Indeed, to reconstruct the inaccessible states of a system, it is necessary to know, a priori whether the variables are observable or not. Observability stands for the possibility of reconstructing the full trajectory from the observed data, that is, from the output trajectory in the uncontrolled case, or from the couple (output trajectory, control trajectory) in the controlled case.

1.2.1 Observability of Nonlinear Systems

Let us consider the class of nonlinear systems described by :

$$\begin{cases} \dot{x} &= f(x, u), \\ y &= h(x, u), \end{cases} \quad (1.3)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$.

Definition 1.3

Observability of nonlinear systems (Geometric Condition)

- Indistinguishability: Let $y_u^0(t) \rightarrow x^0$ and $y_u^1(t) \rightarrow x^1$. Then x^0 and x^1 are indistinguishable if $y_u^0(t) = y_u^1(t)$, $\forall t \geq 0$, $\forall u$. Otherwise, x^0 and x^1 are distinguishable.
- The system (1.3) is observable in x^0 if x^0 is distinguishable from any $x \in \mathbb{R}^n$. In addition, the system (1.3) is observable if $\forall x^0 \in \mathbb{R}^n$, x^0 is distinguishable.

Definition 1.4: Observability of nonlinear systems (rank condition)

We say that system (1.3) is observable if the following rank condition holds^a:

$$\text{rank} \begin{pmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-1} h \end{pmatrix} = n$$

where the expression of $dL_f^k h$ is given by

$$dL_f^k h = \left(\frac{\partial L_f^k h}{\partial x_1}, \frac{\partial L_f^k h}{\partial x_2}, \dots, \frac{\partial L_f^k h}{\partial x_n} \right).$$

^aLie derivative: Let $h : D \rightarrow \mathbb{R}$, $f : D \rightarrow \mathbb{R}^n$, the Lie derivative of h along f is

$$L_f h(x) = \sum_{i=1}^n f_i(x) \frac{\partial h}{\partial x_i}(x).$$

1.2.2 Observability of linear systems

Let us consider the class of linear systems described by :

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \quad (1.4)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control and $y \in \mathbb{R}^p$ is the measured output.

Definition 1.5: Kalman observability criterion

the system (1.4) is said to be observable if and only if

$$\text{rank}(\mathcal{O}(A, C)) \triangleq \text{rank} \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} = n \quad (1.5)$$

$\mathcal{O}(A, C)$ is called Kalman observability matrix (of size $np \times n$).

1.2.3 Observer principle

A state observer is a typically computer-implemented system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system. Observers are needed since the full state cannot be measured or is too expensive to measure in many applications. In addition, some variables in many applications have to be estimated and cannot be measured due to unavailability of sensors at any cost.

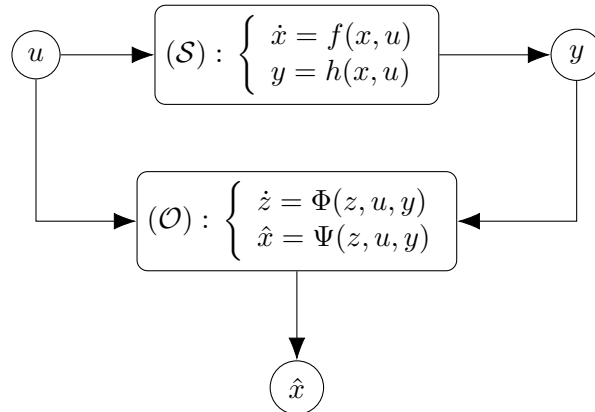


Figure 1.1: State observer principle

Definition 1.6: State Observer

Consider the following dynamical system:

$$(\mathcal{S}) : \begin{cases} \dot{x} = f(x, u), \\ y = h(x, u) \end{cases} \quad (1.6)$$

The dynamical system

$$(\mathcal{O}) : \begin{cases} \dot{z} = \Phi(z, u, y), \\ \hat{x} = \Psi(z, u, y) \end{cases} \quad (1.7)$$

is a local asymptotic observer for system (1.6) if :

- $x(0) = \hat{x}(0) \Rightarrow x(t) = \hat{x}(t) \quad \forall t \geq 0;$
 - $\exists \Omega \subseteq \mathbb{R}^n : x(0) - \hat{x}(0) \in \Omega \Rightarrow \lim_{t \rightarrow +\infty} \|x(t) - \hat{x}(t)\| \rightarrow 0.$
1. $\|x(t) - \hat{x}(t)\| \rightarrow 0$ exponentially \implies Exponential observer.
 2. $\Omega = \mathbb{R}^n \implies$ Global observer.

1.3 Some Backgrounds on High-gain Type Observers

Let us consider the class of nonlinear systems which are diffeomorphic to the form of the system studied in [23]:

$$\begin{cases} \dot{x} = Ax + Bf(x), \\ y = Cx, \end{cases} \quad (1.8)$$

where for $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector and $y(t) \in \mathbb{R}$ is the measured output. The matrices $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$, $B \in \mathbb{R}^{n \times 1}$ and the nonlinear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. We consider the standard Luenberger observer structure:

$$\dot{\hat{x}} = A\hat{x} + Bf(\hat{x}) + L(y - C\hat{x}), \quad (1.9)$$

The estimation error $e = x - \hat{x}$ is given by

$$\dot{e} = (A - LC)e + B(f(x) - f(\hat{x})). \quad (1.10)$$

1.3.1 High-Gain Methodology

Here, we recall the basic high gain observer as in [24]. The advantage of the high gain methodology is that it always guarantee the existence of an exponentially convergent observer, thanks to the tuning of only one parameter that should be chosen large enough. However, the use of a large gain remains a major drawback. Indeed, a high gain observer is very sensitive to output measurement noise because of the value of the tuning parameter which may be huge for higher dimensional systems having nonlinearities with large Lipschitz constants.

Basically, in the high-gain methodology, we write the observer gain L under the form:

$$L := T(\theta)K, \quad \theta \geq 1 \quad (1.11)$$

where

$$T(\theta) := \text{diag}(\theta, \dots, \theta^n) \text{ and } K \in \mathbb{R}^{n \times p}.$$

In addition, the high-gain methodology focuses on the transformed estimation error

$$\hat{e} := T^{-1}(\theta)e \quad (1.12)$$

where $T^{-1}(\theta)$ is the inverse of $T(\theta)$ given by

$$T^{-1}(\theta) = \text{diag}\left(\frac{1}{\theta}, \dots, \frac{1}{\theta^n}\right).$$

Hence, the dynamics of the error \hat{e} is given by

$$\dot{\hat{e}} = \theta(A - KC)\hat{e} + \frac{1}{\theta^n}B\Delta f \quad (1.13)$$

with

$$\Delta f := f(x) - f(x - T(\theta)\hat{e})$$

From the Lipschitz condition (1.2) and the fact that $\theta \geq 1$, we can show as in [25] that there exists a positive scalar constant k_f , independent of θ , so that

$$\|T^{-1}(\theta)B\Delta f\| \leq k_f\|\hat{e}\|. \quad (1.14)$$

Consequently, following the high-gain methodology we have the following theorem:

Theorem 1.3: [24]

If there exist $P > 0$, $\lambda > 0$, Y , and $\theta \geq 1$ such that

$$A^T P + PA - C^T Y - Y^T C + \lambda I < 0 \quad (1.15)$$

$$\theta > \theta_0 = \frac{2k_f \lambda_{\max}(P)}{\lambda} \quad (1.16)$$

then the estimation error e is asymptotically stable with

$$K = P^{-1}Y^T.$$

Proof: For more details about the proof of this theorem, we refer the reader to [24], [25], [26].
□

1.3.2 Enhanced High-Gain Observer

We introduce again the dynamical equations of the system (1.8):

$$\begin{cases} \dot{x} &= Ax + f(x), \\ y &= Cx, \end{cases} \quad (1.17)$$

where the matrices A, C , and the function f have the same structures given by are defined as follows:

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad C \triangleq [1 \ 0 \ \cdots \ 0],$$

$$f(x) \triangleq \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_{n-1}(x_1, x_2, \dots, x_{n-1}) \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix}.$$

It should be noticed that as demonstrated in [9], all uniformly observable systems can be transformed into system (1.8). Several real-world models are or can be transformed into the triangular form [9], [27]. The following references give more details about this family of systems and its practical importance [23] and [28].

in Section 1.3. The enhanced observer structure is proposed in [6]. Such an observer has a more general structure as compared with the standard high-gain observer, which can be regarded as a particular case of enhanced high-gain observer because of a special choice of the design parameters. The more general structure allows for additional degrees of freedom in the selection of the observer parameters. Consider the following state observer:

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + G(\gamma, K)(y - C\hat{x}), \quad (1.18)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$, for $t \in \mathbb{R}$, and

$$G(\gamma, K) \triangleq [\gamma_1 k_1 \quad \gamma_2 k_2 \quad \cdots \quad \gamma_n k_n]^T \triangleq T(\gamma)K,$$

with

$$\begin{aligned} \gamma &\triangleq [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n]^T \in \mathbb{R}_{>0}^n; \\ K &\triangleq [k_1 \quad k_2 \quad \cdots \quad k_n]^T \in \mathbb{R}^n; \\ T(\gamma) &= \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n). \end{aligned}$$

As usually in the high-gain methodology, we consider the transformed error

$$\tilde{x} \triangleq T^{-1}(\gamma)e, \quad (1.19)$$

where $\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$ is the estimation error vector. After developing the computations, it follows that

$$\dot{\tilde{x}} = \gamma_1(A - KC + \Omega(\gamma))\tilde{x} + T^{-1}(\gamma)(f(x) - f(x - T(\gamma)\tilde{x})), \quad (1.20)$$

where

$$\Omega(\gamma) \triangleq \begin{bmatrix} 0 & z_1 & 0 & \cdots & 0 \\ 0 & 0 & z_2 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & z_{n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

with

$$z_i \triangleq \frac{\gamma_{i+1}}{\gamma_1 \gamma_i} - 1, \quad i = 1, 2, \dots, n-1. \quad (1.21)$$

Remark 1.1

The observer structure (1.18) is considered in the goal to get multiple tuning parameters contrarily to the standard observer where only one tuning parameter is considered in the design procedure. The use of such a decomposition leads to the involvement of a new matrix of decision variables in the design, namely the matrix Z , which can be tuned to reduce the high-gain values, as well explained in [6, Section III] by considering a lot of numerical aspects.

1.4 LPV Approach

We consider a class of nonlinear systems without linear state part. Indeed, we will see later that the linear part is not necessary for the design methods that we develop. For simplicity of presentation, we consider, without loss of generality, that the nonlinear function depends only on system state, and the output is linear. The extension to nonlinear output is straightforward. The class of systems we treat in this section is described by the following equations:

$$\begin{cases} \dot{x} = \Psi(x) \\ y = Cx \end{cases} \quad (1.22)$$

where the nonlinear function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be γ_Ψ -Lipschitz, i.e.:

$$\|\Psi(x) - \Psi(y)\| \leq \gamma_\Psi \|x - y\|, \quad \forall x, y \in \mathbb{R}^n \quad (1.23)$$

1.4.1 Preliminaries and Definitions

This part is devoted to a new reformulated Lipschitz property. This reformulation is necessary for our developed LPV approach. A useful definition and two lemmas are presented.

Definition 1.7

Consider two vectors

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n.$$

For all $i = 0, \dots, n$, we define an auxiliary vector $X^{Y_i} \in \mathbb{R}^n$ corresponding to X and Y as follows:

$$\begin{cases} X^{Y_i} = \begin{pmatrix} y_1 \\ \vdots \\ y_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix} & \text{for } i = 1, \dots, n \\ X^{Y_0} = X \end{cases} \quad (1.24)$$

Lemma 1.1: [10]

Consider a function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then, for all

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad \text{and} \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

there exist functions $\psi_j : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, \dots, n$ so that

$$\Psi(X) - \Psi(Y) = \sum_{j=1}^{j=n} \psi_j(X^{Y_{j-1}}, X^{Y_j}) e_n^T(j) (X - Y) \quad (1.25)$$

Proof: The proof consists of rewriting $\Psi(X) - \Psi(Y)$ as

$$\Psi(X) - \Psi(Y) = \sum_{j=1}^{j=n} [\Psi(X^{Y_{j-1}}) - \Psi(X^{Y_j})]$$

Now, defining the functions ψ_j by

$$\psi_j(X^{Y_{j-1}}, X^{Y_j}) = \begin{cases} 0 & \text{if } x_j = y_j \\ \frac{\Psi(X^{Y_{j-1}}) - \Psi(X^{Y_j})}{x_j - y_j} & \text{if } x_j \neq y_j \end{cases} \quad (1.26)$$

we can write

$$\begin{aligned} \Psi(X) - \Psi(Y) &= \sum_{j=1}^{j=n} [\psi_j(X^{Y_{j-1}}, X^{Y_j})] (x_j - y_j) \\ &= \sum_{j=1}^{j=n} [\psi_j(X^{Y_{j-1}}, X^{Y_j}) e_n^T(j)] (X - Y) \end{aligned} \quad (1.27)$$

□

Lemma 1.2: [10]

Considering the function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the two following items are equivalent:

- *Lipschitz property* : Ψ is γ_Ψ -Lipschitz with respect to its argument, i.e.:

$$\|\Psi(X) - \Psi(Y)\| \leq \gamma_\Psi \|X - Y\|, \quad \forall X, Y \in \mathbb{R}^n \quad (1.28)$$

- *Reformulated Lipschitz property* : for all $i, j = 1, \dots, n$, there exist functions

$$\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_{ij}}$ and $\bar{\gamma}_{\psi_{ij}}$, so that $\forall X, Y \in \mathbb{R}^n$,

$$\Psi(X) - \Psi(Y) = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \psi_{ij} H_{ij} (X - Y) \quad (1.29)$$

and

$$\underline{\gamma}_{\psi_{ij}} \leq \psi_{ij} \leq \bar{\gamma}_{\psi_{ij}} \quad (1.30)$$

where

$$\psi_{ij} \triangleq \psi_{ij}(X^{Y_{j-1}}, X^{Y_j}) \quad \text{and} \quad H_{ij} = e_n(i) e_n^T(j)$$

Proof:

1. *Sufficiency:* We start by proving sufficiency. Assume that for all $i, j = 1, \dots, n$, there exist functions

$$\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_{ij}}$ and $\bar{\gamma}_{\psi_{ij}}$, so that (1.29) and (1.30) hold for all $X, Y \in \mathbb{R}^n$. Then, we have

$$\begin{aligned} \|\Psi(X) - \Psi(Y)\| &\leq \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} |\psi_{ij}| \|X - Y\| \\ &\leq \left(\sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \lambda_{ij} \right) \|X - Y\| \end{aligned} \quad (1.31)$$

where $\lambda_{ij} = \max(|\underline{\gamma}_{\psi_{ij}}|, |\bar{\gamma}_{\psi_{ij}}|)$. Hence, the function Ψ is γ_{Ψ} -Lipschitz with

$$\gamma_{\Psi} \leq \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \max(|\underline{\gamma}_{\psi_{ij}}|, |\bar{\gamma}_{\psi_{ij}}|).$$

2. *Necessity:* We know that for all X , we can write

$$\Psi(X) = \begin{pmatrix} \Psi_1(X) \\ \vdots \\ \Psi_n(X) \end{pmatrix} = \sum_{i=1}^{i=n} e_n(i) \Psi_i(X)$$

Consequently, if Ψ is γ_{Ψ} -Lipschitz, then we deduce that there are constants $0 < \gamma_{\Psi_i} \leq \gamma_{\Psi}$ for all $i = 1, \dots, n$ so that each component Ψ_i is γ_{Ψ_i} -Lipschitz. Indeed, we have

$$\begin{aligned} \|\Psi(X) - \Psi(Y)\|^2 &= \sum_{i=1}^{i=n} |\Psi_i(X) - \Psi_i(Y)|^2 \\ &\leq \gamma_{\Psi}^2 \|X - Y\|^2. \end{aligned} \quad (1.32)$$

Inequality (1.32) leads to

$$|\Psi_i(X) - \Psi_i(Y)| \leq \gamma_{\Psi} \|X - Y\|$$

which means that Ψ_i is γ_{Ψ_i} -Lipschitz with $\gamma_{\Psi_i} \leq \gamma_{\Psi}$. From lemma 1, there are functions $\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $j = 1, \dots, n$ so that

$$\Psi_i(X) - \Psi_i(Y) = \sum_{j=1}^{j=n} \psi_{ij}(X^{Y_{j-1}}, X^{Y_j}) e_n^T(j) (X - Y) \quad (1.33)$$

where ψ_{ij} is given as in equation (1.26) of Lemma 1 by replacing Ψ by Ψ_i . Since Ψ_i is γ_{Ψ_i} -Lipschitz, then we have

$$\begin{aligned} |\Psi_i(X^{Y_{j-1}}) - \Psi_i(X^{Y_j})| &\leq \gamma_{\Psi_i} \|X^{Y_{j-1}} - X^{Y_j}\| \\ &= \gamma_{\Psi_i} |x_j - y_j| \end{aligned}$$

which means that

$$-\gamma_{\Psi_i} \leq \psi_{ij} \leq \gamma_{\Psi_i}.$$

This ends the proof. \square

In fact, Lemma (2) provides a best less conservative Lipschitz condition. Indeed, the reformulation (1.29)-(1.30) allows to treat the nonlinearity with a best precision and exploits all the interesting properties of the nonlinearity of the investigated system. For instance, the Lipschitz condition (1.2) does not distinguish between the nonlinearities $\Psi(x) = \sin(x_2)$ and $\Psi(x) = \tanh(x_2)$. However, with the reformulation (1.30), we can see the difference. Indeed, for $\Psi(x) = \sin(x_2)$ we have $\underline{\gamma}_{\psi_{11}} = -1$ and for $\Psi(x) = \tanh(x_2)$, we have $\underline{\gamma}_{\psi_{11}} = 0$.

1.4.2 LPV-based approach

Consider the following Luenberger observer:

$$\dot{\hat{x}} = \Psi(\hat{x}) + L(y - C\hat{x}) \quad (1.34)$$

The dynamic of the estimation error $e = x - \hat{x}$ is then given by:

$$\dot{e} = [\Psi(x) - \Psi(\hat{x})] - LCe \quad (1.35)$$

Since $\Psi(\cdot)$ is γ_{Ψ} -Lipschitz, then following Lemma 2 there are functions

$$\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_{ij}}$ and $\bar{\gamma}_{\psi_{ij}}$, so that

$$\Psi(x) - \Psi(\hat{x}) = \left[\sum_{i=1}^n \sum_{j=1}^n \psi_{ij} H_{ij} \right] e \quad (1.36)$$

and

$$\underline{\gamma}_{\psi_{ij}} \leq \psi_{ij} \leq \bar{\gamma}_{\psi_{ij}} \quad (1.37)$$

where

$$\psi_{ij} \triangleq \psi_{ij}(x_k^{\hat{x}_k-1}, x^{\hat{x}_j})$$

is defined as in (1.26) by replacing Ψ by Ψ_i (the i^{th} component of Ψ). In the sequel, for simplicity of presentation, we use only ψ_{ij} instead of $\psi_{ij}(x_k^{\hat{x}_k-1}, x^{\hat{x}_j})$.

Now, define the matrices

$$\Theta = \left(\psi_{ij} \right)_{ij} \quad (1.38)$$

and

$$\mathcal{A}(\Theta) = \sum_{i=1}^n \sum_{j=1}^n \psi_{ij} H_{ij} \quad (1.39)$$

Consequently, the dynamics (1.35) can be rewritten as

$$\dot{e} = [\mathcal{A}(\Theta) - LC]e \quad (1.40)$$

According to (1.37), the matrix parameter Θ belongs to a bounded convex set \mathcal{H}_n for which the set of vertices is defined by:

$$\mathcal{V}_{\mathcal{H}_n} = \left\{ \Phi \in \mathbb{R}^{n \times n} : \Phi_{ij} \in \left\{ \underline{\gamma}_{\psi_{ij}}, \bar{\gamma}_{\psi_{ij}} \right\} \right\}. \quad (1.41)$$

The following theorem, which provides LMI conditions for observer design of Lipschitz systems.

Theorem 1.4: [10]

The observer (1.34) is asymptotically convergent if there exist a positive definite matrix \mathcal{P} , a matrix \mathcal{R} of appropriate dimension so that the following LMI conditions hold:

$$\mathcal{A}(\Phi)^T \mathcal{P} + \mathcal{P} \mathcal{A}(\Phi) - C^T \mathcal{R} - \mathcal{R}^T C < 0, \quad \forall \Phi \in \mathcal{V}_{\mathcal{H}_n} \quad (1.42)$$

Hence, the observer gain is given by

$$L = \mathcal{P}^{-1} \mathcal{R}^T.$$

Remark 1.2

It is clear that from the computational complexity point of view, the LPV method is the less interesting, view the cost of more demanding LMIs to be solved. In fact, for the proposed LPV method, we have to solve 2^{n^2} LMIs for n dimensional nonlinear vector. But, in spite of that, the LPV method provides less restrictive LMI synthesis conditions. On the other hand, in spite of the cost of more demanding LMIs, this does not play an important role on the feasibility of the proposed LMIs. In general, the computational complexity plays a role in online methods for real time applications.

High-Gain Like Observer Design For a Class of Nonlinear Systems

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2.1 Introduction and Preliminaries

2.1.1 Introduction

In [8] a new structure of observers, called HG/LMI observer, has been developed by combining the standard high-gain methodology with the LPV/LMI technique [10]. This new observer has the advantage to provide lower tuning parameter compared to the previous high-gain observers, without using saturation functions or filtering.

