Applications of Thermofield Dynamics to Quark-Gluon Plasma in equilibrium and non-equilibrium phase state

Merzak Chekerker

Department of Physics

University Mouloud-Mammeri of Tizi-ouzou

A thesis submitted for the degree of

PhilosophiæDoctor (PhD), Doctorat

January 2012

1. President : ZIANE Abdelhamid, University Mouloud-Mammeri of Tizi-Ouzou.

2. Supervisor : LADREM Madjid, ENS de Kouba, Algeria; Taiba University, KSA.

3. External Reviewer: F. C. Khanna, University of Alberta, Canada.

4. External Reviewer: LAMROUS Omar, University Mouloud-Mammeri of Tizi-Ouzou.

5. External Reviewer: MEBARKI Noureddine, University of Constantine, Algeria.

Day of the defense: January 2012

Signature from head of PhD committee:

Abstract

The thermofield dynamics, a real-time formalism for finite temperature quantum field theory (TFD), is used to calculate the rates for e^+e^- reactions at finite temperature and for the pion production. The results indicate that temperature plays an important role in defining the possible hadronic state after the plasma has cooled down. It is also utilised to define the fragmentation function at high energy experiments. The definition of the fragmentation is extended to finite temperature. The results at $T = 0$ and $T \neq 0$ are compared. In general, we find that the fragmentation function decreases in magnitude. At the end, this formulation of Quantum field theory (TFD) as well as LvN approach are extended to define two-point correlators at finite t and T, and time-dependent Green-functions for open systems. We obtain results for two-point correlators, the Green-functions for scalar field and ϕ^4 field theory to second order and for Green-functions for fermions.

Dedicated to

Malika, Zine-eddine, Amine, Bouchra, Zakaria.

Acknowledgements

I owe my thanks to both Professor khanna and Professor Santana for their helpful advice and contributions to the reading, writing and correction of the thesis and to Professor Ladrem with whom I worked who provided me with the full idea of TFD and was very helpful in shaping the present work together with professor Santana. I am al soin debt to both Dr khanna and Dr Santana for many fruthful discussions during the hole period of my stay as a visitor in the university of alberta and their encouragment as well. I'am also in debt and gratitude to Santana's student who provided me with the transcript and full material necessary to the writing of my thesis in it's present form. Special thanks go to Theoretical Physics Institute and Physics Department for facilities and access to laboratories of University of Alberta. Finally, I owe my thanks to 'L'ECOLE NORMALE SUPERIEURE' (ENS de Kouba, Algiers) where all the present work has been initiated; cristallised in Algeria and refined at the University of Alberta (CANADA). I owe Also my thanks to my school 'L'ECOLE NATIONALE POLYTECHNIQUE' (ENP d'El Harrach, Algiers) for my research award. finally, my thanks go to the University 'MOULOUD MAMMERI' of 'Tizi-Ouzou' for its acceptance of my registration at the faculty of Science in the Physics department.

Contents

List of Figures

LIST OF FIGURES

List of Tables

Glossary

DAPI 4',6-diamidino-2-phenylindole; a fluorescent stain that binds strongly to DNA and serves to marks the nucleus in fluorescence microscopy

DEPC diethyl-pyro-carbonate; used to remove RNA-degrading enzymes (RNAases) from water and laboratory utensils

DMSO dimethyl sulfoxide; organic solvent, readily passes through skin, cryoprotectant in cell culture

EDTA Ethylene-diamine-tetraacetic acid; a chelating (two-pronged) molecule used to sequester most divalent (or trivalent) metal ions, such as calcium (Ca^{2+}) and magnesium (Mg^{2+}), copper (Cu^{2+}), or iron (Fe^{2+} / Fe^{3+})

NAME="GraphicsSave" CONTENT="32"¿

1

Introduction.

Nowadays, it is not clear whether a form of deconfined hadronic matter exists in the universe. Such a state could be probably investigated in the laboratory (RHIC experiments) to detect this new hadronic matter called the quark gluon plasma (QGP for short). This is some sort of a soup of quarks and gluons moving at hight temperature[1],[2]. Usual predictions for such a state of matter is performed, using perturbative quantum chromodynamics (QCD) and statistical mechanics models[3],[4],[5],[6] . All experiments with colliders have been carried out within this theoretical framework of quantum field thory. Such experiments are for instance: (i) pp collisions [7]; (ii) Drell-Yang processes [7].(iii) The e^+e^- annihilation process[8][9][10][11][12][13][14]. The latter could be considered as a huge factory of information to investigate quark interaction. It provides us with a good test for QCD at hight energies. For other energy scales, some insight is obtained on the phenomenological models of quark interactions. Moreover, it is the cleanest arena in where we can study in detail electroweak annihilation with an almost asymptotically free initial quark-antiquark pair at very large Q^2 with a QCD color field appropriate for the production of various light quark mesons, the most fundamental and elementary tests of the color field behaviour. Experimentally, the large electron-positron machine (LEP) which was running recently due to its success through a decade of successful operation, providing a wealth of precise data on the electroweak and the strong interactions using multipurpose detectors ALEPH, DELPHI, L3 and OPAL [15][16]. The LEP colliders operated at energies above the Z^0 resonance threshold :

$$\sqrt{s} = E_{cm} \succ M_{Z^0}.c^2$$

where M_{Z^0} is the Z^0 boson mass and c is the velocity of light. Mainly, the basic prediction of the standard model of electroweak interaction, for fermion-antifermion production annihilation around the Z^0 resonances are sketched in this process. The measured cross section for W-pair production $e^+e^- \to W^+W^-(\gamma)$ is pursued. Using QCD events in e^+e^- annihilation, jet events were also studied within the framework of hadronization. Annihilation (e^+e^-) process is the most reliable and suitable process for the study of multiparticle production in jet physics. In accordance with QCD, it can be realized throught the production of γ or Z^0 boson into two quarks

$$e^+e^- \to (Z^0/\gamma) \to q\bar{q}$$

Some hadronisation models were proposed [8]. In such models, hadronization occurs through the e^+e^- annihilation process. They then undergo Q^2 evolution via a virtual photon and fragment into hadrons. Light hadron physics at the B factories gives a good account of clusters of hadronic matter that are obtained in different channels [13]. Annihihilation and QCD give different fragmentation processes (τ production for instance) in the Z^0 channel [11],[12]. A measurement of the total cross section for $e^+e^- \to hadrons$ at $\sqrt{s} = 10.25\ Gev$ was made using the CLEO detector at the Cornell Electron Storage Ring (Ref. [17] of chapter 6). The last but not the least, new results from Novosibirsk and Beijing on the low energy e^+e^- annihilation into hadrons have been described [12]. Brief reviews on e^+e^- annihilation process are given in the books by Kapusta [6], Le bellac [5].

