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Notations and acronyms

1. ***Acronyms :***
   - TDS : Time Delay system.
   - LTI : Linear Time Invariant.
   - FDE : Functional Differential Equation .
   - RDDE : Retarded Delay Differential Equation.
   - NFDE : Neutral Functional Differential Equation .
   - ODE : Ordinary differential Equation.
   - MID : Multiplicity Induced Dominancy.
   - CIR : Crossing Imaginary Root.
   - TDPS : Time Delay Positive System.

2. ***Notations :***
   - $\mathbb{R}$: The set of real numbers.
   - $\mathbb{R}_+$: Positive reals.

- $\mathbb{R}_+^\star$: Non-negative reals.
- $\mathbb{C}$: The set of complex numbers.
- $\mathbb{C}_+$: The right half complex plane.
- $\mathbb{C}_-$: The left half complex plane.
- $\Re(.)$: The real part of a complex number.
- $\Im(.)$: The imaginary part of a complex number.

Personal publications

Below is a list of publications related to the work of this thesis :

- **Journals**
  1. **Souad Amrane**, Fazia Bedouhene, Islam Boussaada and Silviu-Iulian Niculescu. On Qualitative Properties of Low-Degree Quasipolynomials : Further remarks on the spectral abscissa and rightmost-roots assignment. Bull. Math. Soc. Sci. Math. Roumanie Tome, volume : 61, pages : 361-381. 2018.

  2. **Souad Amrane**, Islam Boussaada, Fazia Bedouhene and Silviu-Iulian Niculescu. Some insights on rightmost spectral values assignment for time delay systems. to appear in IFAC-PapersOnline Journal Elsevier. 2020.

- **International Conferences**
  1. **Souad Amrane**, Fazia Bedouhene, Islam Boussaada. On some applications of center manifold theory. 2ème Colloque International sur la théorie des opérateurs, EDP et ses applications, 23-24 novembre 2016 (CITO'2016) à l'université Echahid Hamma lakhdar. EL oued.
- **National Conference**
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CHAPITRE

1

General Introduction

Contents

1.1 History and application of delay differential equation

For a very long time, people in various scientific fields have used ordinary or partial differential equations (since they were born in the late 17th century) to describe systems of which the future state was determined by the present state and the rate of change of the state. With a more and more profound understanding of the world, people gradually found out that in some cases it would be unrealistic not to consider the influence of the past on the present. At the IV International Congress of Mathematicians (Rome, April 10th 1908), Picard [109] pointed out the importance of "heredity" in the modeling of physical systems, namely the influence of the past state on the present or future state.

Volterra seemed to be the first to apply this idea. In (1931), he wrote a fundamental book on the role of hereditary effects on models for the interaction of species. In his

work [111, 112] he formulated some general differential equations incorporating the past states of systems to treat the problems about the hereditary phenomena and to model viscoelasticity in population dynamics. Unfortunately, his results were not drawing enough attention, but it gained much momentum (especially in the Soviet Union) after 1940 due to the consideration of meaningful models of engineering systems and control.
During the 1950's, the theory of functional differential equations has been developed extensively. In 1951, Mishkis published the book Linear differential equations with retarded argument, which is considered to be the first rather complete research in this area. In the '50s and '60s, several books about functional differential equations appeared [6, 7, 75, 101]. They focused on the linear systems with retarded arguments and presented a well-organized theory, which also mathematically supported research in other areas. Now delay differential equations have been applied in various fields to describe behaviours with time lags, for example, in economics [7, 8, 54, 96], in physics [63, 65, 109], in population dynamics [61, 62, 84, 111], in botanic [86, 89, 90], in epidemiology [53, 55, 60, 83], in medical science [4], even in pedagogy [72], [73].

Most research on functional differential equations (FDE) dealt primarily with linear equations and the preservation of stability (or instability) of equilibria under small nonlinear perturbations when the linearization was stable (or unstable). For the linear equations with constant coefficients, it was natural to use the Laplace transform. This led to expansions of solutions in terms of the eigenfunctions and the convergence properties of these expansions.

1.2 System with Time Delay

The problem of investigation of time-delay systems has been exploited over many years. Time-delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. A distinct feature of time-delay systems is that their evolution depends on the information of the past history with a selective memory. Such systems may be modeled using differential equations. The presence of time-delay in dynamical systems may induce complex behavior, and this behavior is not always intuitive. Even if a system's equation is scalar, oscillations may occur.
The study of oscillatory/ non oscillatory solutions of delay differential equation has been the focus of numerous contributions, see for instance [1, 6, 70] and [22] where the strongest deal is with linear delay differential equations with constant coefficients and constant delay. In this case necessary and sufficient conditions are given for all solutions to be oscillatory, for the case of non constant coefficient and non constant delay and for nonlinear equations, typical results give conditions sufficient for all solutions to be oscillatory, or conditions sufficient for some solutions to be non oscillatory see for instance [23] and [77].

Time-delay in control loops are usually associated with degradation of performance and robustness, but at the same time, there are situations where time-delay are used as controller parameters. So, time delay has also been purposely introduced in control design for enhancing stability or performance. Early contributions in this direction can be traced to the work of Smith [13]. Recently, several results have been published in this context, see for example [49].

1.3 Practical Example

Before discussing time delay systems in more details, let us start by giving an example to give the reader a glimpse on how widely time delays may occur in practice.

Example 1.3.1. *To illustrate the widespread presence of time delays, consider the water temperature regulation problem in the shower, which we encounter almost every day and is a simple example of a dynamical system with a propagation delay. This example was taken from [24].*
When taking a shower one has to mix warm and cold water to obtain the right water temperature. However, when one adds, for example, more warm water to increase the perceived temperature of the water, it can take some time before this change is noticeable because the water takes some time to reach the showerhead. An impatient person who does not anticipate this delay, will increase the water temperature too much. This overshoot has to be corrected, but now impatience will cause the water temperature to become too low, etc. Fortunately, experience has told us how to deal with this delay and, hence, how to avoid large temperature deviations. The water temperature regulation problem in the shower is illustrated in Figure 1.1.

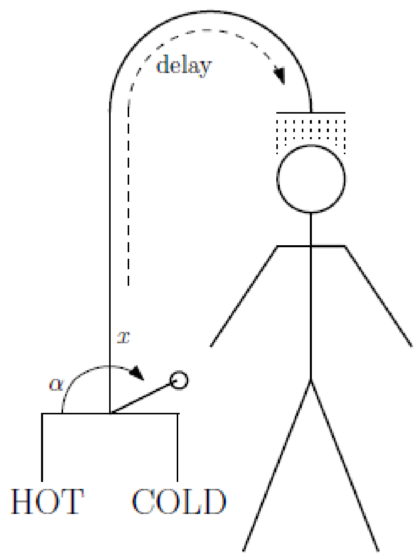


FIGURE 1.1 – The water temperature regulation problem in the shower taken from [24].

The example just given can be described in the form of a mathematical model using FDEs presented as follows :

1. *Let x_d be the desired value we would like the perceived water temperature to reach.*
2. *Let $x(t)$ denote the water temperature in the mixer output.*
3. *Let τ be the time needed by the water to go from the mixer output to the person's head.*
4. *Suppose that the change in water temperature at the mixer output, i.e., where the warm and cold water are mixed, is proportional to the change of the mixer angle α, which influences the ratio of warm and cold water only and not the flow of water, via some constant c.*
5. *Assume that the rate of rotation of the handle is proportional to the deviation in water temperature from x_d perceived by the person via coefficient k (which depends on the person's temperament).*
6. *At time t the person feels the water temperature leaving the mixer at time $t-\tau$, which results in the following equation with the constant delay τ :*

$$\dot{x}(t) = -ck\big[x(t-\tau) - x_d(t)\big], \quad t \in \mathbb{R}. \tag{1.1}$$

1.4 Stability Analysis of Time Delay Systems

The problem of stability analysis of time-delay systems has received considerable attention and many papers dealing with this problem have appeared. In the literature, various stability analysis techniques have been utilized to derive stability criteria for asymptotic stability of the time-delay systems by many researchers. The developed stability criteria are classified often into two classes : time-domain and frequency-domain approaches.

In the first approach, we have the comparison principle based techniques for functional differential equations [25], [27] and respectively the Lyapunov stability approach with the Krasovskii and Razumikhin based methods [77], [28]. The stability problem is thus reduced to one of finding solutions to Lyapunov [29] or Riccati equations [30], solving linear matrix inequalities (LMIs) [31–34] or analyzing eigenvalue distribution of appropriate finite-dimensional matrices [35].
Most results in the literature use matrices for presenting the Lyapunov functional and formulate the stability condition as LMIs, which can be efficiently solved as a convex programming problem. However, the infinite dimensional nature of the Lyapunov functional means any presentation using a small number of decision variables will in general introduce conservativeness.

In the second approach, the stability of LTI time-delay systems depends exactly on the roots of the associated characteristic equation. The system is asymptotically stable if and only if all the roots of the associated characteristic equation are located on the left half

complex plane. This observation leads to the spectrum-based approaches. Due to the fact that the characteristic equation of time-delay systems are not polynomials, computing the rightmost characteristic root is a challenging problem. Several approaches for numerically computing the rightmost characteristic roots exist [99], but intensive computation is required. To avoid these difficulties, different methods have been proposed in the literature, as an example we cite the root locus methods which consists to determine the number of characteristic roots on the right complex half plane for a given parameter domain without knowing the exact locations of characteristic roots, and among this method we have D-decomposition method and τ-decomposition method.
The main idea of D-decomposition approach is to separate the parameter space into disjoint sub-regions. In each sub-region, the number of unstable roots is constant. On the boundary of the sub-regions, imaginary characteristic roots appear. By analyzing the behaviors of these imaginary characteristic roots with respect to small variation of parameters, the system stability can be determined for different sub-regions in the parameter domain. The τ-decomposition methods [26] can be viewed as a special case of the D-decomposition methods as the parameter involved in this case is the delay τ.

Among these methods we can also find a new result discussed in recent studies which focus on the stability and stabilizing controllers design for linear time invariant retarded time delay systems where the stability analysis is done using an interesting property, called Multiplicity Induced-Dominancy (MID in the sequel). As a matter of fact, it is shown that multiple spectral values for Time-delay systems can be characterized using a Birkhoff/Vandermonde based approach ; see for instance [45–47, 52]. More precisely, in [46], it is shown that the admissible multiplicity of the zero spectral value is bounded by the generic Polya and Szegö bound denoted PSB, which is nothing but the degree of the corresponding quasipolynomial (i.e the number of the involved polynomials plus their degree minus one), see for instance [16]. In [47], it is shown that a given crossing imaginary roots with non vanishing frequency never reaches PSB and a sharper bound for its admissible multiplicities is established. Moreover, in [52], the variety corresponding to a multiple root for scalar Time-delay equations defines a stable variety for the steady state. The multiplicity of a root itself is not important as such but its connection with the dominancy of this root is a meaningful tool for control synthesis. An example of a scalar retarded equation with two delays is studied in [47] where it is shown that the multiplicity of real spectral values may reach the PSB. In addition, the corresponding system has some further interesting properties : (i) it is asymptotically stable, (ii) its spectral abscissa (rightmost root) corresponds to this maximal allowable multiple root located on the imaginary axis. Such observations enhanced the outlook of further exhibiting the existing links between the maximal allowable multiplicity of some negative spectral value reaching the quasipolynomial degree and the stability of the trivial solution of the corresponding dynamical system. This property induced from multiplicity appears also in optimization problems since such a multiple spectral value is nothing but the rightmost root, see also [17, 18]. Also notice that the property was already observed in [19], where a tuning strategy is proposed for the design of a delayed Proportional-Integral controller by placing a triple real dominant root for the closed-loop system. However, the dominancy is only checked using a Mikhailov curve and QPmR toolbox, see for instance [20]. To the best of our knowledge, the first time an analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented in [21]. The dominancy property is further explored and analytically shown in the case of second-order systems and a rightmost root assignment based design using delayed state-feedback is proposed in [41, 42] where its applicability in damping active vibrations for a piezo-actuated beam is proved. See also [48, 50] which exhibit an analytical proof for the dominancy of the spectral value with maximal multiplicity for second-order systems controlled via a delayed proportional-derivative controller.

1.5 Objectives of the thesis

This thesis is devoted to stability analysis of linear time delay systems with single delay. The main objectives consists to address the problem of the spectral abscissa characterization and the coexistence of non oscillating modes, which are not necessary multiple for retarded time delay systems. This study is mainly inspired by the recent results of Professor Boussaada see [45], [47], [42], [41], [48], [51], where the property Multiplicity Induced Dominancy (MID) is investigated. So, by this dissertation we investigate the effect of coexistence of real roots on the stability of the trivial solution. In other words, we give necessary and sufficient conditions for the coexistence of sufficiently many negative non oscillating roots, guaranteing the location of the remaining roots in the left half plane.
The dominancy of such non oscillating modes is analytically shown for the considered reduced order time delay systems (scalar and second order delay differential equation), using an adequate factorization, which allows to write the quasipolynomial in an some appropriate integral form and find its right most roots. furthermore, if they are negative, this guarantees the asymptotic stability of the trivial solution. The stability analysis in our case is difficult and complicated compared to the MID case due to the number of parameters to handle.

The drawback of the methods mentioned previously is that in some cases a factorization of the quasipolynomial function is hard to be established, especially for the quasipolynomial with higher order. To overcome this difficulties the dominancy property can be shown using the stability criteria proposed and developed in [110] and generalized later in [82], which allows to compute analytically the number of characteristic roots in the right-half complex plane, (i.e., unstable roots) for time-delay system of retarded type. Such idea is exploited in the context of delayed output feedback by an appropriate partial pole placement for generic linear second order time-invariant retarded system with single delay. Our objective here is to construct an appropriate delay state feedback controller allowing to guarantee the stability of the system in the closed loop. To do this, we use the stability criterion based on the manifold defined by the coexistence of maximal number of negative spectral values and Stépàn-Hassard formula.
To show the importance of the work proposed in this thesis, we summarize the contributions in the following items

1. Develop efficient methods for precise stability of linear time-invariant retarded time delay systems with single delay in frequency domain approach.

2. Acquire a deeper understanding of the connection between stability of the system and the manifold characterizing the coexistence of non oscillating modes of reduced order time delay systems.

3. The destabilizing effect of the delays is well known, but it can also have a stabilizing effect which is exploited in this thesis, by stabilizing a second order system without delay using a delayed control law (Proportional Derivative controller, PD in the sequel).

4. Apply the proposed stability analysis to study the problem of stabilizing the Mach number of a transonic flow in a wind tunnel.

1.6 Outline of the thesis

This thesis consists of three principal chapters. In what follows, more details about the content of each part :

Chapter 2 : Devoted to present some useful preliminaries and to remind of some essential definitions needed to better understand the manuscript. In addition, it contains a description background to time-delay systems as well as an overview of some existing methods in frequency domain approach.

Chapter 3 : This chapter dedicated to the stability analysis of time delay system with single delay, based on the spectrum distribution of the corresponding characteristic equation. The main contribution of this chapter consists in proposing a design approach which is a delayed output feedback where the candidates' parameters result from the manifold defined by the coexistence of an exact number of negative spectral values, which is equal to the degree of the quasipolynomial, which guarantees the asymptotic stability of the system's solutions. So, first we investigate the coexistence of non oscillatory modes for the scalar differential equation. Next, second order delay equation with three real spectral values is considered. The stability of this class of systems is guaranteed using the property of the dominancy which is shown using an adequate factorization. At the end of this chapter, further description of the spectrum description is presented using the W-Lambert function.

Chapter 4 : This part focus on the study of generic second order systems with single delay. Unfortunately, the method discussed in the previous chapter can not be applied for this class of system since we are not able to find an adequate factorization which allows to proof the dominancy. So, to deal with the stability of this kind of system we use an appropriate state feedback controller allowing to guarantee the stability in the closed loop. The dominancy property is shown using the Stépàn-Hassard formula based on the argument principle, which allows to compute the number of roots in the right-half.

CHAPITRE

2 Generalities and previous literature

Contents

2.1 Introduction

The aim of this chapter is to represent both the necessary and crucial reminders for the comprehension of this thesis, and previous literature of the different methods for the stability analysis of time-delay systems which is known also as hereditary systems or systems with memory, with aftereffect, or time lag, etc. This kind of systems have been widely encountered when modeling phenomena in both scientific and engineering disciplines largely due to the time needed to transport material or information. The evolution of time-delay systems depends on the information of the past history with a selective memory. Such systems may be modeled using differential equations on abstract or functional spaces. The most way is to describe time-delay systems as functional differential equations(FDE). So, we start by introducing some properties of functional differential equations, there exists different type of this function, but we are only interested by the study of retarded functional differential equations RFDEs and neutral differential equation NFDEs. We state briefly the stability criteria in time domain approach. Most important results about the zeros characteristic function as well as a survey of the basic methods of stability in frequency domain will be given. Finally, we close this chapter by exposing some results on the oscillations of time delay systems.

2.2 Functional Differential Equation

Functional differential equation find use in mathematical models that assume a specific behavior or phenomenon depends on the present as well as the past state of a system. In other words, past events explicitly influence future result. For this reason, functional differential equation are used in many applications rather than ordinary differential equation.
There exists different types of FDEs that are used in numerous applications, namely retarded functional differential equations (RFDEs), neutral functional differential equations (NFDEs) and advanced functional differential equations (AFDEs). The classification depends on whether the rate of change of the current state of the system depends on past values, future values or both, see [97]. So, we have the following classification.

– When the rate of change of the state depends on the present and past values of the system state, the model corresponds to a retarded functional differential equation (RFDE), and it is described in general as follows :

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau)\big).$$

– When the rate of change of the state depends not only on the present and past values of the system state but also on the earlier value of the state rate change (under appropriate constraints on the corresponding delay difference operator), the corresponding equation is of neutral type, also known as neutral functional differential equation (NFDE), and it can be written as follows :

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), \dot{x}(t-\tau)\big).$$

– When the rate of change of state is determined by future values of the state, the system is described by an advanced functional differential equation (AFDE), and it is given by :

$$\dot{x}(t-\tau) = f\big(t, x(t), x(t-\tau)\big).$$

Notice that, AFDE is rarely used in practical applications.

The delay phenomena may occur on functional differential equation in different ways. In what follows, we introduce some kind of the delays.

2.3 Categories of Delay

There exists two main ways in which the delay is widely used in the literature : in a punctual way, we speak then about systems with punctual (or discrete) delays or in a continuous way leading to the so-called distributed delay systems. So, The way to treat time-delay systems differs according to the type of delay. let us start by :

Systems with discrete delay

Time-delay system with discrete delay that has been largely used is described as follows :

$$\dot{x}(t) = \sum_{i=0}^{n} A_i x(t - \tau_i).$$

This class of system are largely studied over the years and conditions for stability and stabilization have been formulated, see [92], [100] and references therein.
Based on the relation between the delays, we can distinguish two types of discrete delays : commensurate delays and incommensurate delays.

1. **Commensurate :** $\tau_i \in \mathbb{R}$, $i \in \mathbb{N}$ are commensurate if τ_i/τ_j is rational, which corresponds to finding a minimal delay τ such that $\tau_i = i\tau$, then the system becomes as follows :
$$\dot{x}(t) = \sum_{i=0}^{n} A_i x(t - i\tau).$$
The characteristic equation associated to this class of system have the same algebraic properties as the system with single delay see [103]. Hence, the stability problem can be treated similarly to the single delay case. In [88], the authors study the systems with commensurate delay depend coefficient, the stability analysis method are developed based on the corresponding characteristic equation following a generalized τ-decomposition approach.

2. **Incommensurate :** The delays $\tau_i \in \mathbb{N}$ are free parameters. Several studies exists in the literature for stability and stabilization of systems with this kind of delay. In recent results [48], [50] the asymptotic stability of this type of systems is studied using the new property namely multiplicity induced dominancy (MID).

Systems with distributed delay

$\int_{s-\tau}^{t} x(s)ds$: this type of delay is widely studied in the literature. See for example [110] for stability of time delay systems.
Notice that, we are interested only in this thesis by retarded and neutral functional differential equations with discrete delay.

State-dependent delay

The delay is presented as a function of the state of the system, it arise frequently in applications as model of equations and for this reason the study of this type of equation has received a significant amount of attention in the last years, we refer to [66].

2.4 Retarded Delay Differential Equation RDDE

Retarded delay differential equation (RDDE) is an equation where the evolution of the system at a certain time t depends on the state of the system at an earlier time $t-\tau$. This is distinct from ordinary differential equations (ODEs) where the derivatives depend only on the current value of the independent variable. So, a simple ODE might be written as follows :

$$\dot{x}(t) = ax(t),$$

while an example of a simple DDE is

$$\dot{x}(t) = ax(t-\tau).$$

To solve this equation at the moment t, it is obvious that it is necessary to know the value of the system at time $t=-\tau$, but in order to calculate the solution of system at time $t=\tau$ we need to know the value of system at time $t=0$. Consequently, to know the complete solution in the interval $[0,\tau]$, we need to know the whole initial solution (condition) in the interval $[-\tau,0]$. Therefore, in what follows we first have to prescribe such an initial condition. The following results can be found in [9].

2.4.1 Initial value problem and stability definitions

Let us consider the vector space $\mathscr{C}([a,b],\mathbb{R}^n)$ which is a Banach space of continuous function that maps the interval $[a,b]$ to $\mathbb{R}^n$. More precisely, for $[a,b]=[-\tau,0]$, let $\mathscr{C} := \mathscr{C}([-\tau,0],\mathbb{R}^n)$ to denote the state space. For a function $\phi \in \mathscr{C}$, the continuous norm $\|.\|_s$ is given by :

$$\|\phi\|_s = \max_{-\tau<\theta<0} \|\phi(\theta)\|,$$

where $\|.\|$ denotes the Euclidean norm. For any positive number A and continuous function $\phi \in \mathscr{C}([t_0-\tau, t_0+A])$, we use ϕ_t to denote a segment of the function ϕ defined as $\phi_t(\theta) = \phi(t+\theta)$, $\forall\theta \in [-\tau,0]$.
The general form of an autonomous retarded functional differential equation is

$$\dot{x}(t) = f(x_t), \tag{2.1}$$

where the functional $f : \mathscr{C} \to \mathbb{R}^n$ is continuous.
The asymptotic behavior of functional differential equations is an important property in many applications and it is studied using a several methods. Moreover, it is known that the stability of a DDEs is defined as the stability of (so called) null solution and we have the following definitions.

Definition 2.4.1. *[9]. The solution* $x_t = 0$ *of equation* (2.1) *is said to be :*

1. *Stable if and only if* $\forall \varepsilon > 0$, $\forall \phi \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n)$ $\exists \delta > 0$ *such that :*
$$\|\phi\|_s < \delta \Rightarrow \forall t \geq 0 \quad \|x_t(\phi)\|_s < \varepsilon.$$
2. *Asymptotically stable if and only if it is stable and*
$$\lim_{t \mapsto \infty} x(\phi)(t) = 0.$$
3. *Exponentially stable if and only if for any initial condition* $\phi \in \mathscr{C}$, $\exists\, C > 0$ *and* $\gamma > 0$ *such that :*
$$\|x_t(\phi)\|_s \leq Ce^{-\gamma t}\|\phi\|_s.$$

2.4.2 Existence and uniqueness results

In what follows we present some classic results on the existence and uniqueness of solution of the equation (2.1), which can be found in [79] and [110].

Definition 2.4.2. *[110]. A function* f *is said to be Lipschiz function if there exists a positive real constant* K *such that for all* $\phi_1, \phi_2 \in \mathscr{C}$
$$\|f(\phi_1) - f(\phi_2)\| < K\|\phi_1 - \phi_2\|, \quad \forall\, t \in [0, +\infty).$$

Definition 2.4.3. *[79]. For a given* $\phi \in \mathbb{C}$, $t_0 \in \mathbb{R}$, *a solution of the equation* (2.1) *is a function denoted by* $x(t)$ *such that* $x(t) = \phi(t)$ *if* $t \in [t_0 - \tau, t_0]$ *satisfying* (2.1) *for* $t \in [t_0, t_0 + A]$ *with* $A > 0$. *Such a function is said solution of* (2.1) *with initial value* ϕ *at* t_0 *or simply solution through* (t_0, ϕ) *and it is often noted by*
$$x(t) = x(t_0, \phi, f).$$

Theorem 2.4.4. *[79](Existence). suppose* Ω *an open subset in* $\mathbb{R} \times \mathbb{C}$ *and* $f \in \mathbb{C}(\Omega, \mathbb{R}^n)$. *If* $(t_0, \phi) \in \Omega$ *then there exists a solution of equation* (2.1) *passing through* (t_0, ϕ).

Theorem 2.4.5. *[79](Uniqueness). Suppose* Ω *is an open set in* $\mathbb{R} \times \mathbb{C}$ *and* $f : \Omega \to \mathbb{R}^n)$ *is continuous and* $f(t, \phi)$ *is lipschizian in* ϕ *in each compact set in* Ω. *If* $(t, \phi) \in \Omega$ *then there is a unique solution of equation* (2.1) *through* (t_0, ϕ).

Notice that, in this thesis we are only interested by the study of autonomous linear functional differential equations with discrete delays. So we linearize (2.1) about the origin which leads to a linear time delay system of the following form :
$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i), \tag{2.2}$$
where $A_i \in \mathbb{R}^{n \times n}, i = 0, 1 \ldots, m$ are real matrices, $x(t) \in \mathbb{R}^n$ is state variable at time t and $0 < \tau_1 < \tau_2 < \ldots < \tau_m$ represents the time delays. For an initial condition $\phi \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n)$ the solution of (2.2) is defined as follows :
$$x(\phi)(\theta) = \phi(\theta), \quad \theta \in [-\tau_m, 0].$$

The state at time t is given by the function segment $x_t(\phi) \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n)$ defined as :

$$x_t(\phi)(\theta) = x(\phi)(t+\theta), \quad \theta \in [-\tau_m, 0].$$

This solution can be defined in the other way using the operator $T(t)$ which gives :

$$T(t)\phi(\theta) = x_t(\phi)(\theta) = x(\phi)(t+\theta), \quad \theta \in [-\tau_m, 0].$$

This is a strongly continuous semigroup, the infinitesimal generator of which is $A\phi = \frac{d\phi}{d\theta}$ with the domain :

$$D(A) = \left\{\phi \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n) : \frac{d\phi}{d\theta} \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n), \quad \phi(0) = A_0(\phi) + \sum_{k=1}^{m} A_k \phi(-\tau_m)\right\}.$$

2.4.3 Characteristic Equation

The characteristic equation for an homogeneous linear functional differential equation with constant coefficients is obtained from the equation by looking for nontrivial solutions of the form ce^{ts} with $c \in \mathbb{C}^{n\times 1} \setminus \{0\}$. The characteristic function associated to the system (2.2) is given by :

$$\Delta(s, \tau) := \det\left(sI - A_0 - \sum_{i=1}^{m} A_i e^{-s\tau_i}\right). \tag{2.3}$$

Example 2.4.6. *The scalar equation :*

$$\dot{x}(t) = -ax(t) - bx(t-\tau), \tag{2.4}$$

has a no trivial solution of the form ce^{st} if and only if :

$$cse^{st} = ace^{st} + bce^{s(t-\tau)}$$

which implies that :

$$s = a + be^{-s\tau},$$

hence

$$f(s) = s - a - be^{-s\tau} = 0. \tag{2.5}$$

$f(s) = 0$ is called the characteristic equation associated to (2.4).

Definition 2.4.7. *For the delay differential equation DDE* (2.2)*, we have the following definitions :*

– *If $s \in \mathbb{C}$ satisfies the characteristic equation :*

$$\Delta(s, \tau) = 0, \tag{2.6}$$

we say that s is a characteristic root.

– *The set of all solutions of* (2.6) *is called spectrum of DDE* (2.2) *and denoted by :*

$$\sigma := \{s \in \mathbb{C},\ \Delta(s, \tau) = 0\}.$$

– *The spectral abscissa corresponding to the system* (2.2) *is defined as the real part of rightmost eigenvalues of the corresponding supremum i.e*

$$\alpha(\tau) = \sup\{\Re(s) : \Delta(s,\tau) = 0\}.$$

The stability analysis of (2.2) is equivalent to the problem of determining conditions under which all roots of its characteristic equation lie in the left-half complex plane and no sequence of characteristic roots approaches the imaginary axis.
In what follows, we introduce a proposition which plays an important role in the study of continuity properties of the spectrum of time-delay systems and allows to construct an envelope curve around the characteristic roots of the quasipolynomial (2.3). For more details see [9].