Herein, we develop a new state observer design for systems with multi-nonlinearities in triangular form or any system that can be transformed into a triangular structure. The proposed observer has the advantage of allowing more possibilities of choosing the design parameters. The idea consists in combining the enhanced high-gain methodology [6] with the LPV/LMI methodology in order to reduce more the value of the observer gains. This structure has the advantage to use multiple tuning parameters. Indeed, the observer in [8] becomes a particular case of the

proposed observer in this chapter by a special choice of the design parameters. It is shown through a simple example that the proposed technique reduces the peaking phenomenon and decreases the sensitivity to high-frequency measurement noise as compared with the high-gain observer.

Let us consider the class of nonlinear systems (1.8). First, we introduce the following notations:

- $\mathbb{R}_{\geq 0}^n = \{(l_i)_{1 \leq i \leq n} \in \mathbb{R}^n; l_i \geq 0, \forall i = 1, \dots, n\}$,
- $\mathbb{R}_{> 0}^n = \{(l_i)_{1 \leq i \leq n} \in \mathbb{R}^n; l_i > 0, \forall i = 1, \dots, n\}$,
- $\mathbb{R}_{> 0, \uparrow}^n = \{(l_i)_{1 \leq i \leq n} \in \mathbb{R}_{> 0}^n; l_{k+1} > l_k, \forall k = 1, \dots, n-1\}$.

As usually done for this class of systems (1.8), we introduce the following Lipschitz assumption.

Assumption 2.1

The function f satisfies the global Lipschitz condition, i.e., there exists a vector $L_f = (L_i^f)_{1 \leq i \leq n} \in \mathbb{R}_{\geq 0}^n$ such that

$$|f_i(\bar{x}_1 + w_1, \bar{x}_2 + w_2, \dots, \bar{x}_i + w_i) - f_i(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_i)| \leq \sum_{j=1}^i L_i^f |w_j|,$$

for all $\bar{x} = (\bar{x}_i)_{1 \leq i \leq n}$, $w = (w_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ and $i = 1, \dots, n$.

Before formulating the problem, we introduce this lemma, which is necessary for the mathematical developments given in the next sections.

Lemma 2.1: [6]

Let X and Y be two matrices of adequate dimensions. Then the following inequality holds for any symmetric and nonsingular matrix S of appropriate dimension:

$$X^\top Y + Y^\top X \leq \frac{1}{2}(X + SY)^\top S^{-1}(X + SY).$$

2.1.2 Motivating example

The fact that k_f in inequality (1.14) is independent of θ is not necessarily an advantage. Indeed, this depends on how θ would be involved in k_f . Also, the fact that k_f is independent of θ does not come only from the condition $\theta \geq 1$, but essentially from the presence of the last component of x in f . Because of this last component, the parameter θ vanishes from the term $\frac{1}{\theta^n} \Delta f$ for $\theta \geq 1$. This can be shown easily by using the Lipschitz property (1). To illustrate this point and to motivate our study, let us consider a simple three dimensional system. If we take a nonlinear function

$$f(x) = \gamma_f \sin(x_3),$$

then we get from (1)

$$\frac{1}{\theta^3} \|\Delta f\| \leq \frac{\gamma_f}{\theta^3} \times |\theta^3 \hat{x}_3| = \gamma_f |\hat{x}_3| \leq k_f \|\hat{x}\|,$$

where $k_f = \gamma_f$ in this case. However, if we take

$$f(x) = \gamma_f \sin(x_2),$$

then we get

$$\frac{1}{\theta^3} \|\Delta f\| \leq \frac{\gamma_f}{\theta^3} \times |\theta^2 \hat{x}_2| = \frac{\gamma_f}{\theta} |\hat{x}_2| \leq \frac{k_f}{\theta} \|\hat{x}\|.$$

Hence, by replacing in (1.16) k_f by $\frac{k_f}{\theta}$, θ_0 will be reduced to $\sqrt{\theta_0}$, which will reduce significantly the values of the observer gain.

The main result is based on the above idea. Thanks to the LPV/LMI technique combined with the standard high-gain methodology, we will be able to obtain a high-gain observer with a lower gain called HG/LMI observer.

2.1.3 HG/LMI observer

The authors in [8] developed a new high-gain observer design method for nonlinear systems that has a lower gain compared to the standard high-gain observer. This new observer, called HG/LMI observer is obtained by combining the standard high gain methodology with the LMI-based observer design technique. We consider the standard high-gain observer structure:

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(y - C\hat{x}), \quad (2.1)$$

where L is defined as in (1.11).

The dynamics of the transformed error $\hat{\hat{x}}$, defined in (1.12), is then given by:

$$\dot{\hat{\hat{x}}} = \theta(A - KC)\hat{\hat{x}} + T^{-1}(\theta)\Delta f \quad (2.2)$$

with

$$\Delta f := f(x) - f(x - T(\theta)\hat{\hat{x}}).$$

Each nonlinear component f_i can be written under the form

$$\Delta f_i = \sum_{j=1}^{i-j_i} \theta^j \psi_{ij} \hat{x}_j + \sum_{j=1}^{j_i} \theta^{k_i(j)} \psi_{ik_i(j)} \hat{x}_{k_i(j)}, \quad (2.3)$$

where

$$\begin{aligned} k_i(j) &= i - (j_i - j), \\ 0 &\leq j_i \leq i. \end{aligned}$$

It follows that Δf is written as

$$\Delta f = \underbrace{\sum_{i=1}^n \sum_{j=1}^{i-j_i} \theta^j \psi_{ij} e_n(i) \hat{x}_j}_{\Delta f_1} + \underbrace{\sum_{i=1}^n \sum_{j=1}^{j_i} \theta^{k_i(j)} \psi_{ik_i(j)} e_n(i) \hat{x}_{k_i(j)}}_{\text{for LPV/LMI}}. \quad (2.4)$$

$$\dot{\hat{\hat{x}}} = \theta(\mathcal{A}(\Psi^\theta) - KC)\hat{\hat{x}} + T^{-1}(\theta)\Delta f_1, \quad (2.5)$$

where

$$\mathcal{A}(\Psi^\theta) = A + B \sum_{i=1}^n \sum_{j=1}^{j_i} \psi_{ij}^\theta e_n(i) e_n^\top(k_i(j)), \quad (2.6)$$

$$\Psi^\theta = \begin{pmatrix} \psi_{11}^\theta \\ \vdots \\ \psi_{1j_1}^\theta \\ \psi_{21}^\theta \\ \vdots \\ \psi_{2j_2}^\theta \\ \vdots \\ \psi_{nj_n}^\theta \end{pmatrix} \in \mathbb{R}^{\sum_{i=1}^n j_i}, \quad (2.7)$$

$$\psi_{ij}^\theta = \frac{\psi_{ik_i(j)}}{\theta^{1+(j_i-j)}}. \quad (2.8)$$

We define the bounded convex set

$$\mathcal{H}_{j_{\min}}^\sigma = \left\{ \Phi \in \mathbb{R}^{\sum_{i=1}^n j_i} : \frac{\underline{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \leq \Phi_{ij} \leq \frac{\bar{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \right\} \quad (2.9)$$

for which the set of vertices is defined by

$$\mathcal{V}_{\mathcal{H}_{j_{\min}}^\sigma} = \left\{ \Phi \in \mathbb{R}^{\sum_{i=1}^n j_i} : \Phi_{ij} \in \left\{ \frac{\underline{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}}, \frac{\bar{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \right\} \right\}, \quad (2.10)$$

where $\underline{\gamma}_{\gamma_{ik_i(j)}} \leq 0$ and $\bar{\gamma}_{\gamma_{ik_i(j)}} \geq 0$ are respectively, the lower and upper bounds of the bounded parameter $\psi_{ik_i(j)}$.

On the other hand, it is easy to show that there exists a constant $k_{j_{\min}}$ independent from θ such that the following holds:

$$\|\mathbb{T}^{-1}(\theta)\Delta f_1\| \leq \frac{k_{j_{\min}}}{\theta^{\min(j_i)}} \|\hat{x}\|. \quad (2.11)$$

Hence, by analogy, we get the next general theorem valid for systems with multi-nonlinearities.

Theorem 2.1: [8]

If there exist $P > 0$, $\lambda > 0$, Y , and $\sigma > 0$ such that

$$\mathcal{A}(\Psi^\sigma)^T P + P \mathcal{A}(\Psi^\sigma) - C^T Y Y^T C + \lambda I < 0, \forall \Psi^\sigma \in \mathcal{V}_{\mathcal{H}_{j_{\min}}^\sigma}, \quad (2.12)$$

$$\theta^{\frac{1+\min(j_i)}{j_i \neq i}} > \frac{2k_{j_{\min}} \lambda_{\max}(P)}{\lambda}, \quad (2.13)$$

then the estimation error \tilde{x} is asymptotically stable with

$$L = T(\theta) \overbrace{P^{-1}Y^T}^K, \quad (2.14)$$

$$\theta \geq \max \left(\sigma, \left[\frac{2k_{j_{\min}} \lambda_{\max}(P)}{\lambda} \right]^{\frac{1}{1 + \min_{j_i \neq i} (j_i)}} \right). \quad (2.15)$$

On the advantages of HG/LMI observer

Herein we give some clarifications on the advantages of the proposed HG/LMI observer. We will clarify more the role of the so-called "*compromise index*" j_{\min} .

The advantages of the proposed HG/LMI technique can be summarized in the following items:

- The proposed technique can be viewed as a way to reduce the number of LMIs related to the LPV/LMI technique. Indeed, the number of LMI of LPV/LMI method is reduced from 2^n to $2^{j_{\min}}$, which represents the number of vertices in the set $\mathcal{V}_{\mathcal{H}_{j_{\min}}^\sigma}$;
- The observer gain L of the HG/LMI technique is significantly smaller than that returned by the standard high-gain observer because of the power $\frac{1}{1+j_{\min}}$ due to the compromise index j_{\min} ;
- The standard high-gain observer is a particular solution corresponding to $j_{\min} = 0$;
- The LPV/LMI observer is a particular solution corresponding to $j_{\min} = n$;
- Solutions with LPV/LMI method are always guaranteed. Indeed, there always exists $\sigma > 0$ so that the LMIs (2.12) admit solutions because

$$\lim_{\sigma \rightarrow +\infty} (\mathcal{H}_{j_{\min}}^\sigma) = \{0_{\mathbb{R}^{j_{\min}}}\}, \quad (2.16)$$

we have

$$\lim_{\sigma \rightarrow +\infty} (\mathcal{A}(\Psi^\sigma)) = \mathcal{A} \left(\lim_{\sigma \rightarrow +\infty} (\Psi^\sigma) \right) = A. \quad (2.17)$$

This proves that the LPV/LMI technique for this class of systems with the high-gain structure can always provide solutions for the set of sufficient LMIs (2.12) for σ large enough. Indeed, from (2.17) and the continuity of $\mathcal{A}(\Psi^\sigma)$ with respect to σ , if (1.15) holds, we have

$$\begin{aligned} & \lim_{\sigma \rightarrow +\infty} \left(\mathcal{A}(\Psi^\sigma)^T P + P \mathcal{A}(\Psi^\sigma) - C^T Y - Y^T C + \lambda I \right) \\ &= \mathcal{A} \left(\lim_{\sigma \rightarrow +\infty} (\Psi^\sigma) \right)^T P + P \mathcal{A} \left(\lim_{\sigma \rightarrow +\infty} (\Psi^\sigma) \right) \\ & \quad - C^T Y - Y^T C + \lambda I \\ &= A^T P + P A - C^T Y - Y^T C + \lambda I < 0. \end{aligned} \quad (2.18)$$

Then, by Definition of the limit, there exists σ_0 such that (2.12) is satisfied for any $\sigma > \sigma_0$. In addition, using the definition of $\mathcal{A}(\Psi^\sigma)$, we can show that the standard high-gain observer is a particular solution of (2.12). Indeed, from the construction of $\mathcal{A}(\Psi^\sigma)$, we can write

$$\mathcal{A}(\Psi^\sigma) = A + A_\sigma \quad \text{with} \quad \|A_\sigma\| \leq \frac{k_f}{\sigma}.$$

Then, the left term of inequality (2.12) satisfies the following inequality:

$$\begin{aligned} & \left(A^T P + PA - C^T Y - Y^T C + \lambda I \right) + A_\sigma^T P + P A_\sigma \\ & \leq \left(A^T P + PA - C^T Y - Y^T C + \lambda I \right) + \frac{2k_f \lambda_{\max}(P)}{\sigma} I. \end{aligned} \quad (2.19)$$

It follows that if (1.15) is satisfied (which is always feasible due to the observability canonical form), then there exists $\epsilon > 0$ so that

$$A^T P + PA - C^T Y - Y^T C + \lambda I \leq -\epsilon I.$$

Then, (2.19) holds for any $\sigma > \sigma_0 = \frac{2k_f \lambda_{\max}(P)}{\epsilon}$.

Table 2.1 sums up the number of LMIs, n_{LMI} , and the value of θ for each method: standard high-gain (Standard HG); LPV/LMI technique, and the combined HG/LMI method.

Methods	Standard HG	LPV/LMI	HG/LMI
θ	θ_0	σ	$\theta \frac{1}{j_{\min}^{1+j_{\min}}}$
n_{LMI}	1	2^n	$2^{j_{\min}}$

Table 2.1: Illustration of the HG/LMI method

2.2 A More General HG/LMI Observer

To make the section easier to read, we introduce again the dynamical equations of the system (1.17):

$$\begin{cases} \dot{x} &= Ax + f(x), \\ y &= Cx, \end{cases} \quad (2.20)$$

where the matrices A, C , and the function f have the same structures given in Section 1.3.2. This section is devoted to the main contribution of this chapter. Instead of standard high-gain observer structure, we use the enhanced structure previously proposed in [6]. Consider the following state observer [6]:

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + G(\gamma, K)(y - C\hat{x}), \quad (2.21)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$, for $t \in \mathbb{R}$, and

$$G(\gamma, K) \triangleq \begin{bmatrix} \gamma_1 k_1 & \gamma_2 k_2 & \cdots & \gamma_n k_n \end{bmatrix}^T \triangleq T(\gamma)K,$$

with

$$\begin{aligned} \gamma &\triangleq \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix}^T \in \mathbb{R}_{>0}^n; \\ K &\triangleq \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}^T \in \mathbb{R}^n; \\ T(\gamma) &= \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n). \end{aligned}$$

As usually in the high-gain methodology, we consider the transformed error

$$\tilde{x} \triangleq T^{-1}(\gamma)e, \quad (2.22)$$

where $\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$ is the estimation error vector. After developing the computations, it follows that

$$\dot{\tilde{x}} = \gamma_1 \left(A - KC + \Omega(\gamma) \right) \tilde{x} + T^{-1}(\gamma) \left(f(x) - f(x - T(\gamma)\tilde{x}) \right), \quad (2.23)$$

where

$$\Omega(\gamma) \triangleq \begin{bmatrix} 0 & z_1 & 0 & \cdots & 0 \\ 0 & 0 & z_2 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & z_{n-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

with

$$z_i \triangleq \frac{\gamma_{i+1}}{\gamma_1 \gamma_i} - 1, \quad i = 1, 2, \dots, n-1. \quad (2.24)$$

The aim consists in synthesizing the observer parameters γ_i and k_i such that the error \tilde{x} converges exponentially to zero. Usually, this problem is solved by using the standard high-gain observer methodology [23]. However, in some situations (for instance larger Lipschitz constants, high dimension of the systems), it leads to extremely high values of the gains, which render the observer very sensitive to high frequency measurement noise and causes the picking phenomenon in the transient. To overcome this obstacle, various improvements have been established in the literature, proposing high-gain observers with constant or time-varying gains. Limiting our study, in this chapter, to observers with constant gains, some recent methods proposed considerable solutions [6], [14], [8], nevertheless, the problem remains still open for further improvements. In this chapter, we will combine between [6] and [8] to propose a new approach. To tackle this problem, a convenient decomposition of the nonlinearity and introduction of additional parameters are required. This is the goal of the next section.

2.2.1 Preliminary transformations

The idea is to exploit the results of [6] and [8] to improve the solution of the observer gains. Borrowed from [8, Eq. (54)], the first step consists in decomposing the nonlinearity of the system into two parts. After some rearrangements, there exist functions ψ_{ij} , scalars $\underline{\nu}_{\psi_{ik_i(j)}} \leq 0$ and $\bar{\nu}_{\psi_{ik_i(j)}} \geq 0$ such that

$$f(x) - f(x - T(\gamma)\tilde{x}) = \Delta f_1 + \Delta f_2,$$

with

$$\begin{aligned} \Delta f_1 &\triangleq \sum_{i=1}^n \sum_{j=1}^{i-j_i} \gamma_j \psi_{ij} e_n(i) \tilde{x}_j, \\ \Delta f_2 &\triangleq \sum_{i=1}^n \sum_{j=1}^{j_i} \gamma_{k_i(j)} \psi_{ik_i(j)} e_n(i) \tilde{x}_{k_i(j)}, \\ k_i(j) &\triangleq i - (j_i - j), \quad 0 \leq j_i \leq i, \end{aligned}$$

and

$$\underline{\nu}_{\psi_{ik_i(j)}} \leq \psi_{ij} \leq \bar{\nu}_{\psi_{ik_i(j)}}.$$

By analogy to [8], the first term Δf_1 will be handled by the EHGO-approach in [6], while the second one, Δf_2 , will be associated to the linear part and will be processed by the LPV/LMI method [10] as in [8]. Notice that the term $T^{-1}(\gamma)\Delta f_2$ can be rewritten as:

$$T^{-1}(\gamma)\Delta f_2 = \sum_{i=1}^n \sum_{j=1}^{j_i} \frac{\gamma_{k_i(j)}}{\gamma_i} \psi_{ik_i(j)} e_n(i) \tilde{x}_{k_i(j)}.$$

Now, we will introduce some notations needed to rewrite system (2.23) under a suitable structure to apply the ideas of [10] and [8]. Let us introduce the following matrix function:

$$\mathcal{A}(\Psi^\gamma) \triangleq A + \sum_{i=1}^n \sum_{j=1}^{j_i} \psi_{ij}^\gamma e_n(i) e_n^\top(k_i(j)), \quad (2.25)$$

where

$$\Psi^\gamma \triangleq \left(\psi_{11}^\gamma \quad \dots \quad \psi_{ij_i}^\gamma \quad \dots \quad \psi_{nj_n}^\gamma \right)^T \in \mathbb{R}^d, \quad (2.26)$$

and

$$\psi_{ij}^\gamma \triangleq \frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \psi_{ik_i(j)}, \quad d \triangleq \sum_{i=1}^n j_i.$$

Consequently, system (2.23) can be expressed as follows:

$$\dot{\tilde{x}} = \gamma_1 \left(\mathcal{A}(\Psi^\gamma) - KC + \Omega(\gamma) \right) \tilde{x} + T^{-1}(\gamma)\Delta f_1. \quad (2.27)$$

Before stating the observer design conditions ensuring the exponential convergence of the proposed state observer, we start by introducing some preliminary results, which are necessary for the proposed design procedure. We first define the convex set for any fixed $\gamma \in \mathbb{R}_{>0,\uparrow}^n$:

$$\tilde{\mathcal{H}}^\gamma \triangleq \left\{ \Phi \in \mathbb{R}^d : \frac{\gamma_{k_i(j)} \underline{\psi}_{ik_i(j)}}{\gamma_1 \gamma_i} \leq \Phi_{ij} \leq \frac{\gamma_{k_i(j)} \bar{\psi}_{ik_i(j)}}{\gamma_1 \gamma_i} \right\}. \quad (2.28)$$

It is obvious that $\tilde{\mathcal{H}}^\gamma$ is a bounded convex. Indeed, from the fact that the function f is Lipschitz, the scalars $\underline{\psi}_{ik_i(j)}$ and $\bar{\psi}_{ik_i(j)}$ are bounded. On the other hand, since $\gamma \in \mathbb{R}_{>0,\uparrow}^n$, we have $\frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \leq \frac{1}{\gamma_1}$. It follows that

$$\frac{\underline{\psi}_{ik_i(j)}}{\gamma_1} \leq \Phi_{ij} \leq \frac{\bar{\psi}_{ik_i(j)}}{\gamma_1},$$

which means that Φ_{ij} is bounded since $\gamma_1 > 0$.

At this stage, the bounded convex set $\tilde{\mathcal{H}}^\gamma$ is not exploitable in an LMI framework because the set of vertices depends on all the parameters $\gamma_i, i = 1, \dots, n$. In other word, it depends on the decision variables $z_i, i = 1, \dots, n-1$. To overcome this obstacle, we need to define a new bounded and convex hyper-rectangle independent from all these observer parameters. Before introducing such a set, we first state the following lemma.

Lemma 2.2: [20]

Let $\gamma \in \mathbb{R}_{>0,\uparrow}^n$ and z_i given by (2.24). If z_i satisfies $z_i \leq 0$, then there exists $\alpha \in]0, 1]$ such

that inequality (2.29) below holds:

$$\frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \leq \frac{1}{(\alpha \gamma_1)^{1+(j_i-j)}}. \quad (2.29)$$

Proof: From the definition of the variables z_i and the assumption $z_i \leq 0$, we get

$$\gamma_i = \gamma_1^i \prod_{k=1}^{i-1} (z_k + 1), \quad i = 2, \dots, n,$$

and

$$0 < 1 + z_k \leq 1, \quad i = 1, \dots, n-1.$$

It follows that

$$\frac{\gamma_{k_i(j)}}{\gamma_1 \gamma_i} \leq \left[\gamma_1^{1+(j_i-j)} \prod_{k=i-(j_i-j)}^i (z_k + 1) \right]^{-1}, \quad (2.30)$$

and from the Archimedean property, we deduce that there exists $\alpha \in]0, 1]$ so that

$$0 < \alpha \leq z_k + 1 \leq 1. \quad (2.31)$$

Hence, by exploiting (2.31) in (2.30) as $\frac{1}{1+z_k} \leq \frac{1}{\alpha}$ the inequality (2.29) is straightforwardly inferred. \square

Now we are ready to introduce a new bounded convex set parameterized by two scalar variables, namely α given as in Lemma 2 and a new tuning parameter $\sigma > 0$ to be included later in the observer design procedure.