The effect of temperature arises in a number of questions concerning details of the production mechanisms and have been already treated, extensively in ultrarelativistic heavy ion collisions [17],[18],[19]. The first order correction to the decay rate of dilepton pair produced in a thermalized medium are examined using the real time formalism proposed by Baier, Pire,Schiff [20],[21]. A vanishing chemical potential has been assumed for simplicity. Our focus is to analyse the effect of temperature on the decay rate, particularly, for hadron production and $\pi^\pm$ production using real time formalism at the tree level.

Indeed, the motivation is the following :

-our work, first of all, follows Rakhimov and Khanna original work [22] on calculating decay rates for different reactions using the real-time formalism. The plotting of graphical representations (Fig. 1, Fig. 2, of chapter 5; Fig. 4 of chapter 6) of different decay rates phenomena, when temperature increases for each process. Numerous attempts have been made using to account for thermal effects during dilepton production [20],[21]. There is a need to develop of a finite temperature field theory which could provide such answers. Several methods has been proposed. One formalism using Wick rotation was proposed by Matsubara [23],[24],[25],[26], the imaginary-time formalism. An extension of Matsubara work to quantum field theory was carried out by Ezawa,Tomozawa and Umezawa[23],[24]. A real-time formalism, the closed-time path formulation,was proposed by Schwinger [27], Mahanthapa and Bakhshi [28], and Keldysh [29]. This approch uses a closed contour in the complex plane. This requires a doubling of the degrees of freedom of the theory such that the green function is represented by a 2*2 matrices. Umezawa and Takahashi introduced the method of Thermo-field Dynamics (TFD) that introduced a second Hilbert space. Bogoliubov transformation is used to mix states in the two Hilbert spaces. In equilibrium, all three methods are equivalent. Howeve, the objective of real-time theories was to extend the formalism to non-equilibrium processes.This has not been achieved as yet. Our calculation is directed to study total non-polarised decay rates at finite temperature in the laboratory frame for e^+e^- annihilation process giving rise to hadronique matter within a heat bath (QGP). We have obtained results for various centre of mass energies and different chemical potentials. The nature of the hadrons is specified by the value of centre of mass energy $\sqrt{s}$. Two areas are considered : the production of bound states heavy quarks which can be treated by an effective field theory (EFT) (Refs. [8],[15] of chapter 8). The technical calculation uses Wick's theorem developed in [31],[32]. We also consider an example of hadronisation into $\pi^\pm$ (light mesons).

To further investigate the hadronization phenomenon of quarks in QGP, we study specific fragmentation functions at finite temperature. The latter are determined in a universal manner by the direct calculation of appropriate Feynmann diagrams (Refs. [1],[2],[3],[4],[8],[15] of chapter 8). Using this definition, we find anultraviolette divergence in Wilson line during the calculation which can be handled by a renormalisation procedure. We interpret this divergence due to a thermal bath of quarks and gluons moving freely in a quenched medium.The fragmentation function is in part the same as the one found by Ma (Ref. [8] of chapter 8). In the sequence, we perform a direct calculation using Wilson lines (Refs. [32],[34] of chapter 8) in a general gauge and plot the curve in the light cone gauge (Ref. [5] of chapter 8). The results satisfy zero limit (Fig. 8 of chapter 8).

In conclusion, and in the entire analytic work, together with the results obtained for the curves,we beleive that, the probability of hadronisation of quarks and gluons within a QGP is diminishing as the temperature goes up (Fig. 1,Fig. 2 of chapter 5; Fig. 4 of chapter 6; Fig. 8 of chapter 8).

The thesis is organized as follows :

Chapter 1 : we give a brief introduction about our work and the organization of the chapters.

Chapter 2 : in the introductory chapter, we examin the fundamental and remarquable aspect of TFD which is the doubling of the Hilbert space and the corresponding degrees of freedom of the thermal field quantized. It has been suggested by the introduction of the density matrix related to Gibbs ensemble in the introductory chapter of TFD [3],4],[5],[6]. Our work requires the evaluation of statistical mean values which leads us to the calculation of causal green functions using T-products at finite temperature. Because of doubling of the degrees of freedom, these functions transform as matrices [3],[4],[5],[6],[23],[24],[25],[26].

Chapter 3 : we give classical examples of T.F.D : the thermal oscillators based on Umezawa original paper [24]. The material can also be found in [3],[4],[5],[6].

Chapter 4 : this chapter deals essentially with an introduction to TFD stressing on the use of Wicks theorem [5],[25],[31].

Chapter 5 : an introductory chapter about scattering processes at finite temperature is provided. The main material could also be found mainly in [3],[4],[5],[6],[22],[23],[24],[25],[26].

Chapter 6 : We have examine the hadronization phenomenon of quarks and gluons within a heat bath (QGP medium with color confinment). We have elaborated a technique for calculation of the total non-polarized cross section of hadronization of the e^+e^- annihilation process using one photon annihilation at finite T and for different centre of mass energies :$\sqrt{s} = 2\epsilon_0 = 10GeV, 100GeV$ and different chemical potential values :$\mu = 0, 400Mev$. The technique uses Wick's theorem. The medium is non-abelian and without emission of gluons. We work in the light cone region: $x^2 \approx 0$. To calculate σ_{tot} at $T \neq 0$, we follow the lines of reference [22] using Wick's theorem method. Then the total cross section will be redefined over the thermal vaccum $|0(\beta)\rangle$.Then the whole machinary at $T = 0$ can be reproduced at $T \neq 0$.The results during the hadronization phenomenon allow us to exhibit the nature of the hadrons.The curves can be plotted in two dimensions with MATHEMATICA (Figs. 3a-3e of chapter 6). The hadronsformed could be heavy hadrons or light hadrons depending on the energy regimes. The secondremark is that the formation of hadrons is diminished more or less when temperature increases compared to the formation of hadrons within the heat bath. The heavy quarks are distributed over the curves far from the axis and hadronise for centre of mass of high energy far from the axis.

Chapter 7 : we examine an example of hadronization with the formation of a bound state $q\bar{q}$ of a meson. The method used is the real-time method at finite temperature, in other words the canonical method. The result obtained shows as previously that when the temperature increases, the hadronization of quarks into a π meson is lowered. This result is in accordance with the general case.

Chapter 8 : the hadronization phenomenon of quarks is examined within a QGP. We use an analytic model where we elaborate an algebraic method relying on the definition of the fragmentation function of a rapid quark to a hadron, $q\bar{q}$, without spin.The Feynman diagram technics (Refs. [8],[15] of chapter 8) is used to derive fragmentation functions for heavy quarks and heavy quark effective field theories (HQET) (Ref. [15] of chapter 8). We attempt to give an expression for the fragmentation function at finite T for a hadron as a bound state of two heavy quarks without spin. We inside the case of a fast moving quark with an exchange of a hard photon. It should be stressed that in the case of finite temperature, the statistical weights of thermal distributions of Bose-Eimstein and Fermi-Dirac statistics arise naturally in the calculation without need to define them in the hadronic tensor formula [30]. We note that the fragmentation function decreases with the temperature. This result is in accord with the preceding calculations.This result is to be confirmed experimentally. Limit cases are discussed for low and high temperatures in a general gauge and a curve is plotted for the light cone gauge.