Proposition 2.4.8. *[9]. If s is a spectral value corresponding to* (2.2) *then it satisfies :*

$$|s| \leq \|A_0\|_2 + \sum_{i=1}^{m} \|A_i\|_2 \, e^{\Re(s)\tau_i}.$$

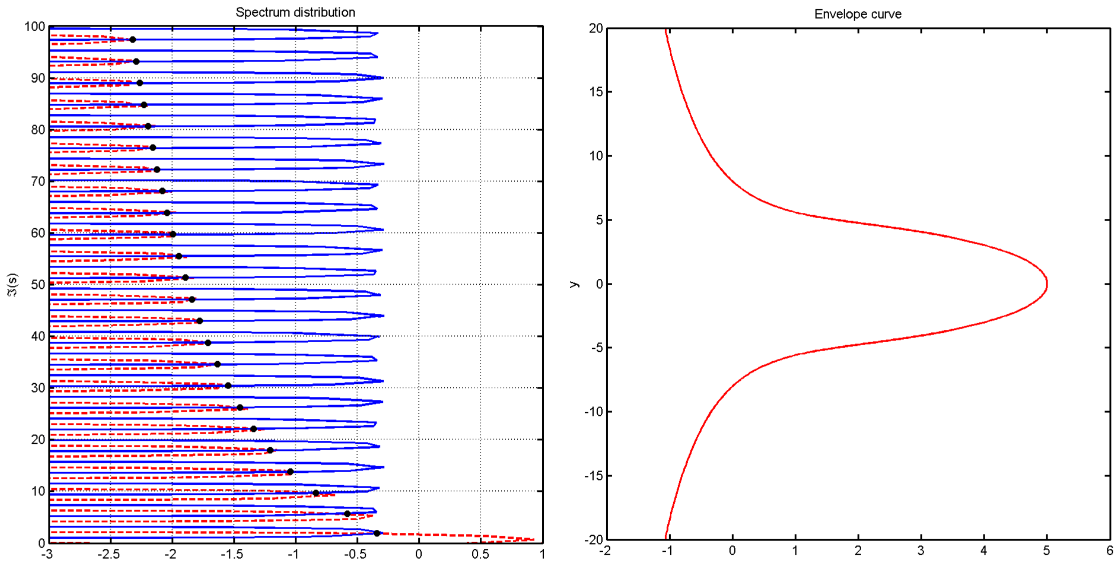


FIGURE 2.1 – (Left) Spectrum distribution of (2.5) for $a = -2$ and $b = 3$.(Right) Envelope of characteristic roots of (2.5) for the same value of a and b, illustrated using QPmR toolbox from [20].

Properties of Characteristic Roots

The presence of the exponential term $e^{-s\tau}$, induced by time-delay term $x(t-\tau)$ make the characteristic equation transcendental (i. e, infinite dimensional), so it has an infinite number of roots. Equation of this type are studied in [103], [9] and we have the following results.

Proposition 2.4.9. *If there exists a sequence* $\{s_k\}_{k\geq 1}$ *of characteristic roots of* (2.2) *such that*

$$\lim_{k\to\infty} |s_k| \to +\infty,$$

then

$$\lim_{k\to\infty} \Re(s_k) \to -\infty.$$

Corollaire 2.4.10. *The roots of* (2.3) *have the following properties :*
- *There are only a finite number of characteristic roots in any vertical strip of complexes plane, given by*
$$\{s \in \mathbb{C},\ \gamma_1 < \Re(s) < \gamma_2\},$$
with $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_1 < \gamma_2$.
- *There exists a number* $\gamma_0 \in \mathbb{R}$ *such that all characteristic roots are confined to the half plane*
$$\{s \in \mathbb{C},\ \Re(s) < \gamma_0\}.$$

According to the previous corollary, we deduce that the spectral abscissa always exists and is finite. So there always exists a rightmost root s such that :

$$\Re(s) = \alpha(\tau).$$

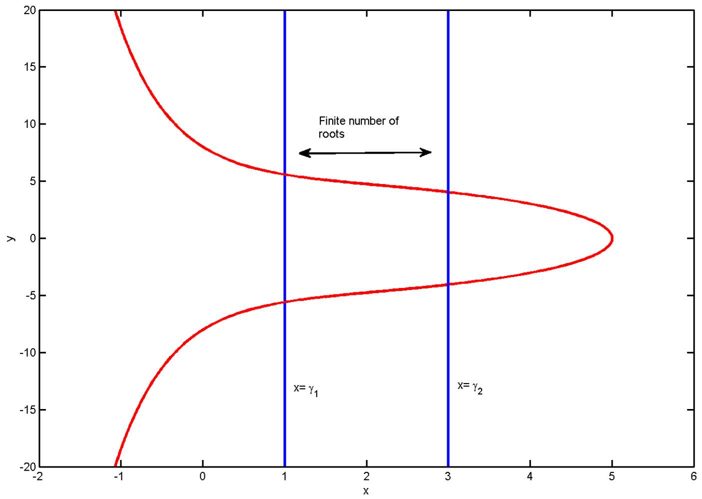


FIGURE 2.2 – Envelope curve of the characteristic equation with $a =$, $b =$, and $\tau = 0.5$

Hence, in term of the supremum the asymptotic stability of RDDE (2.2) is equivalent to that the spectral abscissa is negative i.e

$$\alpha(\tau) < 0.$$

Notice that the spectral abscissa of retarded functional differential equation is continuous, see (Theorem. 1.14 [9]).

Remark 2.4.11. *There is a relationship between the spectra of the linear operator A, $T(t)$ and the characteristic roots that is :*

– *The characteristic roots are the eigenvalues of the operator A, and it is also known that $\sigma(A) = P\sigma(A)$ (point spectrum)*

– *The spectra of A and $T(t)$ are related as follows :*

$$\sigma(T(t)) = \exp(t\,\sigma(A)) \quad \textit{plus possibly } \{0\}.$$

If $s \in \sigma(A)$ then $e^{st} \in P\sigma(T(t))$. Conversely, If $z(t) \in \sigma(T(t))$ and $z(t) \neq 0$ then there exists $s \in \sigma(A)$ such that $z(t) = e^{st}$.

Remark 2.4.12. *The stability analysis of* (2.2) *can be also determined by the spectral radius $r_\sigma(T(1))$. Hence the null solution of* (2.2) *is exponentially stable if and only if*

$$r_\sigma(T(1)) < 1.$$

For more detail see [9] and [103].

2.5 Neutral Functional Differential Equation (NFDE)

In this section, we introduce an another class of equations depending on past as well as present value which involve derivatives with delays as well as the function itself. Such equations are called neutral delay differential equations (NDDEs) (or neutral functional differential equations(NFDEs)). In order to establish this class of systems, we need to introduce the following definition from [77].

Definition 2.5.1. *[77]. Suppose $\Omega \subseteq \mathbb{R} \times \mathbb{C}$ is open with elements (t,ϕ). A function $D : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if D is continuous together with its first and second Frechet derivatives with respect to ϕ and D_ϕ, the derivative with respect to ϕ is atomic at β on Ω.*

The atomicity specifies the manner in which $D(t,\phi)$ varies with ϕ. Now, we can define a general class of neutral delay differential equation. Let us consider a Banach space $\mathscr{C}$ and $\Omega \subseteq \mathbb{R} \times \mathscr{C}$ then this equations can be described as follows :

$$\frac{d}{dt}D(t,x_t) = f(t,x_t), \tag{2.7}$$

where $f : \Omega \to \mathbb{R}^n$, and $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D is atomic at zero. The function D is called difference operator for NDDE(D,f).

Example 2.5.2. *If $\tau > 0$, B is an $n \times n$ constant matrix, $f : \Omega \to \mathbb{R}^n$ and*

$$D(\phi) = \phi(0) - B\phi(-\tau)$$

is continuous then the pair (D, f) defines an NDDE of the form :

$$\frac{d}{dt}[x(t) - Bx(t-\tau)] = f(t,x_t).$$

2.5.1 Existence and uniqueness

In this part, we consider the questions of existence and uniqueness of solutions of NFDEs. This results can be found in [79] and [103].

Definition 2.5.3. *[79]. For a given NFDE(D,f), a function x is said to be a solution of the NFDE(D,f) if there are $t_0 \in \mathbb{R}$, $A > 0$ such that :*

$$x \in \mathscr{C}([t_0-\tau, t_0+A), \mathbb{R}^n), \quad (t, x_t) \in \Omega, \quad t \in (t_0, t_0+A),$$

$D(t, x_t)$ is continuously differentiable and satisfies equation (2.7) on $[t_0, t_0+A)$.
For a given $t_0 \in \mathbb{R}$, $\phi \in \mathscr{C}$ and $(t_0, \phi) \in \Omega$, we say that $x(t_0, \phi, D, f)$ is a solution through (t_0, ϕ) if there is an $A > 0$ such that on $[t_0-\tau, t_0+A)$ and $x_{t_0}(t_0, \phi, D, f) = \phi$.

Theorem 2.5.4. *[79](Existence). If Ω is an open set in $\mathbb{R} \times \mathscr{C}$ and $(t_0, \phi) \in \Omega$, then there exists a solution of NFDE(D,f) through (t_0, ϕ).*

Theorem 2.5.5. *[79](Uniqueness). If $\Omega \subseteq \mathbb{R} \times \mathscr{C}$ is open and $f : \Omega \to \mathbb{R}^n$ is lipschitzian in ϕ on compact sets of Ω, then for any $(t_0, \phi) \in \Omega$ there exists a unique solution of NFDE(D,f) through (t_0, ϕ).*

2.5.2 Characteristic equation

The same thing as the case of retarded functional differential equations, we are interested in a particular class of NFDE(D,f) having a linear difference operator. Thus in the sequel, we consider the following representations.

$$\frac{d}{dt}\left(x(t) + \sum_{k=1}^{m} H_k x(t-\tau_k)\right) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t-\tau_k), \tag{2.8}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t and $0 < \tau_1 < \tau_2 < \ldots < \tau_m$ represent the time-delays. The characteristic equation of (2.8) can be obtained by substituting a simple solution of the form ce^{st} where $c \in \mathbb{C}^{n\times 1}$ 0 in (2.8) which leads to the following equation :

$$\Delta_N(s,\tau) = \det\left(s\left(I - \sum_{k=1}^{m} H_k e^{-s\tau_k}\right) - A_0 - \sum_{k=1}^{m} A_k e^{-s\tau_k}\right). \tag{2.9}$$

The zeros of (2.9) are called the characteristic roots of (2.8). The stability definitions are similar for DDEs of retarded type.
The associated delay difference equation of (2.8) is given by :

$$D(x_t) = x(t) - \sum_{k=1}^{m} H_k x(t-\tau_k) = 0. \tag{2.10}$$

Let :

$$\mathscr{C}_D([-\tau_m, 0], \mathbb{R}^n) = \{\phi \in \mathscr{C}([-\tau_m, 0], \mathbb{R}^n) : D(\phi) = 0\},$$

which a closed subspace of $\mathscr{C}([-\tau_m, 0], \mathbb{R}^n)$. For any initial condition $\phi \in \mathscr{C}_D([-\tau_m, 0], \mathbb{R}^n)$, the solution of (2.10) is defined by

$$y(\phi) : t \in [-\tau_m, 0] \to y(\phi)(t) \in \mathbb{R}^n.$$

Hence the state at time t is given by :

$$y_t(\phi)(\theta) = y(\phi)(t+\theta), \quad \theta \in [-\tau_m, 0].$$

We denote by $T_D(t)$ the corresponding solution operator.
The characteristic equation corresponding to (2.10) is given by :

$$\Delta_D(s, \tau) = \det\Big(I - \sum_{k=1}^{m} H_k e^{-s\tau_k}\Big), \tag{2.11}$$

the zero of (2.11) is called the characteristic roots of (2.10), and its spectral abscissa is defined as follows :

$$\alpha_D = \sup\{\Re(s) : \Delta_D(s, \tau) = 0\}. \tag{2.12}$$

Definition 2.5.6. *The null solution of* (2.8) *is exponentially stable if and only if all characteristic roots are located in the open left half plane and bounded away from the imaginary axis .*

The exponential stability of the null solution of (2.10) is related with the exponential stability of (2.8), this result is summarized in the following proposition :

Proposition 2.5.7. *A necessary condition for exponential stability of the null solution of the neutral equation* (2.8) *is the exponential stability of the null solution of the delay difference equation* (2.10).

2.5.3 Spectrum properties

Recall that in the case of retarded system, the number of characteristic roots lying to the right of such a line are finite. More precisely, there are only a finite number of characteristic roots in any vertical strip of the complex plane, see corollary (2.4.10). For the linear neutral system (2.8) the situation is more complicated. The following proposition from [9] describe the typical behavior of the spectrum.

Proposition 2.5.8. *Suppose s satisfies* (2.11)*, then there exists a sequence of characteristic root of* (2.9) $\{s_n\}_{n\geq 1}$*, such that*

$$\lim_{n\to\infty} \Re(s_n) = \Re(s), \quad \lim_{n\to\infty} \Im(s_n) = \infty.$$

Therefore, contrary to the retarded time delay systems, the characteristic roots of the neutral system contained in a given vertical strip of the complex plane can be infinite.
The spectrum of delay system of neutral type may exhibit such peculiarities, this refers to the fact that even when all characteristic roots are stable for $\tau = 0$, for τ being small and positive infinitely many of these roots may becomes unbounded. In other words, a

small variation of time delay in delay difference equation (2.10) associated with (2.8) may affects the continuity properties of the characteristic roots which leads to infinitely large root variation. This sometimes called behavioral discontinuity, a feature unique to neutral differential equation as compared to delay differential equation.
To develop a general theory it is necessary initially to impose additional restrictions on the delay difference equation (2.10).
This problem is solved in [9] by introducing a theorem which gives sufficient conditions for a continuous spectral abscissa. In particular if the associated difference operator is strongly stable then the spectral abscissa is continuous. For further details on the continuity properties of the spectral abscissa, see [9], [79] and [103].

Definition 2.5.9. *We say that the null solution of* (2.10) *is strongly exponentially stable if it is remains stable when subjected to small variation in the delays.*

For more information in this context we refer to [76]. The following example from [79] shows that a small perturbations on the delay may destroy the stability of delay difference equation.

Example 2.5.10. *Consider the neutral system*

$$\frac{d}{dt}(x(t)-ax(t-1)-bx(t-2))=cx(t)+bx(t-1).$$

The associated delay difference equation :

$$x(t)=ax(t-1)+bx(t-2), \tag{2.13}$$

whose characteristic equation is

$$1-ae^{-s}+be^{-2s}=0.$$

Using a change of variable $X=e^{s}$, the roots of this equation is given by :

$$X=\frac{a\pm\sqrt{a^2+4b}}{2}.$$

For $a=b=\frac{1}{2}$, $|X|^2=\frac{1}{2}$. Therefore, $e^{2s}=\frac{1}{2}$ which means that $s=-\frac{1}{2}\ln(2)<0$ implies that $\Re(s)<0$.
For a given $n>0$, we consider the equation :

$$x(t)=-\frac{1}{2}x(t-1-\frac{1}{2n+3})-\frac{1}{2}x(t-2).$$

$x(t)=\sin(n+\frac{3}{2})\pi t$ is a solution of this equation and it does not approach zero as $t\to\infty$. For a small perturbation $\frac{1}{2n+3}$ in the delay, we take n sufficiently large for each such perturbation, the equation has a solution which does not approach zero. In other words the spectrum α_D is not continuous which means that it is sensitive to infinitesimal delay perturbations.
To show the behavior of the supremum of delay difference equation (2.13), *we plot the characteristic roots in the complex plan for both the nominal delay $\vec{\tau}=(1,2)$ and the perturbed delay $\vec{\tau}=(0.99,2)$. For small delay perturbation, a change in the behavior of the spectrum may occur and this is illustrate in Figure 4.1.*

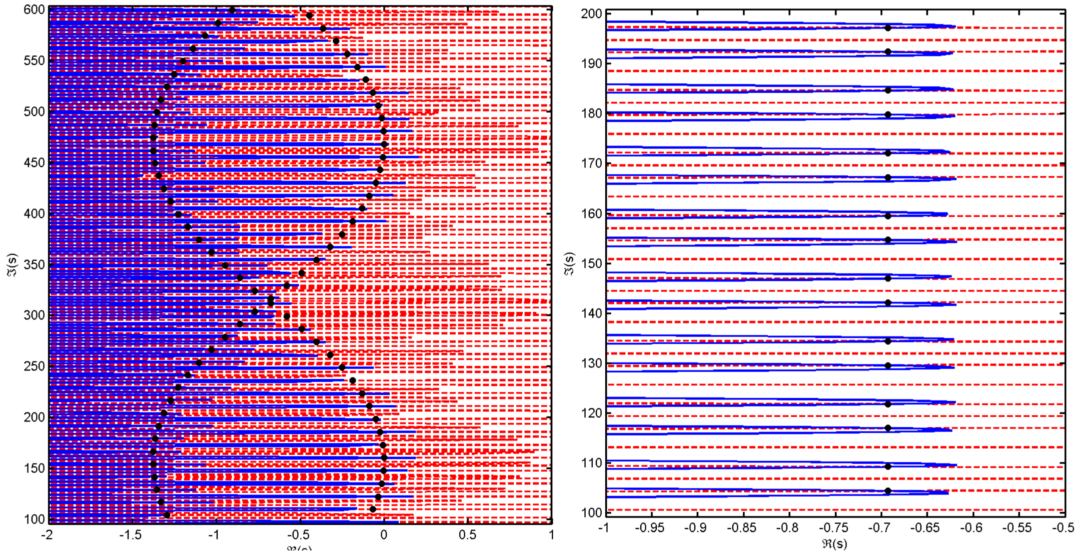


FIGURE 2.3 – (Right) The distribution of the spectrum of equation (2.13) with $a = b = \frac{1}{2}$ and the delay $\vec{\tau} = (1,2)$. (Left) The distribution of the spectrum of equation (2.13) with perturbed delay $\vec{\tau} = (0.99,2)$, illustrated using QPmR toolbox from [20].

2.6 Stability of Time Delay Systems

The main methods of examining the stability of time delay systems can be classified into two types : frequency and time domain methods. Frequency domain methods determine the stability of a system from the distribution of the roots of the corresponding characteristic equation. In the time domain methods many of the stability analysis have been formulated based on Lyapunov's second method using Lyapunov Krasovskii functionals or Lyapunov Razumikhin functions. In this thesis, we focus on methods in the frequency domain for stability analysis. However, it is important to cite some of the methods in the time domain developed in the literature.

2.6.1 Time-Domain Methods

In this part, we give sufficient conditions for the stability of the solution x = 0 of Functional Differential Equations (FDEs) which generalize the second method of Lyapunov for Ordinary Differential Equations (ODEs). We consider the following system :

$$\begin{cases} \dot{x}(t) = & f(t,x(t),x_t), \\ x_{t0}(\theta) = & \phi(\theta) \quad \text{with} \quad \theta \in [-\tau,0]. \end{cases} \tag{2.14}$$

Lyapunov's second method requires the existence of a positive definite function V such that $\dot{V} < 0$ if $x \neq 0$. Clearly this requires a few adaptations on the Lyapunov function since it is no an ordinary function but a functional which depends on certain past values of the argument t which makes the state space infinite dimensional. For this, two extension have been developed in this context, the first introduced by Krasovskii which leads to the use of functional, the second is developed by Razumikhin who proposed to use function rather than functional.

Lyapunov-Krasovskii Stability Theorem

Krasovskii approach consists to search a functional $V(t,x_t)$ that are decreasing along the solutions of equation (2.14). Let $V : \mathbb{R}\times\mathscr{C} \to \mathbb{R}$ be continuous, and $x(t,\phi)$ be a solution of (2.14) at time t. Then, we may calculate the derivative of V with respect t as :

$$\dot{V}(t,\phi) = \lim_{h\to 0^+}\sup \frac{1}{h}\Big[V(t+h,x_{t+h}(t,\phi) - V(t,\phi)\Big].$$

The function $\dot{V}(t,\phi)$ is the upper right-hand Dini's derivative of $V(t,\phi)$ along the solutions of equation (2.14).

Theorem 2.6.1. *(Lyapunov-Krasovskii stability theorem) [79] : Suppose $f : \mathbb{R}\times\mathscr{C} \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathscr{C}$) into bounded sets of $\mathbb{R}^n$, and $u,v,w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R}\times\mathscr{C} \to \mathbb{R}$ such that :*

$$u(|\phi(0)|) \le V(t,\phi) \le v(|\phi|)$$

$$\dot{V}(t,\phi) \le -w(|\phi(0)|)$$

then the solution $x = 0$ of equation (2.14) is uniformly stable.
If $u(s) \to \infty$ as $s \to \infty$, the solutions of equation (2.14) are uniformly bounded.
If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

Razumikhin Stability Theorem

Lyapunov Krasovskii theorem requires the manipulation of functionals, which makes it difficult to apply. This motivated the use of a classical Lyapunov function $V(t,x(t))$ for ordinary differential equations instead of functionals. The Lyapunov functions are much simpler to use and it is natural to explore the possibility of using the rate of change of a function in $\mathbb{R}^n$ to determine sufficient conditions for stability. On the other hand, it is not necessary to require that $\dot{V}(t,x(t)) \le 0$ for all initial conditions, but only for special solutions of the system. This approach is called the Razumikhin theorem.

Theorem 2.6.2. *(Razumikhin theorem) [79] : Suppose $f : \mathbb{R}\times\mathscr{C} \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathscr{C}$) into bounded sets of $\mathbb{R}^n$.*
Suppose that $u,v,w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions. u(s), v(s) positive for $s > 0$, u(0) =0, v(0) =0.
If there is a continuous functions $V : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ such that

$$u(|x|) \le V(t,x) \le v(|x|), \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^n.$$

The following statement hold

1. $\dot{V}(t,\phi(0)) \leq -w(|\phi(0)|)$ *if* $V(t+\theta,\phi(\theta)) \leq V(t,\phi(0)),\theta \in [-\tau,0]$, *then the solution* $x=0$ *of the equation* (2.14) *is uniformly stable.*
2. *If there exists a continuous nondecreasing function* $p:\mathbb{R}^+ \to \mathbb{R}^+$, $p(s) > s$ *for* $s>0$ *such that :* $\dot{V}(t,\phi(0)) \leq -w(|\phi(0)|)$ *if* $V(t+\theta,\phi(\theta)) \leq V(t,\phi(0))$, $\theta \in [-\tau,0]$.

2.6.2 Frequency-Domain Methods

Stability analysis of ordinary differential equation is determined from the roots of the characteristic equation which is a polynomial. The corresponding system is called stable if and only if all of the roots have negative real part, in this case Routh-Hurwitz criterion allows to give precise conditions for stability. For delay differential equations, the stability is also determined by the location of roots of the characteristic function, but as known this function takes the form of quasi-polynomial function, which is transcendental and admits infinitely many roots. Furthermore, the Routh-Hurwitz criteria are not applicable. Many approaches have been taken to determine the stability of functional differential equation.

Roots Locus Methods

Due to the difficulty in computing the characteristic roots of time-delay systems, it is desirable to determine system stability without the exact knowledge of the abscissa of the characteristic equation. So the root locus method consists to determine the values of the parameters for which the characteristic equation has roots on the imaginary axis. Such result is defined in the following theorem.

Theorem 2.6.3. *If the matrices* $A_0,\ldots A_m$ *and the delays* $\tau_1,\ldots,\tau_m$ *are varied then a loss or acquisition of exponential stability of the null solution of* (2.2) *is associated with characteristic roots on the imaginary axis.*

1. **D-Decomposition method :** This approach due to Neimark [102] is a powerful tool for stability analysis when the characteristic roots close to C^+ depend on τ continuously. The idea of this approach is to divide the parameter space into disjoint sub-regions. This method is also known as D-subdivision method. For each point of the boundaries the corresponding characteristic equation has at least one root in the imaginary axis. The points in the interior of each sub-region correspond to the roots of characteristic equation with the same number of zeros with positive real parts. The number of zeros with positive real parts can only change when a zero passes across the imaginary axis, i.e., when the point in the coefficient space passes across the boundary of the region.
   The regions having no zero with positive real part correspond to the set of parameters for which the considered system is asymptotically stable.

   Example 2.6.4. *To illustrate this method, we consider the following equation*

   $$\dot{x}(t) + ax(t) + bx(t-\tau) = 0, \tag{2.15}$$

To apply the D-decomposition method, we need to fixe the delay parameter and we analyze the system with respect to other parameters. The characteristic equation associated to (2.15) *is given by :*

$$\Delta(s,a,b) = s + a + be^{-s\tau} = 0, \tag{2.16}$$

this latter equation admits the root $s = 0$ for

$$a + b = 0. \tag{2.17}$$

Likewise, if equation (2.16) *has a pure imaginary root $s = iw$, then we have*

$$\Delta(iw,a,b) = iw + a + be^{-iw\tau} = 0. \tag{2.18}$$

By separating the real parts and the imaginary parts, we obtain an equation parameterized in w of the boundaries of the D-decomposition :

$$a(w) = -\frac{w\cos(w\tau)}{\sin(w\tau)}, \quad b(w) = \frac{w}{\sin(w\tau)}. \tag{2.19}$$

$b(w)$ can be prolonged by continuity at $w = 0$, and we have

$$\lim_{w\to 0} \frac{w}{\sin(w\tau)} = b(0) = \frac{1}{\tau}.$$

So, we deduce that the point $(a,b) = \left(\frac{-1}{\tau}, \frac{1}{\tau}\right)$, there are a several regions as illustrated in the figure 2.4.

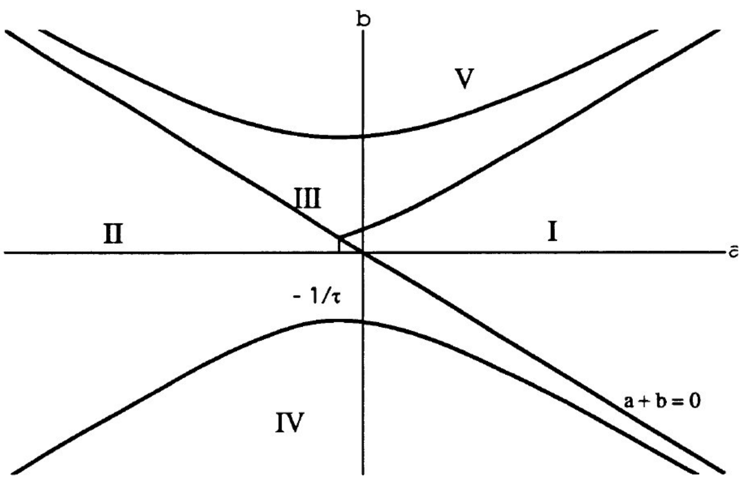


FIGURE 2.4 – D-decomposition method applied to the scalar equation (2.15) taken from [118].

For $b = 0$, the equilibrium $x = 0$ of the system (2.15) *is asymptotically stable if and only if $a > 0$. So, the region I correspond to the set of parameters for which* (2.15) *is asymptotically stable.*

The next step consists to determine the number of zeros with positive real part. To do this, we need to calculated the sign of the differential of the real part of the root calculated at the point of the boundary.
For example, for $s = 0$ we have $a + b = 0$, and at this point we get :

$$\Re\left(\frac{ds}{da}\right) = \frac{-1}{1 - b\tau}. \tag{2.20}$$

Hence, we distinguish two cases :

(a) for $b < \frac{1}{\tau}$, we have $\Re\left(\frac{ds}{da}\right) < 0$. For each pair of parameters (a, b) located in the region II, the system admits a unique characteristic root with a positive real part.

(b) for $b > \frac{1}{\tau}$, the expression (2.20) *has a positive sign, so when we go from the region II to region III, the number of unstable roots increases by 1. We do the same thing for the other regions.*

2. **τ-Decomposition method :** When the parameter p in the D-decomposition method developed in the previous section is the delay parameter, it is known as the τ-decomposition method. So, in this case the stability analysis is done with respect to the delay parameter and the other parameters of the system are assuming to be fixed. The τ-decomposition method proceeds as follows :

   – First, we decompose the delay axis into intervals such that in the interior of each interval, the system has no imaginary roots, therefore the number of unstable roots,i.e root $\in C^+$ is constant for all points of the interval.
   – The second step consists to investigate the change of the number of roots at the end points of the delay intervals computed in the previous step, such that each end point of the interval corresponds to a characteristic root crossing with respect to the imaginary axis. More precisely one has to determine how many characteristic roots will move toward C^+. This step is refereed to as the crossing direction.