Let α and σ be two positive scalars. Define the bounded convex set

$$\mathcal{H}_\alpha^\sigma \triangleq \left\{ \Phi \in \mathbb{R}^d : \frac{\underline{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i-j)}} \leq \Phi_{ij} \leq \frac{\bar{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i-j)}} \right\} \quad (2.32)$$

for which the set of vertices, $\mathcal{H}_\alpha^\sigma$, is given by:

$$\mathcal{V}_{\mathcal{H}_\alpha^\sigma} \triangleq \left\{ \Phi \in \mathbb{R}^d : \Phi_{ij} \in \left\{ \frac{\underline{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i-j)}}, \frac{\bar{\nu}_{\psi_{ik_i(j)}}}{(\sigma\alpha)^{1+(j_i-j)}} \right\} \right\}. \quad (2.33)$$

The next lemma is useful and plays an important role in the design procedure we proposed hereafter.

Lemma 2.3: [20]

Let $\gamma \in \mathbb{R}_{>0,\uparrow}^n$ and $\sigma > 0$ such that $\gamma_1 \geq \sigma$. Let $\alpha \in]0, 1]$ be a positive scalar given by (2.31). Then the following inclusion holds:

$$\tilde{\mathcal{H}}^\gamma \subseteq \mathcal{H}_\alpha^\sigma. \quad (2.34)$$

Proof: The proof is straightforward by using Lemma 2 and the fact that the quantities $\underline{\nu}_{\psi_{ik_i(j)}}$ and $\bar{\nu}_{\psi_{ik_i(j)}}$ are negative and positive, respectively. The inequality $\frac{1}{\gamma_1} \leq \frac{1}{\sigma}$ is also used and substituted in (2.29). \square

The next section is devoted to the stability analysis of the estimation error dynamics. By using Lyapunov arguments and the preliminary results provided above, new high-gain like synthesis conditions will be established.

2.2.2 Stability analysis

To investigate the stability analysis of the estimation error dynamics, we consider the Lyapunov function

$$V(\tilde{x}) = \tilde{x}^\top P \tilde{x},$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite matrix. First, let us consider the change of variable $\tilde{K} = PK$. Therefore, after developing the derivative of the function $V(\tilde{x})$ along the trajectories of (2.27), we obtain

$$\dot{V}(\tilde{x}) = \gamma_1 \tilde{x}^\top \left[\mathcal{A}(\Psi^\gamma)^\top P + P \mathcal{A}(\Psi^\gamma) - C^\top \tilde{K}^\top - \tilde{K} C + \Omega^\top(\gamma) P + P \Omega(\gamma) \right] \tilde{x} + 2\tilde{x}^\top P T^{-1}(\gamma) \Delta f_1. \quad (2.35)$$

Before presenting the stability conditions ensuring the exponential convergence of the estimation error \tilde{x} to zero, we provide some informations on the term Δf_1 . This term will be handled by using the high-gain methodology.

Lemma 2.4: [20]

Under the assumptions of Lemma 2, there exists a positive scalar k_{f_1} such that

$$\|T^{-1}(\gamma) \Delta f_1\| \leq \frac{1}{(\alpha \gamma_1)^{j_{\min}}} k_{f_1} \|\tilde{x}\|, \quad (2.36)$$

where

$$j_{\min} = \min_{\substack{j_i \neq i \\ 1 \leq i \leq n}} j_i.$$

Proof: We have

$$\Delta f_1 = \sum_{i=1}^n \left(\sum_{j=1}^{i-j_i} \gamma_j \psi_{ij} \tilde{x}_j \right) e_n(i),$$

$$T^{-1}(\gamma) \Delta f_1 = \sum_{i=1}^n \left(\sum_{j=1}^{i-j_i} \frac{\gamma_j}{\gamma_i} \psi_{ij} \tilde{x}_j \right) e_n(i).$$

Therefore

$$\|T^{-1}(\gamma) \Delta f_1\|^2 = \sum_{i=1}^n \left(\sum_{j=1}^{i-j_i} \frac{\gamma_j}{\gamma_i} \psi_{ij} \tilde{x}_j \right)^2.$$

Using Assumption 1 and Hölder's inequality, it follows that:

$$\begin{aligned}
\|T^{-1}(\gamma)\Delta f_1\|^2 &\leq \sum_{i=1}^n \left(\sum_{j=1}^{i-j_i} L_j^f |\tilde{x}_j| \frac{\gamma_j}{\gamma_i} \right)^2 \\
&\leq \sum_{i=1}^n \left[\left(\max_{1 \leq j \leq i-j_i} L_j^f \right)^2 \left(\frac{\gamma_{i-j_i}}{\gamma_i} \right)^2 \left(\sum_{j=1}^{i-j_i} |\tilde{x}_j| \right)^2 \right] \\
&\leq \left(\max_{1 \leq j \leq n} L_j^f \right)^2 \sum_{i=1}^n (i-j_i) \left(\frac{\gamma_{i-j_i}}{\gamma_i} \right)^2 \|\tilde{x}\|^2 \\
&\leq \left[k_{f_1} \max_{1 \leq i \leq n} \frac{\gamma_{i-j_i}}{\gamma_i} \right]^2 \|\tilde{x}\|^2,
\end{aligned} \tag{2.37}$$

where

$$k_{f_1} = \bar{l}_f \sqrt{\left(\frac{n(n+1)}{2} - \sum_{i=1}^n j_i \right)}, \quad \bar{l}_f = \max_{1 \leq j \leq n} L_j^f.$$

Since $z_k \leq 0, \forall k = 1, \dots, n-1$, then from Lemma 2, there exists $\alpha \in]0, 1]$ such that:

$$\frac{\gamma_{i-j_i}}{\gamma_1 \gamma_i} \leq \frac{1}{(\alpha \gamma_1)^{1+j_i}}.$$

By putting $j_{\min} = \min_{\substack{j_i \neq i \\ 1 \leq i \leq n}} j_i$, we get

$$\max_{1 \leq i \leq n} \frac{\gamma_{i-j_i}}{\gamma_i} \leq \frac{1}{(\alpha \gamma_1)^{j_{\min}}}.$$

Then inequality (2.36) is inferred. This ends the proof. \square

Now we are ready to state the first theorem which provides sufficient design conditions ensuring exponential convergence of the estimation error to zero.

Theorem 2.2: [20]

Assume there exist $P = P^\top > 0$, $\lambda > 0$, $\tilde{K} \in \mathbb{R}^n$, $\gamma \in \mathbb{R}_{>0, \uparrow}^n$ and $\sigma > 0$ such that:

$$\left\{ \mathcal{A}(\Psi)^\top P + P \mathcal{A}(\Psi) - C^\top \tilde{K}^\top - \tilde{K} C + \Omega^\top(\gamma) P + P \Omega(\gamma) \right\} + \lambda I < 0, \quad \forall \Psi \in \mathcal{V}_{\mathcal{H}_\alpha^\sigma}, \tag{2.38}$$

and

$$\gamma_1 > \max \left(\sigma, \left[\frac{2k_{f_1} \lambda_{\max}(P)}{\lambda \alpha^{j_{\min}}} \right]^{\frac{1}{1+j_{\min}}}, \frac{1}{\alpha} \right). \tag{2.39}$$

Then the estimation error $\tilde{x}(t)$ is exponentially stable.

Proof: From the convexity principle and the inclusion (2.34) for $\gamma_1 \geq \sigma$, if (2.38) hold, we deduce that

$$\dot{V}(\tilde{x}) \leq -\gamma_1 \lambda \|\tilde{x}\|^2 + 2\tilde{x}^\top P T^{-1}(\gamma) \Delta f_1. \tag{2.40}$$

Let $\alpha \in]0, 1]$ satisfying (2.31). Then, from Lemma 4, we have

$$\begin{aligned} 2\tilde{x}^\top P T^{-1}(\gamma) \Delta f_1 &\leq 2\lambda_{\max}(P) \|\tilde{x}\| \left\| T^{-1}(\gamma) \Delta f_1 \right\| \\ &\leq \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}} k_{f_1} \|\tilde{x}\|^2. \end{aligned} \quad (2.41)$$

It follows that

$$\dot{V}(\tilde{x}) \leq - \left(\gamma_1 \lambda - \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}} k_{f_1} \right) \|\tilde{x}\|^2. \quad (2.42)$$

From the definition of $\dot{V}(\tilde{x})$ and after integrating from 0 to t , we obtain

$$\|\tilde{x}(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|\tilde{x}_0\| e^{-\left(\gamma_1 \lambda - \frac{2\lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}} k_{f_1} \right) t}, \quad (2.43)$$

which means that $\tilde{x}(t)$ converges exponentially to zero as $t \rightarrow +\infty$ if

$$\gamma_1 > \lambda - \frac{2k_{f_1} \lambda_{\max}(P)}{(\alpha\gamma_1)^{j_{\min}}} > 0.$$

On the other hand, to guarantee $\gamma \in \mathbb{R}_{n,\uparrow}^n$, we need to have $\gamma_1 \geq \frac{1}{\alpha}$, since γ_1 satisfies (2.31). These conditions on γ_1 lead to inequality (2.39). To sum up, the conditions on γ_1 are required for the following reasons:

1. $\gamma_1 \geq \sigma$ is needed to ensure feasibility of the inequalities (2.38). It is justified by the inclusion (2.34);
2. $\gamma_1 > \frac{1}{\alpha}$ is needed to guarantee $\gamma \in \mathbb{R}_{n,\uparrow}^n$;
3. $\gamma_1 > \left[\frac{2k_{f_1} \lambda_{\max}(P)}{\lambda \alpha^{j_{\min}}} \right]^{\frac{1}{1+j_{\min}}}$ is required to ensure $\tilde{x}(t)$ converges exponentially to zero, as $t \rightarrow \infty$, according to (2.43).

This ends the proof. □

Although Theorem 2 provides sufficient conditions to guarantee the design of the observer parameters K and γ , it still not fully exploitable at this stage because the matrix inequalities (2.38) are not numerically tractable. Indeed, (2.38) are not LMIs and depend on the parameter γ multiplied by the Lyapunov matrix P . To linearize (2.38) and to render it independent of γ , we should separate the coupling $P\Omega(\gamma)$ and use some mathematical tools to make γ vanish from inequality (2.38). To start the linearization procedure, we consider the following decomposition of $\Omega(\gamma)$:

$$\Omega(\gamma) \triangleq \Omega(Z) = A_1 Z A_2,$$

where

$$Z = \text{diag}(z_1, \dots, z_{n-1}) \in \mathbb{R}^{(n-1) \times (n-1)},$$

and

$$A_1 \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times (n-1)}, \quad A_2 \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(n-1) \times n}.$$

Then a simple use of Lemma 1 leads to separate P from $\Omega(Z)$. To satisfy (2.31), we should include the constraints:

$$Z \leq 0, \quad (2.44a)$$

$$(\alpha - 1)I_{n-1} - Z \leq 0. \quad (2.44b)$$

Hence we are ready to state the following main theorem, which provides LMI-based synthesis conditions ensuring the exponential convergence of the observer.

Theorem 2.3: [20]

If for a fixed $\alpha \in]0, 1]$, there exist positive scalars λ, σ , a symmetric positive definite matrix P , diagonal matrices $S > 0$ and $W \leq 0$, and a vector $\tilde{K} \in \mathbb{R}^n$, such that for all $\psi \in \mathcal{V}_{\mathcal{H}_g}$ the following conditions are fulfilled:

$$\begin{bmatrix} \mathcal{A}(\psi)^\top P + P\mathcal{A}(\psi) - C^\top \tilde{K}^\top - \tilde{K}C + \lambda I & (*) \\ A_1^\top P + WA_2 & -2S \end{bmatrix} < 0 \quad (2.45)$$

$$(\alpha - 1)S - W \leq 0, \quad (2.46)$$

then the estimation error $\tilde{x}(t)$ converges exponentially towards zero, as $t \rightarrow +\infty$ if the observer parameters are selected as follows:

$$K = P^{-1}\tilde{K}, \quad Z = S^{-1}W, \quad (2.47a)$$

$$\gamma_1 > \max \left(\sigma, \left[\frac{2k_{f_1} \lambda_{\max}(P)}{\lambda \alpha^{j_{\min}}} \right]^{\frac{1}{1+j_{\min}}}, \frac{1}{\alpha} \right), \quad (2.47b)$$

$$\gamma_i = \gamma_1^i \prod_{k=1}^{i-1} (z_k + 1), \quad i = 2, \dots, n. \quad (2.47c)$$

Proof: First, to get (2.45), we apply Lemma 1 on the inequalities (2.38) of Theorem 2. Indeed, from Lemma 1, we have

$$\begin{aligned} \Omega^\top P + P\Omega &= (A_1 P)^\top (Z A_2) + (Z A_2)^\top (A_1 P) \\ &\leq \frac{1}{2} (A_1^\top P + S Z A_2)^\top S^{-1} (A_1^\top P + S Z A_2). \end{aligned}$$

Then, after using Schur lemma and the change of variables $W = SZ$, $\tilde{K} = PK$, we get (2.45). Also, condition $W \leq 0$ comes from (2.44a). As for the inequality (2.46), it stems from (2.44b) after multiplying it by S . This ends the proof. \square

Remark 2.1: [20]

The proposed observer design method is more general than those proposed in the literature and related to high-gain methodology with constant observer gain. Indeed, for particular cases, the design is reduced to some recent methods. Although our idea is taken from references [6, 8], this combination is not systematic and has generated major mathematical difficulties (see Lemma 2–Lemma 4) necessary for the analysis of the stability of the dynamics of the error. We summarize the particular cases in the following items:

1. If we take $j_i = 0$ and $z_k < 0$, we will get exactly the enhanced high-gain proposed in [6]. Indeed, in such a case, we have $j_{\min} = 0$, $k_{f_1} = k_f$, and $\mathcal{A}(\psi) \equiv A$.
2. Likewise, if we have $j_i = 0$ (then $j_{\min} = 0$) and $z_k = 0$, we get the standard high-gain and Theorem 3 will be reduced to the main theorem of the standard high-gain observer [23].
3. Notice also that the HG/LMI observer proposed in [8] is a particular case of the result in Theorem 3 corresponding to $j_{\min} \geq 1$ and $z_k \equiv 0$.

The design procedure of the proposed observer can be summarized in the following well-structured design Algorithm. The algorithm is solved by using Matlab LMI Toolbox and YALMIP.

Algorithm 1: EHG/LMI Observer design

if Check if Assumption 1 holds. If yes, **then**

Step 1. Set an appropriate value for $\alpha \in]0, 1]$, choose a small $\epsilon > 0$ for the gridding, take $\tau = \frac{1}{2}$, $\lambda = 1$, a sufficiently high value $v_{gain} > 0$ and go to **Step 2**.

Step 2. While $\tau + \epsilon < 1$, take $\tau := \tau + \epsilon$ and return to Step 2;

Step 3. Solve the optimization problem (2.45) and (2.46) with respect to the decision variables P, \tilde{K}, S, W using the gridding method with respect to $\epsilon > 0$.

Step 4. Compute the decision variables P, \tilde{K}, Z and W . Take $\sigma = \frac{\tau}{1-\tau}$; $K = P^{-1}\tilde{K}$; $Z = S^{-1}W$, and compute the gains

$$\gamma_1 = \max \left(\sigma, \left[\frac{2k_f \lambda_{\max}(P)}{\lambda \alpha^{j_{\min}}} \right]^{\frac{1}{1+j_{\min}}}, \frac{1}{\alpha} \right);$$

$$\gamma_i = \gamma_1^i \prod_{k=1}^{i-1} (z_k + 1), \quad i = 2, \dots, n;$$

$$G = T(\gamma)P^{-1}\tilde{K}.$$

if $v_{gain} > \|G\|$ **then**

└ put $v_{gain} = \|G\|$ and go to **Step 2**.

else

└ $G^* = T(\gamma)P^{-1}\tilde{K}$, with G^* as the final result of the procedure.

else

└ The observer gains can not be computed.

Remark 2.2: [20]

1. Algorithm 1 is based on the use of the gridding technique with respect to the parameter $\sigma > 1$. Such a technique consists in scaling $\sigma > 1$ by defining $\tau \in]0.5, 1]$, via the change of variable $\tau = \sigma/(1 + \sigma)$. Then, we assign a uniform subdivisions of the interval $]0.5, 1]$, and solve (2.45) and (2.46) for each subdivision. Note that LMIs (2.45) and (2.46) are always feasible for any $\sigma \geq 1$ (or $\tau \in [0.5, 1[$ and $\epsilon > 0$). This issue has been addressed in [8, Eq. (46)-(47)]. However, the choice of ϵ has an impact on the value of σ , which is explicitly related to the value of the high-gain parameter γ_1 . Therefore, the smaller is ϵ , the smaller is the value of the obtained parameter γ_1 . Of course, this could generate significant computational complexity. This is the reason why we introduce the stop test $v_{gain} \leq \|G\|$.
2. The scalar α is fixed priori. In the case of our illustrative example, one set $\alpha = 0.95$. Indeed, the smaller is the value of $1 - \alpha > 0$, the smaller is the value of the observer parameter γ_1 .

2.2.3 Illustration through an academic example

This part is dedicated to a simple numerical example to show the validity and effectiveness of the proposed design technique. We show the benefits of combining the enhanced high-gain and the LMI observer design method for different values of the compromise index and the Lipschitz constant. We aim also to compare our proposed approach in particular to the one developed recently in [8] and the classical high-gain observer.

Consider the third-order system

$$\begin{cases} \dot{x} = Ax + f(x), \\ y = Cx, \end{cases} \quad (2.48)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0],$$

and

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^3; x \mapsto \left(0, 0, \frac{k_f}{\sqrt{3}} \sum_{i=1}^3 \sin x_i \right).$$

The nonlinearity f satisfies Assumption 1 with $L_1 = L_2 = 0$ and $L_3 = k_f$.

Now, to design the observer parameters, we will follow all the steps of Algorithm 1. Let choose $\alpha = 0.95$ and $v_{gain} = 2e18$. For the gridding, we choose $\epsilon = 10^{-3}$.

According to the values of the index j_{min} , γ_f and σ , we obtain the following gains G^* for different values of j_{min} and $k_f = 1$:

Now we will provide some comparisons between the standard high-gain (HGO), the enhanced high-gain observer (EHGO), the HG/LMI observer, and the proposed technique in Theorem 3. Table 2.3 shows numerical comparisons between the HG/LMI observer proposed in [8] and the classical high-gain observer. The superiority of the method compared to the authors' previous work and in the papers [8] and [6] is clearly illustrated. On the other hand, it was demonstrated

j_{\min}	0	1	2
σ	–	2.3557	1.7548
G^*	$\begin{bmatrix} 24.9 \\ 472.9 \\ \boxed{3739.5} \end{bmatrix}$	$\begin{bmatrix} 7.8474 \\ 29.6747 \\ \boxed{48.6805} \end{bmatrix}$	$\begin{bmatrix} 6.1598 \\ 18.4522 \\ \boxed{24.8470} \end{bmatrix}$

Table 2.2: The gains G^* for different values of j_{\min} and σ

		HGO				EHGO [6]			
j_{\min}	k_f	θ_0	K	γ	K	γ	K	γ	K
0	0.1	2.34	1.06	0.86	0.29	2.34	1.06	0.86	0.29
	1	23.45	1.06	0.86	0.29	14.82	7.46	5.95	1.87
	10	234.51	1.06	0.86	0.29	148.24	7.46	5.95	1.86

HG/LMI Observer [8]							
j_i	k_f	n_{LMI}	σ	$\theta_{j_i}^{\frac{1}{1+j_i}}$	K		
1	0.1	2	1.004	1.004	2.5021	2.9498	1.4199
	1		2.3557	2.3557	4.8587	8.7086	5.8140
	10		9.7527	9.7527	13.9080	28.0327	26.5827
2	0.1	4	1.004	1.0040	2.5930	3.1640	1.6028
	1		1.7548	1.7548	5.4481	10.6211	8.4631
	10		6.0423	6.0423	16.9046	49.8802	72.2843

EHG/LMI Observer Theorem 2							
j_i	k_f	n_{LMI}	σ	$\gamma_1^{\frac{1}{1+j_i}}$	K		
1	0.1	2	1.1834	1.1794	1.62	1.30	0.49
	1		2.9063	2.8924	2.70	3.5133	1.9832
	10		10.4943	10.2405	5.6260	11.1557	10.1883
2	0.1	4	1.008	1.0054	1.7455	1.5160	0.6135
	1		1.9674	1.9565	3.1310	4.7674	3.2630
	10		6.2993	6.1383	7.7352	20.9695	27.7039

Table 2.3: Comparisons between the high-gain observers for different values of k_f and j_{\min} .

that the authors' previous work improves the existing high-gain design techniques, namely the standard high-gain observer and the Astolfi/Marconi observer.

The advantage of the compromise index j_{\min} is shown for different values of k_f . We can see clearly that our method provides a smaller gain as compared with the other methods. For example, for fixed $k_f = 10$, for $j_{\min} = 1$, it suffices to solve 2 LMIs to reduce significantly the gain from $\theta_0 = 234.51$ obtained by classical high-gain into 10.2405. For $j_{\min} = 2$, the same gain is reduced to 6.2993. This shows the importance of the introduction of increasing values of the index j_{\min} . Figure 2.1 shows results of a simulation run in presence of an additive uniform random noise on the output when $t \in [0.5, 1] \cup [3, 3.5]$ which is a Gaussian distributed random signal with mean zero and variance 0.01, initial state equal to $[5 \ 5 \ 5]^T$ and initial estimated state equal to $[-5 \ -5 \ -5]^T$ for all the discussed methods.