Chapter 9 : In this chapter, A real-time-dependent finite-temperature formulation of Quantum field theory (TFD) as well as Liouville von-Neumann approach are utilised to define two-point correlators at finite t and T, and time-dependent Green-functions for open systems. Nonequilibrium TFD is used to obtain results for two-point correlators and the Green-functions for scalar field, ϕ^4 field theory and for fermions.

1. INTRODUCTION.

Bibliography

[1] J. W. Harris and B. Muller, 'The search for the Quark-Gluon Plasma', arXiv:hep-ph/9602235 v2.

[2] A.V. Smilga, 'Physics of Thermal QCD', arXiv:hep-ph/9612347 v3.

[3] F.C. Khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, A. E. Santana, 'Thermal Quantum Field Theory, Algebraic Aspects and Applications',World Scientific Publishing Company (2009).

[4] A. Das, 'Finite Temperature Field Theory', (W.Scientific, Singapour, 1977)

[5] M. Le Bellac, 'Thermal Field Theory (Cambridge University Press, Cambridge, 1996)

[6] J. I. Kapusta, 'Finite Temperature Field Theory' (Cambridge University Press, Cambridge, 1989).

[7] WU-KI TUNG, 'Perturbative QCD and the Parton Structure of the Nucleon', Based on a contribution to: At the frontiers of Particle Physics-Handbook of QCD, vol. 2* 887-971, ed. M. Shifman, Pub. World Scientific Publisher, Singapore, New York, 2001.

[8] Chien-er LEE, Yeou-Wei YANG and Wen-Fa TSAI , 'Hadronization in the e+e- Annihilation: The Model',Chinese Journal of Physics (vol 22, N 03, 1984)

[9] 34th International Conference on light Energy Physics, Philadelphia, P.A, July 29th-August 5th, 2008.

[10] M.E.Peskin,'Introduction to the Standard Model', PiTP summer school, July 2005.

[11] J.D.Bjorken and B.L.Ioffe (Usp.Fiz.Nauk 116, september 15-144 (May 1975))

[12] S.I.Eidelman,'recent results from $e^+e^- \rightarrow$ hadrons 7th International workshop on Tau Lepton physics (Tau 02) Santa Cruz, CA, 10-13 september, 2002.

[13] Andreas Petzold representing the BABAR collaboration : the 2007 Europhysics Conference on Hight Energy Physics, Journal of Physics : Conference Series 110 (2008)022041.

[14] M.E.Peskin, 'Physics at e^+e^- linear colliders 1. e^+e^- annihilation into pairs., March, 2002.

[15] S. Bethke, 'Standard Model Physics at LEP', arXiv:hep-ex/00010223 v1.

[16] S. Bethke, 'Jet Physics at LEP and World Summary of α_s', arXiv:hep-ex/9812026.

[17] See, for example, in Quark Matter, Proceedings of the Sixth International Conference on Ultra-relativistic Nucleus-Nucleus-Quark Matter 1987, edited by H. Satz, H. J. Specht, and R. Stock, in Z. Phys. C-Particles and Fields 38 (1988)

[18] *E. L. Feinberg, Nuovo Cimento 34A, 391 (1976)
*K. Kajantie and H. Miehinen, Z. Phys. C-Particles and Fields 9, 341 (1981); ibid. 14, 357 (1982).
*R. C. Hwa and K. Kajantie, Phys. Rev. D32, 1109 (1985);
*E. V. Shuryak, Yad. Fiz. 28 796 (1978); and Phys. Rep. 61, 71(1980); Ī
*G. Domokos and J. I. Goldman, Phys. Rev. D23, 203 (1981);

[19] L. D. Mc Lerran and T. Toimela, Phys. Rev. D31, 545 (1985).

[20] R. Bair, B. Pire and D. Schiff, 'Dilepton Production at finite temperature, Perturbative Treatment at Order s', BI-TP 88/20 July 1988.

[21] R. Bair, B. Pire and D. Schiff, 'Perturbative aspects of Dilepton Production at finite temperature',LPTHE Orsay 87/57, Septembre 1987.

[22] A.Rakhimov and F.C.Khanna,'Finite temperature amplitudes and reaction rates in thermofield dynamics', Phys. Rev C, volume 64 064907.

[23] Y. Takahashi and H. Umezawa, 'Thermofield dynamics',Collective Phenomena, Vol. 2, pp.55-8 (1975).

[24] Y. Takahashi and H. Umezawa, 'Thermo field Dynamics', International journal of modern Physics B, Vol. 10, Nos. 1314 (1996) 1755-1805.

[25] P;V;Landshoff, 'Introduction to equilibrium thermal field theory', hep-ph/9808362.

[26] M. H. Thoma,'New Developments and Applications of Thermal Field Theory', arXiv:hep-ph/0010164.

[27] J. Schwinger, J. Math. Phys. 2, 407 (1961).

[28] *P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. 4, 1 and 12 (1963)
*K. T. Mahanthappa, Phys. Rev. 126, 329 (1962)

[29] L. V. Keldysh, Zh. Eksp. Toer. Fiz. 47, 1515 (1964) [Sov. Phys. JETP 20, 1018 (1965)]

[30] Jonathan A.Osborn, Enke Wang and Xin-Nian Wang, 'Evolution of Parton Fragmentation Functions at Finite Temperature', arXiv : hep-ph/0212131.

[31] T.S. Evans, D.A.Steer, 'Wick's theorem at finite temperature', arXiv: hep-ph/9601268.

[32] Christian thesis.

[33] C. Itzykson and J. B. Zuber, 'Quantum Field Theory', McGraw-Hill International Editions, (1988).

[34] Jonathan A.Osborn, Enke Wang and Xin-Nian Wang, 'Evolution of Parton Fragmentation Functions at Finite Temperature', arXiv : hep-ph/0212131.

2

Introduction to the Formalism of Thermofield Dynamics (TFD)

In this chapter we present a review of the real-time formalism for quantum field theory at finite temperature. This review is based on the references [1],[2],[3],[4],[5],[6],[7]. We emphasize in the presentation some aspects of the formalism that will be useful in the following chapters. Other aspects of this method can be found in the references above mentioned.