Root crossing direction analysis

For a given critical pair $(jw^\star, \tau^\star)$, where $\tau^\star$ is the value of delay for which the characteristic equation admits root on the imaginary axis. If $s = jw^\star$ is a simple root, then

$$\frac{d\Delta(s,\tau)}{ds}(jw^\star, \tau^\star) \neq 0.$$

By the implicit function theorem see [91], in a neighborhood of $(jw^\star, \tau^\star)$ the characteristic root s is a function of τ denoted here as $s(\tau)$. We have :

$$\frac{ds(\tau^\star)}{d\tau} = -\frac{\partial_\tau \Delta(s,\tau)}{\partial_s \Delta(s,\tau)}(jw^\star, \tau^\star).$$

In order to determine the root crossing direction, the following quantity must be considered,

$$Sgn\left(\Re\left(\frac{d}{d\tau}s(\tau^\star)\right)\right), \tag{2.21}$$

according to this term, we have the following situations :

1. If the term given in (2.21) is positive then the characteristic roots crosses the imaginary axis and moves toward $\mathbb{C}^+$ as τ increases through $\tau^\star$.
2. If this term is negative then the characteristic roots moves toward $\mathbb{C}^-$ and becomes stable.
3. When the right hand side of equation is zero, higher order derivatives needs to be computed, which is introduced in [99] and briefly discussed in [88].

Method of Walton and Marshell (1987)

Method of Walton and Marshell or direct method was introduced in [113] and it is also discussed in [98], it is an algebraic method for the stability of delay differential equation. The idea of this method consists in finding values of the retard $\tau^\star$ for which the characteristic equation has roots on the imaginary axis, and at these points $\tau^\star$ that a change in the asymptotic behavior of the system can produce, we are therefore led to calculate a place of the roots parameterized in τ. To better illustrate the ideas of this methods, we consider systems with a single delay, hence equation (2.2) becomes :

$$\dot{x}(t) = Ax(t) + Bx(t-\tau),$$

whose characteristic equation is given by :

$$\Delta(s,\tau) = \det(sI - A - Be^{-s\tau}).$$

This later can be rewritten as follows :

$$\Delta(s,\tau) = P(s) + Q(s)\,e^{-s\tau}.$$

The procedure of this method is summarized in what follows :

1. Determine the location of the roots of the non-retarded system ($\Delta(s,0)=0$).
2. Calculate the polynomial W obtained from the equation

   $$\Delta(s,\tau) = 0 = \Delta(\bar{s},\tau),$$

   for the imaginary roots $s = jw$. So we have :

   $$\begin{cases} P(-iw) + Q(-iw)e^{-iw\tau} = 0 \\ P(iw) + Q(iw)e^{iw\tau} = 0. \end{cases}$$

   We known that, when the root s lies on the imaginary axis, for some $w \in \mathbb{R}$, $e^{-s\tau}$ lies on the unit circle of the complex plane, hence the exponential term is eliminated from this equation which leads to the following polynomial in w^2 :

   $$F(w^2) = P(jw)P(-jw) - Q(jw)Q(-jw). \tag{2.22}$$

   Notice that the degree of (2.22) is equal to that of $P(s)$. Without loss of generality only positive solutions w are considered.

3. Search a possible real positive roots of $F(w^2) = 0$ which also correspond to roots of $\Delta(s,\tau) = 0$. If there are no solutions of polynomial (2.22) there are no crossing point and no change appears on the stability of the system as the delay is increased. This means that the system remains stable / unstable according to whether the system free of delay stable / unstable respectively.
4. If there exists a critical value $\tau^\star$ for which a zero lie in the imaginary axis then it is necessary to determine in which direction the zero crosses the axis. So we distinguish two situation : when the roots passes from the left to right which leads to destabilizing values and when the roots passes from right to left which gives stabilizing values. Therefore, stability is guaranteed for these values of retard $\tau > \tau^\star$.

Example 2.6.5. *Let us consider the system*

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) = 0, \tag{2.23}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.24}$$

The characteristic equation corresponding to (2.23) *is given by :*

$$s^2 + s + 1 + s e^{-s\tau} = 0. \tag{2.25}$$

1. *The equation*
$$\Delta(s,0) = s^2 + 2s + 1 = (s+1)^2 = 0, \tag{2.26}$$
has a unique solution which means that the system is asymptotically stable for $\tau = 0$.
2. *We have :*
$$q(w^2) = (-w^2 + jw + 1)(-w^2 - jw + 1) - (jw)(-jw) = (w^2 - 1)^2 = 0. \tag{2.27}$$
3. $w^2 = 1$ *is a double root of the later equation. Thus the zeros touch the imaginary axis at* $w = 1$. *So for this value of* w *we have*
$$\Delta(j,\tau) = e^{-j\tau} + 1 = 0, \tag{2.28}$$
hence $\cos(\tau) = -1$ *and* $\sin(\tau) = 0$ *which implies that* $\tau = (2n+1)\pi$ *with* n *positive number.*
4. *The system* (2.23) *is asymptotically stable for all value of* $\tau \neq (2n+1)\pi$.

There exists an interesting relationship between the function $F(w)$ defined in (2.22) and the crossing direction of the characteristic roots. Therefore, the following theorem shows that it is not necessary to calculate $\Re\left(\frac{\partial s}{\partial \tau}\right)$ but that its sign may be deduced from the polynomial given by (2.22).

Theorem 2.6.6. *(Walton and Marshall [113] :)*

$$sgn\left(\Re\left(\frac{\partial s}{\partial \tau}\right)\right) = sgn\left(\frac{\partial F}{\partial w^2}\right),$$

where $s = iw$ for w^2 positive root of equation (2.22).

According to the last equation, as t sweeps through $\tau^\star$ from left, a pair of imaginary roots $\pm jw^\star$ crosses the imaginary axis toward the right half complex plane if $F'(w^\star) > 0$. This pair of roots move toward the left half complex plane if $F'(w^\star) < 0$.

Lambert W-function

Introduced in the 1700s by Lambert and Euler (Corless et al., 1996), the Lambert W has been used in a variety of applications (see [58]) to find solutions for different non-algebraic equation including exponential functions or logarithm as well as to give explicit formulas for nonlinear equations which were previously mostly solved numerically. The Lambert W function is the logarithmic type function defined as the multivalued inverse of the complex function $f(x) = xe^x$, where x is any complex number. Lambert W function is defined to be any function $W(x)$ satisfies :

$$W(x) = e^{W(x)} = x.$$

For each integer k there is one branch denoted by $W_k(x)$, W_0 is known as the principle branch. $W_0(x)$ is real and positive for $x \in (0,\infty)$ and $W_0(x)$, $W_{-1}(x)$ both are real and negative for $x \in (-\frac{1}{e},0)$. All other $W_k(x)$ are complex $\forall x$.

Example 2.6.7. *Solving the equation $x + e^x = 0$. We have :*

$$x + e^x = 0 \quad \Rightarrow \quad x = -e^x,$$

rewriting the later equation under the form :

$$-xe^{-x} = 1,$$

we obtain the following solution :

$$-x = W_0(1) = -0.56714.$$

Relationship between DDEs and Lambert W function :

The Lambert W function, known to be useful for solving scalar first-order DDEs, it has recently been extended to higher order systems of DDEs using the matrix Lambert W function. In their article Asl and Ulsoy [2] connected the roots of (2.5) with solutions to a specific exponential polynomial known as the Lambert W equation ([59]). Let us consider the first order system given by the equation (2.4), recall that the corresponding characteristic equation is described by :

$$s + a + be^{-s\tau} = 0.$$

To solve for the roots of this equation, we rewrite it in the form

$$\tau(s-a)\,e^{-\tau(s-a)} = -b\,\tau\,e^{-\tau\,a}. \tag{2.29}$$

Define a new function W such that :

$$W(b\,\tau\,e^{-a\,\tau}) = \tau(s-a).$$

The function to be solved becomes :

$$W(b\,\tau\,e^{-a\,\tau})\,e^{W(b\,\tau\,e^{-a\,\tau})} = b\,\tau\,e^{-a\,\tau},$$

which is the same form as $W(x)\,e^{W(x)} = x$, the well studied Lambert W function. Hence the roots of (2.5) are written as follows :

$$s = \frac{1}{\tau}W(b\,\tau\,e^{-a\,\tau}) + a.$$

For a general introduction to analysis of time delay system using the Lambert W function we refer to [114].

Method of Hopf bifurcation

In dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden "qualitative" or topological change in its behavior, see [94], [70]. Generally, at a bifurcation value, the local stability properties of equilibria, periodic orbits or other invariant sets changes. In other words, we known that in a dynamical system, a parameter is allowed to vary, then the differential system may change. An equilibrium can become unstable and a periodic solution may appear or a new stable equilibrium may appear making the previous equilibrium unstable. The value of parameter at which these changes occur is known as "bifurcation value" and the parameter that is varied is known as the "bifurcation parameter". In what follows, we present the classic Hopf bifurcation theory for delay differential equations discussed in [3] where the delay plays the role of bifurcation parameter. To explain this better, let us consider the linear system :

$$\dot{x}(t) + ax(t) + bx(t-\tau) = 0, \tag{2.30}$$

where $b > a > 0$. The associated characteristic equation is given by :

$$s + a + be^{-s\tau} = 0. \tag{2.31}$$

A standard requirement for a Hopf bifurcation at such an equilibrium is that a pair of complex conjugate roots of the characteristic equation(2.31) cross the imaginary axis of the complex plane (with nonzero imaginary part) as the parameter τ in the model is varied.
Substituting $s = iw^\star$ in (2.31) we get :

$$\begin{cases} a = -b\cos(w^\star\tau), \\ w^\star = b\sin(w^\star\tau). \end{cases} \tag{2.32}$$

These two equations determine the two unknown $w^\star$ and τ for (2.31) to have two purely imaginary roots. For positive $w^\star$ it will follows from (2.32) that :

$$\cos(w^\star \tau) < 0 \quad \text{and} \quad \sin(w^\star \tau) > 0,$$

and hence we derive that $w^\star \tau$ satisfies :

$$2\pi n + \frac{\pi}{2} < w^\star \tau < 2\pi + \pi.$$

If we solve (2.31) for unknown $w^\star$, τ we obtain :

$$\begin{cases} w^\star = \sqrt{b^2 - a^2}, \\ \tau^\star = \cos^{-1}(-a/b)/w^\star, \end{cases}$$

where $\tau^\star$ represents the hopf bifurcation value and it is sometimes called "delay induced bifurcation". In particular, for values of τ near $\tau^\star$, the trivial solution of (2.30) is asymptotically stable for $\tau < \tau^\star$ and it losses it stability when $\tau > \tau^\star$. We have similar situation for all $\tau^\star + 2k\pi$, $k = 1, 2, \ldots$. Thus, for $\tau = \tau^\star$ the linear variational system (2.30) has a periodic solutions with a period of $\frac{2\pi}{w^\star}$.

Multiplicity Induced Dominancy (MID) approach

The effect of the multiplicity of spectral values on the exponential stability of the retarded functional differential equation is recently studied. This approach consists in studying the connection between the maximal multiplicity of some negative spectral value where the maximal multiplicity correspond to the degree of the quasipolynomial and the stability of trivial solution of the corresponding dynamical system. So, several methods have been proposed to show the dominancy property, for example in [48, 50] the dominancy is shown using an adequate factorization, and in [49], it is shown using Stépan-Hassard formula.

– **Factorization based approach**

1. First order system : The property multiplicity induced dominancy is first introduced in [52] for the scalar differential equation with one delay represented by the following linear system :

$$\dot{x}(t) + ax(t) + bx(t - \tau) = 0. \tag{2.33}$$

   The study of stability of the later system is based on the analytic characterization of the rightmost root of the corresponding characteristic function, which implies the dominancy of one of the spectral values of (2.33) with maximal multiplicity. The analytical proof of this result is established using an adequate factorization of the quasipolynomial function. The following proposition gives conditions on the equations parameters that guarantees the asymptotic stability of the system (2.33).

Proposition 2.6.8. *[50]. For a given delay $\tau \in \mathbb{R}^+$, system* (2.33) *admits a double spectral value at $s = s_0$ if and only if*

$$s_0 = -\frac{a\tau+1}{\tau} \quad \textit{and} \quad b = \frac{e^{-a\tau-1}}{\tau}. \tag{2.34}$$

If $a > -\frac{1}{\tau}$ then the zero solution of system (2.33) *is asymptotically stable and $s = s_0$ is the corresponding rightmost root.*
If $a = -\frac{1}{\tau}$ then $s_0 = 0$ is the only double spectral value of (2.33) *on the imaginary axis, and the set of unstable root is empty.*

2. Second order system : The multiplicity-induced-dominancy is extended for second order retarded differential equations described by :

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) + \beta x(t-\tau) = 0. \tag{2.35}$$

The corresponding characteristic equation after using the change of variable $s = \frac{c_1\lambda}{2}$, (normalized characteristic function) is as follows :

$$\left\{ \begin{array}{c} \tilde{\Delta}(\lambda,\tilde{\tau}) = \lambda^2 + 2\lambda + a_0 + \alpha e^{-\lambda\tilde{\tau}} \\ \text{where} \quad \alpha = \frac{4}{c_1^2}\beta, \quad \tilde{\tau} = \frac{c_1}{2}\tau \quad \text{and} \quad a_0 = 4\frac{c_0}{c_1^2}. \end{array} \right. \tag{2.36}$$

The following theorem gives conditions on the parameter coefficients guaranteing the stability of the corresponding system.

Theorem 2.6.9. *[42].*

(a) The multiplicity of any given root of the quasipolynomial function (2.36) *is bounded by* 3.

(b) The quasipolynomial (2.36) *admits a real spectral value at $\lambda = \lambda_0$ with algebraic multiplicity* 3 *if and only if*

$$\tilde{\tau} = \sqrt{\frac{1}{a_0-1}}, \quad \lambda_0 = -1 - \frac{1}{\tilde{\tau}}, \quad \alpha = -\frac{2e^{-(1+\tilde{\tau})}}{\tilde{\tau}^2}. \tag{2.37}$$

(c) If (2.37) *is satisfied then $\lambda = \lambda_0$ is the rightmost root of* (2.36).

– **Argument principle**
The principle of argument is a well known tool for stability analysis of time delay systems based on the knowledge of the characteristic polynomials, see [103], [105] and [104]. The same principle holds for quasipolynomial of the form :

$$\Delta(s,\tau) = P(s) + Q(s)e^{-s\tau}. \tag{2.38}$$

If the polynomials $P(s)$ and $Q(s)$ satisfy certain conditions, the stability of the system can be easily determined by mapping the contour (C) and using the argument principle to determine if any root s satisfying (2.38) lie inside the contour in which case the system (2.38) is unstable. The argument principle gives the number of root of $\Delta(s,\tau)=0$ inside C by calculating

$$\frac{1}{2\pi}\Delta_C arg\Delta(s,\tau), \tag{2.39}$$

which is equivalent to the counterclockwise encirclement of the origin. The following theorem from [110] gives the number of unstable roots of the characteristic function.
Theorem 2.6.10. *[110]. Let the characteristic function* (2.38) *have no zeros in the imaginary axis, then the number of its zero in the right half plane is determined by :*

$$Z=\frac{1}{2\pi i}\lim_{R\to\infty}\oint_C \frac{\partial_s\Delta(s,\tau)}{\Delta(s,\tau)}ds. \tag{2.40}$$

According to the Cauchy residue theorem or the principle of argument, the integral in (2.40) gives $(Z-P)2\pi i$ where Z and P are the number of zeros and poles (including the multiplicities) respectively within the closed contour (C). The contour $\lim_{R\to\infty}(C)$ encircles the whole right half plane and the characteristic function has no pole.

The result on maximal multiplicity induced dominancy for spectral values for generic retarded second order system with single delay is analytically studied in [48] and [49] using the argument principle. Such that, in [48] propose a parameterized analysis along the contour C, the contour chosen is given in Figure 2.5. For the result in [49], they use Stépàn Hassard formula proposed in [82] which allows to establish necessary and sufficient conditions for asymptotic stability of the zero solution of linear time delay systems.

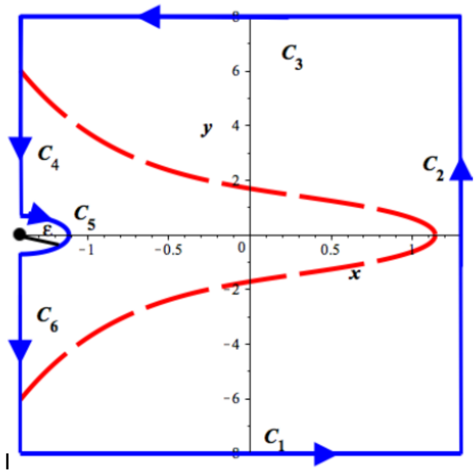


FIGURE 2.5 – The simplified contour used in [48] for applying the argument principle to investigate the dominancy of multiple root.

1. A parameterized based approach : The proposition that follows generalizes the Theorem 2.6.9 which is restricted to $\beta=0$. In this case, the dominancy property is shown using a parameterized analysis.

Proposition 2.6.11. *[48]. Consider the quasipolynomial function*

$$s^2 + c_1 s + c_0 + (\beta_0 + \beta_1 s)e^{-\tau s}. \tag{2.41}$$

The following assertion hold :

(a) The multiplicity of any given root of the quasipolynomial (2.41) *is bounded by* 4*, it can be attained only on the real axis.*

(b) The quasipolynomial (2.41) *admits a real spectral value at $s = s_\pm$ with algebraic multiplicity four if and only if, either*

$$\begin{cases} s_+ = & \frac{-2+\sqrt{-2+c_0\tau^2}}{\tau} \\ \beta_0 = & 2\,\frac{e^{-2+\sqrt{-2+c_0\tau^2}}(-5+\sqrt{-2+c_0\tau^2})}{\tau^2} \\ \beta_1 = & -2\,\frac{e^{-2+\sqrt{-2+c_0\tau^2}}}{\tau}, \quad c_1 = -2\,\frac{\sqrt{-2+c_0\tau^2}}{\tau}, \end{cases} \tag{2.42}$$

or

$$\begin{cases} s_- = & \frac{-2-\sqrt{-2+c_0\tau^2}}{\tau} \\ \beta_0 = & 2\,\frac{e^{-2-\sqrt{-2+c_0\tau^2}}(-5-\sqrt{-2+c_0\tau^2})}{\tau^2} \\ \beta_1 = & -2\,\frac{e^{-2-\sqrt{-2+c_0\tau^2}}}{\tau}, \quad c_1 = 2\,\frac{\sqrt{-2+c_0\tau^2}}{\tau}, \end{cases} \tag{2.43}$$

where τ is arbitrarily chosen satisfying $c_0\tau^2 \geq 2$.

(c) If either (2.42) *and* (2.43) *is satisfied then $s = s_\pm$ is the rightmost root* (2.41)*.*

2. Stépàn-Hassard formula : In [82] the authors gives a formula that count the number of roots in the right half plane of the characteristic equation for linear time delay system with constant coefficients. This formula allows to establish necessary and sufficient conditions for asymptotic stability of zeros solution of linear delay differential equation using the so called Stépàn-Hassard formula.

Definition 2.6.12. *[110]. The curve (g) is called a Bromwich contour if*

$$g = \bigcup_{k=1}^{3} (g_k), \tag{2.44}$$

where (g_k) $(k = 1, 2, 3)$ is given in the complex plane as follows :

$$\begin{aligned} g_1 =& \{s = Re^{i\theta},\, R \in \mathbb{R}^+ \text{ and } \quad \theta : -\frac{\pi}{2} \to \frac{\pi}{2}\}, \\ g_2 =& \{s = iw,\; w : 0 \to R\}, \\ g_3 =& \{s = iw,\; w : 0 \to -R\}, \end{aligned}$$

as illustrated in Figure 2.6.

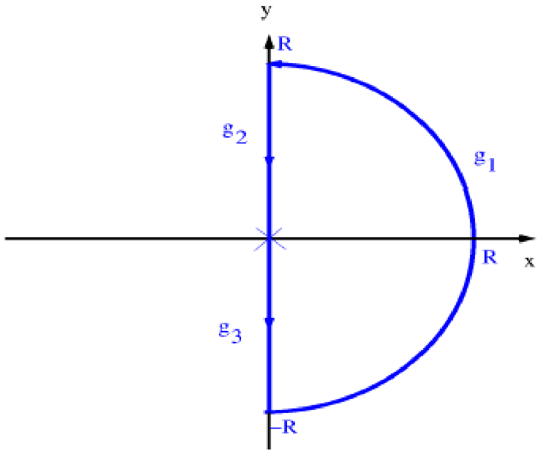


FIGURE 2.6 – The standard Bromwich contour usually used for asymptotic stability investigation tacked from [110].

Theorem 2.6.13 (Hasard's Theorem [82])**.** . *Let $A_1,...,A_m$ be real n by n matrices, and let $\tau_1,\cdots,\tau_m$ be nonnegative reals. Let*

$$\Delta(s,\tau)=\det(sI-\sum_{j=1}^{m}e^{-s\tau_j}A_j). \tag{2.45}$$

Let $\rho_1,\cdots,\rho_J$ be the positive zeros of $R(y)=\Re(i^{-n}\Delta(iy))$, counted by multiplicity and ordered so that $\rho_1\geq\cdots\geq\rho_J>0$. For each $j=1,\cdots,J$ such that $\Delta(i\rho_j)=0$, assume that the multiplicity of $i\rho_j$ a zero of $\Delta(\lambda)$ is the same as the multiplicity of ρ_j as a zero of $R(y)$. Then, the number of roots of the characteristic equation $\Delta(\lambda)=0$ which lie in $\Re(s)>0$, counted by multiplicity, is given by the formula :

$$\frac{n-K}{2}+\frac{1}{2}(-1)^r sgnS^{(k)}(0)+\sum_{j=1}^{J}(-1)^{j-1}sgnS(\rho_j), \tag{2.46}$$

where K is the number of zeros of $\Delta(s,\tau)$ on $\Re(\lambda)=0$, counted by multiplicity, k is the multiplicity of $s=0$ as a root of $\Delta(s,\tau)=0$, and $S(y)=Im(i^{-n}\Delta(iy))$. Furthermore, the count (2.46) *is odd if $\Delta^{(k)}(0)<0$ and is even if $\Delta^{(k)}(0)>0$. If $R(y)$ has no positive zeros, set $r=0$ and omit the summation term in* (2.46). *If $\lambda=0$ is not a root of the characteristic equations, set $k=0$ and interpret $S(0)^{(0)}$ as $S(0)$ and $\Delta(0)^{(0)}$ as $\Delta(0)$.*

The main hypothesis (no zero in the positive half plane) is verified in [78] where the authors employed a rectangle region in the positive half plane inside which all zeros (if any) of $\Delta(s,\tau)$ must lie and numerical quadrature is used to approximate integral of the logarithmic derivative of $\Delta(s,\tau)$ around the boundary of this rectangle. The following example is studied in [78]. Using the Stepàn Hasard formula is much simpler and more efficient then applying the result in [78].

Example 2.6.14. *[82]. Consider the following linear time delay system with two delays*

$$\dot{x}(t) = \delta\big[x(t-\tau_1) + x(t-\tau_2)\big], \tag{2.47}$$

where $\delta = \pi 3^{\frac{-3}{2}}$, $\tau_1 = 1$, and $\tau_2 = 2$, which arises in Hopf bifurcation analysis of a nonlinear delay differential equation considered in [78]. The characteristic equation corresponding to (2.47)

$$\Delta(s) = s + \delta(e^{-s} + e^{-2s}). \tag{2.48}$$

To apply the formula , assume that there exists $w > 0$ such that $s = iw$ is a root of (2.48) *and define*

$$\begin{cases} R(w) = & \Re\big(i^{-1}\Delta(iw,\tau)\big) = w - \delta(\sin(w) + \sin(2w)), \\ S(w) = & \Im\big(i^{-1}\Delta(iw,\tau)\big) = -(\cos(w) + \cos(2w)). \end{cases} \tag{2.49}$$

$\Delta(s,\tau)$ has a pair of zero $\pm iw = \pm i\frac{\pi}{3}$.
$R(w)$ has a positive real zero $w = \frac{\pi}{3}$ and we have $R'(\frac{\pi}{3}) = 1 + \frac{\delta}{2} \neq 0$ which implies that the multiplicity of $\frac{\pi}{3}$ as a root of $R(w)$ is one so $rho_1 = 1$ and $x = 1$. From this we deduce that $\Delta'(\frac{i\pi}{3}) \neq 0$ hence the multiplicity of $\frac{i\pi}{3} \neq 0$ as a root of $\Delta(w)$ is one, and we have $S(\rho_1) = 0$.
Now, we are able to calculate the number of zero of $\Delta(s,\tau)$ in $\Re(s) > 0$

$$z = \frac{n-k}{2} + \frac{1}{2}(-1)^k \, sgn\, S^{(k)}(0), \tag{2.50}$$

in this case, we have $r = 1$, $k = 0$ and $K = 2$ which gives

$$z = \frac{1}{2} - 1 + \frac{1}{2}(-1)\, sgn\, S(0) = 0.$$

We conclude that there are no zeros in the right half plane, which ensure the stability of the corresponding system.

The paper [49] revisit recent result on maximal Multiplicity Induced Dominancy (MID) for spectral values, so it focuses on the effect of multiplicity of spectral values on the exponential stability of the generic second order system. So, they consider the following system :

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = u(t), \tag{2.51}$$

where u is the unknown control and a and b are parameters. The aim of this study is to construct a control u under the following form

$$u(t) = \alpha_0 x(t-\tau) - a_1 \dot{x}(t-\tau), \tag{2.52}$$

such that, the following closed loop system is stable

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) + \alpha_0 x(t-\tau) + \alpha_1 \dot{x}(t-\tau) = 0. \tag{2.53}$$

To do this, they establish conditions on the parameters of the generic control problem. The corresponding characteristic equation is given by :

$$\Delta(,\tau) = P(s) + Q(s)e^{-\tau s}, \tag{2.54}$$

where

$$\begin{cases} P(s) = s^2 + as + a_0, \\ Q(s) = \alpha_1(s) + \alpha_0. \end{cases} \tag{2.55}$$

The following theorem gives conditions on the parameter coefficients parameters guaranteing the stability of the corresponding system.

Theorem 2.6.15. *[49]. Considering equation* (2.54) *the following assertion hold*

- *The multiplicity of any given root of the quasipolynomial function* (2.54) *is bounded by* 4 *it can be attained only on the real axis and under the negativity of* Δ.
- *The quasipolynomial* (2.54) *admits a real spectral value at* $s = s_0$ *with algebraic multiplicity* 4 *if and only if*

$$s_0 = -\frac{a_1 + \sqrt{-2\Delta}}{2}, \tag{2.56}$$

and the system parameters satisfy :

$$\tau = 2\sqrt{-\frac{2}{\Delta}}, \quad \alpha_0 = \frac{5\Delta - a_1\sqrt{-2\Delta}}{4}e^{s_0\tau}, \quad \alpha_1 = -\frac{\sqrt{-2\Delta}}{2}e^{s_0\tau}. \tag{2.57}$$

- *If* (2.57) *is satisfied then* $s = s_0$ *is the spectral abscissa corresponding to* (2.54).
- *If* (2.57) *is satisfied then the trivial solution of the closed loop equation* (2.51) *is asymptotically stable if and only if* $\left(a_1 \geq 0 \text{ and } a_0 > \frac{a_1^2}{4}\right)$ *or* $\left(a_1 < 0 \text{ and } a_0 > \frac{3a_1^2}{8}\right)$.

Dominant Pole of Time Delay Positive Systems

Positive systems constitute a class of systems that has the important property that its state variables are never negative, given a positive initial state. These systems appear frequently in practical applications in biology, network communications, economics and probabilistic systems. A particularity of TDPS is that they are asymptotically stable if and only if its corresponding system delay-free system is asymptotically stable, and this property holds irrespective of the length of delays. The paper [64] focus on the dominant pole analysis of asymptotically stable time-delay positive systems (TDPSs). The authors assert that the exact calculation of dominant pole is possible for this class of systems , which is not the case for general LTI-TDS (which are not necessary positive).

The considered system is given as follows :

$$\Sigma : \begin{cases} \dot{x}(t) & = A_0x(t) + \sum_{i=1}^{N} A_i x(t-\tau_i) \quad (t \geq 0), \\ x(\theta) & = \phi(\theta) \quad (-\tau \leq \theta \leq 0), \end{cases} \tag{2.58}$$

where $A_0 \in M^n$, $A_i \in \mathbb{R}_{++}^{n\times n}$ $i = 1,\ldots,N$ with $\tau := \max_{i=1,\ldots,N} \tau_i$ and $\phi \in \mathbb{C}([-\tau,0],\mathbb{R}^n)$ is the function of initial condition.