Denote by $\hat{x} = [\hat{x}_1 \ \hat{x}_2 \ \hat{x}_3]^T$, $\hat{x}_Z = [\hat{x}_{1,Z} \ \hat{x}_{2,Z} \ \hat{x}_{3,Z}]^T$ and $\hat{x}_{\text{HG}} = [\hat{x}_{1,\text{HG}} \ \hat{x}_{2,\text{HG}} \ \hat{x}_{3,\text{HG}}]^T$

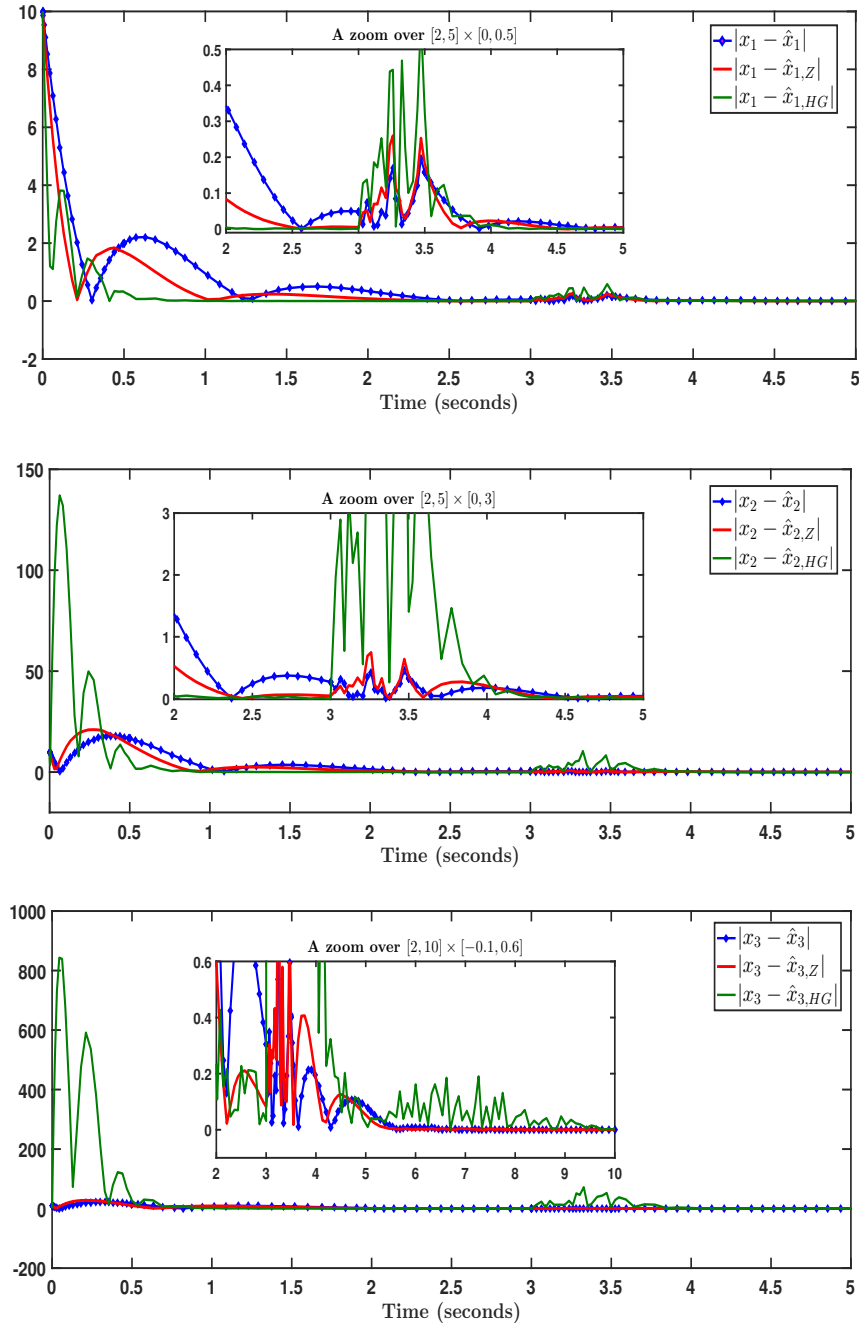


Figure 2.1: Absolute values of the errors for $k_f = 1$, $j_{\min} = 2$, $\alpha = 0.95$ and $\sigma = 1.9674$.

the state estimates for the system (2.48) by using the observer design method proposed in [8], in the present work and the classical high-gain observer, respectively.

The curves of the absolute values $|x - \hat{x}_i|$, $|x - \hat{x}_{i,Z}|$ and $|x - \hat{x}_{i,HG}|$ $i = 1, \dots, 3$ are depicted in Figure 2.1. We can see that the errors converge toward zeros. Note, however, that the transient performance of the new designed observer is better, although the convergence speeds are almost

the same. The proposed new technique is able to avoid the peaking phenomenon. We also see a significant improvement of the sensitivity to high-frequency measurement noise; the estimated states are not so much disturbed as the standard high-gain.

2.2.4 Application to a tumor growth model

This section is devoted to an application of the proposed observer design method on a real-world process, namely a tumor growth model.

The tumor growth model

Angiogenesis is the physiological process through which new blood vessels form from pre-existing vessels, formed in the earlier stage of vasculogenesis.

Angiogenesis is a normal and vital process in growth and development, as well as in wound healing and in the formation of granulation tissue. However, it is also a fundamental step in the transition of tumors from a benign state to a malignant one, leading to the use of angiogenesis inhibitors in the treatment of cancer.

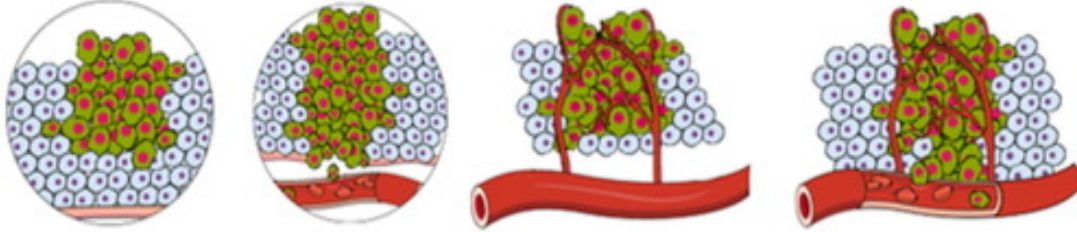


Figure 2.2: Angiogenesis process within a solid tumor

Angiogenesis inhibitors are substances which prevent this formation of new blood vessels, thereby stopping or slowing the growth or spread of tumours.

An angiogenesis inhibitor can be endogenous or come from outside as drug or a dietary component.

Anti-angiogenic therapies are cancer treatments, proposed at first in [29] in the early seventies, consolidated along the nineties by several discoveries on the main principles regulating tumor angiogenesis [20] and widely debated in several theoretical and experimental studies throughout the last decade

The model under investigation is a nonlinear model accounting for angiogenic stimulation and inhibition [30], and is given by the following Ordinary Differential Equations (ODE) system:

$$\begin{cases} \dot{x}_1 &= -\lambda x_1 \ln\left(\frac{x_1}{x_2}\right), \\ \dot{x}_2 &= bx_1 - (\mu + dx_1^{\frac{2}{3}})x_2 - cx_2x_3, \end{cases} \quad (2.49)$$

where the model parameters are

- the tumor volume x_1 [mm^3];
- the carrying capacity of the vasculature x_2 [mm^3];
- The first equation describes the phenomenology of tumor growth slowdown, as the tumor grows and resorts its available support;

- the tumor growth rate constant λ [day^{-1}];
- In the second equation the first term represents the stimulatory capacity of the tumor upon the inducible vasculature (bx_1), the second term accounts for spontaneous loss (μx_2) and for tumor-dependent endogenous inhibition ($dx_1^{\frac{2}{3}}x_2$) of previously generated vasculature; the third term refers to the vasculature inhibitory action performed by an exogenous drug administration (cx_2x_3)
 - the actual control law u [$day^{-1}(mg/kg)$];
 - the vascular birth rate b [day^{-1}];
 - the endothelial cell death (death rate) d [$day^{-1}mm^{-2}$];
 - the sensitivity to the drug c [$day^{-1}(mg/kg)^{-1}$];
 - the spontaneous vascular inactivation rate μ [day^{-1}]. According to the model literature [30], without loss of generality, parameter μ will be set equal to zero in the following .

Being the anti-angiogenic drug not directly administrated in vein, a further compartment is considered to account for drug diffusion:

$$x_3(t) = \int_0^t e^{-\eta(t-t')}u(t')dt', \quad (2.50)$$

with

- $u[day^{-1}(mg/kg)]$, being the actual control law;
- $\eta[day^{-1}]$, being the diffusion rate into serum.

As a matter of fact, the whole system (2.49)-(2.50) (with $\mu = 0$) may be written in a compact ODE form:

$$\begin{cases} \dot{x}_1 &= -\lambda x_1 \ln\left(\frac{x_1}{x_2}\right), \\ \dot{x}_2 &= bx_1 - (\mu + dx_1^{\frac{2}{3}})x_2 - cx_2x_3, \\ \dot{x}_3 &= -\eta x_3 + u. \end{cases} \quad (2.51)$$

From here on, system (2.51) will be also referred to as:

$$\dot{x} = f(x) + g(x)u, \quad (2.52)$$

with $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $g : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 1}$ defined by:

$$\dot{x} = \begin{bmatrix} -\lambda x_1 \ln\left(\frac{x_1}{x_2}\right) \\ bx_1 - dx_1^{\frac{2}{3}}x_2 - cx_2x_3 \\ -\eta x_3 + u \end{bmatrix} \quad (2.53)$$

Transformation of the model

Consider a domain D in \mathbb{R}^3 that excludes $x_1 = 0$ and $x_2 = 0$. Then, the working hypothesis concerning the state feedback linearization is satisfied, since the single-input single-output system has full relative degree in D . Indeed, $\forall x \in D$ it is :

$$L_g h(x) = L_g L_f h(x) = 0 \quad (2.54)$$

$$L_g L_f^2 h(x) = -\lambda c x_1 \neq 0, \quad (2.55)$$

with L_f^k , $k = 1, 2, \dots$ denotes the Lie derivate of order k of scalar function h along the vector field f . As matter of fact, consider the following nonlinear map $\Theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, with $\tilde{x} = \frac{x_1}{x_2}$:

$$z = \Theta(x) = \begin{bmatrix} x_1 \\ L_f h(x) \\ L_f^2 h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ -\lambda x_1 \ln(\tilde{x}) \\ \lambda x_1 \left(\lambda(1 + \ln(\tilde{x})) \ln(\tilde{x}) + b\tilde{x} - dx_1^{\frac{2}{3}} - cx_3 \right) \end{bmatrix} \quad (2.56)$$

Then the inverse Θ^{-1} of Θ is given by:

$$x = \Theta^{-1}(z) = \begin{bmatrix} z_1 \\ z_1 e^{\frac{z_2}{\lambda z_1}} \\ \frac{-z_3}{c\lambda z_1} - \frac{z_2}{cz_1} \left(1 - \frac{z_2}{\lambda z_1} \right) + \frac{b}{c} e^{-\frac{z_2}{\lambda z_1}} - \frac{d}{c} z_1^{\frac{2}{3}} \end{bmatrix} \quad (2.57)$$

Therefore, $z = \Theta(x)$ is a state transformation, and the observable canonical form of system (2.53) in the z -coordinates is:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ L_f^3 h(\Theta^{-1}(z)) + L_g L_f^2 h(\Theta^{-1}(z))u \end{bmatrix}$$

where

$$\begin{aligned} L_f^3 h(\Theta^{-1}(z)) &= \varphi(x) \lambda \tilde{x} \left(2\lambda \ln(\tilde{x}) - \varphi(x) + cx_3 + dx_1^{2/3} + \lambda \right) - \lambda^2 \frac{x_1^2}{x_2} (\lambda \tilde{x} \\ &+ \lambda \left(1 + 2 \ln(\tilde{x}) + (\ln(\tilde{x}))^2 \right) - \left(\frac{5}{3} \right) dx_1^{2/3} + 2b(\tilde{x}) - cx_3) - \lambda c \eta x_1 x_3; \end{aligned} \quad (2.58)$$

with

$$\varphi(x) = dx_1^{\frac{3}{2}} - bx_1 + cx_2 x_3.$$

Simulation results

λ (day^{-1})	b (day^{-1})	d ($day^{-1} mm^{-2}$)	μ (day^{-1})	c ($day^{-1} (mg/kg)^{-1}$)	η (day^{-1})
0.192	5.85	0.00873	0	0.66	1.7

Table 2.4: Model Parameters.

Simulations have been carried out by setting the initial tumor volume $x_1(0)$ equal to 200 mm^3 and by varying the initial value of the carrying capacity $x_2(0)$ in the percentage range

of $[-40; +200]\%$ with respect to the reference value of 625 mm^3 . The initial value of the serum level of the administered angiogenic inhibitor $x_3(0)$ is set equal to zero. Also these initial values are consistent with real data coming from [30]. For the observer initialization, $\hat{x}_1(0)$ is set equal to the measurement of the tumor volume at day zero, $\hat{x}_3(0)$ is set equal to zero, and $\hat{x}_2(0)$ is allowed to vary in the percentage range of $[-40; +200]\%$ with respect to real value $x_2(0)$.

The actual states and their estimates by using observer design method presented in this chapter are plotted. In Figure 2.3, we see that the observer works for the system (2.53) and it is seen that the estimates reach the actual states.

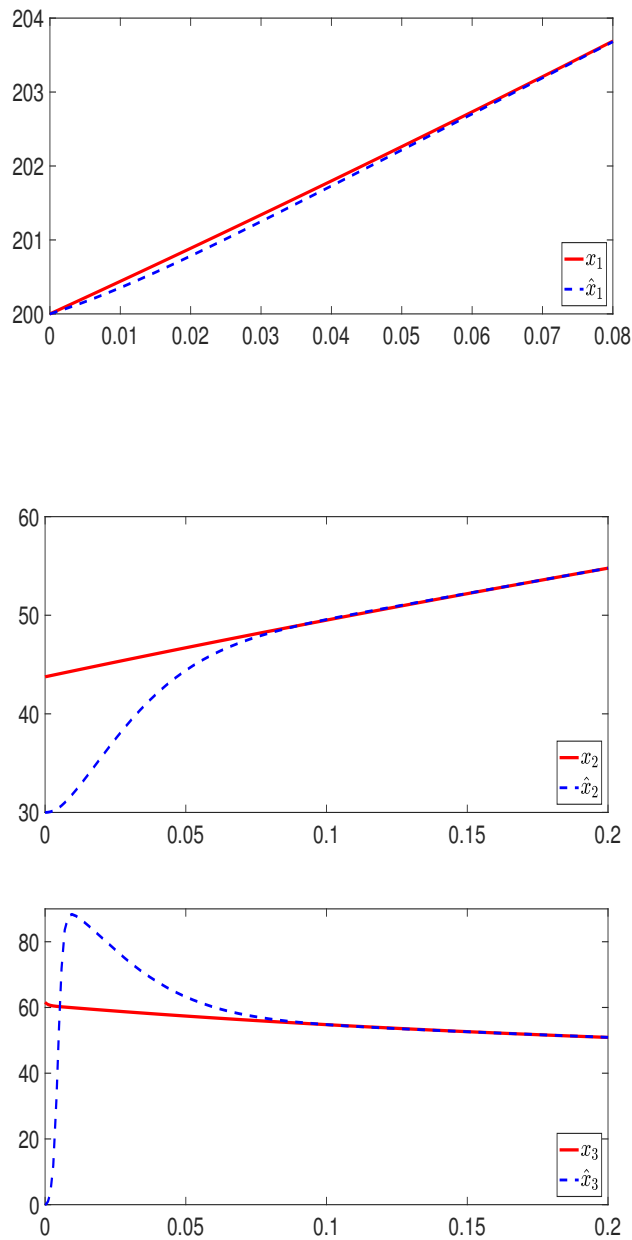


Figure 2.3: Behavior of the states and their estimates.

2.3 Conclusion

What we proposed in this chapter is a general structure that comprises many of methods discussed in the literature that we can recover by making particular choices on the observer design parameters. Especially, we generalized the work presented in [8] with more many possibilities of choosing the design parameters of the gain. The stability of the estimation error is shown using a Lyapunov function after having successfully established new high-gain like synthesis conditions. A numerical academic example was provided in order to demonstrate the performances of the proposed approach. Finally, an application to a tumor growth model is given to show validity and efficiency of the proposed method.

LMI Feasibility Improvement to Design Observers for a Class of Nonlinear Systems

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3.1 Introduction and Preliminaries

The present chapter proposes general LMI conditions to design observers for a class of Lipschitz nonlinear systems. These LMIs reduce the conservatism of those existing in the literature and in other hand, they allow reducing the number of LMIs in the LPV/LMI approach by [31]. For particular classes of systems, it is shown that by introducing suitable linear and nonlinear transformations, the original system is transformed into a new one for which the corresponding LMIs are always feasible. Ensuring feasibility of the LMI conditions for any value of Lipschitz constant of the nonlinearity is an important result in the LMI context for observer design. Indeed, usually, LMIs are considered as only sufficient conditions without guarantee of feasibility, especially for systems with large values of Lipschitz constants.

For simplicity of the presentation, we consider systems where the nonlinearity depends only on the state x . We investigate the class of systems described by the following equations:

$$\begin{cases} \dot{x} = \Gamma(x) + Bu \\ y = Cx \end{cases} \quad (3.1)$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output measurement, and $u \in \mathbb{R}^m$ is the control input. The nonlinear function $\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be γ_Γ -Lipschitz, i.e.:

$$\|\Gamma(x) - \Gamma(y)\| \leq \gamma_\Gamma \|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \quad (3.2)$$

3.2 General LMI Design Method

For optimal clarity, we will present the results step by step. Before introducing the state observer, and in the goal to propose a general LMI design method, we first write the function $\Gamma(\cdot)$ in a detailed form as:

$$\Gamma(x) \triangleq Ax + \Pi \begin{pmatrix} \Gamma_1(\Lambda_1 x) \\ \vdots \\ \Gamma_q(\Lambda_q x) \end{pmatrix} = Ax + \Pi \sum_{i=1}^{i=q} e_q(i) \Gamma_i(\Lambda_i x) \quad (3.3)$$

where $A \in \mathbb{R}^{n \times n}$, $\Pi \in \mathbb{R}^{n \times q}$ and $\Lambda_i \in \mathbb{R}^{n_i \times n}$ are constant matrices, $e_q(i)$ represents the i^{th} vector of the canonical basis of \mathbb{R}^q , and q is the number of nonlinear components.

Usually we use the Luenberger observer

$$\dot{\hat{x}} = A\hat{x} + Bu + \sum_{i=1}^{i=q} \Pi e_q(i) \Gamma_i(\Lambda_i \hat{x}) + L(y - C\hat{x}) \quad (3.4)$$

with a constant gain matrix L . However, in some situations, this observer leads to conservative LMI conditions. For instance, LPV systems with bounded parameters are a particular class of nonlinear Lipschitz systems, and in this particular case of systems, the observer (3.4) with a constant observer gain is not sufficient. On the other word, the LMIs are conservative if a constant gain is considered for LPV systems with known and bounded parameters. To overcome this obstacle, we propose a more general state observer, which covers all these classes of systems, namely, linear systems, LPV systems, and Lipschitz nonlinear systems. To this end, we need to define the sets

$$\mathfrak{P} \triangleq \left\{ (i, j) : i = 1, \dots, q; j = 1, \dots, n_i \right\}, \quad (3.5)$$

$$\mathfrak{F} \triangleq \left\{ (i, j) : \gamma_{ij}(x_k^{\hat{x}_{j-1}}, x_k^{\hat{x}_j}) = g_{ij}(y, \rho, t) \neq \text{constant} \right\}, \quad (3.6)$$

$$\mathfrak{P} \setminus \mathfrak{F} \triangleq \left\{ (i, j) : (i, j) \in \mathfrak{P} \text{ and } (i, j) \notin \mathfrak{F} \right\}, \quad (3.7)$$

and

$$\eta \triangleq \text{card}(\mathfrak{P}) = \sum_{i=1}^{i=q} n_i$$

$$\eta_1 \triangleq \text{card}(\mathfrak{P} \setminus \mathfrak{F}) = \eta - \text{card}(\mathfrak{F}), \quad \eta_2 \triangleq \text{card}(\mathfrak{F}),$$

where $\text{card}(\mathfrak{P})$ is the number of elements of \mathfrak{P} , ρ is a known parameter. For shortness, we denote $g_{ij}(y, \rho, t)$ by g_{ij} .