2.1 Introductory TFD.

The essential quantity in the statistical mechanics in thermal equilibrium is the statistical average of a quantity A, say, over the grand canonical ensemble at temperature T given by

$$\langle A \rangle = Z^{-1}(\beta) Tr \left[A e^{-\beta \aleph} \right], \tag{2.1}$$

where

$$\begin{aligned} \aleph &= H - \mu N, \\ Z(\beta) &= Tr(e^{-\beta \aleph}), \end{aligned}$$

with

$$\beta = (k_{\mathbf{B}} T)^{-1}.$$

Here H is the total hamiltonian of the system, μ is the chemical potential and $k_{\mathbf{B}}$ is the Boltzmann constant. Thermofield dynamics is the operator formalism, which reproduces the thermal average of a statistical ensemble as an expectation value with respect to "temperature dependent vaccum". Our aim will be achieved if we can construct a field theory in which the statistical average of a quantity A is given by some sort of expectation value rather than the trace operation. In other words, we try to construct a representation in which the "vaccum" expectation value coincides with the statistical average defined by

$$\langle A\rangle = Z^{-1}(\beta)Tr\left[e^{-\beta\aleph}A\right] = \langle O(\beta)|A|O(\beta)\rangle \tag{2.2}$$

where, $\mid O(\beta)\rangle$ is the temperature dependent "vaccum" state in a new space to be constructed. We will give a general construction and general examples for this state in the next sections.

2.2 Doubling of degrees of freedom and its meaning in thermofield dynamics.

We will examine one fundamental and remarquable aspect of TFD which is the doubling of degrees of freedom of the thermal fields quantized with the introduction of non-physical fields "the tilde fields" for each thermal field. Our suggestion is to interprete this fictitious field as a hole in the heat bath. This has been suggested by the following definition of the density matrix related to the Gibbs ensemble :

$$\begin{aligned} \rho_\beta &= a^\dagger \rho a \\ \tilde{\rho}_\beta &= a\tilde{\rho}a^\dagger \end{aligned}$$

which is a hermitian operator, and where a and $a^\dagger$ are respectively the creation and annihilation operators. Here, ρ is no more than the projector $\rho = \mid\rangle\langle\mid$ of a state, say, $\mid n\rangle$. We can find a similar interpretation from The Dirac field (the Dirac sea is full of positive holes) or in the semi-conductor theory where the hole liberated by the electron can be considered as a positive charge. The introduction of the thermal " fields" is necessary taking into account the fact that the energy spectra of the system in equilibrium belong to an invariant hamiltonian by a symmetry operation called BOGOLIOUBOV transformation. Under this transformation, MATSUBARA has shown that we should introduce a thermal vacuum to describe statistical mean values. Therefore, we convert a statistical mean value of an operator over different states $|\rangle$ to the mean value over the thermal vacuum

$$\langle A\rangle_{\beta} = \langle O(\beta) \mid A \mid O(\beta)\rangle = Z^{-1}(\beta) \sum_{n} e^{-\beta E_n} \langle n \mid A \mid n\rangle \tag{2.3}$$

where, $Z(\beta) = Tr\left(e^{-\beta \aleph}\right)$ is the partition function allowing to describe a statistical system in equilibrium and $\mid n\rangle$ the state of energy E_n.

The introduction of the BOGOLIOUBOV transformation allows one to write thermal fields under the prescription of a vector of two dimensions (the matter was discussed during a seminar on degree of freedom doubling : CNPDP, Tizi-Ouzou, Algiers : 29-30 Mai 2007).

2.2.1 General Consideration.

The doubling interpretation of the Hilbert space could be seen as follows. We define the thermal vacuum by Eq.(2.3)

$$\langle O(\beta) \mid A \mid O(\beta)\rangle = Z^{-1}(\beta) \sum_{n} \langle n \mid A \mid n\rangle e^{-\beta E_n} \tag{2.4}$$

for an arbitrary observable A, where

$$\aleph \mid n\rangle = E_n \mid n\rangle$$

$$\langle n \mid m\rangle = \delta_{nm}.$$

To construct the thermal vacuum $\mid O(\beta)\rangle$, we need to expand it in terms of $\mid n\rangle$ as

$$\mid O(\beta)\rangle = \sum_{n} f_n(\beta) \mid n\rangle. \tag{2.5}$$

Substituting Eq.(2.5) into Eq.(2.4), we obtain

$$f_n^*(\beta) f_m(\beta) = Z^{-1}(\beta) e^{-\beta E_n} \delta_{nm} \tag{2.6}$$

which is impossible if we considere $f_n(\beta)$ as complex numbers. We observe that this relation could be interpreted as an orthogonality condition in a Hilbert space in which these coefficients could be considered as vectors. In other words, the state $\mid O(\beta)\rangle$ is a vector in a space spanned by $\mid n\rangle$ and $f_n(\beta)$ .So, in order to realise such a representation, we introduce a fictitious system, composed by the ”tilde fields”. The non-physical system is characterized by the Hamiltonian $\tilde{H}$ and the state vectors $\mid \tilde{n}\rangle$

$$\tilde{\aleph} \mid \tilde{n}\rangle = E_n \mid \tilde{n}\rangle$$

$$\langle \tilde{n} \mid \tilde{m}\rangle = \delta_{nm}.$$

The matrix elements of the operator A is

$$\langle \tilde{m}, n \mid A \mid n', \tilde{m}'\rangle = \langle n \mid A \mid n'\rangle \delta_{mm'}$$

and that corresponding to $\tilde{A}$ is

$$\langle \tilde{m}, n \mid \tilde{A} \mid n', \tilde{m}'\rangle = \langle \tilde{m} \mid \tilde{A} \mid \tilde{m}'\rangle \delta_{nn'}.$$

Now, if we put

$$f_n(\beta) = \mid \tilde{n}\rangle e^{-\beta E_n/2} Z^{-1/2}(\beta), \tag{2.7}$$

then, Eq.(2.6) is satisfied

$$\begin{aligned} f_n^*(\beta) f_m(\beta) &= Z^{-1}(\beta) e^{-\beta E_n/2} e^{-\beta E_m/2} \langle \tilde{n} \mid \tilde{m}\rangle \qquad (2.8) \\ &= Z^{-1}(\beta) e^{-\beta E_n} \delta_{nm}. \end{aligned}$$

The state $\mid O(\beta)\rangle$ can be constructed if we substitute Eq.(2.7) into Eq.(2.5)

$$\mid O(\beta)\rangle = Z^{-1/2}(\beta) \sum_n e^{-\beta E_n/2} \mid n, \tilde{n}\rangle. \tag{2.9}$$

We may then verify relation (2.4) directly as

$$\begin{aligned} \langle O(\beta) \mid A \mid O(\beta)\rangle &= Z^{-1}(\beta) \sum_{n,m} e^{-\beta E_n/2} e^{-\beta E_m/2} \times \langle \tilde{n}, n \mid A \mid m, \tilde{m}\rangle \qquad (2.10) \\ &= Z^{-1}(\beta) \sum_n e^{-\beta E_n} \times \langle n \mid A \mid n\rangle. \end{aligned}$$

2.2.2 The meaning of the duplication of degrees of freedom in TFD: thermo-algebras.

The doubling interpretation requires the introduction of an operation of symmetry to the theory. This will provide an algebraic structure to TFD from the point of view of symmetry. The introduction of a new state called the thermal vaccum state, $\mid O(\beta)\rangle$, is necessary.