Now, we need to state some useful results on the properties of positive systems. So, the characteristic equation associated to(2.58) is given by :

$$\det\left(\lambda I - A_0 - \sum_{i=1}^{N} A_i e^{-\lambda_i \tau_i}\right) = 0. \tag{2.59}$$

The set of solutions of (2.59) is denoted by $\Lambda_\Sigma \in \mathbb{C}$, and the dominant pole of the time delay positive system (2.58) is defined by :

$$\kappa(\Sigma) := arg \max_{\lambda \in \Lambda_\Sigma} \Re(\lambda). \tag{2.60}$$

Definition 2.6.16. *[80, 85]. The TDS* (2.58) *is said to be positive if for every* $\phi \in \mathbb{C}([-\tau, 0], \mathbb{R}^n_+)$*, the solution* x *satisfies* $x(t) \in \mathbb{R}^n_+$ *for all* $t \geq 0$*.*

Proposition 2.6.17. *[80, 85]. The TDS* (2.58) *is positive if and only if* $A_0 \in M^n$ *and* $A_i \in \mathbb{R}^{n\times n}_+$ $i = 1, \ldots, N)$*.*

The following theorem gives a characterization of the dominant pole of TDPS.

Theorem 2.6.18. *[64]. Consider the TDPS* Σ *described by* (2.58) *and assume* Σ *is asymptotically stable, i.e.,*

$$A = \sum_{i=0}^{N} A_i \in M^n \cap H^n. \tag{2.61}$$

Then, the dominant pole $\kappa(\Sigma)$ *defined by* (2.60) *is given by :*

$$\kappa(\Sigma) = -\alpha^\star < 0, \tag{2.62}$$

where $\alpha^\star \in \mathbb{R}_{++}$ *is a unique solution of*

$$\lambda_F\left(\alpha^\star I + A_0 + \sum_{i=1}^{n} A_i e^{\alpha^\star \tau_i}\right) = 0.$$

Moreover, we have :

$$\kappa(\Sigma) \geq \kappa(\Sigma_0) = \lambda_F(A), \tag{2.63}$$

irrespective of $\tau_i \in \mathbb{R}_{++}$ $(i = 1, \ldots, N)$*.*

Other stability criteria

In this part, we discussed briefly some other stability criteria, which are based on the stability analysis of characteristic functions. This criteria is also briefly studied in [103] and [110].

Pontryagin criterion : The characteristic equation of linear delay system with a single or commensurate delay can be transformed into the following form

$$\Delta(s, e^s) = \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}\, s^j\, e^{ks}. \tag{2.64}$$

The main idea behind the Pontryagin criterion [106], [107] can be summarized as follows : suppose (2.64) has principal term (i.e. $a_{pq} \neq 0$). Let $F(w)$ and $G(w)$ denote the real and the imaginary part, respectively of the quasipolynomial $\Delta(s, e^s)$ Then :

– If all the roots of $\Delta(s,e^s)$ are in $\mathbb{C}$, then the roots of $F(w)$ and $G(w)$ are real simple, alternate and
$$F'(w)G(w) - F(w)G'(w) > 0 \quad \forall\, w \in \mathbb{R}. \tag{2.65}$$

– Conversely, all the roots of $\Delta(s,e^s)$ are in $\mathbb{C}^-$ if one of the next conditions is satisfied.

1. All the roots of $F(w)$ and $G(w)$ are real simple alternate and the inequality (2.65) is satisfied for at least one $w \in \mathbb{R}$.
2. All the roots of $F(w)$ (or $G(w)$) are real simple and for each root the inequality (2.65) holds.

Although, this method has strong limitations and it may become very complicated for systems with more delays than one.
Cheboterev criterion : is the most generalization of Routh-Hurwitz criterion for quasi-polynomial (2.64). However [57], its application as an analytical criterion is not effective. Practically, since it has a drawback of computing a large number of determinants.
Yesupovisch criterion : This criterion can be used for quasipolynomial (2.64) in the special form
$$\Delta(s,e^s) = P(s) + Q(s)e^{-s\tau}, \tag{2.66}$$
where P and Q are polynomials, see [110]. It is clear that only linear time delay system with single delay can be investigated in this way. In fact, the idea of this method is to transform the stability analysis to test sign of some function with respect to the real axis.

2.7 Oscillation and non-oscillation in time delay systems

Oscillatory solutions of differential equations with or without delay have been frequently encountered in many physical and mechanical systems and biological processes described by a mathematical model. In 1836, when investigating the thermal conductivity, Jacques Charles Sturm posed oscillation problems of the second-order differential equation
$$\ddot{x}(t) + q(t)x(t) = 0, \tag{2.67}$$
in the real domain, which initiated a whole new direction in the qualitative theory of differential equations. His innovative idea was to deduce oscillatory properties of unknown solutions of a given differential equation from known ones of an another. For further insight, we refer to [87].
We say that an object is in oscillation, if it passes continuously from one side to the other of a position called the equilibrium position.
If we consider for example a pendulum, in this case the mass passes alternately from one side to the other of the equilibrium position, which is at the lowest point of the movement of the pendulum. If we assume that $x = 0$ is the equilibrium position this means that the position of the object takes alternately positive and negative values.
An oscillatory solution is generally defined as follows. See [70] and [1].

Definition 2.7.1. *A non trivial solution x is said to be oscillatory, if there exists a sequence $\{t_m\}$ as $m \to \infty$ such that $x(t_m) = 0$ $(m1.2.3\ldots$, and x is said to be non oscillatory if there exists a $T \in [t_0, \infty)$ such that $|x(t)| > 0$ for $t > T$.*

The essence of the oscillation theory lies in establishing conditions for the existence of oscillatory (non oscillatory) solutions and/or the convergence to zero, in studying the laws of distribution of the zeros, in obtaining the lower bounds for the distance between consecutive zeros, in studying the number of zeros in a given interval, as well as in examining the relationship between the oscillatory properties of solutions and corresponding oscillatory processes in systems of diversified physical nature. Since Sturm's pioneering work, the oscillation theory has become an indispensable mathematical tool for many sciences and high technologies.

Igor Gumowski [74] was the first to propose an analytical study of periodical solutions, starting from DDE of first order, with constant delay, and delay function of the state variable. The effect of delay on the oscillatory and nonoscillatory behavior of delay differential equation

$$\dot{x}(t) + ax(t - \tau) = 0, \tag{2.68}$$

can be summarized in the following proposition.

Theorem 2.7.2. *Let $a \in (0, \infty)$ then all nontrivial solutions of* (2.68) *are oscillatory*

$$a\, e\, \tau > 1, \tag{2.69}$$

and (2.68) *has a nonoscillatory solution if*

$$a\, e\, \tau \leq 1.$$

Such a result is a very special case of a large class of DDEs studied by Gopalsamy [69]. As in the previous section, the study of the influence of parameters on the appearance of oscillating solutions is based on a fundamental result about the relationship between linear DDE (1.6) and the position of the characteristic roots of equation (1.7) in the complex plan . This result is given by the following proposition from [69].

Proposition 2.7.3. *All nontrivial solutions of (1.3.1) are oscillatory if and only if the associated characteristic equation*

$$s + ae^{-s\tau} = 0,$$

has no real roots.

Theorem that follows gives conditions in oscillation of solution of linear time delay systems with several delays, see [69].

Theorem 2.7.4. *For $a_j, \tau_j \in (0, \infty)$, all solutions of the equation*

$$\dot{x}(t) + \sum_{j=1}^{m} a_j x(t - \tau_j) = 0, \tag{2.70}$$

are oscillatory if and only if :

$$-s+\sum_{j=1}^{m} a_j e^{s\tau_j} > 0 \quad \text{for all} \quad \mathrm{s} \in (0,\infty). \tag{2.71}$$

Condition (2.71) is easily seen to be equivalent to $a\,e\,\tau > 1$ when $n = 1$.
Now, we present some previous literature on the known result of oscillatory solution of first order linear equation. Hence many authors have considered the delay differential equation with positive coefficient.

$$\dot{x}(t) + p(t)x(t-\tau) = 0. \tag{2.72}$$

The result of Ladas [95] has been inspirational to several authors working in oscillation of delay differential equations, where they proved that every solution of (2.72) oscillates if :

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e} \quad \text{and} \quad \liminf_{t\to\infty} \int_{t-\frac{\tau}{2}t} p(s)ds > 0. \tag{2.73}$$

Integral condition like (2.73) have been employed by many others in the study on f oscillatory properties of various functional differential equation. For example, see the papers [71], [95], and [115].
In Cahlon 1998 [56] a new property has been introduced called generic oscillation and generic non oscillation defined as a topological assessment of the extent of oscillatory or non oscillatory solutions of a linear delay differential equation. Specifically, they generalize theorem (2.7.2) and theorem (2.7.4) and give concrete condition for the solution of (2.70) to be generically oscillatory or generically non oscillatory.

Theorem 2.7.5. *Let* $\tau_j \geq 0$ $(j = 1,\ldots,n)$.

1. *Suppose that* $a_j > 0$ $(j = 1,\ldots,n)$ (2.71) *holds then all solution of* (2.70) *are oscillatory.*
2. *Suppose that* $a_j \in \mathbb{R}$ $(j = 1,\ldots,n)$ *and*

$$-s_0 + \sum_{j=1}^{n} |a_j| e^{s_0\tau_j} < 0.$$

   for $s_0 > 0$ *then the solution of* (2.70) *are generically nonoscillatory.*

3. *Suppose that* $a_j > 0$ $(j = 1,\ldots,n)$ *and some* $\tau_i > 0$*. If*

$$-s_0 + \sum_{i=1}^{n} a_i e^{s_0\tau_i} = 0,$$

   For some $s_0 > 0$ *and*

$$-s + \sum_{j=1}^{n} a_j e^{s\tau_j} \geq 0,$$

   for all $s > 0$*,then the solutions of* (2.70) *are generically non oscillatory.*

This result is the first study on the generic oscillations which can yield more insight into solutions of DDEs than the standard oscillation theorem.

2.8 Conclusion

This chapter gave an overview of some important stability criteria encountered in the context of linear time delay systems.

CHAPITRE 3

Spectral abscissa and rightmost-roots assignment for time delay systems

Contents

3.1 Introduction

The objective of this chapter centers on the stability and stabilizing-controllers design for linear time-invariant retarded time-delay systems. The study of conditions on the equation parameters that guarantees the exponential stability of solutions is a question of ongoing interest and remains an open problem especially when the systems are of high order or having multiple and/or distributed delays. The starting point of this study is an interesting property Multiplicity Induced Dominancy (MID) which ensure the stability of the considered system. So, in this chapter, we aim to formulate conditions on the coefficients parameters that guarantees the coexistence of certain number of non oscillating spectral values which are not necessary multiple. The design approach we propose here is merely a delayed-output-feedback where the candidates' parameters result from the manifold defined by the coexistence of an exact number of negative spectral values, which guarantees the asymptotic stability of the system's solutions. The dominancy of such non oscillating modes is analytically shown for the considered reduced order Time-delay systems (scalar and second order system) using an adequate factorization. This chapter is organized as follows : first, we recall some important facts on the spectrum distribution for retarded Time-delay systems. Next the coexistence of non oscillatory modes for the scalar differential equation with a single delay is investigate. After this second order delay equation with three real spectral values is considered. Some concluding remarks ends the chapter.

3.2 Prerequisites and Problem Statement

In this section, we present some important facts and properties of the spectrum distribution of Time-delay systems. Let consider the generic n-order system with a single time delay :

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau). \tag{3.1}$$

Here τ is a positive constant delay and the matrices $A_j \in M_n(\mathbb{R})$ for $j = 0 \dots 1$. It is well known that the asymptotic behavior of the solutions of (3.1) is determined from the spectrum designating the set of the roots of the associated characteristic function (denoted in the sequel $\Delta(s,\tau)$). Namely, the characteristic function corresponding to system (3.1) is a quasipolynomial $\Delta : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ of the form :

$$\Delta(s,\tau) = \det(sI - A_0 - A_1 \mathrm{e}^{-\tau s}). \tag{3.2}$$

Asymptotic stability of the trivial solution and oscillatory behavior of (3.1) are known. In particular, the zero solution of this equation is asymptotically stable if and only if all roots of (3.2) lie in the left half plane, and all solutions of (3.1) are non oscillatory if and only if (3.2) have a real roots.
We start by introducing a proposition which plays an important role in the study of continuity properties of the spectrum of retarded Time-delay systems. For more details see [9, page 10]

Proposition 3.2.1. *[9] If s is a spectral value corresponding to system* (3.1) *then it satisfies*

$$|s| \leq \|A_0 + A_1 e^{-s\tau}\|_2. \tag{3.3}$$

The above proposition allows to construct an envelope curve around the characteristic roots of the quasipolynomial (3.2).
The following result was first introduced and claimed in the problems collection published in 1925 by G. Pólya and G. Szegö. In the fourth edition of their book [16, Problem 206.2, page 144 and page 347], G. Pólya and G. Szegö emphasize that the proof was obtained by N. Obreschkoff in 1928 using the principle argument, see [11]. Such a result gives a bound for the number of quasipolynomial's roots in any horizontal strip. As a consequence, a bound for the number of quasipolynomial's real roots can be easily deduced.

Theorem 3.2.2 ([16]). *Let $\tau_1, \ldots, \tau_N$ denote real numbers such that $\tau_1 < \tau_2 < \ldots < \tau_N$ and $d_1, \ldots, d_N$ positive integers such that $d_1 + d_2 + \ldots + d_N = D$. Let $f_{i,j}(s)$ stand for the function $f_{i,j}(s) = s^{i-1}\exp(\tau_j s)$, for $1 \le i \le d_j$ and $1 \le jN$. Let $\sharp$ be the number of zeros of the function*

$$f(s) = \sum_{\substack{1\le j\le N\\ 1\le i\le d_j}} c_{i,j} f_{i,j}(s) \tag{3.4}$$

that are contained in the horizontal strip $\alpha \le \mathrm{Im}(z) \le \beta$. Assuming that

$$\sum_{1\le k\le d_1} |c_{k,1}| > 0 \quad \text{and} \quad \sum_{1\le k\le d_N} |c_{k,N}| > 0$$

then

$$\frac{(\tau_N - \tau_1)(\beta-\alpha)}{2\pi} - D + 1 \le \sharp \le \frac{(\tau_N - \tau_1)(\beta-\alpha)}{2\pi} + D + N - 1. \tag{3.5}$$

Setting $\alpha = \beta = 0$, the above theorem yields $\sharp_{PS} \le D + N - 1$ where D stands for the sum of the degrees of the polynomials involved in the quasipolynomial function f and N designates the associated number of polynomials. This gives a sharp bound for the number of s real roots.

3.3 On the coexistence of real spectral value and their dominancy for scalar delay equation

Consider the simple scalar differential equation with one delay representing a biological model discussed by K.L.Cooke in [10]. It describes a vector disease dynamics where [] the infected host population $x(t)$ is governed by :

$$\dot{x}(t) + ax(t) + bx(t-\tau) = 0, \tag{3.6}$$

here $b > 0$ designates the contact rate between infected and uninfected populations and it is assumed that the infection of the host recovery proceeds exponentially at a rate $b > 0$, see also [12] for more insights on the modeling and stability results.
The characteristic equation associated to (3.6) is as follows :

$$\Delta(s,\tau) := s + a + b\exp(-s\tau) = 0. \tag{3.7}$$

Theorem 3.3.1. *For a given delay $\tau > 0$, the system* (3.6) *admits two distinct real spectral values at $s = s_2$ and $s = s_1$, with $s_2 < s_1$, if and only if*

$$\begin{cases} a = a(s_1, s_2, \tau) & := \dfrac{s_2\exp(-s_1\tau) - s_1\exp(-s_2\tau)}{\exp(-s_2\tau) - \exp(-s_1\tau)}; \\ b = b(s_1, s_2, \tau) & := \dfrac{s_1 - s_2}{\exp(-s_2\tau) - \exp(-s_1\tau)}. \end{cases} \tag{3.8}$$

– *Moreover, both spectral values* s_2 *and* s_1 *of* (3.6) *are negative, if and only if there exists a critical delay* τ^{**},
$$\tau^{**} = \frac{\ln|s_2| - \ln|s_1|}{s_1 - s_2} \tag{3.9}$$
such that $a(s_1, s_2, \tau) \geq 0$ *for all* $\tau \geq \tau^{**}$.
– *The spectral value* s_1 *is nothing but the spectral abscissa corresponding to* (3.6). *Furthermore, the zero solution of* (3.6) *is asymptotically stable.*

Remark 3.3.2. *From a control theory point of view, the assignment of such real negative spectral values subjects to choose them in the interval* $]-\infty, -a[$.

Proof : According to the Theorem 3.2.2, see also [16], the number of real roots for (3.7) is two, hence we are interested to investigate the existence of two distinct negative spectral values, s_2 and s_1 with $s_2 < s_1$. The values of a and b are calculated by solving the system :
$$\begin{cases} s_2 + a + b\exp(-s_2\tau) = & 0; \\ s_1 + a + b\exp(-s_1\tau) = & 0. \end{cases} \tag{3.10}$$
We obtain immediately the values of a and b given in (3.8), as functions of s_2 and s_1 and τ. These values are unique for each fixed $\tau > 0$. Observe that $b(s_1, s_2, \tau) > 0$ for every $\tau > 0$, while the parameter $a(s_1, s_2, \tau)$ changes the sign according to the delay τ and the position of s_1 and s_2 on the real line. On the other hand, both $s_1 + a$ and $s_2 + a$ are not null, otherwise, $b = 0$ and then $s_1 = s_2$. Which is impossible. Now, from (3.10), we have
$$\frac{s_1 + a}{s_2 + a} = \exp(-\tau(s_1 - s_2)).$$
This means that $s_2 + a$ and $s_1 + a$ have necessarily same sign, and the following expression of τ :
$$\tau = -\frac{\ln(s_1 + a) - \ln(s_2 + a)}{s_1 - s_2} \tag{3.11}$$
is well-defined. Applying the Mean Value Theorem to the function $t \mapsto \ln(t + a)$, $t \in [s_2, s_1]$. This ensures the existence of $c \in]s_2, s_1[$ such that
$$\tau = -\frac{1}{c + a}. \tag{3.12}$$
Since $\tau > 0$ so $c + a < 0$. Consequently $s_2 + a < a + c < 0$. This means that $s_2 < -a$. Likewise for s_1, since $\text{sign}(s_1 + a) = \text{sign}(s_2 + a)$, we get $s_1 < -a$.
To show the negativeness of s_2 and s_1, we use the variations of the mapping $\tau \mapsto a(s_1, s_2, \tau)$. An easy calculation shows that a is a continuous and increasing function from $-\infty$ to $-s_1$. So, if equation $a(s_1, s_2, \tau) = 0$ admits a root $\tau^{**} > 0$, then $a(s_1, s_2, \tau) \geq 0$, for all $\tau \geq \tau^{**}$. Consequently, $s_1 < 0$. Conversely, if $s_1 < 0$ then $a(s_1, s_2, \tau) \geq 0$, for all $\tau \geq \tau^{**}$.
The study of the stability of the system (3.6) is based on the dominancy of s_1. To do this, we use an adequate factorization of the quasipolynomial $\Delta(s, \tau)$ from the characteristic

equation (3.7). This later can be written as follows :

$$\begin{aligned}\Delta(s,\tau) &= (s-s_1)\left(1-\frac{(a+s_1)-\exp(-(s-s_1)\tau)}{s-s_1}\right)\\ &= (s-s_1)\left(1-(a+s_1)\Big(\frac{-\exp(-(s-s_1)\tau)}{s-s_1}+\frac{1}{s-s_1}\Big)\right).\end{aligned} \tag{3.13}$$

Thus,

$$\Delta(s,\tau) = (s-s_1)\left(1-\tau(a+s_1)\int_0^1 \exp(-\tau(s-s_1)t)dt\right). \tag{3.14}$$

To prove that s_1 is the right most root (dominancy), we assume that there exists some $s_0 = \zeta + j\eta$, a root of (3.7) such that $\zeta > s_1$. Then

$$\begin{aligned}1 &= \tau(a+s_1)\int_0^1 \exp(-\tau(s_0-s_1)t)dt\\ &= \mathrm{Re}\left(\tau(a+s_1)\int_0^1 \exp(-\tau(\zeta+j\eta-s_1)t)dt)\right)\\ &\le \tau|a+s_1|\left|\int_0^1 \exp(-\tau(\zeta+j\eta-s_1)t)dt\right|\\ &\le \tau|a+s_1|\int_0^1 \exp(-\tau(\zeta-s_1)t)dt.\end{aligned} \tag{3.15}$$

From (3.12), we have

$$\tau < \frac{1}{-(a+s_1)} = \frac{1}{|a+s_1|}. \tag{3.16}$$

Thus

$$\tau \mid a+s_1 \mid < 1. \tag{3.17}$$

Moreover, since $\zeta - s_1 > 0$, we have

$$\int_0^1 \exp(-\tau(\zeta-s_1)t)dt < 1. \tag{3.18}$$

Inequality (3.15) can not be satisfied simultaneously with (3.17) and (3.18). Which proves that the hypothesis $\zeta > s_1$ is inconsistence. So s_1 is the spectral abscissa corresponding to (3.6), guarantying the asymptotic stability of the system (3.6). □

Theorem 3.3.1 shows that the co-existence of two spectral negative values s_1 and s_2 is guaranteed by the positivity of the quantity $a(s_1,s_2,\tau)$, for $\tau > 0$. Inequality $a(s_1,s_2,\tau) \ge 0$ is satisfied if and only if

$$\tau \ge \tau^{**} = \frac{\ln|s_2| - \ln|s_1|}{s_1 - s_2}. \tag{3.19}$$

with $a(s_1,s_2,\tau^{**}) = 0$.

Remark 3.3.3. *1. In recent results [45, 47], where the question of the effect of multiple spectral values on the dynamics of time-delay systems is investigated. So, we are interested in the behavior and the geometric structure of the corresponding envelope, in which case* $a(s_0,\tau) = -s_0 - \frac{1}{\tau}$*; and* $b(s_0,\tau) = \frac{1}{\tau}\exp(\tau s_0)$*. The connected structure of the envelope is then observed, with the appearance of an invariant node with respect to the delay, at the spectral abscissa* s_0*, for every* $\tau \geq \tilde{\tau}$*. This critical value* $\tilde{\tau}$ *corresponds to the value that vanishes* $a(s_0,\tau)$*, namely,* $\tilde{\tau} = -\frac{1}{s_0}$*, see Figure 3.*

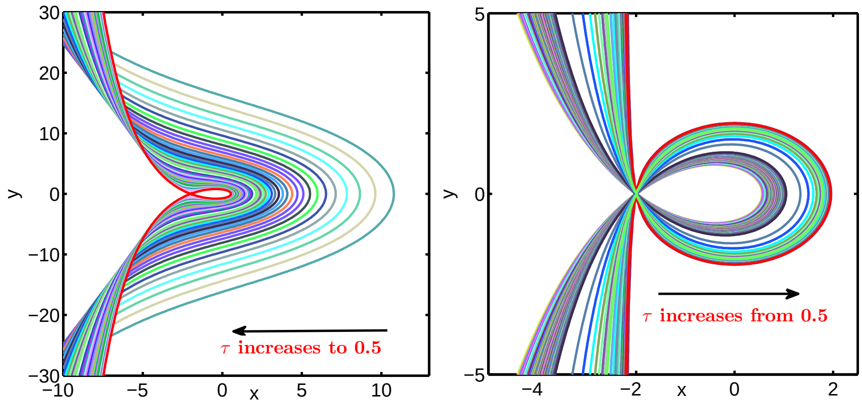


FIGURE 3.1 – Envelope curve of the characteristic equation (3.7). Case when $s_0 = -2$ is a double root. The critical delay $\tau = 0.5$ corresponds to the value that vanishes the coefficient $a = 2 - \frac{1}{\tau}$, see [52].

2. *Interestingly, when considering two real distinct spectral values, varying the value of delay* τ *may induce a change in the geometry of the envelope curve. The connected structure of the envelope is lost. Indeed, as* τ *increases reaching some critical delay* τ^**, the connected structure of the envelope is preserved. Exceeding this critical value* τ^**, there is appearance of two connected components, moving away more and more until a constant distance, reached for every delay* $\tau \geq \tau^{**}$*, see Figure 3.2.*
3. *Such geometry is observed in [14], when the analytical study and synthesis of rightmost eigenvalues of* $\dot{x}(t) = Ax(t-\tau)$ *is considered.*

The description of this kind of curve, when considering two simples spectral values s_2, s_1, with $s_2 < s_1 < 0$, is given in the following theorem.

Theorem 3.3.4. *Let*

$$C^\tau := C(s_2,s_1,\tau) = \left\{(x,y)\,;\ \sqrt{x^2+y^2} = (|a(s_2,s_1,\tau)| + |b(s_2,s_1,\tau)|\exp(-x\tau))\right\} \qquad (3.20)$$

be the envelope curve associated to the quasipolynomial (3.7).
By browsing the delay τ *from a small value until a large value, the following assertions holds for* (3.7) *:*

1. *For any* $\tau > 0$*, the envelope curve* C^τ *intersects one times the real axis, in a positive abscissa,* $x^+(\tau)$*.*

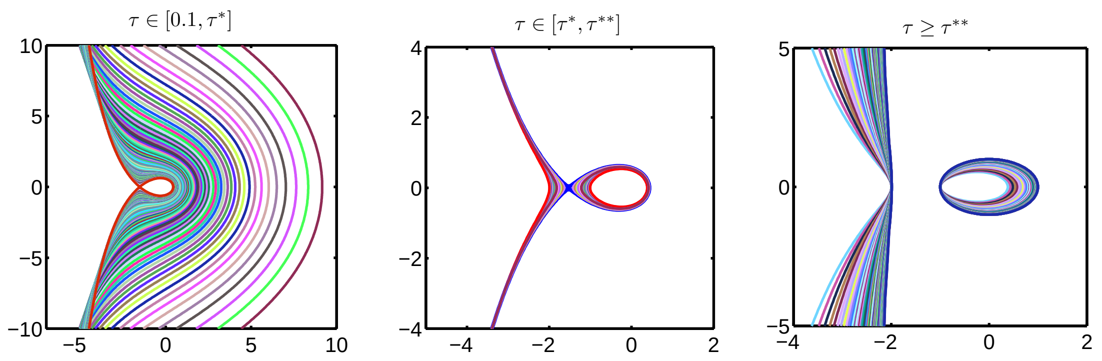


FIGURE 3.2 – Envelope curve of the characteristic equation (3.7). Case of co-existence of two simple real roots $s_1 = -1$, $s_2 = -2$.

2. *There exists $\tau^*, \tau^{**} > 0$, with $\tau^* < \tau^{**}$, such that :*
   - *When $\tau = \tau^*$, the curve C^{τ^*} contains a node, $(x(\tau^*), 0)$, in the region $x < 0$.*
   - *For $\tau < \tau^*$, there is only one intersection of the envelope curve with the real axis.*
   - *For $\tau \in]\tau^{**}, \tau^*[$, the curve C^τ splits into two disjoint segments C_1^τ and C_2^τ, which intersect the the region $]s_2, s_1[$ at points $(x_1^-(\tau), 0)$ and $\underline{(x_2^-(\tau), 0)}$, respectively.*
   - *There exists $\bar{x}(\tau^{**}) < s_2$ such that for every $\tau < \tau^{**}$, $\overline{\overset{\circ}{C^{\tau^{**}}}} \subset \overline{\overset{\circ}{C^\tau}}$ in the region $x > \bar{x}(\tau^{**})$, and for every $\tau > \tau^{**}$, $\overline{\overset{\circ}{C^\tau}} \subset \overline{\overset{\circ}{C^{\tau^{**}}}}$.*
   - *For $\tau \geq \tau^{**}$, the curve C^τ splits into two disjoint segments C_1^τ and C_2^τ, with the property that $(s_2, 0) \in \bigcap_{\tau \geq \tau^{**}} C_1^\tau$ and $(s_1, 0) \in \bigcap_{\tau \geq \tau^{**}} C_2^\tau$ and*

$$d(C_1^\tau, C_2^\tau) = |s_2 - s_1|.$$

3. *As $\tau \to \infty$,*
   - *$x_1^-(\tau) \to s_2$, $x_2^-(\tau) \to s_1$ and $x^+(\tau) \to -s_1$. The two first limits are reached from $\tau = \tau^{**}$, see Figure 3.4.*
   - *C^∞, the envelope limit, is the circle centered at the origin with the rayon $r = |s_1|$.*

Proof 1.