3.2.1 More general observer

Now, we can consider the following general observer which considers the presence of known LPV parameters:

$$\dot{\hat{x}} = A\hat{x} + Bu + \sum_{i=1}^{i=q} \Pi e_q(i) \Gamma_i(\hat{\vartheta}_i) + \mathbb{L}(t)(y - C\hat{x}) \quad (3.8a)$$

$$\dot{\hat{\vartheta}}_i = \Lambda_i \hat{x} \quad (3.8b)$$

$$\mathbb{L}(t) = L + \sum_{(i,j) \in \mathfrak{F}} g_{ij} \bar{L}_{ij}. \quad (3.8c)$$

The dynamics of the estimation error $e = x - \hat{x}$ is given by:

$$\dot{e} = (A - \mathbb{L}(t)C)e + \sum_{i=1}^{i=q} \Pi e_q(i) [\Gamma_i(\vartheta_i) - \Gamma_i(\hat{\vartheta}_i)] \quad (3.9)$$

As $\Gamma_i(\cdot)$ is γ_{Γ_i} -Lipschitz, then from Lemma 1 there exist functions

$$\gamma_{ij} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \longrightarrow \mathbb{R}$$

and constants $\underline{\gamma}_{ij}$ and $\bar{\gamma}_{ij}$, such that

$$\Gamma_i(\vartheta_i) - \Gamma_i(\hat{\vartheta}_i) = \left[\sum_{i=1}^{i=q} \sum_{j=1}^{j=n_i} \gamma_{ij}(\vartheta_i^{\hat{\vartheta}_i^{j-1}}, \vartheta_i^{\hat{\vartheta}_i^j}) \Lambda_i^j \right] e \quad (3.10)$$

and

$$\underline{\gamma}_{ij} \leq \gamma_{ij} \leq \bar{\gamma}_{ij}, \quad (3.11)$$

where $\Lambda_i^j = e_{n_i}^\top(j) \Lambda_i$. In the sequel, for the sake of brevity, we use γ_{ij} instead of $\gamma_{ij}(\vartheta_i^{\hat{\vartheta}_i^{j-1}}, \vartheta_i^{\hat{\vartheta}_i^j})$. By defining the matrix

$$\Psi \triangleq (\Psi_{ij}) \in \mathbb{R}^{q \times n_{\max}}, \quad (3.12)$$

where

$$\Psi_{ij} = \begin{cases} \gamma_{ij}, & i = 1, \dots, q; j \leq n_i \\ 0, & i = 1, \dots, q; j > n_i \end{cases} \quad (3.13)$$

and the affine matrix

$$\mathcal{A}(\Psi) \triangleq A + \left[\sum_{i=1}^{i=q} \sum_{j=1}^{j=n_i} \Psi_{ij} \Pi e_q(i) \Lambda_i^j \right],$$

where $n_{\max} = \max_i(n_i)$, the dynamics of the estimation error becomes

$$\dot{e} = [\mathcal{A}(\Psi) - \mathbb{L}(t)C]e. \quad (3.14)$$

It is worth noting that the matrix $\mathcal{A}(\Psi)$ can be rewritten under the form:

$$\begin{aligned} \mathcal{A}(\Psi) &= \sum_{(i,j) \in \mathfrak{F} \setminus \mathfrak{F}} \left[\frac{\theta}{\eta_1} A + \Psi_{ij} \Pi e_q(i) \Lambda_i^j \right] \\ &+ \sum_{(i,j) \in \mathfrak{F}} \left[\frac{(1-\theta)}{\eta_2} A + g_{ij} \Pi e_q(i) \Lambda_i^j \right]. \end{aligned} \quad (3.15)$$

To start, let us define the following matrix

$$\mathbb{A}_{ij}^\mu(\theta, \eta) \triangleq A + \frac{\eta}{\theta} [\mu \Pi e_q(i) \Lambda_i^j]. \quad (3.16)$$

which simplifies the statement of the next theorem.

Before stating the theorem, we assume without loss of generality that the system is not completely linear ($\mathfrak{P} \neq \emptyset$). Now, we state the general theorem.

Theorem 3.1: [21]

The observer (3.8) is asymptotically convergent if there exist a positive definite matrix \mathcal{P} , matrices \mathcal{R}_{ij} , $(i, j) \in \mathfrak{P} \setminus \mathfrak{F}$ and $\bar{\mathcal{R}}_{ij}$, $(i, j) \in \mathfrak{F}$ of appropriate dimensions such that the following matrix inequalities hold:

1. *Inequality related to nonlinear terms:*

$$\begin{aligned} & \left[\mathbb{A}_{ij}^\mu(\theta, \eta_1) \right]^T \mathcal{P} + \mathcal{P} \left[\mathbb{A}_{ij}^\mu(\theta, \eta_1) \right] \\ & \quad - C^T \mathcal{R}_{ij} - \mathcal{R}_{ij}^T C < 0 \\ & \quad \forall \mu \in \left\{ \underline{\gamma}_{ij}, \bar{\gamma}_{ij} \right\}, (i, j) \in \mathfrak{P} \setminus \mathfrak{F}, \end{aligned} \quad (3.17)$$

2. *Inequality related to LPV terms:*

$$\begin{aligned} & \left[\mathbb{A}_{ij}^\mu(1 - \theta, \eta_2) \right]^T \mathcal{P} + \mathcal{P} \left[\mathbb{A}_{ij}^\mu(1 - \theta, \eta_2) \right] \\ & \quad - C^T \left(\mathcal{R}_{ij} + \mu \bar{\mathcal{R}}_{ij} \right) - \left(\mathcal{R}_{ij} + \mu \bar{\mathcal{R}}_{ij} \right)^T C < 0, \\ & \quad \forall \mu \in \left\{ \underline{\gamma}_{ij}, \bar{\gamma}_{ij} \right\}, (i, j) \in \mathfrak{F}. \end{aligned} \quad (3.18)$$

Hence, the observer gains L and \bar{L}_{ij} are given by:

$$L = \frac{\theta}{\eta_1} \sum_{(i,j) \in \mathfrak{P} \setminus \mathfrak{F}} \mathcal{P}^{-1} \mathcal{R}_{ij}^T + \frac{(1 - \theta)}{\eta_2} \sum_{(i,j) \in \mathfrak{F}} \mathcal{P}^{-1} \mathcal{R}_{ij}^T \quad (3.19)$$

$$\bar{L}_{ij} = \frac{(1 - \theta)}{\eta_2} \sum_{(i,j) \in \mathfrak{F}} \mathcal{P}^{-1} \bar{\mathcal{R}}_{ij}^T. \quad (3.20)$$

Proof: The proof uses the standard quadratic Lyapunov function $V(e) = e^T \mathcal{P} e$. The key idea of the results consists in using the form (3.15) and the following decomposition of the constant gain L :

$$L = \frac{\theta}{\eta_1} \sum_{(i,j) \in \mathfrak{P} \setminus \mathfrak{F}} L_{ij} + \frac{(1 - \theta)}{\eta_2} \sum_{(i,j) \in \mathfrak{F}} L_{ij}. \quad (3.21)$$

Calculating the derivative of $V(e)$ along the trajectories of the estimation error $e(t)$, we obtain

the equation given in (3.22).

$$\begin{aligned} \dot{V}(e) = & \frac{\theta}{\eta_1} \sum_{(i,j) \in \mathfrak{P} \setminus \mathfrak{F}} \left[\overbrace{\left(\mathbb{A}_{ij}^{\gamma_{ij}}(\theta, \eta_1) - L_{ij}C \right)^T \mathcal{P} + \mathcal{P} \left(\mathbb{A}_{ij}^{\gamma_{ij}}(\theta, \eta_1) - L_{ij}C \right)}^{\Delta_{ij}^{\gamma_{ij}}(\theta, \eta_1)} \right] \\ & + \frac{(1-\theta)}{\eta_2} \sum_{(i,j) \in \mathfrak{F}} \left[\overbrace{\left(\mathbb{A}_{ij}^{g_{ij}}(1-\theta, \eta_2) - \left(L_{ij} + g_{ij} \frac{\eta_2}{(1-\theta)} \bar{L}_{ij} \right) C \right)^T \mathcal{P} + \mathcal{P} \left(\mathbb{A}_{ij}^{g_{ij}}(1-\theta, \eta_2) - \left(L_{ij} + g_{ij} \frac{\eta_2}{(1-\theta)} \bar{L}_{ij} \right) C \right)}^{\bar{\Delta}_{ij}^{g_{ij}}(1-\theta, \eta_2)} \right]. \end{aligned} \quad (3.22)$$

Using the change of variables

$$\mathcal{R}_{ij} = L_{ij}^T \mathcal{P} \quad \text{and} \quad \bar{\mathcal{R}}_{ij} = \frac{\eta_2}{(1-\theta)} \bar{L}_{ij}^T \mathcal{P}$$

we deduce from (3.17) and (3.18) that $\Delta_{ij}^{\gamma_{ij}}(\theta, \eta_1) < 0$ and $\bar{\Delta}_{ij}^{g_{ij}}(1-\theta, \eta_2) < 0$, respectively. Notice that $\Delta_{ij}^{\gamma_{ij}}(\theta, \eta_1)$ and $\bar{\Delta}_{ij}^{g_{ij}}(1-\theta, \eta_2)$ are defined in (3.22). This leads to $\dot{V}(e) < 0$, which ends the proof. \square

3.2.2 Some remarks

It is clear that the proposed general observer covers systems with both purely nonlinear and LPV terms.

Linear systems

The purely linear case can be handled separately for simplification of the presentation and for avoiding any ambiguity. To summarize, in the linear case ($\mathfrak{P} = \emptyset$), the observer design is reduced to the solvability of the classical Lyapunov inequality:

$$A^T \mathcal{X} + \mathcal{X} A - C^T \mathcal{Y} - \mathcal{Y}^T C < 0 \quad (3.23)$$

with $L = \mathcal{X}^{-1} \mathcal{Y}^T$.

LPV systems

On the other hand, if we assume that we have a complete LPV system, then the design is reduced to solve only inequalities (3.18). Indeed, in such a case, the inequalities (3.17) vanish because $\mathfrak{F} = \mathfrak{P}$ and then $\mathfrak{P} \setminus \mathfrak{F} = \emptyset$. So to reduce the effect of the term $\frac{\eta_2}{1-\theta}$ and to enhance the feasibility, we take $\theta = 0$. In this case, the observer gain L is reduced to

$$L = \frac{1}{\eta_2} \sum_{(i,j) \in \mathfrak{F}} \mathcal{P}^{-1} \mathcal{R}_{ij}^T$$

and

$$\bar{L}_{ij} = \frac{1}{\eta_2} \sum_{(i,j) \in \mathfrak{F}} \mathcal{P}^{-1} \bar{\mathcal{R}}_{ij}^T.$$

Nonlinear systems without LPV terms

In the case where $\mathfrak{F} = \emptyset$, the observer design is reduced to inequalities (3.17) because the inequalities (3.18) related to the LPV terms vanish. Then, the observer gain L is given by:

$$L = \frac{1}{\eta_1} \sum_{(i,j) \in \mathfrak{P} \setminus \mathfrak{F}} \mathcal{P}^{-1} \mathcal{R}_{ij}^T$$

for $\theta = 1$.

About the linear components

Another important point that deserves mention here is the presence of all the linear components in both (3.17) and (3.18). Indeed, the constant matrix A is added to these two last inequalities in the aim to avoid loss of detectability condition, which is necessary for the feasibility of the LMIs. Hence the use of the decomposition

$$A = \theta A + (1 - \theta)A. \quad (3.24)$$

The first term, θA , is included in the nonlinear part, and the second one, $(1 - \theta)A$, is associated to the LPV part. Without including A , a necessary condition for the feasibility of the proposed LMIs is the detectability of the pair $(\mathbb{A}_{ij}^\mu(\theta, \eta), C)$ for all (i, j) and μ . However, in general since C is a constant matrix, this condition is difficult to satisfy without a constant part independent of (i, j) and especially μ . That is why we added the matrix A to $\mathbb{A}_{ij}^\mu(\theta, \eta)$.

LMI reduction

One more important issue in Theorem 1 is that the number of LMIs to be solved is reduced to $2 \sum_{i=1}^q n_i$ instead of $2^{i=1}^q n_i$ required by the LPV/LMI method in [31]. This is due to the introduction of the form (3.15) and by isolating the sums in the Lyapunov developments.

Superiority compared to T-S representation approach

By the T-S based transformation approaches, the system is transformed into a Takagi-Sugeno structure and then the observer is based on the transformed system [32], [33], [34]. However, such a transformation creates an uncertain system, which is not suitable from stability analysis viewpoint because the observer takes the structure of the transformed system without the uncertainty. Such a manipulation leads to LMI conditions which may be feasible only if the system (3.1) is exponentially stable [35]. This strong and very conservative constraint is not required by our proposed approach where all the transformations are applied on the estimation error and its dynamics.

3.3 LMI Feasibility Improvement

Although the LMIs proposed in the previous section are more general than those existing in the literature, they are only sufficient conditions. Hence, further improvement are possible to reduce more the conservatism. This is the goal of this section, where new techniques are proposed to enhance feasibility of the LMIs and to increase the possibility of designing an observer for Lipschitz nonlinear systems.

3.3.1 Improvement by introducing linear transformation

To enhance the feasibility of the proposed LMI conditions (3.17) and (3.18), one of the solutions consists in transforming the system state by using a linear coordinates transformation. Such a change of coordinates may lead to a new system with smaller values of the bounds $\underline{\gamma}_{ij}$ and $\bar{\gamma}_{ij}$. Indeed, feasibility of (3.17) and (3.18) depends strongly on the bounds $\underline{\gamma}_{ij}$ and $\bar{\gamma}_{ij}$.

By transforming the system by $z = \mathbb{T}x$, where $\mathbb{T} \in \mathbb{R}^{n \times n}$ is a nonsingular matrix, we obtain a new system as:

$$\begin{cases} \dot{z} = A_{\mathbb{T}}z + \mathbb{T}Bu + \Pi_{\mathbb{T}} \sum_{i=1}^{i=q} \Gamma_{\mathbb{T}i}(\Lambda_{\mathbb{T}i}z) \\ y_{\mathbb{T}} = C_{\mathbb{T}}z \end{cases} \quad (3.25)$$

All parameters of the new system (3.25) are indexed by \mathbb{T} . For instance, the bounds $\underline{\gamma}_{ij}$ and $\bar{\gamma}_{ij}$, will be replaced by $\underline{\gamma}_{\mathbb{T}ij}$ and $\bar{\gamma}_{\mathbb{T}ij}$, respectively. Hence, we apply straightforwardly Theorem 1 on the new system (3.25).

We can summarize the strategy in the following procedure.

- (i) Find a non singular matrix \mathbb{T} , that allows to decrease the bounds $\underline{\gamma}_{\mathbb{T}ij}$ and $\bar{\gamma}_{\mathbb{T}ij}$, and go to step (ii);
- (ii) Apply *Theorem 1* on the transformed system (3.25), and go to step (iii);
- (iii) If the *LMIs* (3.17) and (3.18) provide solutions, then stop. Else, return to step (i) to choose another matrix \mathbb{T} to decrease more the bounds $\underline{\gamma}_{\mathbb{T}ij}$ and $\bar{\gamma}_{\mathbb{T}ij}$.

3.3.2 Improvement by using nonlinear transformation

Another solution to reduce conservatism of the LMIs consists in finding a specific nonlinear transformation to transform the system into a new coordinates. The new system can have different dimension than the original system. It is not possible to develop a systematic and general method providing an adequate transformation, which reduces the bounds $\underline{\gamma}_{ij}$ and $\bar{\gamma}_{ij}$. Systematic results can be developed only for some particular classes of systems. Contrarily to the case of linear transformation, a nonlinear transformation may reduce the number of LMI to be solved by canceling some nonlinear components.

Let us consider the diffeomorphism Φ defined by

$$\begin{aligned} \Phi : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x \rightarrow z &= \Phi(x) \end{aligned} \quad (3.26)$$

which transforms the system (3.1) into the following form:

$$\begin{cases} \dot{z} = Az + \Pi \Gamma_1(z) \\ y = Cz \end{cases} \quad (3.27)$$

for which the corresponding bounds $\underline{\gamma}_{\Phi ij}$ and $\bar{\gamma}_{\Phi ij}$ are smaller than those of the original system, namely $\underline{\gamma}_{ij}$ and $\bar{\gamma}_{ij}$. In this case, the feasibility of (3.17)-(3.18) corresponding to the new system (3.27) may be improved. Indeed, feasibility depends also on the matrices A and C . For some classes of nonlinear systems, we can always guarantee feasibility of the LMIs for any bounds $\underline{\gamma}_{\Phi ij}$ and $\bar{\gamma}_{\Phi ij}$. This is the aim of the next two subsections.

Systems in canonical form

Here we will study the case where system (3.1) can be transformed by (3.26) into the triangular form:

$$\begin{cases} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ \Gamma_1(z) \end{bmatrix} \\ y = z_1 \end{cases} \quad (3.28)$$

which can be written under the form (3.27) with A , Π , and C have the companion structure [36]. Note that a more general class of systems with a nonlinearity $f_{i\Phi}(z_1, \dots, z_i)$ in each component of the system can be considered, without loss of generality. However, for sake of brevity, we investigate (3.28) with only a single nonlinearity in the last component of the system.

Now introduce the second transformation

$$\zeta = \mathbb{T}z, \text{ where } \mathbb{T}(\tau) \triangleq \text{diag} \left(\frac{1}{\tau}, \dots, \frac{1}{\tau^n} \right) \quad (3.29)$$

which transforms (3.28) into

$$\dot{\zeta} = \tau A \zeta + \frac{1}{\tau^n} \Pi \Gamma_1(\mathbb{T}^{-1}(\tau)\zeta). \quad (3.30)$$

Let us consider the following state observer corresponding to (3.30):

$$\dot{\hat{\zeta}} = \tau A \hat{\zeta} + \frac{1}{\tau^n} \Pi \Gamma_1(\mathbb{T}^{-1}(\tau)\hat{\zeta}) + L \left(y - C \mathbb{T}^{-1}(\tau)\hat{\zeta} \right) \quad (3.31)$$

where L , independent from τ , is the constant observer gain to be determined. Then the dynamics of the estimation error $e = \zeta - \hat{\zeta}$ is expressed as

$$\dot{e} = \tau (A - LC) e + \Pi \Delta \Gamma_1 \quad (3.32)$$

where

$$\Delta \Gamma_1 \triangleq \frac{1}{\tau^n} \left[\Gamma_1(\mathbb{T}^{-1}(\tau)\zeta) - \Gamma_1(\mathbb{T}^{-1}(\tau)\hat{\zeta}) \right]. \quad (3.33)$$

Applying Lemma 1, there exist functions

$$\psi_j : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

and constants $\underline{\gamma}_j$ and $\bar{\gamma}_j$, such that

$$\Delta \Gamma_1 = \left[\sum_{j=1}^{j=n} \frac{\psi_j}{\tau^{n-j}} \Lambda_1^j \right] e \quad (3.34)$$

and

$$\underline{\gamma}_j \leq \gamma_j \leq \bar{\gamma}_j, \quad (3.35)$$

where $\Lambda_1^j \triangleq e_n^\top(j)$, the j^{th} element of the canonical basis of \mathbb{R}^n .

In this particular case, with $\theta = 1$, the affine matrix $\mathcal{A}(\Psi)$ defined in (3.15) is reduced to

$$\mathcal{A}(\Psi) = \sum_{j=1}^n \left[\frac{1}{n} A + \frac{1}{\tau^{1+(n-j)}} \Psi_j \Pi \Lambda_1^j \right] \quad (3.36)$$

where $\Psi = [\psi_1, \dots, \psi_n]^\top$. It follows that the dynamics of the estimation error becomes

$$\dot{e} = \tau [\mathcal{A}(\Psi) - LC] e \quad (3.37)$$

By analogy to the previous section, we define the matrix

$$\mathbb{A}_j^\mu(\tau, n) \triangleq A + \frac{n\mu}{\tau^{1+(n-j)}} \Pi \Lambda_1^j. \quad (3.38)$$

Consequently, we can state the following corollary as a particular case of the general Theorem 1.

Corollary 3.1: [21]

Let $\mathcal{P} = \mathcal{P}^\top > 0$ and \mathcal{R} be matrices of appropriate dimensions, and $\tau > 0$ is a scalar, such that the following LMI conditions hold:

$$\left[\mathbb{A}_j^\mu(\tau, n) \right]^\top \mathcal{P} + \mathcal{P} \left[\mathbb{A}_j^\mu(\tau, n) \right] - C^\top \mathcal{R}_j - \mathcal{R}_j^\top C < 0, \quad \forall \mu \in \{\underline{\gamma}_j, \bar{\gamma}_j\}, j = 1, \dots, n. \quad (3.39)$$

Then the observer (3.31) corresponding to (3.30), with

$$L = \frac{1}{n} \sum_{j=1}^n \mathcal{P}^{-1} \mathcal{R}_j^\top,$$

converges exponentially towards zero. Moreover, the estimated state $\hat{x} = \Psi^{-1}(\mathbb{T}^{-1}(\tau)\hat{\zeta})$ converges exponentially to the state x of the original system (3.1).

Proof: The proof is omitted. It is a particular case of Theorem 1. \square

More importantly, we have the following result on the feasibility of the LMIs (3.39).

Proposition 3.1: [21]

LMIs (3.39) are always feasible for any fixed and finite values of the bounds $\underline{\gamma}_j$ and $\bar{\gamma}_j$.

Proof: Since (A, C) is observable, then there always exist matrices $\mathcal{P} = \mathcal{P}^\top > 0$ and \mathcal{R} of appropriate dimensions such that

$$A^\top \mathcal{P} + \mathcal{P} A - C^\top \mathcal{R} - \mathcal{R}^\top C < 0.$$

On the other hand, from (3.38), for all $j = 1, \dots, n$, we have

$$\lim_{\tau \rightarrow +\infty} \left(\mathbb{A}_j^\mu(\tau, n) \right) = A.$$

Then from continuity of $\mathbb{A}_j^\mu(\tau, n)$ with respect to τ , there exists $\tau^* > 0$ large enough such that the LMI (3.39) holds for any $\tau \geq \tau^*$. \square

Remark 3.1

High-gain observer design is a particular case of the proposed methodology. Indeed, as can be seen in the proof of Proposition 1, where a sufficiently large value of τ guarantee exponential convergence of the estimation error.

Remark 3.2: [21]

For the class of feedforward systems, we can also guarantee feasibility of the corresponding LMI conditions for any values of the bounds $\underline{\gamma}_j$ and $\bar{\gamma}_j$.

3.4 Conclusion

In this note, we proposed a generalization of previous work on observer design for a class of Lipschitz nonlinear systems. A general observer with general LMIs are proposed, which lead to LMI synthesis conditions encompassing LMI methods available in the literature for the same class of systems. Then these LMIs are improved to reduce more the conservatism by introducing convenient linear and nonlinear transformations to bring the original system into new coordinates where the corresponding LMIs allow higher values of Lipschitz constant of the nonlinearity of the system. Finally, it is shown that for a particular class of systems, it is always possible to build an observer by solving a set of always feasible LMIs. This property of guarantee of solutions is important in LMI context for observer design in nonlinear systems, where usually only sufficient conditions without guaranteeing solutions are provided in the literature.