2.2.3 Symmetry generators and observables.

The set of kinematical variables, say V , is a vector space of mappings in a certain Hilbert space. The space V is composed of two vector spaces : $V = V_{obs} \oplus V_{gen}$ where V_{obs} stands for the observables operators and V_{gen} for generators of the symmetry. In classical theory, both operators describe the same quantity and are identical. In quantum field theory, the symmetry generators and observables are quite different. For instance, as we know, the generators of rotation $L_3 = ix_1\partial/\partial x_2 - ix_2\partial/\partial x_1$ and of space translation $P_1 = -i\partial/\partial x_1$ are considered as observables, the angular momentum and linear momentum respectively. An infinitesimal rotation α around the x_3 axis applied to the momentum P_1 is

$$e^{(i\alpha L_3)}P_1e^{(-i\alpha L_3)} \simeq (1 + i\alpha L_3)\, P_1\, (1 - i\alpha L_3) = P_1 + i\alpha[L_3, P_1].$$

The commutator, expressing the effect of the rotation on the momentum is given by

$$[L_3, P_1] = L_3P_1 - P_1L_3 = iP_2.$$

Similarly, we write in general,

$$[L_i, P_j] = i\epsilon_{ijk}P_k. \tag{2.11}$$

This shows that the generator of rotation changes the observable to another observable $i\epsilon_{ijk}P_k$. This operator should be thought of as a generator of symmetry not as an observable. An another example is the Lie algebra of the rotation group

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{2.12}$$

In this case, the operator L_i has a double meaning : a generartor of rotation and angular momentum. Actually, there is no dynamical constraints as to consider the observable and the generator of symmetry by the same variable. We are free to consider the separation between the two different objects. An example for this separation may be quoted in the following : H may be considered as the Hamiltonian of the theory , and $\hat{H}$ as the Liouvillian, the time evolution operator.

2.2.4 Doubling of the Lie Algebra.

Let us denote by $l = \{a_i, i = 1, \cdots, s\}$ the set of generators that span a Lie algbera over $\mathbb{R}$, the real field. In this set, there exists a Lie product denoted by

$$[a_i, a_j] = C_{ij}^k a_k,$$

where the sum over repeated indices is assumed. The c-numbers C_{ij}^k characterize the nature of the symmetry group. Further, the Lie product fulfills the condition of antisymmetry

$$[a_i, a_j] = -[a_j, a_i],$$

and the Jacobi identity,

$$[a_i, [a_j, a_k]] + [a_k, [a_i, a_j]] + [a_j, [a_k, a_i]] = 0.$$

For the space of generators of l , we write

$$[\hat{A}_i, \hat{A}_j] = iC_{ij}^k \hat{A}_k, \tag{2.13}$$

where $\hat{A}_i \in V_{gen}$.Althought Eq. (2.13) provides a representation for l the hat operators, operators A have to be taken into account as well. Therefore we have additional commutation relations among A and $\hat{A}$

$$[\hat{A}_i, A_j] = iD_{ij}^k A_k, \tag{2.14}$$

$$[A_i, A_j] = iE_{ij}^k A_k, \tag{2.15}$$

whereC_{ij}^k, D_{ij}^k and E_{ij}^k are the structure constants to be fixed. Equations (2.14) and (2.15) denote a Lie algebra l_T which is a semidirect product of two subalgebras, $l_T = V_{gen} \otimes V_{obs}$. The hat operators $\hat{A}_i$ act on the observables A_j as an infinitesimal action of a symmetry resulting in another observable given by $iD_{ij}^k A_k$.we can see this by the following reasoning. The action of symmetry is specified by

$$e^{i\alpha \hat{A}_i} A_j e^{-i\alpha \hat{A}_i} = A_j(\alpha).$$

Taylor expansion around $\alpha \simeq 0$ to first order, gives

$$A_j(\alpha) = A_j + \alpha \left(\frac{\partial A_j(\alpha)}{\partial \alpha} \right)_{\alpha=0}$$

moreover

$$e^{i\alpha\hat{A}_i}A_je^{-i\alpha\hat{A}_i} \simeq A_j + i\alpha[\hat{A}_i, A_j].$$

Taking $\alpha \longrightarrow 0$, we have

$$[\hat{A}_i, A_j] = -i\left(\frac{\partial A_j(\alpha)}{\partial\alpha}\right)$$

Therefore, the action of symmetry is specified by the term

$$\left(\frac{\partial A_j(\alpha)}{\partial\alpha}\right)$$

which is another observable. This allows us to determine the constants D_{ij}^k thus specifying the commutation relations between the hat operators of symmetry and the observables. Let's take the angular momentum algebra of a spin system. We know that a representation of $su_T(2)$ is given by

$$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k$$

$$[\hat{S}_i, S_j] = i\epsilon_{ijk}S_k$$

$$[S_i, S_j] = i\epsilon_{ijk}S_k$$

where $\hat{S}_i$ are the hat operators of symmetry and S_i are the operators of spin. As a consequence of the commutation relations, we have $C_{ij}^k = D_{ij}^k = E_{ij}^k = \epsilon_{ij}^k$. With this example, we can assume that for our algebra l_T : $C_{ij}^k = D_{ij}^k = E_{ij}^k$. Gathering these results , we write :

$$\begin{aligned}
[\hat{A}_i, \hat{A}_j] &= iC_{ij}^k\hat{A}_k, \\
[\hat{A}_i, A_j] &= iC_{ij}^kA_k, \\
[\hat{A}_i, A_j] &= iC_{ij}^kA_k.
\end{aligned} \tag{2.16}$$

These relations define a new representation in a thermal Hilbert space $\aleph_T$. This representation will be called *Thermal Lie algebra* [1].

2.2.5 Tild conjugation rules.

In our work, we need, in most cases, to evaluate causal green functions at finite temperature. Because of the doubling of degrees of freedom, these functions have matrix representations in two dimensions of Nambu fields.