To find the point of intersection of the curve with the x-axis, we solve this system of two equation obtained from (3.20) using the W-Lambert function [15]

$$\begin{cases} |a| + |b|e^{-x\tau} - x = 0 \\ |a| + |b|e^{-x\tau} + x = 0. \end{cases} \tag{3.21}$$

Using a change of variable $z = -x\tau$ we obtain

$$|s_1 - s_2|e^z - \frac{z}{\tau}|e^{-s_2\tau} - e^{-s_1\tau}| + |s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}| = 0 \tag{3.22}$$

$$|s_1 - s_2|e^z + \frac{z}{\tau}|e^{-s_2\tau} - e^{-s_1\tau}| + |s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}| = 0. \tag{3.23}$$

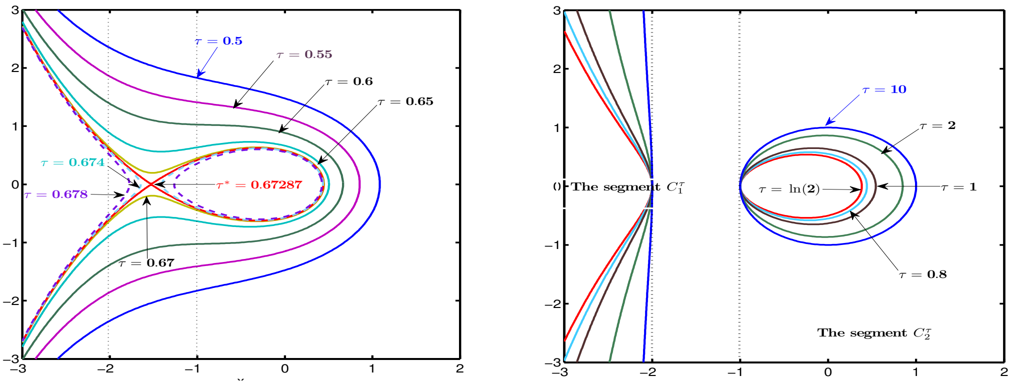


FIGURE 3.3 – (Left) Decreasing behavior of the envelope curve versus $\tau < \tau^{**}$. (Right) Increasing behavior of the envelope curve versus $\tau \geq \tau^{**}$. Case $s_1 = -1$ and $s_2 = -2$

The above two equation are under the following form $\gamma e^z + \beta + \sigma = 0$, *where* $\gamma = |s_1 - s_2|$, $\beta = \pm\frac{e^{-s_2\tau}-e^{-s_1\tau}}{\tau}$ *and* $\sigma = |s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|$.
The existence of real solutions depends on the sign of the discriminant $\Delta(\tau) = \frac{\gamma}{\beta} e^{-\frac{\sigma}{\beta}}$ *and its position with respect to* $-e^{-1}$.

1. *For* $\tau > 0$, *we search the positive solution, so we study the equation* (3.23)*. The discriminant of this equation is given by :*

$$\Delta^+(\tau) = \frac{\tau|s_1 - s_2|}{|e^{-s_2\tau} - e^{-s_1\tau}|} \exp\left(-\frac{\tau|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}\right). \tag{3.24}$$

Hence equation (3.23) *has a positif real solution*

$$z_1^+(\tau) = -W_0(\Delta^+) - \frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|},$$

thus

$$x^+(\tau) = \frac{1}{\tau} W_0(\Delta^+) + \frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}, \tag{3.25}$$

which implies that the envelope curve intersects only one times the real axis in the right half plan at $x^+(\tau)$.

2. *Now we study the equation* (3.22)*, where the discriminant is written as follows :*

$$\Delta^-(\tau) = -\frac{\tau|s_1 - s_2|}{|e^{-s_2\tau} - e^{-s_1\tau}|} \exp\left(\frac{\tau|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}\right). \tag{3.26}$$

Since $\Delta^-(\tau) < 0$, *we need to evaluate the sign of* $\Delta^-(\tau) + e^{-1}$. *Note that the function* $\tau \mapsto \Delta^-(\tau) + \exp(-1)$ *is continuous on* $\mathbb{R}^+$. *In addition, as* $\tau \to 0$, $\Delta^-(\tau) + \exp(-1) \to -\exp(1) + \exp(-1) < 0$. *We have then*

$$\Delta^-(\tau^{**}) = -\ln\left(\frac{s_1}{s_2}\right)\left(\left(\frac{s_1}{s_2}\right)^{-\frac{s_1}{s_2-s_1}} - \left(\frac{s_1}{s_2}\right)^{-\frac{s_2}{s_2-s_1}}\right)^{-1} + \mathrm{e}^{-1}.$$

The variations of

$$F(t) := -\ln(t)\left(t^{-\frac{t}{1-t}} - t^{-(1-t)^{-1}}\right)^{-1} + \mathrm{e}^{-1},\ t \in \mathbb{R}_+^* - \{1\}$$

show that $\Delta^-(\tau^{**}) + e^{-1} > 0, \forall t \in \mathbb{R}_+^* - \{1\}$. Thanks to the intermediate values theorem, there exists $\tau^* \in]0, \tau^{**}[$ such that $\Delta^-(\tau^*) + e^{-1} = 0$.

– *For $\tau = \tau^*$, $\Delta^-(\tau^*) = -e^{-1}$, so* (3.22) *admits a (double) negative real solution*

$$x_d^-(\tau^*) = \frac{1}{\tau^*}W_0(-e^{-1}) - \frac{|s_2e^{-s_1\tau} - s_1e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|} = -\frac{1}{\tau^*} - \frac{|s_2e^{-s_1\tau} - s_1e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|},$$

which allows to the appearance of a node at the point $(x(\tau^*), 0)$. This equation admits also a positive solution $(x^+(\tau), 0)$ given by (3.25)*, from which we deduce that there are two intersections with the real axis.*

– *For $\tau < \tau^*$, there exists one positive solution $(x^+(\tau), 0)$ given by* (3.25)*, which corresponds to unique intersection with the half real axis, $x > 0$.*

– *For $\tau \in]\tau^*, \tau^{**}[$, we have $\Delta^-(\tau) + e^{-1} > 0$, then equation* (3.22) *admits two negative real solutions*

$$\begin{aligned} z_1(\tau) &= -W_0\left(\Delta^-\right) + \frac{\tau\,|s_2e^{-s_1\tau} - s_1e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}, \\ z_2(\tau) &= -W_{-1}\left(\Delta^-\right) + \frac{\tau\,|s_2e^{-s_1\tau} - s_1e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}, \end{aligned}$$

thus

$$x_1^-(\tau) = \frac{1}{\tau}W_0\left(\Delta^-\right) - \frac{|s_2e^{-s_1\tau} - s_1e^{-s_2\tau}|}{|e^{-\tau s_2} - e^{-s_1\tau}|}, \tag{3.27}$$

$$x_2^-(\tau) = \frac{1}{\tau}W_{-1}\left(\Delta^-\right) - \frac{|s_2e^{-s_1\tau} - s_1e^{-s_2\tau}|}{|e^{-s_2\tau} - e^{-s_1\tau}|}, \tag{3.28}$$

this implies that the curve C^τ intersects two times the real axis, from which we deduce that the curve splits into two disjoints segments C_1^τ and C_2^τ which intersect the region $]s_2, s_1[$ at points $(x_1^-(\tau), 0)$ and $(x_2^-(\tau), 0)$ respectively. Details about the property $C_2^\tau \cap C_2^\tau = \emptyset$ will be presented bellow.

– *Here we consider the case where $\tau < \tau^{**}$. Recall that the sign of $a(s_2, s_1, \tau)$ depends on the parameter $\tau > 0$, while $b(s_2, s_1, .) > 0$. So*

$$|s_2\exp(-s_1\tau) - s_1\exp(-s_2\tau)| = \begin{cases} 0 & \text{if} \quad \tau = \tau^{**} \\ s_2\exp(-s_1\tau) - s_1\exp(-s_2\tau) & \text{if} \quad \tau > \tau^{**} \\ s_1\exp(-s_2\tau) - s_2\exp(-s_1\tau) & \text{if} \quad \tau < \tau^{**} \end{cases}$$

Hence

$$\frac{d}{d\tau}\left(|a(s_2, s_1, \tau)|\right) = \begin{cases} e^{-\tau s_2}e^{-\tau s_1}\dfrac{(s_2 - s_1)^2}{(e^{-\tau s_2} - e^{-\tau s_1})^2} & \text{if} \quad \tau > \tau^{**} \\ -e^{-\tau s_2}e^{-\tau s_1}\dfrac{(s_2 - s_1)^2}{(e^{-\tau s_2} - e^{-\tau s_1})^2} & \text{if} \quad \tau < \tau^{**} \end{cases}$$

$$\frac{d}{d\tau}\left(|b\left(s_2,s_1,\tau\right)|\exp(-x\tau)\right)=e^{-x\tau}\frac{\left(s_1-s_2\right)}{\left(e^{-\tau s_2}-e^{-\tau s_1}\right)}\left(\frac{s_2e^{-\tau s_2}-s_1e^{-\tau s_1}}{e^{-\tau s_2}-e^{-\tau s_1}}-x\right)$$

Note that $s_2e^{-\tau s_1}-s_1e^{-\tau s_2}=\left(1-\theta\tau\right)e^{-\theta\tau}\left(s_2-s_1\right)$, *for some* $\theta\in\left]s_2,s_1\right[$, *so*

$$\bar{x}(\tau)\triangleq\frac{s_2e^{-\tau s_1}-s_1e^{-\tau s_2}}{e^{-\tau s_2}-e^{-\tau s_1}}<0,\forall\tau>0.$$

In addition, $\tau\mapsto\bar{x}(\tau)$ *is continuous and increasing on* $]0,\tau^{**}[$, *with*

$$\lim_{\tau\to\tau^{**}}\bar{x}\left(\tau\right)=s_2-\left(s_1-s_2\right)\frac{\left(\dfrac{s_2}{s_1}\right)^{\dfrac{s_1}{s_2-s_1}}}{\left(\dfrac{s_2}{s_1}\right)^{\dfrac{s_2}{s_2-s_1}}-\left(\dfrac{s_2}{s_1}\right)^{\dfrac{s_1}{s_2-s_1}}}=\bar{x}\left(\tau^{**}\right)<s_2.$$

Thus $\tau\mapsto|b\left(s_2,s_1,\tau\right)|\exp(-x\tau)$ *is decreasing on* $\mathbb{R}^{+*}$, *for every* $x>\bar{x}\left(\tau^{**}\right)$. *This means that when* $x>\bar{x}\left(\tau^{**}\right)$, *the function*

$$\tau\mapsto|a\left(s_2,s_1,\tau\right)|+|b\left(s_2,s_1,\tau\right)|\exp(-x\tau)$$

is continuous and decreasing on $]0,\tau^{**}[$. *We deduce that*

$$\forall\tau\in\left]0,\tau^{**}\right[;\ C\left(s_2,s_1,\tau^{**}\right)\subset C\left(s_2,s_1,\tau\right).$$

On the other hand, the variations of $x\mapsto|a(s_2,s_1,\tau)|+|b(s_2,s_1,\tau)|\exp\left(-x\tau\right)$ *show that for all* $x\in\mathbb{R}-[s_2,s_1]$ *and for all* $\tau>\tau^{**}$

$$|b(s_2,s_1,\tau^{**})|\exp\left(-x\tau^{**}\right)-\big(|a(s_2,s_1,\tau)|+|b(s_2,s_1,\tau)|\exp\left(-x\tau\right)\big)<0,\tag{3.29}$$

from which we deduce that

$$\forall\ \tau>\tau^{**};\ C\left(s_2,s_1,\tau^{**}\right)\subset C\left(s_2,s_1,\tau\right).\tag{3.30}$$

– *Now we study the case of* $\tau>\tau^{**}$. *We have*

$$\begin{aligned}\lim_{\tau\to\tau^{**}}C\left(s_2,s_1,\tau\right)&=\bigcap_{\tau<\tau^{**}}C\left(s_2,s_1,\tau\right)=C\left(s_2,s_1,\tau^{**}\right)\\&=\left\{(x,y);\ \sqrt{x^2+y^2}=M(s_2,s_1,x)\right\}\end{aligned}$$

where

$$M(s_2,s_1,x)=\frac{s_1-s_2}{\left(\dfrac{s_2}{s_1}\right)^{-\dfrac{s_2-x}{s_1-s_2}}-\left(\dfrac{s_2}{s_1}\right)^{-\dfrac{s_1-x}{s_1-s_2}}}.$$

Observe that

$$M(s_2,s_1,s_2)\ =\ -s_2;$$

$$M(s_2,s_1,s_1) \quad = \quad -s_1.$$

Hence the boundary of $C(s_2,s_1,\tau^{**})$ *contains both* $(s_1,0)$ *and* $(s_2,0)$. *We get more, that is*

$$(s_2,0) \in \bigcap_{\tau \geq \tau^{**}} C_1^{\tau} \quad \text{and} \quad (s_1,0) \in \bigcap_{\tau \geq \tau^{**}} C_2^{\tau}. \tag{3.31}$$

On the other hand, since

$$M(s_2,s_1,\alpha s_1 + (1-\alpha)s_2) = -s_2\left(\frac{s_1}{s_2}\right)^{\alpha}$$

for every $\alpha \in]0,1[$, *we can show that*

$$C(s_2,s_1,\tau^{**}) \cap]s_2,s_1[= \emptyset.$$

Indeed, if not there exists $z \in]s_2,s_1[$, *such that* $|z| \leq M(s_2,s_1,z)$. *Writing* $z = \alpha s_1 + (1-\alpha)s_2$, *for some* $\alpha \in]0,1[$, *we have*

$$-s_1 < |z| \leq -s_2\left(\frac{s_1}{s_2}\right)^{\alpha} < -s_2\left(\frac{s_1}{s_2}\right) = -s_1.$$

This implies that

$$-s_1 < -s_1$$

a contradiction.
This reasoning allows us to show that

$$C(s_2,s_1,\tau^{**}) \cap \left\{(x,y) \in \mathbb{R}^2,\ s_2 < x < s_1\right\} = \emptyset. \tag{3.32}$$

In fact, in the contrary case, we will get

$$-s_1 \leq \sqrt{s_1^2 + y^2} < \sqrt{x^2 + y^2} = |z| \leq -s_2\left(\frac{s_1}{s_2}\right)^{\alpha} < -s_1.$$

Impossible. This means that the envelope curve $C(s_2,s_1,\tau^{**})$ *is splits into two disjointed parts* $C^{x<s_2}(s_2,s_1,\tau^{**})$ *and* $C^{x>s_1}(s_2,s_1,\tau^{**})$, *separated by the band* $\left\{(x,y) \in \mathbb{R}^2,\ s_2 < x\right.$
As a consequence, we get that distance

$$\begin{aligned} d\left(C^{x<s_2}(s_2,s_1,\tau^{**}), C^{x>s_1}(s_2,s_1,\tau^{**})\right) &= d\left(C^{x<s_2}(s_2,s_1,\tau), C^{x>s_1}(s_2,s_1,\tau)\right) \\ &= \inf_{\substack{z_1 \in C^{x<s_2}(s_2,s_1,\tau^{**}) \\ z_2 \in C^{x>s_1}(s_2,s_1,\tau^{**})}} d(z_1,z_2) \\ &= s_1 - s_2,\ \forall \tau > \tau^{**}. \end{aligned}$$

d being the Euclidean distance.

3. *Now, we deal with the asymptotic behavior of the envelope curve,* $x_1^-(\tau)$, $x_2^-(\tau)$, *and* $x^+(\tau)$ *as* $\tau \to \infty$.

– *Since* $\Delta^-(\tau) \in \left]-e^{-1}, 0\right[$, *we have* $W_0(\Delta^-(\tau))$ *satisfies :*

$$-1 = W_0\left(-e^{-1}\right) < W_0\left(\Delta^-(\tau)\right) < W_0(0) = 0,$$

thus

$$\lim_{\tau\to\infty} \frac{1}{\tau} W_0\left(\Delta^-\right) = 0.$$

From this we deduce that

$$\lim_{\tau\to\infty} x_1^-(\tau) = s_1.$$

On the other hand, since we have

$$d(C_1^\tau, C_2^\tau) = s_1 - s_2, \forall \tau > \tau^{**},$$

we deduce that

$$\lim_{\tau\to\infty} x_2^-(\tau) = s_2.^{1} \tag{3.33}$$

Similarly, we prove that $\lim_{\tau\to\infty} x^+(\tau) = -s_1$. *Indeed, there is always* $\tau_0 > 0$ *such that*

$$\frac{|s_2 e^{-s_1\tau} - s_1 e^{-s_2\tau}|}{|e^{-\tau s_2} - e^{-s_1\tau}|} > -\frac{s_1}{2}, \text{ for all } \tau \geq \tau_0.$$

This means that $x^+(\tau)$ *in* (3.25) *given via the WLambert function satisfies*

$$x^+(\tau) > -\frac{s_1}{2}, \quad \forall \tau \geq \tau_0.$$

Using the fact that

$$0 < \frac{e^{-\tau x^+(\tau)}}{|e^{-\tau s_2} - e^{-s_1\tau}|} < \frac{e^{\frac{s_1\tau}{2}}}{|e^{-\tau s_2} - e^{-s_1\tau}|}, \quad \forall \tau \geq \tau_0.$$

we deduce that $\dfrac{e^{-\tau x^+(\tau)}}{|e^{-\tau s_2} - e^{-s_1\tau}|} = 0$. *Letting* τ *tend to* ∞ *in the following expression :*

$$x^+(\tau) = \frac{e^{-\tau x^+(\tau)}}{|e^{-\tau s_2} - e^{-s_1\tau}|} + \frac{|s_1 e^{-\tau s_2} - s_2 e^{-s_1\tau}|}{|e^{-\tau s_2} - e^{-s_1\tau}|} \tag{3.34}$$

we obtain

$$\lim_{\tau\to\infty} x^+(\tau) = |s_1|.^{2}$$

1. As a consequence, using (3.28), we get the following property of the WLambert function : $\lim_{\tau\to\infty} \frac{1}{\tau} W_{-1}(\Delta^-(\tau)) = 0$.

2. As a consequence, we get the following property $\lim_{\tau\to\infty} \frac{1}{\tau} W_0(\Delta^+) = 0$.

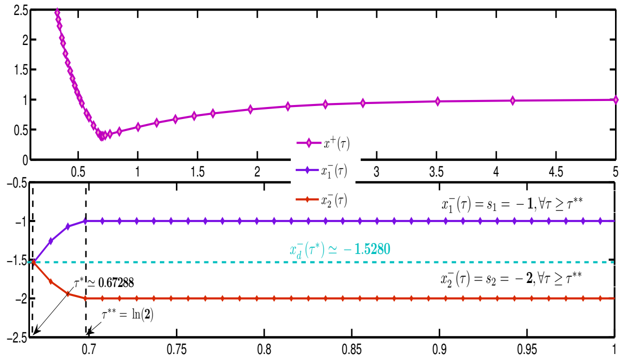


FIGURE 3.4 – Asymptotic behavior of the negative roots of the envelope curve. $s_2 = -1$, $s_1 = -2$

– *Let us consider the equation of the envelope curve* (3.20). *In the region $x > s_1$, we have*

$$\lim_{\tau \to \infty} |b(s_1, s_2, \tau)| \exp(-x\tau) = 0,$$

and

$$\lim_{\tau \to \infty} |a(s_1, s_2, \tau)| = |s_1|.$$

Hence, we deduce that as $\tau \to \infty$ the envelope curve (3.20) *becomes*

$$x^2 + y^2 = s_1^2,$$

which corresponds to a circle C^∞ with the ray $r = |s_1|$. Concerning the region $x < s_2$, simulation results show that for τ sufficiently large, the segment C_1^τ disappears. The question of finding the upper bound of τ for which $\overline{C_1^\tau}$ is exactly the half-plan $x \leq s_2$ remains open.

3.4 Second-order systems

Second-order linear systems capture the dynamic behavior of many natural phenomena, and have found wide applications in a variety of fields, such as vibration and structural analysis. The equation of this system is given as follows :

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + \alpha x(t - \tau) = 0. \tag{3.35}$$

The characteristic equation associated to (3.35) is given by :

$$\Delta(s, \tau) = s^2 + as + b + \alpha \exp(-s\tau) = 0. \tag{3.36}$$

Theorem 3.4.1. *The system* (3.35) *admits three distinct real spectral values s_3, s_2 and s_1 with $s_3 < s_2 < s_1$ if and only if the parameters a, b and α satisfy*

$$\begin{cases} a = a(s_1,s_2,s_3,\tau) := \dfrac{1}{Q} \displaystyle\sum_{\substack{i,j,k\in\Lambda \\ i<j, i\neq j\neq k}} (-1)^{i+j}\,(s_i^2 - s_j^2)\exp(-s_k\tau) \\ b = b(s_1,s_2,s_3,\tau) := -\dfrac{1}{Q} \displaystyle\sum_{\substack{i,j,k\in\Lambda \\ i<j, i\neq j\neq k}} (-1)^{i+j}\, s_i s_j (s_i - s_j)\exp(-s_k\tau) \\ \alpha = (s_1,s_2,s_3,\tau) := -\dfrac{1}{Q} \displaystyle\prod_{\substack{i,j\in\Lambda \\ i<j}} (s_i - s_j) \end{cases} \tag{3.37}$$

where

$$Q := Q(s_1,s_2,s_3,\tau) = \sum_{\substack{i,j,k\in\Lambda \\ i<j, k\neq i,j}} (-1)^{i+j}\,(s_i - s_j)\exp(-s_k\tau).$$

In this case, α is necessarily negative.

– *The spectral value s_1 is negative if and only if there exists $\tau_0 > 0$ such*

$$a(s_1,s_2,s_3,\tau_0) + s_2 = 0.$$

– *s_1 is the spectral abscissa of* (3.35). *This guarantees the asymptotic stability of the system.*

Proof 2. *The parameters given in* (3.37) *can be easily obtained by solving the system*

$$\begin{cases} s_3^2 + as_3 + b + \alpha \mathrm{e}^{-s_3\tau} = 0 \\ s_2^2 + as_2 + b + \alpha \mathrm{e}^{-s_2\tau} = 0 \\ s_1^2 + as_1 + b + \alpha \mathrm{e}^{-s_1\tau} = 0. \end{cases} \tag{3.38}$$

Now, let us study the sign of α. Recall that

$$\alpha = -\frac{(s_1 - s_2)(s_2 - s_3)(s_1 - s_3)}{Q(s_1,s_2,s_3,\tau)}. \tag{3.39}$$

We apply twice the Mean Value Theorem to the denominator $Q(s_1,s_2,s_3,\tau)$, we get :

$$\begin{aligned} Q(s_1,s_2,s_3,\tau) &= s_2 e^{-\tau s_3} - s_1 e^{-\tau s_3} + s_1 e^{-\tau s_2} - s_3 e^{-\tau s_2} + s_3 e^{-\tau s_1} - s_2 e^{-\tau s_1} \\ &= (s_2 - s_1)\left(e^{-\tau s_3} - e^{-\tau s_1}\right) + (s_3 - s_1)\left(e^{-\tau s_1} - e^{-\tau s_2}\right) \\ &= -\tau(s_2 - s_1)(s_3 - s_1)\int_0^1 e^{-\tau(ts_1 + (1-t)s_3)}dt \\ &\quad +\tau(s_3 - s_1)(s_2 - s_1)\int_0^1 e^{-\tau(ts_1 + (1-t)s_2)}dt \\ &= -\tau(s_2 - s_1)(s_3 - s_1)\int_0^1 \left(e^{-\tau(ts_1+(1-t)s_3)} - e^{-\tau(ts_1+(1-t)s_2)}\right)dt \\ &= \tau^2(s_2 - s_1)(s_3 - s_1)(s_3 - s_2)\int_0^1\int_0^1 (1-t)\,e^{-\tau(ts_1+(1-t)(\theta s_3+(1-\theta)s_2))}d\theta dt. \end{aligned}$$

This means that

$$\alpha(s_1,s_2,s_3,\tau) = \frac{-1}{\tau^2 \widetilde{\mathscr{Q}}(s_1,s_2,s_3,\tau)} \tag{3.40}$$

where

$$\widetilde{\mathscr{Q}}(s_1,s_2,s_3,\tau) = \int_0^1 \int_0^1 (1-t)\, e^{-\tau(ts_1+(1-t)(\theta s_3+(1-\theta)s_2))} d\theta dt > 0.$$

Now, let us show the negativeness of $s_1,\ s_2,\ s_3$*. From* (3.38) *and* (3.40)*, we get that*

$$\begin{cases} (s_3-s_2)[s_3+s_2+a] = -\alpha\,[e^{-\tau s_3}-e^{-\tau s_2}] > 0 \\ (s_2-s_1)[s_2+s_1+a] = -\alpha\,[e^{-\tau s_2}-e^{-\tau s_1}] > 0 \\ (s_3-s_1)[s_3+s_1+a] = -\alpha\,[e^{-\tau s_3}-e^{-\tau s_1}] > 0. \end{cases}$$

Since $s_3 < s_2 < s_1$*, we obtain the following equations :*

$$\begin{cases} s_3+s_2 < -a \\ s_2+s_1 < -a \\ s_3+s_1 < -a \end{cases} \tag{3.41}$$

System (3.41) *is reduced to the single inequality* $s_2+s_1 < -a$*, from which we deduce that* $s_3 < s_2 < -\frac{a}{2}$*, and* $s_1 < -a-s_2$*. Since* $\tau \mapsto a(s_1,s_2,s_3,\tau)+s_2$ *is continuous and increasing from* $-\infty$ *to* $-s_1$ *when* τ *varies in* $\mathbb{R}^{+*}$*, this means that* $a(s_1,s_2,s_3,\tau)+s_2$ *takes positive values if (and only if)* $s_1 < 0$*. This means that the equation* $a(s_1,s_2,s_3,\tau)+s_2 = 0$ *has a (unique) root,* $\tau_0 > 0$*.*
Now, to study the stability of the system, we need to study the dominancy of s_1 *by using an adequate factorization of the quasipolynimial* (3.36)*, so we have :*

$$\begin{aligned} s^2+as+b+\alpha\exp(-\tau s) &= (s-s_2)(s-s_1)\left[\frac{s^2+as+b+\alpha\exp(-\tau s)}{(s-s_2)(s-s_1)}\right] \\ &= (s-s_2)(s-s_1)P(s,s_1,s_2,\tau). \end{aligned}$$

The factor $P(s) := P(s,s_1,s_2,\tau)$ *can be rewritten under the more suitable form*

$$\begin{aligned} P(s) &= 1+\frac{(a+s_2+s_1)s+b-s_2s_1}{(s-s_2)(s-s_1)}+\frac{\alpha\exp(-\tau s)}{(s-s_2)(s-s_1)} \\ &= 1-\frac{s_2^2+as_2+b}{(s_1-s_2)(s-s_2)}+\frac{s_1^2+as_1+b}{(s_1-s_2)(s-s_1)}+\frac{\alpha\exp(-\tau s)}{(s_2-s_1)(s-s_2)}-\frac{\alpha\exp(-\tau s)}{(s_2-s_1)(s-s_1)} \\ &= 1+\frac{\alpha\exp(-\tau s)}{(s_2-s_1)(s-s_2)}-\frac{\alpha\exp(-\tau s_2)}{(s_2-s_1)(s-s_2)}+\frac{\alpha\exp(-\tau s_1)}{(s_1-s_2)(s-s_1)}-\frac{\alpha\exp(-\tau s)}{(s_2-s_1)(s-s_1)}. \end{aligned}$$

Using the integral form of the remainder in Taylor's Theorem, we get

$$\begin{aligned} P(s) &= 1+\frac{\alpha\exp(-\tau s_2)(\exp(-\tau(s-s_2))-1)}{(s_2-s_1)(s-s_2)}+\frac{\alpha\exp(-\tau s_1)(1-\exp(-\tau(s-s_1)))}{(s_2-s_1)(s-s_1)} \\ &= 1+\frac{\alpha\exp(-\tau s_2)\left(-\tau(s-s_2)\int\limits_0^1 e^{-\tau(s-s_2)t}dt\right)}{(s_2-s_1)(s-s_2)}-\frac{\alpha\exp(-\tau s_1)\left(-\tau(s-s_1)\int\limits_0^1 e^{-\tau(s-s_1)t}dt\right)}{(s_2-s_1)(s-s_1)} \end{aligned}$$

$$= \quad 1+\frac{\alpha\tau}{(s_2-s_1)}\int_0^1 \exp(-\tau t s)\left(\exp(-\tau(1-t)s_1)-\exp(-\tau(1-t)s_2)\right)dt.$$

Using again the following integral representation

$$\exp(-\tau(1-t)s_1)-\exp(-\tau(1-t)s_2)=-\tau(s_1-s_2)\int_0^1 \exp(-\tau(1-t)(\theta s_2+(1-\theta)s_1))\,d\theta$$

we get

$$P(s,s_1,s_2,\tau)=1+\alpha\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t(s-s_1))\exp(-\tau[s_1+\theta(1-t)(s_2-s_1)])\,d\theta dt.$$

To prove the dominancy property, namely s_1 is the rightmost root of (3.36)*, let us assume that there exists some $s_0=\zeta+j\eta$ a root of* (3.36) *such that $\zeta>s_1$. Then $P(s_0,s_1,s_2,\tau)=0$ and for any $t>0$, one has :*

$$\exp(-\tau[t(\zeta-s_1)])<1.$$

So

$$\begin{aligned}
1 &= -\alpha\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t(s_0-s_1))\exp(-\tau[s_1+\theta(1-t)(s_2-s_1)])\,d\theta dt\\
&= \mathrm{Re}\left(-\alpha\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t(s_0-s_1))\exp(-\tau[s_1+\theta(1-t)(s_2-s_1)])\,d\theta dt\right)\\
&\le |\alpha|\,\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t(\zeta-s_1))\exp(-\tau[s_1+\theta(1-t)(s_2-s_1)])\,d\theta dt\\
&< |\alpha|\,\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau[(1-\theta(1-t))s_1+\theta(1-t)s_2])\,d\theta dt. \qquad (3.42)
\end{aligned}$$

Using the relation $1-\theta(1-t)=t+(1-t)(1-\theta)$, we get

$$\exp(-\tau(1-\theta(1-t))s_1)=\exp(-\tau t s_1)\exp(-\tau(1-t)(1-\theta)s_1).$$

Moreover, since $s_3<s_1$, we obtain, for every $t\in(0,1)$ and every $\theta\in(0,1)$, the following estimation

$$\exp(-\tau(1-t)(1-\theta)s_1)<\exp(-\tau(1-t)(1-\theta)s_3).$$

Summing up, using (3.40)*, inequality* (3.42) *becomes*

$$1<|\alpha|\,\tau^2\int_0^1\int_0^1 (1-t)\exp(-\tau t s_1)\exp(-\tau(1-t)[(1-\theta)s_3+\theta s_2])\,d\theta dt=|\alpha|\,\tau^2\widetilde{\mathscr{Q}}=1,$$

which is inconsistent. This proves the dominancy of s_1. The proof of Theorem 3.4.1 is achieved.