Finite-Time Estimation of a Class of Switched Systems

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4.1 Introduction

Time-delay systems are widely investigated in the literature and they attracted the interest of the community for many reasons [37–41]. They are often encountered in several engineering systems design [42], [43], [44], [45]. Due to the emergence of new technologies, time-delay is

unavoidable in many situations. For instance, in Networked Control Systems (NCSs), although NCSs offer enhanced flexibility, low cost, reduced weight, they are simple to install and maintain, and highly reliable [46], the communication delays included in the feedback control loops makes the analysis and design of NCSs complicated.

State estimation plays a crucial role in practical applications whose states are difficult to measure or even unmeasurable because sensors are noisy or unavailable due to various physical and economic settings. If the state estimation problem for linear systems is completely solved, the presence of delayed output measurements renders the challenge complex. Tremendous research have been conducted in the literature and several methods have been established. The problem becomes considerably more complex when we face switched systems and singular systems. In a general way, the well-know tools used to analyse convergence of the estimation algorithms are Lyapunov-Krasovskii functional [47]; Razumikhin theorem [48]; and Halanay inequalities [49].

Surprisingly, even if the estimation problem for systems with delayed output measurements is an active research topic, a very few attention have been paid to exact and finite-time estimation. The idea is borrowed from [50,51] which focus only on finite estimation of linear systems without delay. However, systems operating in real environments are often affected by delays, as said earlier. Hence it is more accurate and realistic to take into account the delay in the dynamic representation of the system.

In this chapter, we will propose new exact and finite-time estimation algorithms for some classes of systems with delayed output measurements. By using backward substitution technique, together with some assumptions, we will develop two estimation procedures. The first one is based on the use of explicit computation of the system state by exploiting delayed outputs and inputs of the system. As for the second algorithm, it is based on the combination of two asymptotic state observers. The estimation algorithms are then used in output feedback control loop ensuring asymptotic stabilization of the system state by simple eigenvalues assignment.

The algorithms are then generalized to a class of switched linear systems with known switching mode. General exact and finite-time estimation algorithms are then obtained by using only eigenvalues assignment (which is usually not sufficient for this class of systems) to ensure invertibility of a certain key matrix guaranteeing the exact finite-time reconstruction of the system state. The integration of these estimation algorithms, for this class of systems, in output feedback stabilization loops require feasibility of a set of Linear Matrix Inequalities (LMIs) to guarantee asymptotic stabilization. Further extensions are explored, namely switched singular systems, and unknown input estimation for switched systems for which there are no available results in the literature on finite-time estimation. It turns out that these extensions entail complex mathematical developments and that the solutions are not straightforward extensions of their linear counterparts. Finally, an application application to reference trajectory tracking is presented.

The proposed estimation algorithms are illustrated on three numerical examples, namely:

- The estimation and stabilization algorithms are illustrated on a linearized and descritized model of the horizontal plane motion of an autonomous underwater vehicle;
- In the case of switched systems, we used an academic example to validate the estimation algorithms, combined with an output feedback controller for the stabilization step;

- The algorithms are applied on a two masses coupled two springs mechanical model of dimension four, where we consider that the output matrix of the system switches between four modes;
- Finally, reference trajectory tracking results are applied to a regulation of blood glucose problem.

4.2 Preliminary Results for Linear Systems

Before presenting the main results on switched systems and further extensions, we will start by providing preliminary results for linear systems.

4.2.1 Problem formulation

Consider the following system with delayed output:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_{k-d}, \end{cases} \quad (4.1)$$

where x_k is the state of the system; u_k is the control input vector, and y_k is the output vector, $d > 0$ is a constant delay in the measurements.

The system above includes delayed terms, which in practice can make the estimation of the state difficult to evaluate. To overcome this problem, we propose two Finite-time estimation methods: The first one consists of a direct approach which involves some manipulations on the system expression. The second method is based on the use of two combined asymptotic state observers.

4.2.2 Direct estimation approach

The purpose here is to provide an exact estimation algorithm of x_k in presence of a known delay in the output y_k .

Let L and K be two matrices such that $(A - LC)$ and $(A - KC)$ are Hurwitz. Then, the system (4.1) can be written as:

$$x_{k+1} = (A - LC)x_k + Ly_k + Bu_k \quad (4.2a)$$

$$x_{k+1} = (A - KC)x_k + Ky_k + Bu_k \quad (4.2b)$$

The following lemma will be used to prove our main results. For the proof, we refer the reader to [50, 51].

Lemma 4.1: [50, 51]

Assume that the pair (A, C) is observable. Then there exist L , K , and $m \geq 1$ such that the matrix

$$\mathbb{E}_m = (A - LC)^{-m} - (A - KC)^{-m} \quad (4.3)$$

is invertible.

The explicit formula giving the exact estimation or the true value x_k for any $k > m + d$ is presented in the following theorem.

Theorem 4.1: [22]

There exist L , K and $m \geq 1$ such that the solution of (4.1) satisfies $\forall k > m + d$:

$$\begin{aligned} x_k = & A^d \mathbb{E}_m^{-1} \sum_{j=1}^m \left[(A - LC)^{j-m-1} L - (A - KC)^{j-m-1} K \right] y_{k-j} \\ & + A^d \mathbb{E}_m^{-1} \sum_{j=1}^m \left[(A - LC)^{j-m-1} - (A - KC)^{j-m-1} \right] B u_{k-d-j} + \sum_{j=1}^d A^{j-1} B u_{k-j}, \end{aligned} \quad (4.4)$$

where \mathbb{E}_m is defined in (4.3).

Proof: To simplify the presentation, the proof is given in two steps.

- **Step 1:** the first step consists in estimating the delayed state. To this end, let z_k stands for the shifted version of x_k , i.e.:

$$z_k \triangleq x_{k-d}. \quad (4.5)$$

The dynamics of the state z_k is then written as follows:

$$\begin{cases} z_{k+1} = A z_k + B u_{k-d} \\ y_k = C z_k. \end{cases} \quad (4.6)$$

Following Lemma 1 and from [50,51], there exist L , K and $m \geq 1$ such that $\forall k > m + d$, z_k is given by the following expression:

$$\begin{aligned} z_k = & \mathbb{E}_m^{-1} \sum_{j=1}^m \left[(A - LC)^{j-m-1} L - (A - KC)^{j-m-1} K \right] y_{k-j} \\ & + \mathbb{E}_m^{-1} \sum_{j=1}^m \left[(A - LC)^{j-m-1} - (A - KC)^{j-m-1} \right] B u_{k-d-j}. \end{aligned} \quad (4.7)$$

For more details on the expression (4.7), we refer the reader to [50,51, Theorem 1]. Indeed, for a system without delayed state as (4.6), the exact estimation in finite-time is borrowed from [50,51]. Then the detailed proof is omitted.

- **Step 2:** Having estimated z_k , the second step is to recover the state variable x_k at time k . The expression of x_k is rewritten by backward substitution, yielding

$$\begin{aligned} x_k &= A x_{k-1} + B u_{k-1} \\ &\vdots \\ &= A^d \underbrace{x_{k-d}}_{z_k} + \sum_{j=1}^d A^{j-1} B u_{k-j}. \end{aligned}$$

Hence, by substituting (4.7) in (4.8), we get (4.4).

□

4.2.3 Finite-Time Estimation Using Two Observers

We propose a two observers-based estimation technique which consists in combining two asymptotic observers to develop an exact finite-time estimation of the states of system (4.1). Consider the following two state observers corresponding to (4.6) given by:

$$\xi_{k+1} = A\xi_k + Bu_{k-d} + L(y_k - C\xi_k) \quad (4.8)$$

and

$$\eta_{k+1} = A\eta_k + Bu_{k-d} + K(y_k - C\eta_k), \quad (4.9)$$

where L and K are two gain matrices to be determined.

Theorem 4.2: [22]

There exist L , K , and $m \geq 1$ such that the solution of system (4.1) satisfies $\forall k > m + d$:

$$x_k = A^d \mathbb{E}_m^{-1} \left[(A - LC)^{-m} \xi_k - \xi_{k-m} - (A - KC)^{-m} \eta_k + \eta_{k-m} \right] + \sum_{j=1}^d A^{j-1} Bu_{k-j}. \quad (4.10)$$

Proof: Similarly to (4.7), the expressions of ξ_k and η_k can be written as follows:

$$\begin{aligned} \xi_k &= (A - LC)^m \xi_{k-m} + \sum_{j=1}^m (A - LC)^{j-1} (Ly_{k-j} + Bu_{k-d-j}) \\ \eta_k &= (A - KC)^m \eta_{k-m} + \sum_{j=1}^m (A - KC)^{j-1} (Ky_{k-j} + Bu_{k-d-j}) \end{aligned}$$

Remember (4.7), then, we obtain the following relation

$$(A - LC)^{-m} \xi_k - \xi_{k-m} - (A - KC)^{-m} \eta_k + \eta_{k-m} = \mathbb{E}_m z_k \quad (4.11)$$

Then by substituting (4.11) in (4.8), the relation (4.10) is inferred. \square

Remark 4.1: [22]

Notice that since the delay d is a constant integer, then it is possible to use system state augmentation to get a generalized state. Hence we can apply the result of [50, 51] to the augmented system. However, since the dimension of the new system will be increased to $n(d + 1)$, which may increase considerably the finite-time m satisfying invertibility of the matrix \mathbb{E}_m .

4.2.4 Output Feedback Stabilization

In this section, we will exploit the estimation algorithms in output feedback stabilization schemes. Both the proposed estimation architectures can be used in the closed-loop system, as they provide an exact estimation of the system state, x_k . Even if system (4.1) is linear, its stabilization

by a static output feedback controller $u_k = -Fy_k$ leads to a complicated synthesis of the controller gain F . Indeed, although the problem remains open in the case of non delayed outputs, $y_k = Cx_k$, because it leads to the design of the gain F such that the eigenvalues of $A - BFC$ are inside the unit circle [52], [53], it becomes more complicated in case of delayed outputs [54]. However, due the exact and finite-time computation of the system state x_k , the stability analysis of the closed-loop system will lead to the stability of the system

$$x_{k+1} = (A - BF)x_k \quad (4.12)$$

for which a simple eigenvalues assignment can provide the controller gain F . Equation (4.12) is obtained for $k > m + d$ is the following feedback controller is used:

$$\begin{aligned} u_k &= -F\hat{x}_k \\ \hat{x}_k &= A^d \mathbb{E}_m^{-1} \left[(A - LC)^{-m} \xi_k - \xi_{k-m} - (A - KC)^{-m} \eta_k + \eta_{k-m} \right] + \sum_{j=1}^d A^{j-1} B u_{k-j}, \quad \forall k > m + d; \\ \hat{x}_k &= -F\xi_k, \quad \forall k \leq m + d, \end{aligned} \quad (4.13)$$

where ξ_k and η_k are given by (4.8).

Remark 4.2: [22]

To avoid repetition and to cumbersome notations, we introduced only the controller (4.12) based on the estimation algorithm using two combined observers given in the previous section. The first algorithm based on explicit computation of the system state can also be used in a closed-loop and leads to the same structure (4.12) because (4.4) provides exact estimation of x_k in finite-time.

4.2.5 Application to reference trajectory tracking

Having a desired trajectory, the question is how to push a physical system to a desired trajectory via closed-loop control. In this section, we will use the exact estimation algorithms proposed previously in the control loop for tracking a desired reference trajectory. Let x_k^r be an admissible trajectory of the system (4.1), i.e. there exists a nominal input u_k^r such that

$$x_{k+1}^r = Ax_k^r + Bu_k^r. \quad (4.14)$$

The problem of reference trajectory tracking consists in determining a control law which stabilize asymptotically the tracking error [55]

$$e_{k+1}^r \triangleq x_{k+1} - x_{k+1}^r = Ae_k^r + B(u_k - u_k^r). \quad (4.15)$$

$$\begin{aligned} u_k &= -F(\hat{x}_k - x_k^r) + u_k^r \\ \hat{x}_k &= A^d \mathbb{E}_m^{-1} \left[(A - LC)^{-m} \xi_k - \xi_{k-m} - (A - KC)^{-m} \eta_k + \eta_{k-m} \right] \\ &\quad + \sum_{j=1}^d A^{j-1} B u_{k-j}, \quad \forall k > m + d \\ \hat{x}_k &= -F\xi_k, \quad \forall k \leq m + d, \quad k > m + d \end{aligned} \quad (4.16)$$

where ξ_k and η_k are given by (4.8).

From the finite-time estimation algorithms proposed in sections 4.2.2 and 4.2.3, we have $\hat{x}_k = x_k$ for all $k \geq m + d$. Therefore, by injecting the u_k command given by (4.16), into (4.15), we will get the dynamics tracking error:

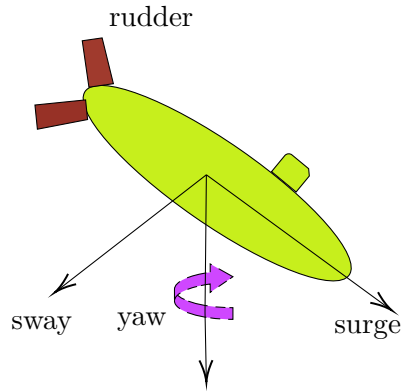
$$e_{k+1}^r = (A - BF)e_k^r, \quad (4.17)$$

whose asymptotic stability is guaranteed if the matrix F is chosen such that the eigenvalues of the matrix $A - BF$ have negative real parts.

4.2.6 Illustrative examples

Example 1: Application to an autonomous underwater vehicle

We study a real world example taken from [56]. Autonomous underwater vehicles (AUVs) are robotic submarines that can be used for a variety of studies of the underwater environment. The vertical and horizontal dynamics of the vehicle must be controlled to remotely operate the AUV.



AUV technology has rapidly grown over the last few decades. This growth is motivated by the increasing number of missions made possible thanks to this technology. A linearized and discretized model of the horizontal plane motion of the vehicle is described under the form (4.1) with the following matrices:

$$A \triangleq \begin{bmatrix} 0.9932 & -0.03434 & 0 \\ -0.009456 & 0.9978 & 0 \\ -0.0002368 & 0.04994 & 1 \end{bmatrix}; B \triangleq \begin{bmatrix} 0.002988 \\ -0.0115 \\ 0.0002875 \end{bmatrix}; C \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad (4.18)$$

The variables of interest in horizontal motion are the sway speed and the yaw angle. The vehicle motion model uses the states: $x_k = [x_k^1 \ x_k^2 \ x_k^3]^\top$, where

- x_k^1 is the sway speed,
- x_k^2 is the yaw angle,
- x_k^3 is the yaw rate, and u_k is the rudder deflection.

Finite-time estimation in open-loop

In the simulation, let $\hat{x}_k = [\hat{x}_k^1 \ \hat{x}_k^2 \ \hat{x}_k^3]^\top$ and $\hat{x}_k = [\hat{x}_k^{1'} \ \hat{x}_k^{2'} \ \hat{x}_k^{3'}]^\top$ be the state estimates for the system (4.18) by using the estimation methods given in Theorem 1 and Theorem 2, respectively. We choose $m = 3$, $d = 2$ and the initial values, $x_0 = [1 \ 2 \ 3]^\top$, $\hat{x}_0 = [10 \ -10 \ 5]^\top$, $\hat{x}_0 = [15 \ 15 \ 15]^\top$. The input used for simulation is $u = 2 \times 10^3 \sin(k)$. We use pole-assignment to the desired eigenvalues $[-0.1 \ 0.3 \ 0.4]$, $[-0.02 \ 0.2 \ -0.01]$. Respectively, the associated

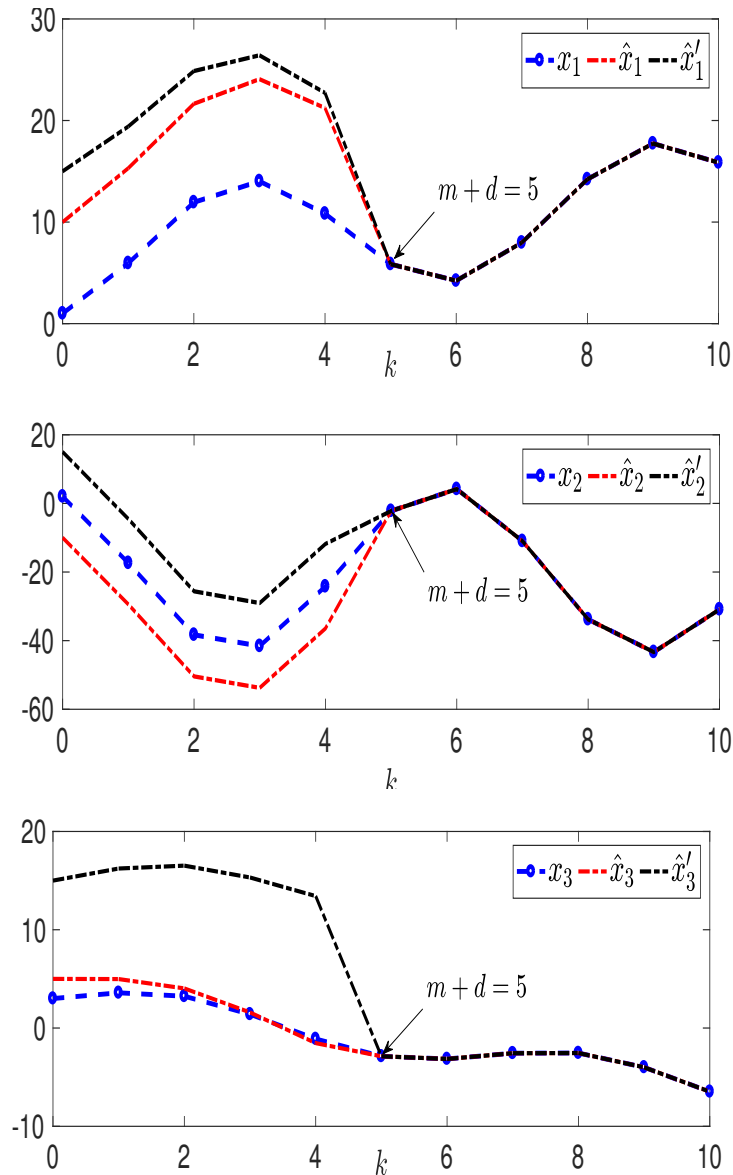


Figure 4.1: Simulation using direct estimation approach (Theorem 1) and two-observers based estimation (Theorem 2)

feedback gain matrices, guaranteeing existence and invertibility of the matrix E_3 with $k \in \mathbb{N}^*$,

are found to be equal to

$$L = \begin{bmatrix} 1.0333 & -0.4561 \\ -6.4953 & 8.6822 \\ -0.4908 & 1.3577 \end{bmatrix}, K = \begin{bmatrix} 1.2476 & -0.3850 \\ -7.1910 & 11.2854 \\ -0.3533 & 1.5734 \end{bmatrix}.$$

Fig. 4.1 shows the curves of the actual states x_k and their estimated states \hat{x}_k and \hat{x}'_k . We see that both trajectories reach exactly the actual states in a finite-time.

Output feedback stabilization

In this part, we keep the values of L and K calculated in the previous estimation step. By assigning the eigenvalues $[0.8 \ 0.75 \ 0.9]$ to $A - BF$, the controller gain matrix F is found to be $F = [-1156.9 \ -370.4 \ -909]$.

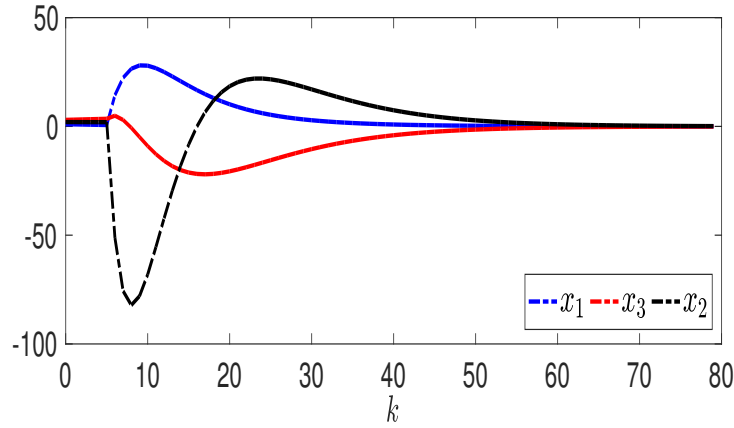


Figure 4.2: The stabilized states by using the output feedback controller (4.16).

The evolution of the states under the output feedback stabilization law (4.16) is depicted in Fig. 4.2. Simulation shows good stabilization of the states.

Example 2: regulation of blood glucose level

Several chronic diseases require the regulation of the patient's blood levels of a specific drug or hormone. For example, some diseases involve the failure of the body's natural closed-loop control of blood levels of nutrients. Most prominent among these is the disease diabetes, where the production of the hormone insulin that controls blood glucose levels is impaired.

Automatic blood glucose regulation in diabetes patients is an extensive and dynamic field of research from decades. Numerous efforts have been made for mathematical modeling of diabetics for glucose insulin regulation mechanism [57–59].

To design a closed-loop insulin delivery system, a sensor is used to measure the concentration of blood glucose (CBG). This measurement is converted to digital form and fed to the control computer. According to the information from sensor, the controller decides the insulin infusion rate (IR) to normalize glucose level. Then, the controller drives a pump that injects the insulin into the patient's blood. Figure 4.3 shows close loop insulin infusion mechanism.

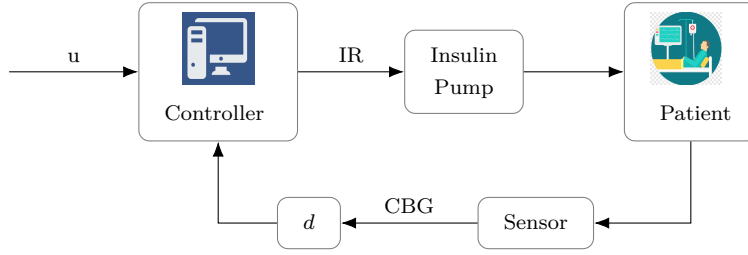


Figure 4.3: Closed loop blood glucose control scheme.