2. INTRODUCTION TO THE FORMALISM OF THERMOFIELD DYNAMICS (TFD)

Heisenberg fields are dependent on temperature via Bogoliubov transformations. The calculation is done through additionel dual fields called the tild conjugation fields as introduced above. The tild conjugation is introduced in the following manner. Let :

$$\pounds(x) = \pounds\left(\psi^{\dagger}(x), \psi(x)\right)$$

the classical lagrangian. It is defined under the following rules :

$$\widetilde{O_1 O_2} = \tilde{O}_1 \tilde{O}_2$$

$$c_1 \widetilde{O_1 + c_2} O_2 = c_1^{*} \tilde{O}_1 + c_2^{*} \tilde{O}_2$$

$$\tilde{\tilde{O}} = \eta_F O, where \eta_F = \left\{ \begin{array}{ll} -1 & \text{for fermions} \\ 1 & \text{for bosons} \end{array} \right\}$$

and ,

$$\tilde{\pounds}(x) = \pounds\,(\psi^{\dagger} \widetilde{(x), \psi}(x)) = \pounds\left(\tilde{\psi}^{\dagger}(x)^{*}, \tilde{\psi}^{*}(x)\right)^{*}$$

The lagrangian density is :

$$\hat{\pounds}(x) = \pounds(x) - \tilde{\pounds}(x)$$

and the total hamiltonian :

$$\hat{H} = H - \tilde{H}.$$

the dynamical equations for the fields are :

$$\Lambda(\partial)\,\psi(x) = j\left(\psi^{\dagger}(x), \psi(x)\right)$$

$$\Lambda^{*}(\partial)\,\tilde{\psi}(x) = j\left(\tilde{\psi}^{\dagger}(x)^{*}, \tilde{\psi}^{*}(x)\right)^{*}.$$

In the infinit past, the fields behave like quasi-particle fields $\phi(x)$ in the following manner :

$$\psi(x)\, t \to -\infty \longrightarrow Z^{\frac{1}{2}}(-i\bigtriangledown)\,\phi(x)$$

$$\tilde{\psi}(x)\, t \to -\infty \longrightarrow Z^{\frac{1}{2}}(-i\bigtriangledown)\,\tilde{\phi}(x),$$

with ,

$$\Lambda(\partial)\,\phi(x) = 0 and \Lambda(\partial)^{*}\,\tilde{\phi}(x) = 0.$$

where $\bigtriangledown$ is the Laplacian and $\Lambda(\partial)$ is an operator to be described in the next subsection. Now, to complete the formalism we go to the general case for our *thermo-algebra.* If we define

$$\tilde{A} = A - \hat{A},$$

we find that the thermal algebra may be written

$$[A_i, A_j] = iC_{ij}^k A_k,$$

$$[\tilde{A}_i, \tilde{A}_j] = -iC_{ij}^k \tilde{A}_k,$$

$$[A_i, \tilde{A}_j] = 0,$$

This shows that a doubling has been introduced as a direct product in the vector space of the thermal Hilbert space $\aleph_T$ describing the observables and the generators of symmetry. This tild conjugation rule can be considered as a mapping in $V = V_{obs} \oplus V_{gen}$, say, $J : V \longrightarrow V$, such that

$$JAJ = \tilde{A},$$

fulfilling the following conditions

$$\begin{aligned}
(A_i A_j)\tilde{} &= \tilde{A}_i \tilde{A}_j, \\
(c_1 A_i + c_2 A_j)\tilde{} &= c_1^* \tilde{A}_i + c_2^* \tilde{A}_j \\
(A_i^\dagger)\tilde{} &= (\tilde{A}_i)^\dagger, \\
(\tilde{A}_i)\tilde{} &= A_i \\
[A_i, \tilde{A}_j] &= 0. \qquad (2.17)
\end{aligned}$$

These properties are called *tild conjugation rules.*

2.2.6 Thermal propagators and Green Functions in TFD.

The finite temperature field theory has been used to define response functions in many-body systems, particularly mean value of physical observables such as energy, free energy, entropy,etc . This requires the development of a technical calculation for physical mean values, therefore we should use, for instance, thermal propagators in real time formalism to calculate the forward scattering at two loops. We consider in the following, the causal green function :

$$G_c^{\alpha\beta}(x-y) = \theta(x_0 - y_0)\langle O(\beta) \mid \phi^{\alpha}(x)\phi^{\beta\dagger}(y) \mid O(\beta)\rangle \mp \theta(x_0 - y_0)\langle O(\beta) \mid \phi^{\beta\dagger}(y)\phi^{\alpha}(x) \mid O(\beta)\rangle \tag{2.18}$$

where $\theta(x)$ is the heavisid function and $\mid O(\beta)\rangle$ is the vaccum at finite temperature. The momentum representation of the green function is :

$$G^{\alpha\beta}(x-y) = \frac{i}{(2\pi)^4}\int d^4p \exp(-ip(x-y))\, G^{\alpha\beta}(p). \tag{2.19}$$

Most of the material of this section rely on references [6],[7],[8]. Further developments could be found in references [1], [2] ,[3],[4].

We turn our attention to the Heisenbeg equation. Let $H(\psi)$ denote the Hamiltonian of the system which consists of an interacting Heisenberg field ψ. Then, $\tilde{H}$ is obtained from H by the tild operation

$$\tilde{H} = H^*\left(\tilde{\psi}^*\right).$$

The total Hamiltonian $\hat{H}$ is given by

$$\hat{H} = H - \tilde{H}.$$

The canonical equation

$$i\hbar\frac{\partial}{\partial t}\psi = [\psi, H]$$

leads to

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi} = -\left[\tilde{\psi}, \tilde{H}\right]$$

which implies that $\hat{H}$ is the Hamiltonian of the fields ψ and $\tilde{\psi}$. When $\pounds(\psi)$ is the Lagrangian which, throught the canonical formalism, leads to the Hamiltonian H, then $\tilde{\pounds} = \pounds^*\left(\tilde{\psi}^*\right)$ is the lagrangian which leads to the Hamiltonian $\tilde{H}$. The total lagrangian is : $\hat{\pounds} = \pounds - \tilde{\pounds}$

When the system contains a large number of particles described by the field ψ, $\pounds$ has the form $\pounds = \pounds_s + \mu N$ where μ is the chemical potential and N is the number operator of the original particles. Thus, the Hamiltonian H also contains the chemical potential term (μN), and the energy is affected by temperature. Let's consider some simple examples and examine the Heisenberg equation of free fields at finite temperature. Suppose a free field φ satisfying $\Lambda(\partial)\varphi = 0$, where the operator $\Lambda(\partial)$ appears to have three forms. Therefore, we distinguish three cases of free fields.