Remark 3.4.2. *The geometric structure of envelope curve of the quasipolynomial (4.2), defined by*

$$\sqrt{x^2+y^2} - \|A_0\|_2 - \|A_1\|_2 \, e^{-\tau x} = 0, \tag{3.43}$$

with $A_0 = \begin{pmatrix} -a & 1 \\ -b & 0 \end{pmatrix}$ *and* $A_1 = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}$, *may loose its connection as observed in the first order equation, depending on the distribution of the roots* s_1, s_2, s_3 *and the delay* τ. *More precisely, three cases can be observed according to the distance of the root* s_3 *with respect to the centered circle, of radius R, with*

$$R^2 = \|A_0\|_2^2 = \frac{1}{2}\left(\sqrt{(a^2+(b-1)^2)(a^2+(b+1)^2)} + a^2 + b^2 + 1\right) \tag{3.44}$$

1. *If* $s_3 < -R$, *exceeding some critical value of the delay* $\tau = \tilde{\tau}$ *gives rise to the birth of two disconnected components as illustrated in Figure 3.5 (left), where the arrows indicate the motion direction of the envelope as* τ *increases. For* τ *sufficiency large, the right component is the centered circle, of radius R, while the left component approaches the half-plane delimited by* $x = s_3$. *The two components are separated by the distance* $-R - s_3$.

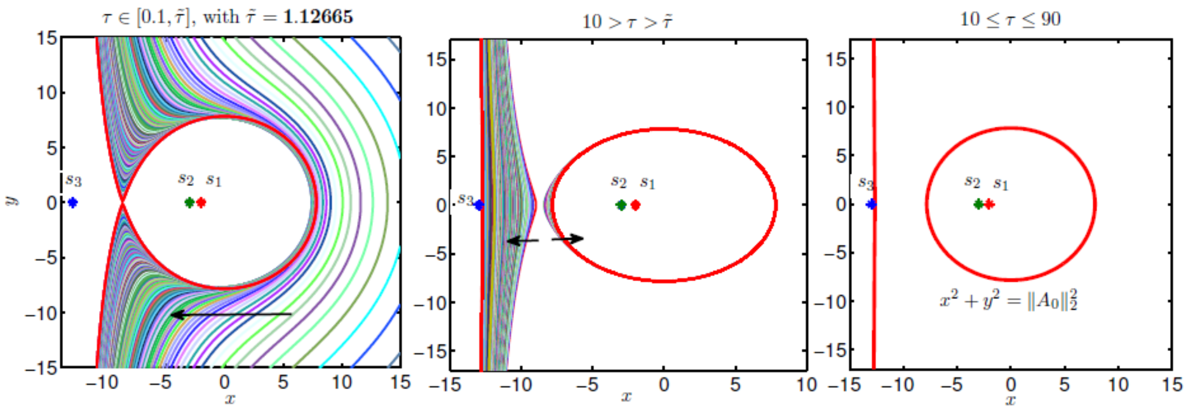


FIGURE 3.5 – Envelope curve of the characteristic equation (3.36). Case $s_1 = -2$, $s_2 = -3$, $s_3 = -13$.

2. *If* $s_3 > -R$, *the connected structure of the envelope is preserved independently from the delay* τ . *For* τ *sufficiency large, the envelope takes the shape of a vertical* (℧) *never closing and which is carried by the centered circle of radius R and delimited by the vertical line* $x = s_3$ *as can be seen in Figure 3.6 (left).*

3. *If* $s_3 = -R$, *the connected structure of the envelope is still preserved as in the second case. Here the structure* (℧) *is also observed. This structure is deformed when the delay increases. Indeed, the angular points are getting closer and closer, until forming a node at* s_3, *see Figure 3.6 (right).*

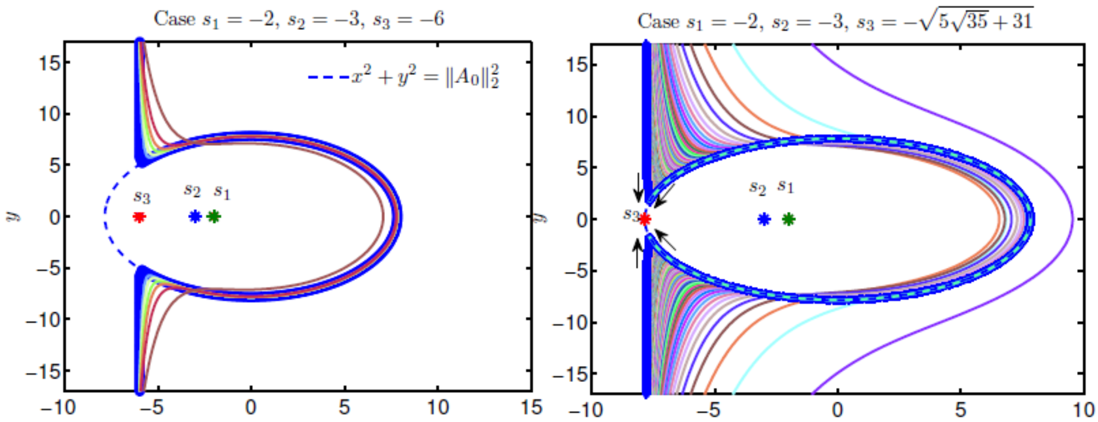


FIGURE 3.6 – Envelope curve of the characteristic equation (3.36). Case $s_1 = -2$, $s_2 = -3$, $s_3 \geq -R$.

3.5 Concluding remarks

The property of *multiplicity induced-dominancy* for spectral value of time-delay systems of retarded type is extended through this work. More precisely, it is shown, for reduced order Time-delay systems, that assigning PS_B real spectral values (not necessarily multiple root) make them the rightmost roots of the corresponding quasipolynomial. Furthermore, if they are set to be negative, this guarantees the asymptotic stability of the trivial solution. This new result emphasizes a new delayed controller-design based on the trivial solution's decay rate assignment.

CHAPITRE 4

Some insights on rightmost spectral values assignment for generic second order delay system

Contents

4.1 Introduction

The main objective of this chapter consists to investigate the stability and stabilization of some generic linear second order time-invariant retarded system with single delay. This study is mainly inspired by [49]where a property called *Multiplicity-Induced-Dominancy* (MID) is emphasized for generic second order delay system that we consider in this chapter, which consists in characterizing the exponential decay rate of the trivial solution. The main idea of this study is to built an appropriate delayed-state-feedback controller allowing to guarantee the stability of the system in the closed loop using an appropriate partial pole placement. Recently in [81] it is shown that under appropriate conditions, the coexistence of the maximal number of negative distinct roots guarantees their dominancy. In this case an adequate factorization is found for the scalar equation and second order system with single delay allowing to write the quasipolynomial in an integral operator form. But in some cases such a factorization is hard to be established, especially for the quasi-polynomials with higher degrees which is the case for the equation studied in this chapter. The argument principle based stability criteria allows to calculate analytically the number of unstable characteristic roots for retarded time-delay system avoiding the direct calculating the corresponding improper integral. So this chapter focus on the application of the Stépàn-Hassard approach to show the dominancy of spectral values which are not necessarily multiple. To do this, a conditions on the system parameters guaranteeing the co-existence of the maximal number of real roots is established, and the argument principle is applied to shown the dominancy of real spectral values in the particular case of equidistributed roots .

4.2 Main Results

Let us focus on the second order system

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = u(t), \tag{4.1}$$

where u is the unknown control. Assume that the system (4.1) is unstable in the uncontrolled case, namely when $u(t) = 0$. Our aim is to built a delayed-state-feedback controller under the form :

$$u(t) = -\alpha_0 x(t-\tau) - \alpha_1\dot{x}(t-\tau), \tag{4.2}$$

which allows to stabilize solutions of control system (4.1). The characteristic quasi-polynomial function corresponding to the generic control problem is described as follows :

$$\Delta(s,\tau) = P(s) + Q(s)e^{-s\tau} = 0, \tag{4.3}$$

where $P(s)$ and $Q(s)$ are polynomials in s with degree of $Q(s)$ is less then the degree of $P(s)$. In our case, we consider

$$P(s) = s^2 + a_1 s + a_0 s \quad \text{and} \quad Q(s) = \alpha_1 s + \alpha_0. \tag{4.4}$$

4.2.1 On qualitative properties of s_1 as a root of (4.3)

The following theorem gives conditions on the parameter coefficients that allows assigning a maximum number of spectral values of the second order system (4.1)-(4.2). This result is useful in control theory.

Proposition 4.2.1. *The following assertions hold :*
– *The quasipolynomial* (4.3) *admits four distinct real spectral values* s_1*,* s_2*,* s_3 *and* s_4 *with*

$s_4 < s_3 < s_2 < s_1$ if and only if the parameters a_1, a_2 , α_1 and α_0 satisfy

$$\left\{\begin{array}{ll} a_1(\tau) = & \frac{1}{Q(\tau)}\begin{vmatrix} 1 & s_1^2 & s_1e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2^2 & s_2e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3^2 & s_3e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4^2 & s_4e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}, \\ a_0(\tau) = & \frac{1}{Q(\tau)}\begin{vmatrix} s_1^2 & s_1 & s_1e^{-s_1\tau} & e^{-s_1\tau} \\ s_2^2 & s_2 & s_2e^{-s_2\tau} & e^{-s_2\tau} \\ s_3^2 & s_3 & s_3e^{-s_3\tau} & e^{-s_3\tau} \\ s_4^2 & s_4 & s_4e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}, \\ \alpha_1(\tau) = & \frac{1}{Q(\tau)}\begin{vmatrix} 1 & s_1 & s_1e^{-s_1\tau} & s_1^2 \\ 1 & s_2 & s_2e^{-s_2\tau} & s_2^2 \\ 1 & s_3 & s_3e^{-s_3\tau} & s_3^2 \\ 1 & s_4 & s_4e^{-s_4\tau} & s_4^2 \end{vmatrix}, \\ \alpha_0(\tau) = & \frac{1}{Q(\tau)}\begin{vmatrix} 1 & s_1 & s_1^2 & s_1e^{-s_1\tau} \\ 1 & s_2 & s_2^2 & s_2e^{-s_2\tau} \\ 1 & s_3 & s_3^2 & s_3e^{-s_3\tau} \\ 1 & s_4 & s_4^2 & s_4e^{-s_4\tau} \end{vmatrix}, \end{array}\right. \tag{4.5}$$

where

$$Q(\tau) = \begin{vmatrix} 1 & s_1 & s_1e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2 & s_2e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3 & s_3e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4 & s_4e^{-s_4\tau} & e^{-s_4\tau} \end{vmatrix}. \tag{4.6}$$

– *The spectral value s_1 is negative if and only if there exists $\tau_0 > 0$ such that*

$$a_1(\tau_0) + s_2 = 0. \tag{4.7}$$

Proof : – According to the G. Pólya and G. Szegö Theorem [16], the number of real roots of (4.3) is four, hence we investigate the existence of four distinct real spectral values $s_1 > s_2 > s_3 > s_4$. The coefficients of the quasi-polynomial function (4.3) described by (4.5) and (4.6) are obtained by solving the following system

$$s_i^2 + a_1 s_i + a_0 + (\alpha_1 s_i + \alpha_0)\exp(-s_i\tau) = 0,\ i = 1\ldots 4, \tag{4.8}$$

that we can represents under the more suitable form :

$$\begin{bmatrix} 1 & s_1 & s_1e^{-s_1\tau} & e^{-s_1\tau} \\ 1 & s_2 & s_2e^{-s_2\tau} & e^{-s_2\tau} \\ 1 & s_3 & s_3e^{-s_3\tau} & e^{-s_3\tau} \\ 1 & s_4 & s_4e^{-s_4\tau} & e^{-s_4\tau} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \alpha_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} -s_1^2 \\ -s_2^2 \\ -s_3^2 \\ -s_4^2 \end{bmatrix}. \tag{4.9}$$

System (4.8) admits a unique solution because the Vondermonde-type matrix in $Q(\tau)$ is invertible, for every $\tau > 0$. Indeed, its determinant is positive since $\tau \mapsto Q(\tau)$ is increasing from 0 to ∞, with $Q(0) = 0$. To get the formulas (4.5), we just apply the Cramer's rule.

– The negativity of the root s_1 is shown through the variation of the function

$$\tau \mapsto a_1(\tau) + s_2. \tag{4.10}$$

So, for all values of $\tau \in \mathbb{R}^{+*}$, this later function is continuous and increasing from $-\infty$ to $-s_1$, from which we deduce that $\tau \mapsto a_1(\tau) + s_2$ takes a positive values if and only if $s_1 < 0$. See [81].

□

Due to the computation complexity of these time-delay dependant coefficients, we consider only the case of equidistance roots. This allows also to decrease the number of parameters to handle. So, we suppose that the roots of (4.3) are such that :

$$d = |s_1 - s_2| = |s_2 - s_3| = |s_3 - s_4|. \tag{4.11}$$

Substituting the formula (4.11) in (4.5) and (4.6) the above parameter coefficients are rewritten as follows :

$$\begin{cases} a_1(\tau) = & \dfrac{1}{e^{d\tau} - 1}\left(2s_1 - 5d + de^{d\tau} - 2s_1 e^{d\tau}\right) \\ a_0(\tau) = & \dfrac{1}{\left(e^{d\tau} - 1\right)^2}\left(6d^2 - ds_1 e^{2d\tau} + 6ds_1 e^{d\tau} - 5ds_1 + s_1^2 e^{2d\tau} - 2s_1^2 e^{d\tau} + s_1^2\right) \\ \alpha_1(\tau) = & 2de^{-2d\tau}\dfrac{e^{\tau s_1}}{e^{-d\tau} - 1} \\ \alpha_0(\tau) = & \dfrac{-2de^{-2d\tau}e^{\tau s_1}}{\left(e^{-d\tau} - 1\right)^2}\left(3d - s_1 + s_1 e^{-d\tau}\right) \end{cases} \tag{4.12}$$

The following theorem gives conditions on the negativeness of s_1.

Theorem 4.2.2. *The spectral values s_1 is negative if and only if one of the following equivalent conditions is satisfied :*

1. *The spectral value s_1 is explicitly given by :*

$$s_1 = -4\frac{d}{e^{d\tau} - 1}.$$

2. *The delay τ takes the value*

$$\tau^* = \frac{1}{d}\ln\frac{1}{s_1}(-4d + s_1).$$

3. *The distance d between the real roots satisfies*

$$d = \left[\left\{\frac{1}{4}s_1 - \frac{1}{\tau}\mathrm{LambertW}\left(1, \frac{\tau s_1}{4}\mathrm{e}^{\frac{\tau s_1}{4}}\right) \mid 1 \in \mathbb{Z}\right\} \setminus \left\{\frac{2i\pi l}{\tau} \mid l \in \mathbb{Z}\right\}\right] \cap]0, +\infty[.$$

Proof : We use the characterization of the negativeness of s_1 given by equation (4.7). Observe that, as $\tau \to \infty$

$$a_1(\tau) = \frac{1}{e^{d\tau}-1}\left(2s_1 - 5d + de^{d\tau} - 2s_1e^{d\tau}\right) \to d - 2s_1,$$

with the property that $\tau \mapsto a_1(\tau)$ is bounded by $-s_1$. So if there exists τ^* such that $a_1(\tau^*) + s_1 - d = 0$, we obtain that $s_1 < 0$, and vis versa. Now, to get the value of s_1, τ^* and d respectively, we just have to solve the equation

$$4d - s_1 + s_1e^{d\tau} = 0. \tag{4.13}$$

Easy computation gives the value of s_1 and $\tau^\star$. For the value of d, it can be calculated using the Lambert W function since the unknown (d) appears both outside and inside the exponential function. The Lambert W function is defined as the multivalued function that satisfies

$$x = W(x)e^{W(x)},$$

for any complex number x. Equivalently, it may be defined as the inverse of the complex function $f(x) = xe^x$.
In order to solve equation (4.13) for d we rewrite it as follows :

$$\left(d - \frac{s_1}{4}\right)e^{-\tau d} = -\frac{s_1}{4},$$

this implies that

$$-\tau\left(d - \frac{s_1}{4}\right)e^{-\tau\left(d-\frac{s_1}{4}\right)} = \frac{s_1\tau}{4}e^{\frac{s_1\tau}{4}}.$$

From the later equation, we have

$$\mathrm{LambertW}\left(\frac{\tau \mathrm{s}_1}{4}\mathrm{e}^{\frac{\tau \mathrm{s}_1}{4}}\right) = -\mathrm{d}\tau + \frac{\tau \mathrm{s}_1}{4}. \tag{4.14}$$

Hence

$$d = \left[\left\{\frac{1}{4}s_1 - \frac{1}{\tau}\mathrm{LambertW}\left(1, \frac{\tau \mathrm{s}_1}{4}\mathrm{e}^{\frac{\tau \mathrm{s}_1}{4}}\right) \mid 1 \in \mathbb{Z}\right\} \setminus \left\{\frac{2i\pi l}{\tau} \mid l \in \mathbb{Z}\right\}\right] \cap]0, +\infty[.$$

□

4.2.2 Some properties of the quasipolynomial (4.3).

Parametrization and scaling :

To show dominancy property for s_1, we need to make some transformations, namely, scaling the spectral values of the quasipolynomial function (4.3). Using the change of variables $z = s - s_1$, and substituting the value of new coefficients in (4.3) we get the scaled quasipolynomial function

$$\tilde{\Delta}(z,\tau) = z^2 + \frac{1}{r - 4r^2 + 6r^3 - 4r^4 + r^5}\Big(d(5r - 16r^2 + 18r^3 - 8r^4 + r^5)z + 6d^2(r - 2r^2 + r^3)$$
$$+ \left(2d(1 - 3r + 3r^2 - r^3)z - 6d^2(r - 2r^2 + r^3)\right)e^{-\tau z}\Big),$$

where $r = e^{\tau d}$.
The last function can be reduced to the following expression :

$$\tilde{\Delta}(z,\tau) = z^2 + \frac{d(r-5)}{(r-1)}z + \frac{6d^2}{(r-1)^2} - \left(\frac{2d}{r(r-1)}z + \frac{6d^2}{(r-1)^2}\right)e^{-\tau z}. \quad (4.15)$$

The zeros of (4.3) are just the scaled zeros of (4.15). So s_1 is a root of (4.3) is equivalent to state that 0 is a zeros of (4.15). See Figure 4.1. Hence the dominancy proof of 0 is equivalent to the dominancy proof of s_1.

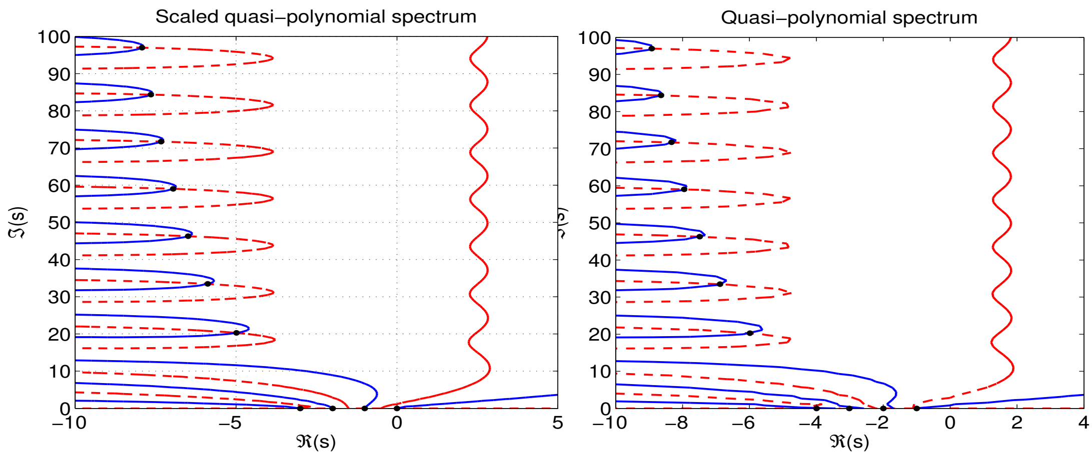


FIGURE 4.1 – (Left) The distribution of the spectrum corresponding to scaled equation (4.15). (Right) The distribution of the spectrum corresponding to the equation (4.3). Case $s_1 = -1$ and $d = 1$. The dominancy property is preserved in the scaled quasi-polynomial function. The roots distribution is illustrated using QPmR toolbox.

Characterizing imaginary roots of the scaled equation

We introduce a deflation which eliminates the roots on the imaginary axis. So, let us first investigate nonzero imaginary roots for (4.15). Assume that there exists $w > 0$ such that : $z = iw$ is a root of (4.15). Let define :

$$\begin{cases} R(w) &= \Re(i^{-2}(\tilde{\Delta}(iw,\tau))), \\ S(w) &= \Im(i^{-2}(\tilde{\Delta}(iw,\tau))). \end{cases}$$

Hence, we get :

$$\begin{cases} R(w) = & w^2 - \frac{6d^2}{(r-1)^2} + \frac{2d}{r(r-1)} w\sin(w\tau) + \frac{6d^2}{(r-1)^2}\cos(w\tau) = 0, \\ S(w) = & -\frac{d(r-5)}{(r-1)} w + \frac{2d}{r(r-1)} w\cos(w\tau) - \frac{6d^2}{(r-1)^2}\sin(w\tau) = 0. \end{cases} \tag{4.16}$$

Using Some algebraic computations for the equation (4.16) we eliminate the trigonometric functions as follows

$$\begin{cases} \cos(w\tau) & = \frac{r}{2}\frac{-w^2(r-1)^2(2r+5)+18d^2r}{(r-1)^2w^2+9d^2r}, \\ \sin(w\tau) & = \frac{(r-1)w}{2d(r-1)^2w^2+9d^2r^2}\Big(-r(r-1)^2w^2 - 3d^2r^2(r-5) + 6d^2r\Big). \end{cases}$$

Using the standard trigonometric identity

$$\cos^2(w\tau) + \sin^2(w\tau) = 1,$$

we obtain 0 and some others solutions which are eliminated since we are considered the assumption that w is a real positive. Hence, the deflated function which is integrated on the Bromwich contour is given by

$$\hat{\Delta}(z,\tau) = \frac{\tilde{\Delta}(z,\tau)}{z}. \tag{4.17}$$

On the non-existence of positive real roots of the function R

Proposition 4.2.3. *The following property is fulfilled. The function $w \mapsto R(w)$ has no positive real roots in the interval $[w^{\star\star}, +\infty[$, where $w^{\star\star}$ is given as follows :*

$$w^{\star\star} = \max(w_1^\star, w_2^\star), \tag{4.18}$$

where

$$w_1^\star = \frac{-\arctan\left(\frac{2}{3}\frac{\sqrt{3r^2+1}}{r^2}\right) + \pi}{\tau} \quad \textit{and}$$
$$w_2^\star = \frac{d}{r(r-1)}\left(1 + \sqrt{2(1+6r^2)}\right).$$

Proof :

Finding the value of $w_1^\star$:

Recall that we are dealing with transcendental equation of the form

$$R(w) = w^2 + \frac{2d\,\sin(w\tau)}{r(r-1)}w + \frac{6d^2\cos(w\tau)}{(r-1)^2} - \frac{6d^2}{(r-1)^2} = 0, \tag{4.19}$$

we notice that the equation (4.19) can be written under the form of a second order polynomial on w with variable coefficients, so the solutions of this later are obtained by calculating the discriminant δ which is given in this case by :

$$\delta = \frac{-d^2}{r^2(r-1)^2}(\cos^2(w\tau) + 6r^2\cos(w\tau) - 6r^2 - 1). \tag{4.20}$$

Equation (4.19) admits two non-explicit solutions if we have

$$-\cos^2(w\tau) - 6r^2\cos(w\tau) + 6r^2 + 1 > 0. \tag{4.21}$$

To check if the condition (4.21) is fulfilled, we study the variation of

$$w \mapsto -\cos^2(w\tau) - 6r^2\cos(w\tau) + 6r^2 + 1. \tag{4.22}$$

To do it we pose $X = \cos(w\tau)$ we obtain

$$-X^2 - 6r^2X + 6r^2 + 1 = 0. \tag{4.23}$$

The solutions of the equation (4.23) are $X_2 = 1$ and $X_1 = -6r^2 - 1$. Since $-1 < \cos(w\tau) \leq 1$, the quantity (4.22) is non-negative for any $d > 0$, $r > 1$ and $w \geq 0$.
The non-explicit solutions of (4.19) are written as follows :

$$w = f_1^{\tau,d}(w) = -\frac{d\sin(w\tau)}{r(r-1)} + \sqrt{\frac{d^2\left(-6\cos(w\tau)r^2 - (\cos(w\tau))^2 + 6r^2 + 1\right)}{r^2(r-1)^2}}, \tag{4.24}$$

$$w = f_2^{\tau,d}(w) = -\frac{d\sin(w\tau)}{r(r-1)} - \sqrt{\frac{d^2\left(-6\cos(w\tau)r^2 + (\cos(w\tau))^2 + 6r^2 + 1\right)}{r^2(r-1)^2}}. \tag{4.25}$$

Finding the number of root of the equation (4.19) is equivalent to finding the number of points of intersection of the line $y = w$ and the graph of $f_1^{\tau,d}$ and $f_2^{\tau,d}$. We have

$$f_2^{\tau,d}(w) = -\frac{d}{r(r-1)}\left(\sin(w\tau) + \sqrt{-\cos^2(w\tau) - 6r^2\cos(w\tau) + 6r^2 + 1}\right), \tag{4.26}$$

this quantity defines a $\frac{2\pi}{\tau}$-periodic function. Evaluating the sign of $f_2^{\tau,d}$ on the interval $]0, 2\frac{\pi}{\tau}[$ gives

1. For $w\tau \in]0, \pi[$, we have $\sin(w\tau) > 0$
and $\sqrt{-\cos^2(w\tau) - 6r^2\cos(w\tau) + 6r^2 + 1} > 0$ which implies that $f_2^{\tau,d}(w) < 0$.
2. For $w\tau \in [\pi, 2\pi]$, suppose that :
$\sin(w\tau) + \sqrt{-\cos^2(w\tau) - 6r^2\cos(w\tau) + 6r^2 + 1} < 0$, thus

$$\sqrt{-\cos^2(w\tau) - 6r^2\cos(w\tau) + 6r^2 + 1} \quad < -\sin(w\tau),$$

this means that

$$1 - \cos(w\tau) < 0,$$

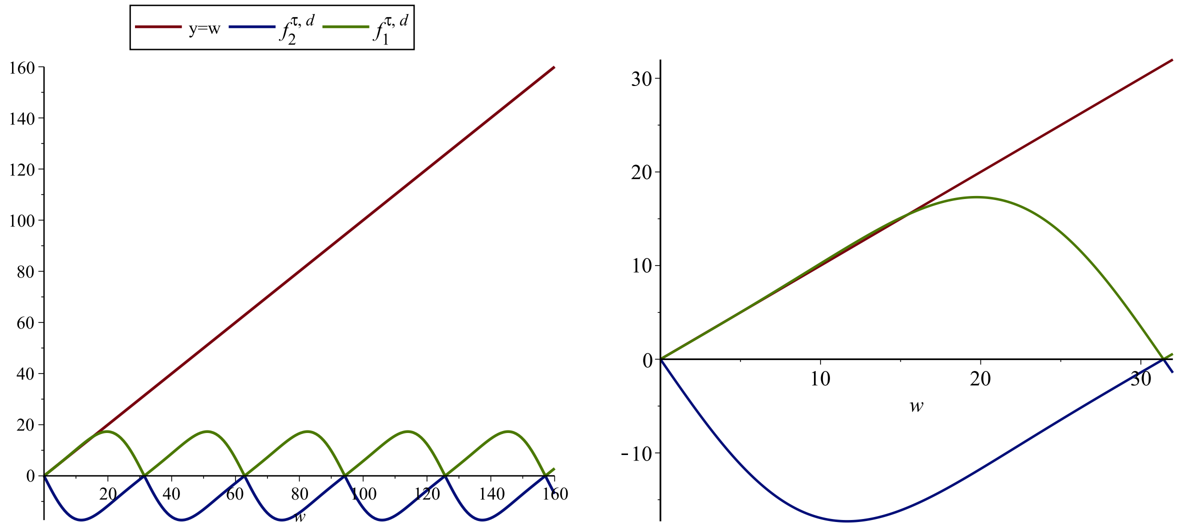


FIGURE 4.2 – (Left) Variation of the functions $w \to f_1^{\tau,d}(w)$ and $w \to f_2^{\tau,d}(w)$ for $\tau = 0.1$ and $d = 1$. (Right) Zoom of the functions $f_1^{\tau,d}$ and $f_2^{\tau,d}$.

we obtain a contradiction, which ensures that
$f_2^{\tau,d}(w) < 0$. See Figure 4.2. So, we eliminate the equation (4.25) since we are dealing with the positive value of w. So the study of the equation (4.19) is equivalent to the study of the equation $f_1^{\tau,d}(w) = 0$. The equation (4.24) allows to obtain a certain bound $w^\star$ for the solution of (4.19). Indeed, $w \mapsto f_1^{\tau,d}$ being bounded, more precisely, we have :

$$|f_1^{\tau,d}(w)| \leq \frac{d}{r(r-1)}\left(1+\sqrt{2(1+6r^2)}\right) = w_1^\star, \tag{4.27}$$

so for all values of $w > w^\star$ the equation (4.19) have no solutions.