A simplified linearized model of insulin regulatory mechanism ([56]) is given by (4.1), with matrices A, B and C given by

$$A \triangleq \begin{bmatrix} -0.04 & -4.4 & 0 \\ 0 & -0.025 & 1.3 \times 10^{-5} \\ 0 & 0.09 & 0 \end{bmatrix}; B \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix}; C \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.19)$$

The state variables are:

- blood glucose concentration x_1 (mg/dl);
- accumulation of blood insulin x_2 ;
- blood insulin concentration x_3 mg/dl .

The controls are:

- rate of glucose infusion u_1 $mg/dl/min$;
- rate of insulin infusion u_2 $mg/dl/min$.

Simulations have been carried out by setting $m = 3$, $d = 2$ and the initial values, $x_0 = [1 \ 2 \ 3]^T$, $\hat{x}_0 = [10 \ -10 \ 5]^T$, $\hat{x}_0 = [15 \ 15 \ 15]^T$. We use pole-assignment to the desired eigenvalues $[-0.1 \ 0.3 \ 0.4]$, $[-0.02 \ 0.2 \ -0.01]$. Respectively, the associated feedback gain matrices, guaranteeing existence and invertibility of the matrix E_3 with $k \in \mathbb{N}^*$, are found to be equal to

$$L = \begin{bmatrix} 1.3513 & -1.1950 \\ -0.0203 & 0.0368 \\ -0.0177 & 0.7237 \end{bmatrix}, K = \begin{bmatrix} 1.4847 & -0.0149 \\ -0.0271 & 0.0007 \\ -0.0136 & 1.0203 \end{bmatrix}.$$

By assigning the eigenvalues $[0.8 \ 0.75 \ 0.9]$ to $A - BF$, the controller gain matrix F is found to be $F = [-1156.9 \ -370.4 \ -909]$. We aim at attaining the desired level of glucose $100 \ mg/dl$ via the designed control law (4.14). The desired trajectory is $x_k^r = [100 \ 0 \ 0]^T$. Resolving equation (4.14) leads to the nominal control $u_k^r = [104 \ 0]^T$.

The obtained result is presented in Figure 4.4. The blood level is stabilized at $100 \ mg/dl$ after $28 \ min$.

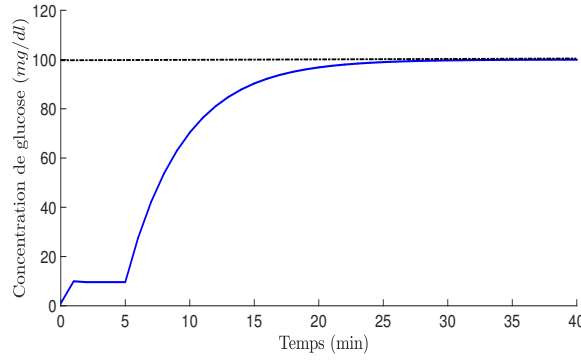


Figure 4.4: Reference trajectory tracking of $x_k^r(1) = 1$.

4.3 Estimation for a Class of Switched Linear Systems

The estimation algorithms provided in Section 4.2 can be extended to switched linear systems with known switching mode. This section is devoted to this extension. To this end, we consider the discrete-time switched linear systems described by:

$$\begin{cases} x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k \\ y_k = C_{\sigma_k} x_{k-d} \end{cases} \quad (4.20)$$

where σ_k is the switching mode, assumed to be explicitly unknown but its instantaneous values are known in real-time, with $\sigma_k \in \mathcal{I}_\sigma \triangleq \{1, \dots, n_\sigma\}$, where n_σ is the number of modes. The following theorem provides new results on exact estimation of linear discrete-time switched systems with delayed output.

4.3.1 The estimation algorithms

The estimation algorithms are summarized in the following theorem.

Theorem 4.3: [22]

Assume that matrices L_i and K_i , $i \in \mathcal{I}_\sigma$, and an integer $m \geq 1$, are selected such that the matrix $\mathbb{E}_m(k)$, defined by

$$\mathbb{E}_m(k) \triangleq \left[\prod_{i=1}^m (A_{\sigma_{k-i}} - L_{\sigma_{k-i}} C_{\sigma_{k-i}}) \right]^{-1} - \left[\prod_{i=1}^m (A_{\sigma_{k-i}} - K_{\sigma_{k-i}} C_{\sigma_{k-i}}) \right]^{-1} \quad (4.21)$$

exists and invertible. Then the two following items provide exact finite-time estimation of the state of system (4.20).

- *Explicit solution:* The solution of system (4.20) can be explicitly computed by (4.22).

$$\begin{aligned}
x_k = & \prod_{s=1}^d A_{\sigma_{k-s}} \mathbb{E}_m^{-1}(k-d) \sum_{j=1}^m \left[\left(\prod_{i=1}^m (A_{\sigma_{k-d-i}} - L_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) \right)^{-1} \right. \\
& \prod_{i=1}^{j-1} (A_{\sigma_{k-d-i}} - L_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) L_{\sigma_{k-d-j}} - \left. \left(\prod_{i=1}^m (A_{\sigma_{k-d-i}} - L_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) \right)^{-1} \right. \\
& \left. + \prod_{i=1}^{j-1} (A_{\sigma_{k-d-i}} - K_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) K_{\sigma_{k-d-j}} \right] y_{k-j} + \prod_{s=1}^d A_{\sigma_{k-s}} \mathbb{E}_m^{-1}(k-d) \\
& + \sum_{j=1}^m \left[\left(\prod_{i=1}^m (A_{\sigma_{k-d-i}} - L_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) \right)^{-1} \prod_{i=1}^{j-1} (A_{\sigma_{k-d-j}} - L_{\sigma_{k-d-j}} C_{\sigma_{k-d-j}}) \right. \\
& \left. - \left(\prod_{i=1}^m (A_{\sigma_{k-d-i}} - K_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) \right)^{-1} \prod_{i=1}^{j-1} (A_{\sigma_{k-d-j}} - K_{\sigma_{k-d-j}} C_{\sigma_{k-d-j}}) \right] \\
& B_{\sigma_{k-d-j}} u_{k-d-j} + \sum_{j=1}^d \prod_{s=1}^{j-1} A_{\sigma_{k-s}} B_{\sigma_{k-j}} u_{k-j}, \quad \forall k > m+d \tag{4.22}
\end{aligned}$$

with the following convention for $j = 1$:

$$\prod_{i=1}^{j-1} (A_{\sigma_{k-d-i}} - K_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) = \prod_{i=1}^{j-1} (A_{\sigma_{k-d-j}} - L_{\sigma_{k-d-j}} C_{\sigma_{k-d-j}}) = \prod_{s=1}^{j-1} A_{\sigma_{k-s}} = 1.$$

- *Two-observers based estimation:* The extended dynamical system:

$$\begin{aligned}
\xi_{k+1} &= A_{\sigma_{k-d}} \xi_k + B_{\sigma_{k-d}} u_{k-d} + L_{\sigma_{k-d}} (y_{k-d} - C_{\sigma_{k-d}} \xi_k) \\
\eta_{k+1} &= A_{\sigma_{k-d}} \eta_k + B_{\sigma_{k-d}} u_{k-d} + K_{\sigma_{k-d}} (y_{k-d} - C_{\sigma_{k-d}} \eta_k)
\end{aligned}$$

$$\begin{aligned}
\hat{x}_k = & \prod_{s=1}^d A_{\sigma_{k-s}} \mathbb{E}_m^{-1}(k-d) \left[\left(\prod_{i=1}^m (A_{\sigma_{k-d-i}} - L_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) \right)^{-1} \xi_k \right. \\
& \left. - \xi_{k-m} - \left(\prod_{i=1}^m (A_{\sigma_{k-d-i}} - K_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}}) \right)^{-1} \eta_k + \eta_{k-m} \right] \\
& + \sum_{j=1}^d \prod_{s=1}^{j-1} A_{\sigma_{k-s}} B_{\sigma_{k-j}} u_{k-j}, \quad \forall k > m+d \tag{4.24}
\end{aligned}$$

Equation (4.24) is an observer for system (4.20), which converges in finite time $m+d$.

Proof: It is easy to follow by using the methodology of Section 4.2. Indeed, using the state transformation (4.5), the dynamics of the state (4.20) is then written as follows:

$$\begin{cases} z_{k+1} = A_{\sigma_{k-d}} z_k + B_{\sigma_{k-d}} u_{k-d} \\ y_k = C_{\sigma_{k-d}} z_k \end{cases} \tag{4.25}$$

Since the matrices A_{σ_k} , B_{σ_k} , and C_{σ_k} are time-varying, then we obviously get

- $\prod_{i=1}^m \left(A_{\sigma_{k-d-i}} - L_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}} \right)$ instead of $(A - LC)^m$;
- $\prod_{i=1}^m \left(A_{\sigma_{k-d-i}} - K_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}} \right)$ instead of $(A - KC)^m$.

By analogy, we get similar assignments for the remaining terms. \square

4.3.2 Output feedback stabilization

As in the linear case, the estimation algorithms can be integrated in a closed-loop system by using a feedback controller based on the system state estimation. By analogy to the linear case, we propose the controller (4.26).

$$\begin{aligned}
u_k &= -F_{\sigma_k} \hat{x}_k \\
\hat{x}_k &= \prod_{s=1}^d A_{\sigma_{k-s}} \mathbb{E}_m^{-1}(k-d) \left[\left(\prod_{i=1}^m \left(A_{\sigma_{k-d-i}} - L_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}} \right) \right)^{-1} \xi_k - \xi_{k-m} \right. \\
&\quad \left. - \left(\prod_{i=1}^m \left(A_{\sigma_{k-d-i}} - K_{\sigma_{k-d-i}} C_{\sigma_{k-d-i}} \right) \right)^{-1} \eta_k + \eta_{k-m} \right] + \sum_{j=1}^d \prod_{s=1}^{j-1} A_{\sigma_{k-s}} B_{\sigma_{k-j}} u_{k-j}, \quad \forall k > m+d; \\
\hat{x}_k &= -F_{\sigma_k} \xi_k, \quad \forall k \leq m+d.
\end{aligned} \tag{4.26}$$

where ζ_k and η_k are given by (4.24). This leads to the closed-loop system:

$$x_{k+1} = \left(A_{\sigma_k} - B_{\sigma_k} F_{\sigma_k} \right) x_k, \quad \forall k > m+d. \tag{4.27}$$

Since (4.27) is a switched system, then eigenvalues assignment is not sufficient to guarantee stability of the system. To design the gain matrices F_{σ_k} , we need to use Lyapunov analysis to get sufficient conditions ensuring asymptotic stability of the switched system (4.27). The use of quadratic Lyapunov function $V(x_k) \triangleq x_k^\top P^{-1} x_k$ leads to the following sufficient conditions:

$$\begin{bmatrix} -P & P(A_i - B_i F_i)^\top \\ (A_i - B_i F_i)P & -P \end{bmatrix} < 0 \tag{4.28}$$

which can be converted to the following LMI:

$$\begin{bmatrix} -P & P A_i^\top - \bar{F}_i B_i^\top \\ A_i P - B_i \bar{F}_i^\top & -P \end{bmatrix} < 0 \tag{4.29}$$

where the controller gain is given by $F_i = \bar{F}_i^\top P$ for $i = 1, \dots, n_\sigma$. More details on the stability, stabilization, or observer design of switched linear systems in discrete-time can be found in [60], [61], [62], [63] and the references therein.

4.3.3 Example: a switched system

Consider a third-order system under the form (4.20) with σ_k defined as

$$\sigma_k = \begin{cases} 1 & \text{if } |\sin(\frac{\pi}{10}k)| \leq 0.25 \\ 2 & \text{otherwise,} \end{cases}$$

and

$$A_1 = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1.1 \end{bmatrix};$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix};$$

$$C_1 = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}.$$

By using pole placement technique, we obtain the gains

$$K_1 = \begin{bmatrix} 0.1100 \\ -0.0240 \\ 0.1200 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.0509 \\ 0.0030 \\ 1.6518 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} -0.2128 \\ -0.0008 \\ 0.5464 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.0621 \\ -0.0006 \\ 1.5942 \end{bmatrix},$$

with $m = 3$ and $d = 2$ the matrix $E_m(k)$ in (4.21) exists and found invertible for any $k \geq 0$. In the simulation, let $\hat{x}_k = [\hat{x}_k^1 \quad \hat{x}_k^2 \quad \hat{x}_k^3]^\top$ be the state estimates for the system by using the estimation formula (4.22). The used input is $u = 2 \times 10^3 \sin(k)$. The initial values are $x_0 = [1 \quad 1 \quad 1]^\top$ and $\hat{x}_0 = [1 \quad 1 \quad 1]^\top$. Fig. 4.5 shows that the estimation \hat{x}_k given by (4.22) reaches exactly the actual state x_k in finite-time.

4.3.4 More Illustrations: Application to a mechanical system

In this section, we will apply the proposed algorithms on the estimation and stabilization of switched systems on the mechanical model given in [64] and depicted in Fig. 4.6. In the two-mass-spring mechanical system shown below, we consider a mass M_1 sliding on a horizontal surface and attached to a vertical surface through a spring S_1 . The mass M_2 is related to the mass M_1 by a spring S_2 . The mass M_1 is subjected to an external force u which move the system from rest.

Description of the dynamical model

The continuous-time dynamics of the system is given by the following equation:

$$\begin{cases} x(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t-d) \end{cases} \quad (4.30)$$

where $x(t) = [x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)]^\top$ is the state vector, $u(t)$ is the control input, and $y(t)$ is the output measurement vector. The components of $x(t)$ contain the physical quantities related to the mechanical model:

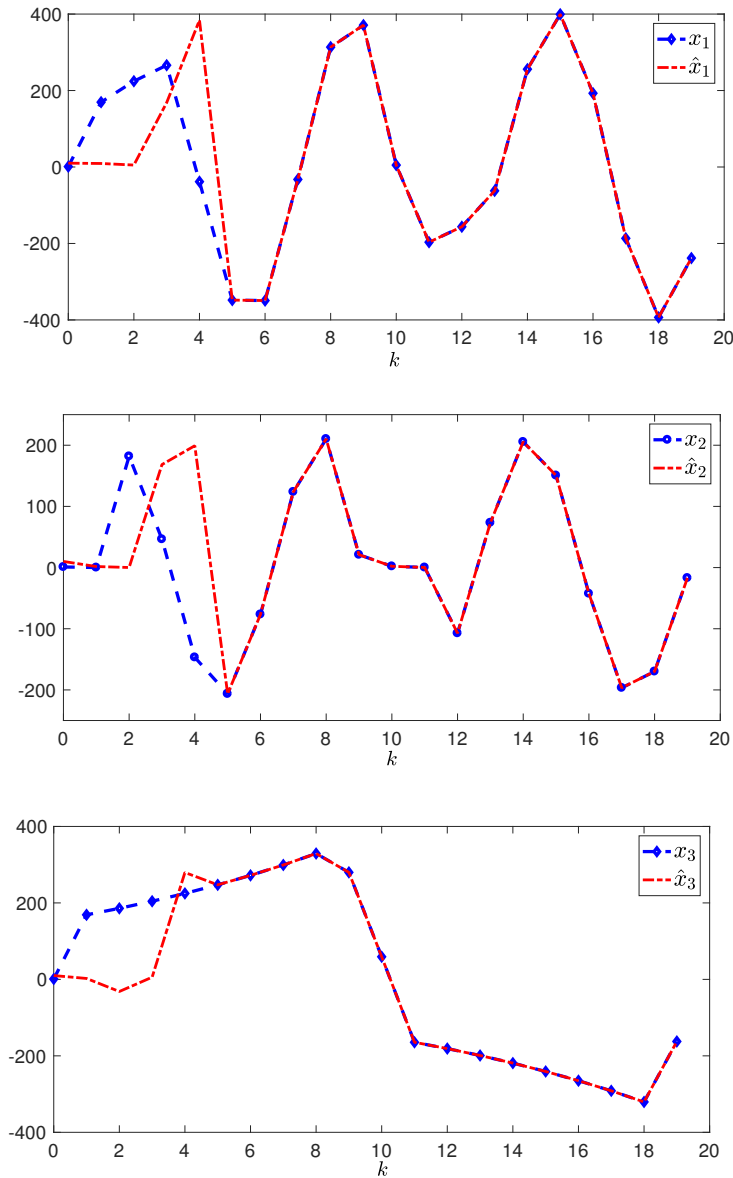


Figure 4.5: Simulation using direct estimation approach (Theorem 3)

- x_1 and x_2 are respectively the position and the velocity of the object M_1 ;
- x_3 and x_4 is the position and the velocity of of the object M_2 , respectively.

The matrices A and B are constants and given as follows:

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & 1 \\ -\frac{(k_1+k_2)}{m_1} & -\frac{b_1}{m_1} & -\frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & -\frac{b_2}{m_2} \end{bmatrix}, B \triangleq \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix},$$

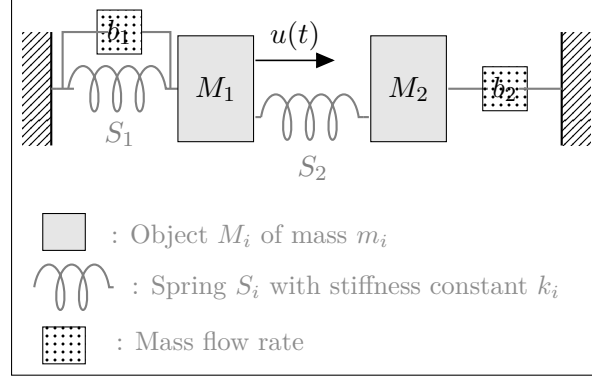


Figure 4.6: Two Masses coupled two springs Model

where the numerical values of the parameters are:

$$\begin{aligned} k_1 &= 0.0642 \text{ Kg/s}^2, \quad k_2 = 0.1925 \text{ Kg/s}^2, \\ m_1 &= 5.1962 \text{ Kg}, \quad m_2 = 1.7321 \text{ Kg}, \\ b_1 &= 0.8660 \text{ Kg/s}, \quad b_2 = 0.1732 \text{ Kg/s}. \end{aligned}$$

The matrix $C_{\sigma(t)}$ takes values in the set $\{C_1; C_2; C_3; C_4\}$ with

$$\begin{aligned} C_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \\ C_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

We assume that the choice among these values is random with equal probability.

Estimation in open-loop

First, we will apply the estimation algorithm (4.22) on the model (4.30) without feedback stabilization. Then the input $u(t)$ is taken to be equal to a sinusoidal force, i.e: $u(t) = k_u \sin(\omega(t))$, where, for each simulation run, k_u and ω are randomly chosen in $[0 \ 4]$ and $[0.1 \ 0.6]$ rad/s, respectively. The corresponding discrete-time state-space equation to (4.30) is obtained by using Euler first-order approximation for the derivative with a sampling time $T_e = 0.1s$. Using the pole placement technique, we get the following matrices $K_i, L_i, i = 1, \dots, 4$ for which the matrix $\mathbb{E}_m(k)$ is found invertible.

$$\begin{aligned} K_1 &= \begin{bmatrix} 1.0113 \\ 0.0108 \\ 0.0047 \\ 0.0301 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.1169 \\ 0.9867 \\ -0.0009 \\ 0.0003 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0.0042 \\ -0.0043 \\ 1.0044 \\ 0.0042 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 0.1199 \\ -0.0000 \\ 0.1162 \\ 0.9976 \end{bmatrix}, \\ L_1 &= \begin{bmatrix} 1.0491 \\ -0.0129 \\ 0.2805 \\ 0.2938 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.3569 \\ 1.0175 \\ -4.0069 \\ 0.3044 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0.9983 \\ -0.2060 \\ 1.0149 \\ 0.2022 \end{bmatrix}, \quad L_4 = \begin{bmatrix} 3.4747 \\ 0.0189 \\ 0.6915 \\ 0.9975 \end{bmatrix}. \end{aligned}$$

Fig. 4.7 shows the validity of the estimation algorithm since the estimated states reach the actual states in a finite-time $m + d = 6$.