Fermion field. For the fermion field, the Heisenberg operator is

$$\Lambda(\partial) = i\frac{\partial}{\partial t} - \varpi(-i\bigtriangledown)$$

and the doublet representation of the fields is

$$\begin{pmatrix} \phi^1(x) \\ \phi^2(x) \end{pmatrix} = \begin{pmatrix} \phi(x) \\ \tilde{\phi}^\dagger(x) \end{pmatrix}$$

where

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p \left\{ \alpha(p)\, c(\varpi(p)) + \tilde{\alpha}^\dagger(p)\, d(\varpi(p)) \right\} \exp(-i\varpi(p)t + ipx)$$

$$\tilde{\phi}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p \left\{ \alpha(p)\, c(\varpi(p)) - \alpha^\dagger(p)\, d(\varpi(p)) \right\} \exp(i\varpi(p)t - ipx)$$

the annihialtion and creation operators satisfy the following commutation relations

$$\left\{ \alpha(p), \alpha^\dagger(p') \right\} = \left\{ \tilde{\alpha}(p), \tilde{\alpha}^\dagger(p') \right\} = \delta(p - p')$$

The operators $\alpha(p)$ and $\tilde{\alpha}(p)$ are related to $a(\beta, p)$ and $\tilde{a}(\beta, p)$ throught the Bogoliubov transformation. The energy $\varpi(p)$ is positive definite for bosons. For fermions, it could be negative for certain values of $\mathbf{p}$

$$\alpha(p) = a(\beta, p)\,\theta(\varpi(p)) - \tilde{b}(\beta, p)\,\theta(-\varpi(p))$$

$$\tilde{\alpha}^\dagger(p) = \tilde{a}^\dagger(\beta, p)\,\theta(\varpi(p)) + b^\dagger(\beta, p)\,\theta(-\varpi(p)).$$

The causal green function is given by

$$S_c^{\alpha\beta}(p) = \begin{pmatrix} S_c^{11}(p) & S_c^{12}(p) \\ S_c^{21}(p) & S_c^{22}(p) \end{pmatrix}$$

A remarquable fact is that the last equation can be cast in the following simple form

$$S_c^{\alpha\beta}(p) = U(\varpi(p)) \begin{pmatrix} \frac{1}{p_0 - \varpi(p) + i\delta} & 0 \\ 0 & \frac{1}{p_0 - \varpi(p) - i\delta} \end{pmatrix} U^\dagger(\varpi(p)) \qquad (2.20)$$

where the elements of the matrix are

$$S_c^{11}(p) = \frac{c^2(\varpi(p))}{p_0 - \varpi(p) + i\delta} + \frac{d^2(\varpi(p))}{p_0 - \varpi(p) - i\delta},$$

$$S_c^{12}(p) = \frac{-c(\varpi(p))\, d(\varpi(p))}{p_0 - \varpi(p) + i\delta} + \frac{c(\varpi(p))\, d(\varpi(p))}{p_0 - \varpi(p) - i\delta}$$

$$S_c^{21}(p) = \frac{-c(\varpi(p))\, d(\varpi(p))}{p_0 - \varpi(p) + i\delta} + \frac{c(\varpi(p))\, d(\varpi(p))}{p_0 - \varpi(p) - i\delta},$$

$$S_c^{22}(p) = \frac{d^2(\varpi(p))}{p_0 - \varpi(p) + i\delta} + \frac{c^2(\varpi(p))}{p_0 - \varpi(p) - i\delta}$$

and

$$U(\varpi(p)) = \begin{pmatrix} c(\varpi(p)) & d(\varpi(p)) \\ -d(\varpi(p)) & c(\varpi(p)) \end{pmatrix} \tag{2.21}$$

$c(\varpi(p))$ and $d(\varpi(p))$ satisfy Fermi-Dirac statistics

$$c^2(\varpi(p)) = \frac{e^{\beta\varpi(p)}}{e^{\beta\varpi(p)}+1}, d^2(\varpi(p)) = \frac{1}{e^{\beta\varpi(p)}+1} \tag{2.22}$$

Real Boson field. In the case of a boson field of real type, the Heisenberg operator becomes

$$\Lambda(\partial) = i\frac{\partial}{\partial t} - \varpi(-i\bigtriangledown)\, with, \varpi(\vec{p}) \succ 0, \eta_F = 1.$$

and we introduce the same notation as before for the doublet field :

$$\begin{pmatrix} \phi^1(x) \\ \phi^2(x) \end{pmatrix} = \begin{pmatrix} \phi(x) \\ \tilde{\phi}^\dagger(x) \end{pmatrix}.$$

Under the bogoliubov transformation , the fields are

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p \left\{ c_B(\varpi(p))\, \alpha(p) + d_B(\varpi(p))\, \tilde{\alpha}^\dagger(p) \right\} \exp(-i\varpi(p)t + ipx)$$

$$\tilde{\phi}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3p \left\{ c_B(\varpi(p))\, \alpha(p) + d_B(\varpi(p))\, \alpha^\dagger(p) \right\} \exp(i\varpi(p)t - ipx)$$

where $\varpi(p)$ is positive definite as in the first case.The causal green function is given by :

$$\triangle_c^{\alpha\beta}(p) = \begin{pmatrix} \triangle_c^{11}(p) & \triangle_c^{12}(p) \\ \triangle_c^{21}(p) & \triangle_c^{22}(p) \end{pmatrix}$$

where the matrix elements are

$$\triangle_c^{11}(p) = \frac{c_B^2(\varpi(p))}{p_0 - \varpi(p) + i\delta} - \frac{d_B^2(\varpi(p))}{p_0 - \varpi(p) - i\delta}$$

$$\triangle_c^{12}(p) = \frac{c_B(\varpi(p))\, d_B(\varpi(p))}{p_0 - \varpi(p) + i\delta} - \frac{c_B(\varpi(p))\, d_B(\varpi(p))}{p_0 - \varpi(p) - i\delta}$$

$$\triangle_c^{21}(p) = \frac{c_B(\varpi(p))\, d_B(\varpi(p))}{p_0 - \varpi(p) + i\delta} - \frac{c_B(\varpi(p))\, d_B(\varpi(p))}{p_0 - \varpi(p) - i\delta}$$

$$\triangle_c^{22}(p) = \frac{d_B^2(\varpi(p))}{p_0 - \varpi(p) + i\delta} - \frac{c_B^2(\varpi(p))}{p_0 - \varpi(p) - i\delta}$$

and the causal green function can be put in a simple form throught the Bogoliubov transformation

$$\triangle_c^{\alpha\beta}(p) = U_B(\varpi(p)) \begin{pmatrix} \frac{1}{p_0-\varpi(p)+i\delta} & 0 \\ 0 & \frac{-1}{p_0-\varpi(p)-i\delta} \end{pmatrix} U_B^{\dagger}(\varpi(p)) \tag{2.23}$$

where

$$U_B(\varpi(p)) = \begin{pmatrix} c_B(\varpi(p)) & d_B(\varpi(p)) \\ d_B(\varpi(p)) & c_B(\varpi(p)) \end{pmatrix}. \tag{2.24}$$

$c_B(\varpi(p))$, and $d_B(\varpi(p))$ are related to Bose-einstein statistics throught

$$c_B^2(\varpi(p)) = \frac{e^{\beta\varpi(p)}}{e^{\beta\varpi(p)} - 1}, d_B^2(\varpi(p)) = \frac{1}{e^{\beta\varpi(p)} - 1}. \tag{2.25}$$

Complex Boson field. In this case, the free bosonic field is complex, therefore the energy $\varpi(p)$ could be either positive definite or negative and the Heisenberg operator becomes

$$\Lambda(\partial) = \left(i\frac{\partial}{\partial t}\right)^2 - \varpi^2(-i\triangledown).$$