Finding the value of $w_2^\star$:
It can be calculated using the variation of the function $w \mapsto f_1^{\tau,d}(w)$. We have

$$\frac{d\,f_1^{\tau,d}}{d\,w}(w) = \frac{d}{r\,(r-1)}\left(-\cos(w\tau)\,\tau + 1/2\,\frac{2\cos(w\tau)\sin(w\tau)\,\tau + 6\,r^2\sin(w\tau)\,\tau}{\sqrt{-(\cos(w\tau))^2 - 6\,r^2\cos(w\tau) + 6\,r^2 + 1}}\right),$$

this equation admits two roots given by :

$$w_1 = \frac{\arctan\left(\frac{2}{3}\frac{\sqrt{3r^2+1}}{r^2}\right) - \pi}{\tau},$$
$$w_2 = \frac{-\arctan\left(\frac{2}{3}\frac{\sqrt{3r^2+1}}{r^2}\right) + \pi}{\tau}.$$

The value of w_1 can be eliminated since it is negative for all values of r and d. Hence in what follows we exploit the value of w_2 which is always positive for any τ and d. We substitute this value in the quantity $f_1^{\tau,d}(w) - w$ we obtain :

$$f_1^{\tau,d}(w) - w = A(r)B(r), \tag{4.28}$$

where $A(r)$ and $B(r)$ are given respectively as follows :

$$A(r) = \frac{1}{\tau r(r-1)(3r^2+2)},$$

$$B(r) = \frac{3r^2+2}{r^2}\Bigg((r^4-r^3)\Big(\arctan\frac{2}{3}\sqrt{\frac{3r^2+1}{r^2}} - \pi\Big) + \sqrt{2}r^2\ln(r)$$
$$\Big(\frac{3r^2+2}{r^2}(27r^6+36r^4+18r^2+2) + 81r^6+104r^4+36r^2\Big)^{\frac{1}{2}}\Bigg) - 2\ln(r)\sqrt{3r^2+1}.$$

Since $\tau > 0$ and $r > 1$ the quantity $A(r)$ is always positive. The study of the variation of the function $r \mapsto B(r)$ given in the formula (4.28) shows that this quantity is negative. See Figure 4.3. We conclude that $f_1^{\tau,d}(w) - w < 0$ which means that from this value of w

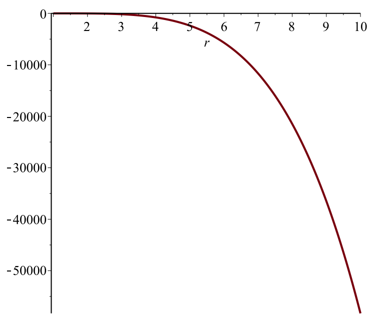


FIGURE 4.3 – Variation of the function $r \mapsto B(r)$.

the line $y = w$ can not intersect the graph of $f_1^{\tau,d}$ implying the non-existence of solutions of the function $f_1^{\tau,d}$. This value is given by :

$$w = w_2^\star = \frac{-\arctan\left(\frac{2}{3}\frac{\sqrt{3r^2+1}}{r^2}\right) + \pi}{\tau}. \tag{4.29}$$

At the end, the bound for the non-existence of positive real roots for the function R is given as follows :

$$w^{\star\star} = \max(w_1^\star, w_2^\star).$$

□

The dominancy proof of s_1 is based on the application of Stépàn-Hassard's formula and the argument principle.

Remark 4.2.4. *Notice that, the bound for the non-existence of positive real roots for the function R can be reformulated depending only on the system parameters. the result is presented in the following proposition.*

Proposition 4.2.5. *The following property is fulfilled. The function $w \mapsto R(w)$ has no positive real roots for the values of $\tau > \tau^\star$, where $\tau^\star$ is given as follows :*

$$\tau^\star = -\frac{\sqrt{3}}{6d}\left(2\,\mathrm{W}\Big(-1, -\frac{1}{6}\pi\sqrt{3}\exp\Big(\frac{-1}{6}\sqrt{3}\pi\Big)\Big)\sqrt{3} + \pi\right).$$

Proof 3. *Recall that :*

$$R(w) = w^2 - \frac{6d^2}{(r-1)^2} + \frac{2d}{r(r-1)} w\sin(w\tau) + \frac{6d^2}{(r-1)^2}\cos(w\tau). \tag{4.30}$$

In order to find the number of positive real roots of the function R, we try to evaluate this function on the interval $\left[\frac{k\pi}{\tau}, \frac{(k+1)\pi}{\tau}\right]$, with $k > 0$.

– ***For k=1**, firstly we substitute the value $w = \frac{k\pi}{\tau}$ in the function R, we obtain*

$$R(w) = \left(\frac{\pi}{\tau} - \frac{\sqrt{12}d}{(e^{\tau d}-1)}\right)\left(\frac{\pi}{\tau} + \frac{\sqrt{12}d}{(e^{\tau d}-1)}\right). \tag{4.31}$$

We notice that the quantity $\left(\frac{\pi}{\tau} + \frac{\sqrt{12}d}{(e^{\tau d}-1)}\right)$ is always positive for all values of τ and d. Concerning the other quantity, we let

$$G(\tau) = \left(\frac{\pi}{\tau} - \frac{\sqrt{12}d}{(e^{\tau d}-1)}\right) \quad \text{with} \quad \tau, d > 0, \tag{4.32}$$

this function vanishes for the given values of τ

$$\tau^\star = -\frac{\sqrt{3}}{6d}\left(2\,\mathrm{W}\Big(-1, -\frac{1}{6}\pi\sqrt{3}\exp\Big(\frac{-1}{6}\sqrt{3}\pi\Big)\Big)\sqrt{3} + \pi\right). \tag{4.33}$$

In order to plot and study the variation of the function G, we rewrite it as a function of a single variable r using the change of variable $r = e^{\tau d}$, so we obtain :

$$\widetilde{G}(r) = \pi(r-1) - \sqrt{12}\ln(r),$$

hence we see that :

$$\begin{cases} \text{For} \quad \tau \le \tau^\star \quad \widetilde{G}(r) < 0 \quad \text{which means that} \quad R(w) < 0. \\ \text{For} \quad \tau > \tau^\star \quad \widetilde{G}(r) > 0 \quad \text{implying that} \quad R(w) > 0. \end{cases} \tag{4.34}$$

Secondly, we inject the value of $w = \frac{(k+1)\pi}{\tau}$ in R, we obtain

$$R\left(\frac{2\pi}{\tau}\right) = \frac{4\pi^2}{\tau^2} > 0.$$

Therefore, according to the mean value theorem, the function R admits at least one roots on the interval $[\frac{\pi}{\tau}, \frac{2\pi}{\tau}]$ for the values of $\tau \le \tau^\star$.

– ***For*** $k > 1$ *we distinguish two cases :*

1. *k is even, we obtain*

$$R(w) = \left(\frac{k\pi}{\tau}\right)^2 > 0 \quad \textit{for all values of } \tau \textit{ and } d.$$

2. *k is odd, we get :*

$$R(w) = \left(k\pi(r-1) - \sqrt{12}\ln(r)\right)\left(k\pi(r-1) + \sqrt{12}\ln(r)\right).$$

The quantity $\left(k\pi(r-1) + \sqrt{12}\ln(r)\right)$ *is always positive. We let*

$$\widetilde{G}_k(r) = k\pi(r-1) - \sqrt{12}\ln(r). \tag{4.35}$$

The later function vanishes for the following value of τ

$$\tau = -\frac{\sqrt{3}}{6d}\left(2\,\mathrm{W}\left(-1, \frac{1}{6}\pi\sqrt{3}\exp\left(\frac{-k}{6}\sqrt{3}\pi\right)\right)\sqrt{3} + k\pi\right) \tag{4.36}$$

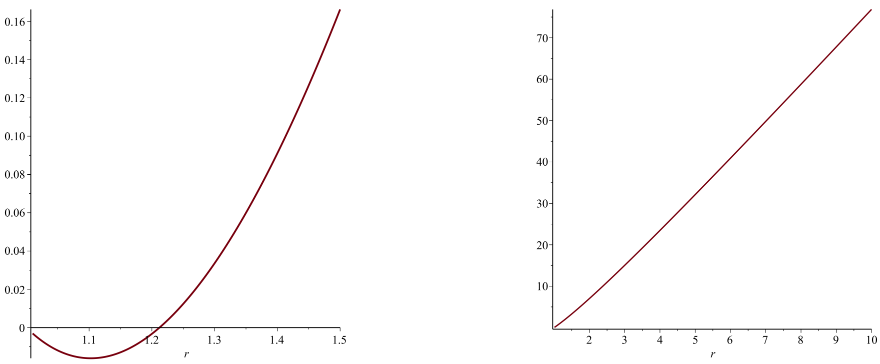


FIGURE 4.4 – (Left) Variation of the functions $r \to G_k(r)$ for $k = 1$. (Right) Variation of the function G_k for $k > 1$, more precisely for $k = 3$.

Notice that the value of τ *given by* (4.36) *is always negative for all values of* k *and* d*. Since we are dealing with the positive values of* τ*, the obtained value is omitted. The variation of the function* $\widetilde{G}_k$ *shown that* $\widetilde{G}_k(r) > 0$ *for all values of* τ *and* d *see Figure 4.4, which means that* $R(w)$ *is always positive. From the mean value theorem, we conclude that the function* R *has no roots on the interval* $\left[\frac{k\pi}{\tau}, \frac{(k+1)\pi}{\tau}\right]$ *for* $k > 1$*.*

In conclusion, the function R *has no positive real roots for the values of* $\tau > \tau^\star$ *with* $\tau^\star$ *is given by* (4.33).

4.2.3 Application of Stépàn-Hassard's formula

The continuation of the study consists in applying the formula (2.46) given in [82] to show dominancy of s_1.

Theorem 4.2.6. *Under the condition given in Proposition 4.2.3, the spectral value s_1 is the spectral abscissa of Equation* (4.3).

Proof : Using the formula (2.46) we calculate the number Z of unstable roots of the system (4.1).

Before applying the formula (2.46), we need first evaluate the sign of $S^{(\kappa)}(0)$. Recall that $\kappa = 1$, and

$$S(w) = d(r-1)2w\cos(w\tau) - dr(r-1)(r-5)w - d^2r6\sin(w\tau), \tag{4.37}$$

we have $S(0) = 0$ and elementary computation gives the derivative :

$$S'_{d,r}(0) = -\frac{d\left(6dr\tau + r^3 - 6r^2 + 3r + 2\right)}{r(r-1)^2}.$$

The study of variation of the function

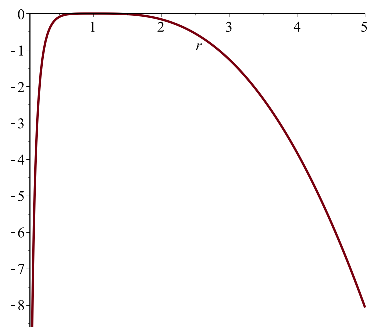


FIGURE 4.5 – Variation of the function $S'_{d,r}(0)$

$$r \mapsto -\left(6\ln(r) + r^2 - 6r + 3 + \frac{2}{r}\right),$$

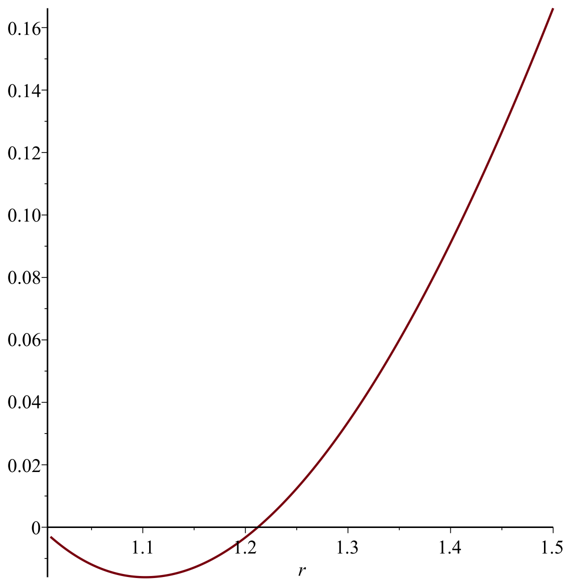
shows that this equation admits a single root $r = 1$ and it is always negative. See figure (4.5). This implies that $S'_{d,r}(0) < 0$.
We know that from a certain bounds $w = w_2^{\star\star}$ given in Proposition 4.2.3, the equation (4.19) has no positive real roots, so we have $J = 0$ and the summation term can be omitted, hence the number of roots in the positive half plan is as follows :

$$Z = \frac{2-1}{2} + \frac{1}{2}(-1)^0(-1) = \frac{1}{2} - \frac{1}{2} = 0.$$

In conclusion, we deduce that the dominancy of $z = 0$ as a root of the deflated quasipolynomial function $\hat{\Delta}(z, \tau)$ given by (4.17) is equivalent to the dominancy of $s = s_1$ as a root of the quasipolynomial function $\Delta(s, \tau)$ given by (4.3). This ensures the stability of the corresponding system (4.1), see appendix for further details.

□

Remark 4.2.7. *The variations of the functions R and S with respect to w depend on the variations of τ and d. These later show that R and S may admit real positive roots, especially for sufficiently small values of τ and d, see Figure 4.6. However the characterization of these roots remain complicated due to the computations complexity. Since we are dealing with pole assignment method the value of d can be fixed, for example if we take $d = 1$, for all values of τ such that $\tau > 0.28$, the function R have no roots.*



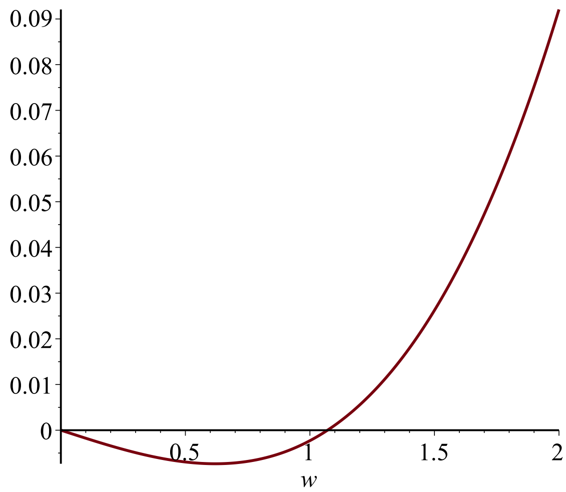


FIGURE 4.6 – (Left)Location of roots of R for $\tau = 0.1$ and $d = 1$. (Right) Sign of S for the same value of τ and d.

4.3 Illustrative example

As an illustration of the presented result, we consider the problem of stabilizing the Mach number of a transonic flow in a wind tunnel. The analysis of transonic flows is a challenging problem in compressible fluid dynamics, since a full model of the flow would involve considering the Navier-Stokes equations in a three-dimensional domain and boundary controls for temperature and pressure regulation. A simplified model was considered in [117] in order to analyze the response of the Mach number of the flow to changes in the guide vane angle. Instead of using a PDE model, propagation phenomena are modeled through a time delay, leading to the following time-delay system

$$\begin{cases} \kappa m'(t) + m(t) = k\vartheta(t-\tau_0), \\ \vartheta''(t) + 2\zeta\omega\vartheta'(t) + \omega^2\vartheta(t) = \omega^2 u(t), \end{cases} \tag{4.38}$$

in which m, ϑ, and u represent perturbations of the Mach number of the flow, the guide vane angle, and the input of the guide vane actuator, respectively, with respect to steady-state values. The parameters κ and k depend on the steady-state operating point and are assumed to be constant as long as m, ϑ, and u remain small, and satisfy $\kappa > 0$ and $k < 0$. The parameters $\zeta \in (0,1)$ and $w > 0$ come from the design of the guide vane angle actuator and are thus independent of the operating point. The time delay τ_0 is assumed to depend only on the temperature of the flow. Notice that, in the absence of control ($u(t) = 0$), the open-loop system (4.38) is exponentially stable. The design of the stabilizing feedback for (4.38) improving its stability properties has been considered in [116] and [48]. The design we propose here is a delayed PD controller which can be written :

$$u(t) = \alpha_0 x(t-\tau) + \alpha_1 \dot{x}(t-\tau_1). \tag{4.39}$$

In closed loop system, the corresponding characteristic equation reads

$$\Delta(s,\tau_1) = (s-a)((s\alpha_1 + \alpha_0 t)e^{-s\tau_1} + 2\omega s\zeta + \omega^2 + s^2). \tag{4.40}$$

Since the parameter a is fixed then one focus on the second factor only,

$$\Delta_4(s,\tau_1) = s^2 + 2\omega s\zeta + \omega^2 + (s\alpha_1 + \alpha_0)e^{-s\tau_1}. \tag{4.41}$$

To assign four equidistributed rightmost real spectral values $s_k = -k$ for $k \in [1,\ldots,4]$. one uses (4.12). For instance, for $\omega = 1.2$ and dampin8g factor $\zeta = 0.472$ it is sufficient to set :

$$\alpha_0 = -\frac{3087}{6050}, \quad \alpha_1 = -\frac{343}{3630}, \quad \text{and} \quad \tau_1 = \ln\frac{22}{7}. \tag{4.42}$$

Figure 4.7 exhibits the spectrum distribution corresponding to (4.41).

4.4 Conclusion

A new dominancy result of the spectral values which are not necessarily multiple is established through this paper for a class of second order time delay system with single delay. The stability analysis is based on the argument principle and the Stépàn-Hassard formula which allows to count the number of unstable roots of the corresponding characteristic equation. This study is not achieved since we are considered only the case where the real part of the quasipolynomial function have no roots. The other case remains under investigation.

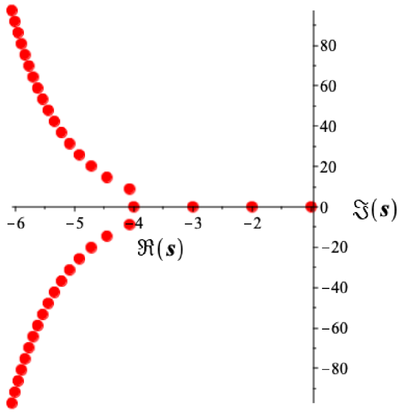


FIGURE 4.7 – Zeros distribution of quasipolynomial (4.41) with $\omega = 1.2$, $\zeta = 0.472$ and the controller gains given by (4.42)

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Appendix : Dominancy proof using the argument principle

The first step of the dominancy proof using the argument principle is to translate the quasipolynomial function $\Delta(s,\tau)$, in order to reduce the study in a standard contour called Bromwich contour. According to the approach of [82], the next step consists in establishing asymptotic properties of scaled quasipolynomial function $\widetilde{\Delta}(z,\tau)$ and $\partial_z\widetilde{\Delta}(z,\tau)/\widetilde{\Delta}(z,\tau)$. We then deflated $\widetilde{\Delta}(z,\tau)$ to remove any zeros on the imaginary axis, and define the contour of integration. Next we apply the argument principle to count the zeros in the right half plane of the deflated function $\widehat{\Delta}(z,\tau)$ and work out the asymptotic behavior of the logarithmic residue as the chosen contour is increased.

Asymptotic behavior : The characteristic function (4.15) can be written in the following form :

$$\widetilde{\Delta}(z,\tau) = z^2 + d_1(z,\tau)z + d_2(z,\tau), \tag{4.43}$$

with
$$d_1(z,\tau)=\widetilde{a}_1+\widetilde{\alpha}_1 e^{-z\tau} \quad \text{and} \quad d_2(z,\tau)=\widetilde{a}_0+\widetilde{\alpha}_0 e^{-z\tau},$$
where
$$\begin{aligned}\widetilde{a}_1 =&\frac{d\,(r-5)}{(r-1)}, \quad \widetilde{\alpha}_1=\frac{2d}{r(r-1)},\\ \widetilde{a}_0 =&\frac{6d^2}{(r-1)^2}, \quad \widetilde{\alpha}_0=\frac{6d^2}{(r-1)^2}.\end{aligned}$$
Then
$$\partial_z\widetilde{\Delta}(z,\tau)=[2+\partial_z d_1(z,\tau)]z+o(1), \tag{4.44}$$
and
$$\frac{\partial_z\widetilde{\Delta}(z,\tau)}{\widetilde{\Delta}(z,\tau)}=z^{-1}[2+\partial_z d_1(z,\tau)]+o(|z|^{-2}). \tag{4.45}$$
Deflation : The deflation of the characteristic function allows to eliminate the root on the imaginary axis. We know that there exists a single root on the imaginary axis $z_0=0$ of multiplicity one. We put
$$P(z)=(z-z_0),$$
which gives :
$$\widehat{\Delta}(z,\tau)=\frac{\widetilde{\Delta}(z,\tau)}{P(z)}=\frac{\widetilde{\Delta}(z,\tau)}{z}. \tag{4.46}$$

Integration contour C_R :

Let C_R denote the Bromwich contour $g_1\cup g_2\cup g_3$ for any $R\in\mathbb{R}$, where
$$\begin{aligned}g_1 =&\{z=Re^{i\theta},\theta:-\frac{\pi}{2}\to\frac{\pi}{2}\},\\ g_2 =&\{z=iw,w:R\to 0\},\\ g_3 =&\{z=iw,w:0\to -R\}.\end{aligned}$$

For $R\to\infty$ all zero of $Q(s,\tau)$ in $\Re(s)>0$ are inside C_R. By the argument principle, the number of such zeros, counted by multiplicity is given by :
$$Z=\frac{1}{2\pi i}\oint_{C_R}\frac{\partial_z\widehat{\Delta}(z,\tau)}{\widehat{\Delta}(z,\tau)}dz. \tag{4.47}$$

The later quantity can be rewritten in the Bromwich contour as follows :
$$Z=\frac{1}{2\pi i}\int_{g_1}\frac{\partial_z\widehat{\Delta}(z,\tau)}{\widehat{\Delta}(z,\tau)}dz+\frac{1}{2\pi i}\int_{g_2\cup g_3}\frac{\partial_z\widehat{\Delta}(z,\tau)}{\widehat{\Delta}(z,\tau)}dz. \tag{4.48}$$

We will next evaluate both integrals given in (4.48).
Integral over g_1 and its limit as $R\to\infty$

For sufficiently large R, $\widetilde{\Delta}(z,\tau)$ and $P(z)$ are non zero for z on g_1 and from (4.48) we have

$$\frac{1}{2\pi i}\int_{g_1}\frac{\partial_z\widehat{\Delta}(z,\tau)}{\widehat{\Delta}(z,\tau)}dz = \frac{1}{2\pi i}\int_{g_1}\frac{\partial_z\widetilde{\Delta}(z,\tau)}{\widetilde{\Delta}(z,\tau)}dz - \frac{1}{2\pi i}\int_{g_1}\frac{P'(z)}{P(z)}dz.$$

We have

$$\begin{aligned}\frac{1}{2\pi i}\int_{g_1}\frac{\partial_z\widetilde{\Delta}(z,\tau)}{\widetilde{\Delta}(z,\tau)}dz &= \frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z^{-1}[2+\partial_z(z,\tau)]+o(|z|^{-2})dz\\ &= \frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{Re^{i\theta}}[2+d_1'(Re^{i\theta},\tau]+o(|Re^{i\theta}|^{-2})\Big)Rie^{i\theta}d\theta\\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[2+d_1'(Re^{i\theta},\tau)]d\theta+o(R^{-1}).\end{aligned}$$

In the later integral we have $|d_1'(z,\tau)|$ is bounded for $\tau>0$, thus $\lim_{R\to\infty} d_1'(Re^{i\theta})=0$ for $\theta\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$. by dominated convergence

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{g_1}\frac{\partial_z\widetilde{\Delta}(z,\tau)}{\widetilde{\Delta}(z,\tau)}dz = 1. \tag{4.49}$$

The second integral is as follows :

$$\frac{1}{2\pi i}\int_{g_1}\frac{P'(z)}{P(z)}dz = \frac{1}{2\pi i}\int_{g_1}\frac{1}{z}dz = \frac{1}{2}. \tag{4.50}$$

Finally, from (4.49) and (4.50) the integral over g_1 is given by :

$$\frac{1}{2\pi i}\int_{g_1}\frac{\partial_z\widehat{\Delta}(z,\tau)}{\widehat{\Delta}(z,\tau)}dz = \frac{1}{2} = \frac{n-K}{2}. \tag{4.51}$$

Integral over $g_2\cup g_3$ and its limit as $R\to\infty$ We introduce functions $M(w)$ and $N(w)$ which are the real and imaginary part of the deflated quasipolynomial function $\widehat{\Delta}(z,\tau)$ respectively, hence

$$\begin{aligned} M(w) &= \Re\big(i^{K-n}\widehat{\Delta}(iw)\big) = \Re\Big(i^{-1}\frac{\widetilde{\Delta}(iw)}{iw}\Big) = \frac{1}{w}R(w),\\ N(w) &= \Im\big(i^{K-n}\widehat{\Delta}(iw)\big) = \Im\Big(i^{-1}\frac{\widetilde{\Delta}(iw)}{iw}\Big) = \frac{1}{w}S(w).\end{aligned}$$

From this, we have

$$M(w)+iN(w) = \frac{R(w)+iS(w)}{w}. \tag{4.52}$$

Setting the polar representation

$$M(w)+iN(w) = A(w)e^{i\phi(w,\tau)}, \tag{4.53}$$

where

$$A(w) = \sqrt{M^2(w) + N^2(w)} \quad \text{and} \quad \phi(w) = \arctan\frac{N(w)}{M(w)},$$

then we have :

$$\frac{\partial_z\widehat{\Delta}(z,\tau)}{\widehat{\Delta}(z,\tau)} = \frac{\partial_s A(w)}{A(w)} + i\partial_s\phi(w). \tag{4.54}$$

Thus

$$\frac{1}{2\pi i}\int_{g_2\cup g_3}\frac{\partial_z\widehat{\Delta}(z,\tau)}{\widehat{\Delta}(z,\tau)}dz = \frac{1}{2\pi i}\int_{-R}^{R}\Big[\frac{\partial_s A(w)}{A(w)} + i\partial_s\phi(w)\Big]dw = \frac{1}{\pi}\big(\phi(0) - \phi(R)\big).$$

Since $\phi'(w)$ is even and $\partial_s A(w)/A(w)$ is odd, according to the result in [82]. Finally the integration over the contour $g_2 \cup g_3$ becomes

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{g_1}\frac{\partial_z\widehat{\Delta}(z,\tau)}{\widehat{\Delta}(z,\tau)}dz = \frac{\phi(0)}{\pi}. \tag{4.55}$$

Calculation of $\phi(0)$
$M(w)$ and $N(w)$ is given respectively as follows

$$\begin{aligned} M(w) =& \frac{1}{w}\left[w^2 - \frac{6d^2}{(r-1)^2} + \frac{2d\,\sin(w\tau)}{r(r-1)}w + \frac{6d^2\cos(w\tau)}{(r-1)^2}\right],\\ N(w) =& \frac{1}{w}\left[-\frac{d(r-5)}{(r-1)}w + \frac{2d\,\cos(w\tau)}{r(r-1)}w - \frac{6d^2\,\sin(w\tau)}{(r-1)^2}\right]. \end{aligned}$$

Both cases represent the indetermination $0/0$. Using l'hospital's rule we obtain $M(0)$ and $N(0)$ as follows

$$\begin{aligned} M(0) &= \lim_{w\to 0}\frac{1}{1!}\left[2w - 6\frac{d^2\sin(\tau w)\,\tau}{(r-1)^2} + 2\frac{d\sin(\tau w)}{r(r-1)} + 2\frac{dw\cos(\tau w)\,\tau}{r(r-1)}\right],\\ N(0) &= \lim_{w\to 0}\frac{1}{1!}\left[2\frac{d\cos(\tau w)}{r(r-1)} - 2\frac{dw\sin(\tau w)\,\tau}{r(r-1)} - 6\frac{d^2\cos(\tau w)\,\tau}{(r-1)^2} + \frac{(-r+5)\,d}{r-1}\right], \end{aligned}$$

which means that

$$\begin{aligned} M(0) &= 0,\\ N(0) &= -\frac{d\left(6d\tau r + r^3 - 6r^2 + 3r + 2\right)}{(r-1)^2 r} < 0. \end{aligned}$$

The later conditions satisfy condition (b) of ([82] : page 228). Recall that in our study we have two cases : R have two real roots and R have no roots. The number of roots of R coincides with the number of roots of M which is the real part of the deflated quasipolynomial function. Let us consider the case where M has no positive roots, so the change in $\phi(w)$ on the interval $[0,\infty[$ is given by the formula :

$$0 - \phi(0) = -\frac{\pi}{2}\,sgn N(0) \quad \text{implying that} \quad \phi(0) = -\frac{\pi}{2}. \tag{4.56}$$

At the end according to (4.51) and (4.55) one obtains :

$$\lim_{R\to\infty} \oint_{C_R} \frac{\partial_z \widetilde{\Delta}(z,\tau)}{\widetilde{\Delta}(z,\tau)} dz = \frac{1}{2} - \frac{1}{2} = 0, \tag{4.57}$$

from which we deduce the dominancy, implying the stability of the corresponding system.