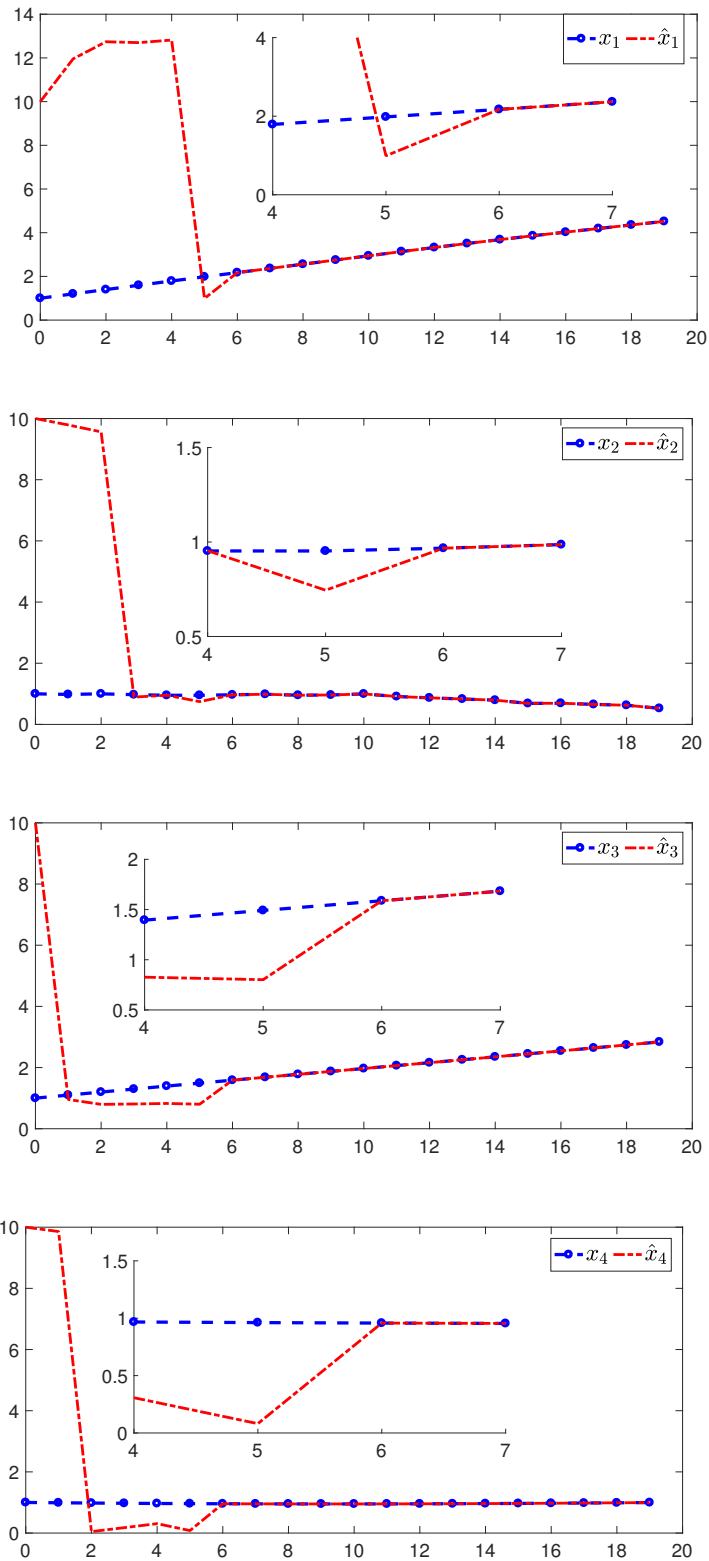


Figure 4.7: Simulation using direct estimation approach.

Stabilization by output feedback

The output feedback stabilization law (4.26) is applied on the model (4.30). We use the same values of $K_i, L_i, i = 1, \dots, 4$ found in the previous estimation step. The controller gain matrix F is found to be $F = \begin{bmatrix} 420 & 104.6 & 3512.9 & 6149.5 \end{bmatrix}$. The exact finite-time estimation based controller (4.26) ensures asymptotic stabilization of system. The proposed algorithm guarantees high performance in the stabilization of the states, as depicted in Figure 4.8.

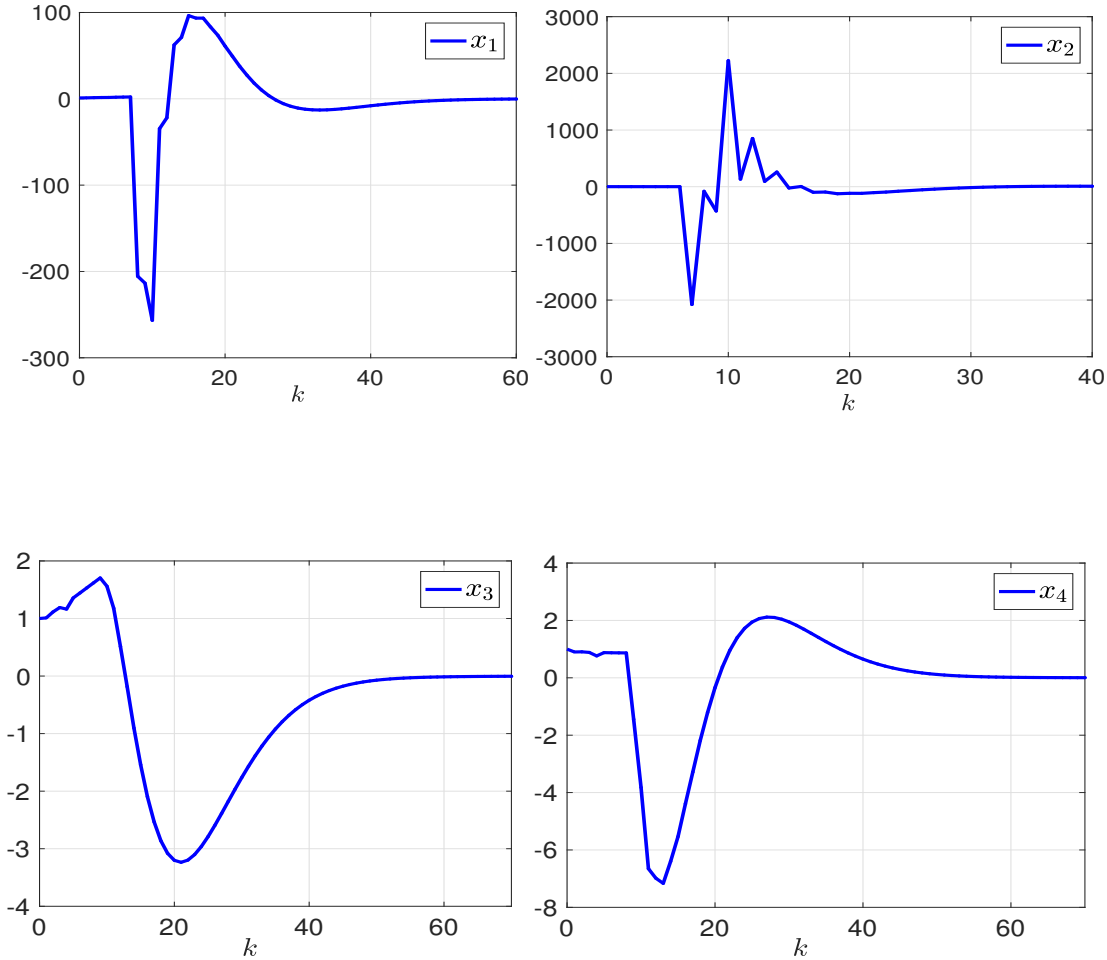


Figure 4.8: The stabilized states under the output feedback controller (4.16).

4.4 Extension to Switched Linear Singular Systems

In this section, we propose an extension of the proposed methodology to a class of switched descriptor systems. For brevity and for avoiding repetition, this section will be concerned by the explicit solution algorithm only. The two-combined observers based estimation algorithm can be obtained by using the same arguments.

4.4.1 System description

Consider the descriptor systems described by the following equations:

$$\begin{cases} E_{\sigma_k}^\zeta \zeta_{k+1} = A_{\sigma_k}^\zeta \zeta_k + B_{\sigma_k}^\zeta v_k \\ y_k = C_{\sigma_{k-d}}^\zeta \zeta_{k-d}, \end{cases} \quad (4.31)$$

where $E_\zeta, A_\zeta, B_\zeta$ and C_ζ are constant matrices of appropriate dimensions. The state vector is $\zeta_k \in \mathbb{R}^{n_\zeta}$ and the control input vector is $v_k \in \mathbb{R}^{m_\zeta}$. Let us introduce the following rank condition, which is standard in descriptor systems theory.

Assumption 4.1: [22]

The matrices $E_{\sigma_k}^\zeta$ and $C_{\sigma_k}^\zeta$ satisfy the following condition:

$$\text{rank} \left(\begin{bmatrix} E_{\sigma_k}^\zeta \\ C_{\sigma_{k+1}}^\zeta \end{bmatrix} \right) = n_\zeta, \quad \forall k \geq 0. \quad (4.32)$$

Condition (4.32) is necessary to guarantee existence of an observer for (4.31).

Before starting the mathematical developments, it is worth to notice that in this case of singular systems, the estimation of the system state ζ_k is not realizable at this stage. Herein, we will provide only exact estimation of the state vector ζ_{k-d} for any $k \geq d + m$.

4.4.2 Transformation of the system and estimation

Consider the transformation

$$\zeta_k^d \triangleq \zeta_{k-d}. \quad (4.33)$$

Since ζ_k^d satisfies (4.31), then we have

$$\begin{cases} E_{\sigma_{k-d}}^\zeta \zeta_{k+1}^d = A_{\sigma_{k-d}}^\zeta \zeta_k^d + B_{\sigma_{k-d}}^\zeta v_{k-d} \\ y_k = C_{\sigma_{k-d}}^\zeta \zeta_k^d. \end{cases} \quad (4.34)$$

System (4.34) can be rewritten under the form

$$\begin{bmatrix} E_{\sigma_{k-d}}^\zeta \\ C_{\sigma_{k+1-d}}^\zeta \end{bmatrix} \zeta_{k+1}^d = \begin{bmatrix} A_{\sigma_{k-d}}^\zeta \\ 0 \end{bmatrix} \zeta_k^d + \begin{bmatrix} 0 \\ \mathbb{I}_p \end{bmatrix} y_{k+1} + \begin{bmatrix} B_{\sigma_{k-d}}^\zeta \\ 0 \end{bmatrix} v_{k-d}. \quad (4.35)$$

From Assumption 1, the matrix

$$\begin{bmatrix} E_{\sigma_{k-d}}^\zeta \\ C_{\sigma_{k+1-d}}^\zeta \end{bmatrix}^\top \begin{bmatrix} E_{\sigma_{k-d}}^\zeta \\ C_{\sigma_{k+1-d}}^\zeta \end{bmatrix}$$

is invertible. After pre-multiplying (4.35) by $\begin{bmatrix} E_{\sigma_{k-d}}^\zeta \\ C_{\sigma_{k+1-d}}^\zeta \end{bmatrix}^\top$, and using the notation

$$\begin{bmatrix} P_{\sigma_k}^\zeta & Q_{\sigma_k}^\zeta \end{bmatrix} \triangleq \left(\begin{bmatrix} E_{\sigma_{k-d}}^\zeta \\ C_{\sigma_{k+1-d}}^\zeta \end{bmatrix}^\top \begin{bmatrix} E_{\sigma_{k-d}}^\zeta \\ C_{\sigma_{k+1-d}}^\zeta \end{bmatrix} \right)^{-1} \begin{bmatrix} E_{\sigma_{k-d}}^\zeta \\ C_{\sigma_{k+1-d}}^\zeta \end{bmatrix}^\top$$

we get :

$$\zeta_{k+1}^d = P_{\sigma_k}^\zeta A_{\sigma_{k-d}}^\zeta \zeta_k^d + \begin{bmatrix} P_{\sigma_k}^\zeta B_{\sigma_{k-d}}^\zeta & Q_{\sigma_k}^\zeta \end{bmatrix} \begin{bmatrix} v_{k-d} \\ y_{k+1} \end{bmatrix} \quad (4.36)$$

which is exactly under the form (4.25) with

$$\begin{aligned} A_{\sigma_k} &\triangleq P_{\sigma_k}^\zeta A_{\sigma_{k-d}}^\zeta, \\ B_{\sigma_k} &\triangleq \begin{bmatrix} P_{\sigma_k}^\zeta B_{\sigma_{k-d}}^\zeta & Q_{\sigma_k}^\zeta \end{bmatrix}, \\ C_{\sigma_k} &\triangleq C_{\sigma_{k-d}}^\zeta, \\ u_k &\triangleq \begin{bmatrix} v_{k-d} \\ y_{k+1} \end{bmatrix}. \end{aligned} \quad (4.37)$$

Theorem 4.4: [22]

Assume that matrices L_i and K_i , $i \in \mathcal{I}_\sigma$, and $m \geq 1$, are selected such that the matrix $\mathbb{E}_m(k)$, defined by

$$\mathbb{E}_m(k) \triangleq \left[\prod_{i=1}^m (A_{\sigma_{k-i}} - L_{\sigma_{k-i}} C_{\sigma_{k-i}}) \right]^{-1} - \left[\prod_{i=1}^m (A_{\sigma_{k-i}} - K_{\sigma_{k-i}} C_{\sigma_{k-i}}) \right]^{-1} \quad (4.38)$$

is invertible. Then the solution of (4.31) satisfies (4.39).

$$\begin{aligned} \zeta_k^d &= \mathbb{E}_m^{-1}(k) \sum_{j=1}^m \left[\left(\prod_{i=1}^m (A_{\sigma_{k-i}} - L_{\sigma_{k-i}} C_{\sigma_{k-i}}) \right)^{-1} \prod_{i=1}^{j-1} (A_{\sigma_{k-i}} - L_{\sigma_{k-i}} C_{\sigma_{k-i}}) L_{\sigma_{k-j}} \right. \\ &\quad \left. - \left(\prod_{i=1}^m (A_{\sigma_{k-i}} - L_{\sigma_{k-i}} C_{\sigma_{k-i}}) \right)^{-1} \prod_{i=1}^{j-1} (A_{\sigma_{k-i}} - K_{\sigma_{k-i}} C_{\sigma_{k-i}}) K_{\sigma_{k-j}} \right] y_{k-j} \\ &\quad + \mathbb{E}_m^{-1}(k) \sum_{j=1}^m \left[\left(\prod_{i=1}^m (A_{\sigma_{k-i}} - L_{\sigma_{k-i}} C_{\sigma_{k-i}}) \right)^{-1} \prod_{i=1}^{j-1} (A_{\sigma_{k-j}} - L_{\sigma_{k-j}} C_{\sigma_{k-j}}) \right. \\ &\quad \left. - \left(\prod_{i=1}^m (A_{\sigma_{k-i}} - K_{\sigma_{k-i}} C_{\sigma_{k-i}}) \right)^{-1} \prod_{i=1}^{j-1} (A_{\sigma_{k-j}} - K_{\sigma_{k-j}} C_{\sigma_{k-j}}) \right] B_{\sigma_{k-j}} u_{k-j}, \quad \forall k > m + d, \end{aligned} \quad (4.39)$$

where A_{σ_k} , B_{σ_k} and C_{σ_k} are given in (4.37). With the following convention for $j = 1$:

$$\prod_{i=1}^{j-1} (A_{\sigma_{k-i}} - K_{\sigma_{k-i}} C_{\sigma_{k-i}}) = \prod_{i=1}^{j-1} (A_{\sigma_{k-j}} - L_{\sigma_{k-j}} C_{\sigma_{k-j}}) = 1.$$

Remark 4.3: [22]

At this stage, we provided only a delayed estimation of the state ζ_k , namely ζ_k^d . The

difficulty to estimate ζ_k raises from the fact that the backward substitution technique cannot be applied on (4.31) because of the descriptor form of the system and since the dynamics equation (4.36) is valid only for $k - d$ and depends on y_{k+1} . This issue is a challenge we aim to solve as a future work.

4.4.3 Case of unknown input estimation

In this section, we propose an extension of the proposed methodology to a class of switched descriptor systems. Consider the systems described by the equations:

$$\begin{cases} x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} v_k + H_{\sigma_k} \mu_k \\ y_k = C_{\sigma_k} x_{k-d} + D_{\sigma_k} \mu_{k-d} \end{cases} \quad (4.40)$$

where $x_k \in \mathbb{R}^{n_x}$, $v_k \in \mathbb{R}^{n_v}$, $\mu_k \in \mathbb{R}^{n_\mu}$ and $d > 0$ are the state, control input, unknown input vectors of the system and a constant delay in the measurements, respectively. The matrix D_{σ_k} satisfies the following rank condition:

$$\text{rank}(D_{\sigma_k}) = n_\mu. \quad (4.41)$$

The objective is to find an estimator which estimates the vector state x_k and the unknown input μ_k . System (4.40) can be transformed into (4.31) with

$$\begin{aligned} E_{\sigma_k}^\zeta &\triangleq \begin{bmatrix} \mathbb{I}_{n_x} & 0 \end{bmatrix}, \quad A_{\sigma_k}^\zeta \triangleq \begin{bmatrix} A_{\sigma_k} & H_{\sigma_k} \end{bmatrix}, \\ C_{\sigma_k}^\zeta &\triangleq \begin{bmatrix} C_{\sigma_k} & D_{\sigma_k} \end{bmatrix}, \quad B_{\sigma_k}^\zeta \triangleq B_{\sigma_k}, \quad \zeta_k \triangleq \begin{bmatrix} x_k \\ \mu_k \end{bmatrix}. \end{aligned} \quad (4.42)$$

Since (4.41) implies (4.32), then, we can apply the results of Section 4.4.2 to construct a delayed estimation of the augmented state, namely $\zeta_k^d \triangleq \zeta_{k-d}$.

Remark 4.4: [22]

Since the dynamics of x_k is completely given in the first equation of (4.40), then in some situations we can build the system state x_k by using backward substitution and the exact estimation of ζ_{k-d} . Indeed, in the particular case $H_{\sigma_k} = 0$, we can construct the state x_k , $\forall k > m + d$, as follows:

$$x_k = \prod_{s=1}^d A_{\sigma_{k-s}} E_{\sigma_k}^\zeta \zeta_k^d + \sum_{j=1}^d \prod_{s=1}^{j-1} A_{\sigma_{k-s}} B_{\sigma_{k-j}} v_{k-j}, \quad (4.43)$$

where A_{σ_k} and B_{σ_k} are those given in (4.40).

4.5 Conclusion

In this chapter, we proposed two finite-time estimation algorithms for several classes of systems, namely linear systems, switched systems, and descriptor systems. The main contributions, compared to previous work in [50, 51], is the extension to systems with delayed outputs and switched systems. Both extensions are not straightforward because they need some mathematical

transformations and assumptions. The estimation algorithms are then used in closed-loop to design output feedback controller, where in switched systems case we need feasibility of a set of LMI conditions to design the controller gain matrices. An application to reference trajectory tracking is also presented. The results are applied on a linearized autonomous underwater vehicle model. In addition, the results concerning switched systems are applied on an academic example and on a four dimensional mechanical model. Finally, reference trajectory tracking results are applied to a regulation of blood glucose problem. an application.



Conclusion and perspectives

Research prospects

In this work we have established several results on the designs of observers of Liptchitzian nonlinear systems. Numerical examples were given to show the effectiveness of the proposed observers. Many questions still open in this area. As future scopes on the results presented in this manuscript.

1. We aim to extend the result obtained in Chapter 4 to systems with time-varying delays in the output measurements and to systems with uncertainties and disturbances. This will lead to the development of approximated or interval estimation instead of exact estimation.
2. We aim to extend the proposed techniques in Chapter 3 to other classes of nonlinear systems, namely feedforward systems. We also plan to reduce the conservatism of the LMIs for certain classes of systems by introducing observers with switched-gains. For the lack of space and due to the page limitation, an application to a nonlinear anaerobic digestion process is not presented in this paper to validate and highlight the results. Detailed comparisons with existing methods in the literature will be addressed in a deeper journal version of this work.



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Abstract

This work deals with the observer design for a class of nonlinear Lipschitz systems. The contribution of this thesis can be decomposed into three parts:

In the first part we develop new high-gain type observer design methods for nonlinear systems, this new designs provide lower gains compared to the high-gain observer. Firstly, we combine the improved high-gain methodology with the LMI-based observer design technique to build a more general observer that allows us to exploit the benefits of both approaches. A numerical example is given to show the effectiveness of the proposed observer with different values of the Lipschitz constant and of the compromise index.

The second part is devoted to the analysis of the feasibility of LMIs conditions within the framework of the design of observers for a class of nonlinear systems. Using some mathematical matrix decompositions, general LMI conditions ensuring the exponential convergence of the estimation error are provided. Thanks to linear and/or nonlinear transformations, these LMIs are enhanced from feasibility viewpoint. Finally, for a particular class of systems, it is shown that the proposed LMIs are always feasible, which overcomes weakness of LMI approaches related to feasibility.

The third part we deal with the problem of exact finite-time estimation for a class of switched discrete-time linear systems with delayed-output measurements. Before tackling switched systems, preliminary results on exact and finite-time estimation of linear systems with delayed outputs are established. Then an extension to a class of switched systems with delayed output measurements is provided. Two estimation algorithms are proposed. The first one is based on explicit computation of the system state by using a combination of delayed inputs/outputs, while the second algorithm uses two asymptotic observers linked by a mathematical relation. Numerical examples are given to illustrate the validity and effectiveness of the proposed algorithms.

Keywords : Observers ; Linear Matrix Inequalities (LMIs) ; Lyapunov stability ; finite time estimation

Résumé

Ce travail est consacré à la conception d'observateurs pour une classe de systèmes non linéaires Lipschitziens. Les contributions de cette thèse se décomposent en trois parties :

La première partie, présente une méthode de conception d'observateurs utilisant la méthodologie du grand-gain améliorée combinée avec la technique de conception d'observateurs basée sur les LMIs pour concevoir un observateur plus général permettant d'exploiter les avantages des deux approches. Un exemple numérique est donné pour montrer l'efficacité de l'observateur proposé avec différentes valeurs de la constante de Lipschitz qui caractérise la non-linéarité du système.

La deuxième partie est consacré à l'analyse de la faisabilité des conditions LMIs dans le cadre de la conception d'observateurs pour une classe de systèmes non linéaires. En utilisant certaines décompositions matricielles mathématiques, des conditions LMIs générales assurant la

convergence exponentielle de l'erreur d'estimation sont fournies. Grâce à des transformations linéaires et/ou non linéaires, ces LMIs sont améliorées du point de vue faisabilité. Enfin, pour une classe particulière de systèmes, il est démontré que les LMIs proposées sont toujours faisables, ce qui surmonte les faiblesses des approches basées LMIs liées à l'existence de solutions qui permettent la synthèse d'un observateur.

La troisième partie traite le problème de l'estimation exacte en temps fini pour une classe de systèmes linéaires en temps discret avec mesures de sortie retardées. Ensuite, une extension à une classe de systèmes à commutation avec mesures de sortie retardées est fournie. Deux algorithmes d'estimation sont proposés. Le premier est basé sur le calcul explicite de l'état du système en utilisant une combinaison d'entrées / sorties retardées, tandis que le second algorithme utilise deux observateurs asymptotiques liés par une relation mathématique. Des exemples numériques sont donnés pour illustrer la validité et l'efficacité des algorithmes proposés.

Mots- clés : observateurs ; inégalités matricielles linéaires (LMIs) ; stabilité de Lyapunov, estimation en temps fini.