General conclusion and some research perspectives

This thesis is dedicated to the stability analysis of retarded time delay systems. The research is principally focused on the stability of reduced order linear time delay system with single delay in frequency domain approach, where the stability of time-delay systems depends exactly on the roots of the associated characteristic equation. The system is asymptotically stable if and only if all the roots of the associated characteristic equation are located on the left half complex plane. In this context, we have seen how to formulate necessary and sufficient conditions for the coexistence of non oscillating roots for reduced order time delay system (scalar and second order delay differential equation) guaranteing the location of the remaining roots in the left half plane, which ensure the stability of the corresponding system. The dominancy of such non oscillating mode has been studied using an adequate factorization.
A new dominancy result of the spectral values has been established for generic linear second order time delay system. The stability criteria based on the manifold defined by the coexistence of the maximal number of negative spectral values, and constructing an appropriate delayed state feedback controller allowing to guarantee the stability of the system in the closed loop. For this class of system, the dominancy property is shown using the argument principle and stépàn Hasard formula which allows to count the number of unstable roots of the corresponding characteristic equation.

Perspectives :

Many questions remain in this area. As future research perspectives on the results presented in this manuscript, we intend to develop and improve the following points :

– Consider the case where the real part of the quasipolynomial function associated to the generic second order system has a real roots.

– Try to find an adequate factorization to study the stability of generic second order delay system.

– Generalize this study for higher order delay systems.

– Extend the property of coexistence of non oscillating modes to study the stability of neutral time-delay differential equation.

Bibliographie

[1] Ravi P Agarwal, Martin Bohner, and Wan-Tong Li. *Nonoscillation and oscillation theory for functional differential equations*, volume 267. CRC Press, 2004.

[2] Farshid Maghami Asl and A Galip Ulsoy. Analysis of a system of linear delay differential equations. *J. Dyn. Sys., Meas., Control*, 125(2) :215–223, 2003.

[3] Majid Bani-Yaghoub. Analysis and applications of delay differential equations in biology and medicine. *arXiv preprint arXiv :1701.04173*, 2017.

[4] Harvey Thomas Banks. *Modeling and control in the biomedical sciences*, volume 6. Springer Science & Business Media, 2013.

[5] Alfredo Bellen and Marino Zennaro. Stability properties of interpolants for runge–kutta methods. *SIAM journal on numerical analysis*, 25(2) :411–432, 1988.

[6] Richard Ernest Bellman and Kenneth L Cooke. Differential-difference equations. 1963.

[7] Richard Ernest Bellman and John M Danskin. A survey of the mathematical theory of time-lag, retarded control, and hereditary processes. 1954.

[8] Raouf Boucekkine, Omar Licandro, and Christopher Paul. Differential-difference equations in economics : on the numerical solution of vintage capital growth models. *Journal of Economic Dynamics and Control*, 21(2-3) :347–362, 1997.

[9] Michiels, Wim and Niculescu, Silviu-Iulian. Stability and stabilization of time-delay systems : an eigenvalue-based approach. SIAM, 2007.

[10] Cooke, Kenneth L. Stability analysis for a vector disease model. The Rocky Mountain Journal of Mathematics. JSTOR, 9(1), 31–42, 1979.

[11] Obreschkoff, N. Nullstellen linearer kombinationen von exponentialfunktionen, Jber. der Deutsch. Math. Verein, 37 :81–84.1928

[12] Ruan, Shigui. Delay differential equations in single species dynamics. Delay differential equations and applications, pages 477–517, Springer, 2006.

[13] Tallman, G and Smith, O Analog study of dead-beat posicast control. IRE Transactions on Automatic Control, 4(1) :14–21,1958.

[14] Qiao, Wei and Sipahi, Rifat. A linear time-invariant consensus dynamics with homogeneous delays : analytical study and synthesis of rightmost eigenvalues. SIAM Journal on Control and Optimization, 51(5) :3971–3992, 2013.

[15] S. Yi and P-W. Nelson and G. Ulsoy. Analysis and control of time delayed systems via the Lambert W function. IFAC Proceedings Volumes, 41(2) :13414–13419, 2008.

[16] Pólya, George and Szegö, Gábor. Problems and Theorems in Analysis : Series, Integral Calculus, Theory of Functions ; Translation [from the German] Springer, 1972.

[17] Michiels, Wim and Engelborghs, Koen and Vansevenant, Patrick and Roose, Dirk. Continuous pole placement for delay equations.
Journal : IFAC Proceedings Volumes. Springer, 33(22), 145–150, 2000.

[18] Vandewalle, Stefan and Michiels, Wim and Verheyden, Koen and Vanbiervliet, Joris. A nonsmooth optimisation approach for the stabilisation of time-delay systems
ESAIM : Control, Optimisation and Calculus of Variations, 14(3), 478–493, 2008.

[19] Ramírez, Adrián and Mondié, Sabine and Garrido, Rubén and Sipahi, Rifat. Design of proportional-integral-retarded (PIR) controllers for second-order LTI systems.
IEEE Transactions on Automatic Control, 61(6), 1688–1693, 2015.

[20] Vyhlidal, Tomas and Zítek, Pavel. Mapping based algorithm for large-scale computation of quasi-polynomial zeros.
IEEE Transactions on Automatic Control, 54(1), 171–177, 2009.

[21] Hayes, ND36426. Roots of the transcendental equation associated with a certain difference-differential equation.
Journal of the London Mathematical Society. Wiley Online Library, 1(3), 226–232. 1950.

[22] Liu, Zeqing and Chen, Lin and Kang, Shin Min and Cho, Sun Young. Existence of nonoscillatory solutions for a third-order nonlinear neutral delay differential equation.
Abstract and Applied analysis, organization : Hindawi. 2011.

[23] Wong, James SW. Oscillation and nonoscillation of solutions of second order linear differential equations with integrable coefficients.
Transactions of the American Mathematical Society, 197–215, 1969.

[24] Kolmanovskii, V and Myshkis, A Introduction to the theory and applications of functional differential equations Springer Science & Business Media, 463, 2013.

[25] Niculescu, S-I and de Souza, Carlos E and Dugard, Luc and Dion, J-M Robust exponential stability of uncertain systems with time-varying delays. journal=IEEE Transactions on Automatic Control, 43(5), 743–748, 1998.

[26] Lee, Muo S and Hsu, CS. On the τ-decomposition method of stability analysis for retarded dynamical systems. SIAM Journal on Control, 7(2), 242–259, 1969.

[27] Richard, JP and Goubet-Bartholomeüs, A and Tchangani, Ph A and Dambrine, M. Nonlinear delay systems : Tools for a quantitative approach to stabilization. Springer, 218–240, 1998.

[28] Kolmanovskii, Vladimir Borisovich and Nosov, Valerij Romanovič. Stability of functional differential equations. Elsevier, 180, 1986.

[29] Su, Juing-Huei Further results on the robust stability of linear systems with a single time delay. Systems & Control Letters, 23(5), 375–379, 1994.

[30] Niculescu, SI and De Souza, CE and Dion, JM and Dugard, Luc. Robust stability and stabilization of uncertain linear systems with state delay : Single delay case. Proc. IFAC Workshop on Robust Control Design, 469–474, 1994.

[31] Boyd, Stephen and El Ghaoui, Laurent and Feron, Eric and Balakrishnan, Venkataramanan. Linear matrix inequalities in system and control theory. SIAM, 1994.

[32] Li, Xi and de Souza, Carlos E LMI approach to delay-dependent robust stability and stabilization of uncertain linear delay systems. Proceedings of 1995 34th IEEE Conference on Decision and Control, 4, 3614–3619, 1995.

[33] Niculescu, S-I and Nito, A Trofino and Dion, J-M and Dugard, Luc. Delay-dependent stability of linear systems with delayed state : An LMI approach. Proceedings of 1995 34th IEEE Conference on Decision and Control, 1495–1496, 1995.

[34] Gu, Keqin. Discretized LMI set in the stability problem of linear uncertain time-delay systems. International Journal of Control. Taylor & Francis., 68(4), 923–934, 1997.

[35] Su, Juing-Huei. The asymptotic stability of linear autonomous systems with commensurate time delays. IEEE Transactions on Automatic control, 40(6), 1114–1117, 1995.

[36] Gu, Keqin A further refinement of discretized Lyapunov functional method for the time-delay systems. Proceedings of the 2001 American Control Conference.(Cat. No. 01CH37148), 3998–4003, 2001.

[37] Gu, Keqin Complete quadratic Lyapunov-Krasovskii functional : limitations, computational efficiency, and convergence. Advances in analysis and control of time-delayed dynamical systems, 1–19,2013.

[38] Peet, Matthew M. A convex reformulation of the controller synthesis problem for MIMO single-delay systems with implementation in SOS. American Control Conference (ACC), 5127–5134, 2017.

[39] Seuret, Alexandre and Gouaisbaut, Frédéric Complete quadratic Lyapunov functionals using Bessel-Legendre inequality. 2014 European Control Conference (ECC), 448–453, 2014.

[40] Peet, Matthew M. LMI parametrization of Lyapunov functions for infinite-dimensional systems : A framework. American Control Conference, 359–366, 2014.

[41] Boussaada, Islam and Niculescu, Silviu-Iulian and Tliba, Sami and Vyhlídal, Tomáš. On the coalescence of spectral values and its effect on the stability of time-delay systems : Application to active vibration control.
Journal : Procedia IUTAM. Elsevier, 22, 75–82. 2017.

[42] Boussaada, Islam and Unal, Hakki Ulas and Niculescu, Silviu-Iulian. Further remarks on the effect of multiple spectral values on the dynamics of Time-delay systems : A Birkhoff-based approach. 2016.

[43] Niculescu, Silviu-Iulian and Dion, Jean-Michel and Dugard, Luc and Li, Huaizhong. Stability of linear systems with delayed state An LMI approach. Journal Européen des Systèmes Automatisés (JESA), 31(6), 955–969, 1997.

[44] Niculescu, Silviu-Iulian and Verriest, Erik I and Dugard, Luc and Dion, Jean-Michel. Stability and robust stability of time-delay systems : A guided tour. Stability and control of time-delay systems, 1–71, 1998.

[45] Islam Boussaada and Silviu-Iulian Niculescu. Computing the codimension of the singularity at the origin for delay systems : The missing link with birkhoff incidence matrices. In *21st International Symposium on Mathematical Theory of Networks and Systems*, pages 1–8, 2014.

[46] Boussaada, Islam and Niculescu, Silviu-Iulian. Characterizing the codimension of zero singularities for time-delay systems. Journal : Acta Applicandae Mathematicae. Springer, 145(1) :47–88, 2016.

[47] Islam Boussaada and Silviu-Iulian Niculescu. Tracking the algebraic multiplicity of crossing imaginary roots for generic quasipolynomials : A vandermonde-based approach. *IEEE Transactions on Automatic Control*, 61(6) :1601–1606, 2015.

[48] Islam Boussaada and Silviu-Iulian Niculescu. On the dominancy of multiple spectral values for time-delay systems with applications. *IFAC-PapersOnLine*, 51(14) :55–60, 2018.

[49] Islam Boussaada, Silviu-Iulian Niculescu, Ali El Ati, Redamy Pérez-Ramos, and Karim Liviu Trabelsi. Multiplicity-induced-dominancy in parametric second-order delay differential equations : Analysis and application in control design. *ESAIM : Control, Optimisation and Calculus of Variations*, 2019.

[50] Islam Boussaada, Silviu-Iulian Niculescu, and Karim Trabelsi. Towards a decay rate assignment based design for time-delay systems with multiple spectral values. 2018.

[51] Islam Boussaada, Sami Tliba, Silviu-Iulian Niculescu, Hakki Ulaş Ünal, and Tomáš Vyhlídal. Further remarks on the effect of multiple spectral values on the dynamics of time-delay systems. application to the control of a mechanical system. *Linear Algebra and its Applications*, 542 :589–604, 2018.

[52] Islam Boussaada, Hakki Ulaş Unal, and Silviu-Iulian Niculescu. Multiplicity and stable varieties of time-delay systems : A missing link. 2016.

[53] Fred Brauer and Carlos Castillo-Chávez. Basic ideas of mathematical epidemiology. In *Mathematical Models in Population Biology and Epidemiology*, pages 275–337. Springer, 2001.

[54] Harald Bugmann. A review of forest gap models. *Climatic Change*, 51(3-4) :259–305, 2001.

[55] HM Byrne. The effect of time delays on the dynamics of avascular tumor growth. *Mathematical biosciences*, 144(2) :83–117, 1997.

[56] Baruch Cahlon and Darrell Schmidt. Generic oscillations for delay differential equations. *Journal of mathematical analysis and applications*, 223(1) :288–301, 1998.

[57] Nikolai Grigor'evich Chebotarev and Naum Natanovich Meiman. The routh-hurwitz problem for polynomials and entire functions. *Trudy Matematicheskogo Instituta imeni VA Steklova*, 26 :3–331, 1949.

[58] Robert M Corless, Gaston H Gonnet, David EG Hare, David J Jeffrey, and Donald E Knuth. On the lambertw function. *Advances in Computational mathematics*, 5(1) :329–359, 1996.

[59] Robert M Corless, David J Jeffrey, and Donald E Knuth. A sequence of series for the lambert w function. In *Proceedings of the 1997 international symposium on Symbolic and algebraic computation*, pages 197–204, 1997.

[60] Rebecca V Culshaw, Shigui Ruan, and Glenn Webb. A mathematical model of cell-to-cell spread of hiv-1 that includes a time delay. *Journal of mathematical biology*, 46(5) :425–444, 2003.

[61] W Jack Cunningham. A nonlinear differential-difference equation of growth. *Proceedings of the National Academy of Sciences of the United States of America*, 40(8) :708, 1954.

[62] Gregory Dunkel. Single species model for population growth depending on past history. In *Seminar on Differential Equations and Dynamical Systems*, pages 92–99. Springer, 1968.

[63] William Krasny Ergen. Kinetics of the circulating-fuel nuclear reactor. *Journal of Applied Physics*, 25(6) :702–711, 1954.

[64] Ebihara, Yoshio and Peaucelle, Dimitri and Arzelier, Denis and Gouaisbaut, Frédéric Dominant pole analysis of stable time-delay positive systems, Journal :IET Control Theory & Applications,8(17) :1963–1971, 2014.

[65] Hartmut Erzgräber, Bernd Krauskopf, and Daan Lenstra. Compound laser modes of mutually delay-coupled lasers. *SIAM Journal on Applied Dynamical Systems*, 5(1) :30–65, 2006.

[66] Bartha, Maria. Periodic solutions for differential equations with state-dependent delay and positive feedback. Journal : Nonlinear Analysis. Theory, Methods & Applications, 53(6) :839–857, 2003.

[67] Minyue Fu, Andrzej W Olbrot, and Michael P Polis. Robust stability for time-delay systems : the edge theorem and graphical tests. *IEEE Transactions on Automatic Control*, 34(8) :813–820, 1989.

[68] Minyue Fu, Andrzej W Olbrot, and Michael P Polis. The edge theorem and graphical tests for robust stability of neutral time-delay systems. *Automatica*, 27(4) :739–741, 1991.

[69] K Gopalsamy. Stability and osilation in delay differential equation in delay differential equations of population dynamics, 1992.

[70] Kondalsamy Gopalsamy. *Stability and oscillations in delay differential equations of population dynamics*, volume 74. Springer Science & Business Media, 2013.

[71] Myron K Grammatikopoulos, EA Grove, and G Ladas. Oscillations of first-order neutral delay differential equations. *Journal of mathematical analysis and applications*, 120(2) :510–520, 1986.

[72] Stephen Grossberg. Nonlinear difference-differential equations in prediction and learning theory. *Proceedings of the National Academy of Sciences of the United States of America*, 58(4) :1329, 1967.

[73] Stephen Grossberg. A prediction theory for some nonlinear functional-differential equations i. learning of lists. *Journal of Mathematical Analysis and Applications*, 21(3) :643–694, 1968.

[74] Igor Gumowski. *Oscillatory evolution processes : quantitative analyses arising from applied science*. Manchester University Press, 1989.

[75] Aristide Halanay and Aristide Halanay. *Differential equations : Stability, oscillations, time lags*, volume 6. Elsevier, 1966.

[76] Jack K Hale and Sjoerd M Verduyn Lunel. Strong stabilization of neutral functional differential equations. *IMA Journal of Mathematical Control and Information*, 19(1_and_2) :5–23, 2002.

[77] Jack K Hale, Sjoerd M Verduyn Lunel, Lunel S Verduyn, and Sjoerd M Verduyn Lunel. *Introduction to functional differential equations*, volume 99. Springer Science & Business Media, 1993.

[78] B. D. Hassard, N. D. Kazarinoff, and Y.-H. Wan. Theory and applications of Hopf bifurcation. Cambridge Univ. Press, London, 1981.

[79] K HALE Jack. Theory of functional differential equations. *Applied Mathematical Sciences*, 3, 1981.

[80] Rami, M Ait and Helmke, Uwe and Tadeo, Fernando. Positive observation problem for linear time-delay positive systems. Mediterranean Conference on Control & Automation, pages :1–6, 2007.

[81] Amrane, Souad and Bedouhene, Fazia and Boussaada, Islam and Niculescu, Silviu-Iulian On Qualitative Properties of Low-Degree Quasipolynomials : Further remarks on the spectral abscissa and rightmost-roots assignment. Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, 61(109),361–381,2018.

[82] BD Hassard. Counting roots of the characteristic equation for linear delay-differential systems. *Journal of Differential Equations*, 136(2) :222–235, 1997.

[83] Frank Hoppensteadt and Paul Waltman. A problem in the theory of epidemics. *Mathematical Biosciences*, 9 :71–91, 1970.

[84] G Evelyn Hutchinson et al. Circular causal systems in ecology. *Ann. NY Acad. Sci*, 50(4) :221–246, 1948.

[85] Haddada, Wassim M and Chellaboinab, VijaySekhar Stability theory for nonnegative and compartmental dynamical systems with time delay. Systems & Control Letters, 51(5) :355–361, 2004.

[86] D Israelsson and A Johnsson. A theory for circumnutations in helianthus annuus. *Physiologia Plantarum*, 20(4) :957–976, 1967.

[87] Irena JADLOVSKÁ. Fundamental concepts in the theory of oscillation.

[88] Chi Jin, Keqin Gu, Islam Boussaada, and Silviu-Iulian Niculescu. Stability analysis of systems with delay-dependant coefficients : A two-parameter approach. In *2017 American Control Conference (ACC)*, pages 5726–5731. IEEE, 2017.

[89] A Johnsson and HG Karlsson. A feedback model for biological rhythms : I. mathematical description and basic properties of the model. *Journal of theoretical Biology*, 36(1) :153–174, 1972.

[90] Anders Johnsson and Donald Israelsson. Application of a theory for circumnutations to geotropic movements. *Physiologia Plantarum*, 21(2) :282–291, 1968.

[91] K Knopp. Theory of functions, parts i and ii, translated to english by f. bagemihl, 1996.

[92] Vladimir B Kolmanovskii. On the liapunov-krasovskii functionals for stability analysis of linear delay systems. *International Journal of Control*, 72(4) :374–384, 1999.

[93] Allan M Krall. Stability criteria for feedback systems with a time lag. *Journal of the Society for Industrial and Applied Mathematics, Series A : Control*, 2(2) :160–170, 1964.

[94] Yuri A Kuznetsov. *Elements of applied bifurcation theory*, volume 112. Springer Science & Business Media, 2013.

[95] Gerasimos Ladas and C Qian. Oscillation in differential equations with positive and negative coefficients. *Canadian Mathematical Bulletin*, 33(4) :442–451, 1990.

[96] Michael C Mackey. Commodity price fluctuations : price dependent delays and nonlinearities as explanatory factors. *Journal of economic theory*, 48(2) :497–509, 1989.

[97] Martha Belem Saldivar Márquez, Islam Boussaada, Hugues Mounier, and Silviu-Iulian Niculescu. Neutral-type time-delay systems : Theoretical background. In *Analysis and Control of Oilwell Drilling Vibrations*, pages 57–82. Springer, 2015.

[98] John Edward Marshall. Time-delay systems : stability and performance criteria with applications. Prentice Hall, 1992.

[99] Wim Michiels and Silviu-Iulian Niculescu. Stability, control, and computation for time-delay systems : an eigenvalue-based approach, volume 27. Siam, 2014.

[100] Wim Michiels, Vincent Van Assche, and S-I Niculescu. Stabilization of time-delay systems with a controlled time-varying delay and applications. *IEEE Transactions on Automatic Control*, 50(4) :493–504, 2005.

[101] N Minorsky. Nonlinear oscillations. inc, 1962.

[102] Ju Neimark. D-subdivisions and spaces of quasi-polynomials. *Prikladnaya Matematika i Mekhanika*, 13(5) :349–380, 1949.

[103] Silviu-Iulian Niculescu. *Delay effects on stability : a robust control approach*, volume 269. Springer Science & Business Media, 2001.

[104] Libor Pekar, Radek Matusu, Petr Dostálek, and Jan Dolinay. The nyquist criterion for lti time-delay systems. *Recent Researches in Automatic Control*, pages 27–29, 2011.

[105] Libor Pekař and Roman Prokop. Argument principle based stability conditions of a retarded quasipolynomial with two delays. *Last Trends on Systems*, 1 :276–281, 2010.

[106] Lev Semionovitch Pontryagin. On the zeros of some elementary transcendental functions. *Amer. Math. Soc. Transl*, 2(1) :95–110, 1955.

[107] LS Pontryagin. On the zeros of some transcedental functions. *American Mathematical Society Translations*, 8 :19–20, 1958.

[108] Evgeniĭ Popov. *The dynamics of automatic control systems*.

[109] J Sieber and B Krauskopf. Complex balancing motions of an inverted pendulum subject to delayed feedback control. *Physica D : Nonlinear Phenomena*, 197(3-4) :332–345, 2004.

[110] Gábor Stépán. *Retarded dynamical systems : stability and characteristic functions*. Longman Scientific & Technical, 1989.

[111] Vito Volterra. Sur la théorie mathématique des phénomenes héréditaires. *Journal de mathématiques pures et appliquées*, 7 :249–298, 1928.

[112] Vito Volterra. *Leçons sur la théorie mathématique de la lutte pour la vie*. Gauthier-Villars et cie., 1931.

[113] K Walton and JE Marshall. Direct method for tds stability analysis. In *IEE Proceedings D-Control Theory and Applications*, volume 134, pages 101–107. IET, 1987.

[114] Sun Yi, Patrick W Nelson, and A Galip Ulsoy. *Time-delay systems : analysis and control using the Lambert W function*. World Scientific, 2010.

[115] BG Zhang and K Gopalsamy. Oscillation and nonoscillation in a nonautonomous delay-logistic equation. *Quarterly of applied mathematics*, 46(2) :267–273, 1988.

[116] Andrzej Z. Manitius. Feedback controllers for a wind tunnel model involving a delay : Analytical design and numerical simulation. IEEE Transactions on Automatic Control,29(12) :1058–1068, 1984.

[117] Armstrong, Ernest S and Tripp, John S. An application of multivariable design techniques to the control of the National Transonic Facility, 1981.

[118] Dambrine, Michel. Contribution à l'étude de la stabilité des systèmes à retards. 1994.

Résumé

Notre étude concerne une approche du domaine fréquentiel dans l'analyse de stabilité et stabilisation via une conception de contrôleur d'ordre réduit pour les systèmes à retard linéaire invariant dans le temps. Plus précisément, nous abordons le problème de caractérisation de l'abscisse spectrale et la coexistence de modes non oscillants pour ces équations differentielle fonctionelles. L'approche de conception que nous proposons est simplement une rétroaction de sortie retardée où les paramètres résultent de la variété définie par la coexistence d'un nombre exact de valeurs spectrales négatives, ce qui garantit la stabilité asymptotique des solutions du système. La dominance de ces modes non oscillants est montrée analytiquement pour les systèmes à retard d'ordre réduit en utilisant deux approches différentes, La première méthode consiste à trouver une factorisation adéquate, permettant d'écrire le quasipolynôme sous forme d'intégrale. La deuxième consiste à appliquer la formule de Stépàn Hasard permettant de calculer le nombre de racines instables du système correspondant.

Mots-clés: Systèmes à retard ; Approche fréquentielle ; Conception du contrôleur ; Principe d'argument ; Placement de pôles ; oscillation ; Formule de Stépàn-Hasard.

Abstract

Our study concerns a frequency domain approach for the stability analysis and stabilization via reduced-order controller design for linear time-invariant retarded Time-delay systems. More precisely, we address the problem of the spectral abscissa characterization and the coexistence of non oscillating modes for such functional differential equations. The design approach we propose is merely a delayed-output-feedback where the candidates' parameters result from the manifold defined by the coexistence of an exact number of negative spectral values, which guarantees the asymptotic stability of the system's solutions. The dominancy of such non oscillating modes is analytically shown for the considered reduced order Time-delay systems using two different approaches, the first method consists to find an adequate factorization, allowing to write the quasipolynomial function in an some appropriate integral operator form. The second consists to apply Stépàn Hasard formula allowing to count the number of unstable roots of the corresponding system.

Keywords: Time-delay systems ; Frequency-domain approach, Controller design ; Argument principle ; Pole placement ; Oscillation ; Stépàn-Hasard formula.