We suppose real fields for simplicity

$$\phi^{\dagger}(x) = \phi(x).$$

Hence

$$\phi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3p}{(2\varpi(p))^{\frac{1}{2}}} \left\{ b(p) \exp(-i\varpi(p)t + ipx) + b^{\dagger}(p) \exp(i\varpi(p)t - ipx) \right\}$$

$$\tilde{\phi}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3p}{(2\varpi(p))^{\frac{1}{2}}} \left\{ \tilde{b}(p) \exp(i\varpi(p)t - ipx) + \tilde{b}^{\dagger}(p) \exp(-i\varpi(p)t + ipx) \right\}$$

where the physical operators $b(p)$ and $\tilde{b}(p)$ transform throught the Bogoliubov transformation with thermal operators $B(p)$ and $\tilde{B}(p)$

$$b(p) = c_B(\varpi(p))\, B(p) + d_B(\varpi(p))\, \tilde{B}^{\dagger}(p)$$

$$\tilde{b}(p) = c_B(\varpi(p))\tilde{B}(p) + d_B(\varpi(p))B^{\dagger}(p),$$

and the anticommutation relations for the annihilation and creation operators are,

$$\left\{B(p), B^{\dagger}(p')\right\} = \left\{\tilde{B}(p), \tilde{B}^{\dagger}(p')\right\} = \delta(p - p').$$

We introduce the column vector notation

$$\begin{pmatrix} \phi^1(x) \\ \phi^2(x) \end{pmatrix} = \begin{pmatrix} \phi(x) \\ \tilde{\phi}(x) \end{pmatrix}$$

therefore the causal green function is given in a simple form via Bogoliubov tansformation as

$$\triangle_c^{\alpha\beta}(p) = U_B(\varpi(p))\begin{pmatrix} \frac{1}{p_0^2-(\varpi(p)-i\delta)^2} & 0 \\ 0 & \frac{-1}{p_0^2-(\varpi(p)+i\delta)^2} \end{pmatrix} U_B^{\dagger}(\varpi(p)) \tag{2.26}$$

where ,

$$U_B(\varpi(p)) = \begin{pmatrix} c_B(\varpi(p)) & d_B(\varpi(p)) \\ d_B(\varpi(p)) & c_B(\varpi(p)) \end{pmatrix} \tag{2.27}$$

and $c_B(\varpi(p))$ and $d_B(\varpi(p))$ satisfy Bose-einstein statistics

$$c_B^2(\varpi(p)) = \frac{e^{\beta\varpi(p)}}{e^{\beta\varpi(p)} - 1}, d_B^2(\varpi(p)) = \frac{1}{e^{\beta\varpi(p)} - 1}. \tag{2.28}$$

In QCD, It should be pointed out that it is convenient to rewrite the standard propagators in TFD, $\triangle(k)$ for bosons (gluons) and $S(p)$ for fermions (quarks) , as follows [7]

for bosons a matrix form could be

$$\triangle(k) = \begin{pmatrix} \frac{g_{\mu\nu}}{k^2+i\epsilon} - \frac{\left(1-\lambda^{-1}\right)k_\mu k_\nu}{\left(k^2+i\epsilon\right)^2} & 0 \\ 0 & -\frac{g_{\mu\nu}}{k^2-i\epsilon} + \frac{\left(1-\lambda^{-1}\right)k_\mu k_\nu}{\left(k^2-i\epsilon\right)^2} \end{pmatrix}$$

$$-2\pi i\delta\left(k^2\right)\frac{g_{\mu\nu}}{e^{\beta|k_0|} - 1}\begin{pmatrix} 1 & e^{\frac{\beta|k_0|}{2}} \\ e^{\frac{\beta|k_0|}{2}} & 1 \end{pmatrix}$$

which could be put in a simple form throught Bogoliubov transformation

$$\triangle(k) = U_B(k_0, \beta)\bar{\triangle}(k)U_B^{\dagger}(k_0, \beta) \tag{2.29}$$

where

$$\bar{\triangle}(k) = \frac{\tau}{(k_0 + i\delta\tau)^2 - \vec{k}^2}\left[g_{\mu\nu} - \frac{\left(1-\lambda^{-1}\right)k_\mu k_\nu}{(k_0 + i\delta\tau)^2 - \vec{k}^2}\right], \tag{2.30}$$

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$U_B(k_0,\beta) = \begin{pmatrix} \cosh\theta(k_0,\beta) & \sinh\theta(k_0,\beta) \\ \sinh\theta(k_0,\beta) & \cosh\theta(k_0,\beta) \end{pmatrix}, \tag{2.31}$$

the thermal Bose-Einstein distribution is

$$\sinh^2\theta(k_0,\beta) = \frac{1}{e^{\beta|k_0|}-1}, \tag{2.32}$$

$\beta = \frac{1}{k_B T}$, k_B and T are the Boltzmann constant and the temperature of the quark gluon system respectively;

for quarks The Green function can be obtained as

$$S(p) = (\bar{p}+m)\left[\begin{pmatrix} \frac{1}{p^2-m^2+i\epsilon} & 0 \\ 0 & \frac{1}{p^2-m^2-i\epsilon} \end{pmatrix}\right.$$

$$\left. +2\pi i\delta\left(p^2-m^2\right)\frac{1}{e^{\beta|p_0|}+1}\begin{pmatrix} 1 & -\epsilon(p_0)\,e^{\frac{\beta|p_0|}{2}} \\ -\epsilon(p_0)\,e^{\frac{\beta|p_0|}{2}} & 1 \end{pmatrix}\right]$$

which can be simplified throught the Bogoliubov transformation as follows

$$S(p) = U_F(\vec{p},\beta)\,\bar{S}(p)\,U_F^\dagger(\vec{p},\beta) \tag{2.33}$$

where

$$\bar{S}(p) = \frac{I}{\bar{p}-m+i\delta\tau};\;\; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$U_F(\vec{p},\beta) = \begin{pmatrix} \cos\theta(|\vec{p}|,\beta) & \epsilon(p_0)\sin\theta(|\vec{p}|,\beta) \\ -\epsilon(p_0)\sin\theta(|\vec{p}|,\beta) & \cos\theta(|\vec{p}|,\beta) \end{pmatrix}, \tag{2.34}$$

where the thermal Fermi-Dirac distribution is

$$\sin^2\theta(|\vec{p}|,\beta) = \frac{1}{e^{\beta p_0}+1} \tag{2.35}$$

and

$$\epsilon(p_0) = \theta(p_0) - \theta(-p_0) \ , \ \theta(p_0) = \left\{ \begin{array}{ll} 1, & p_0 \succ 0, \\ 0, & p_0 \prec 0. \end{array} \right\}$$

The matrices U_B and U_F are the Bogoliubov transformation matrices for bosons and fermions respectively.

Bibliography

[1] F.C. Khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, A. E. Santana, 'Thermal Quantum Field Theory, Algebraic Aspects and Applications',World Scientific Publishing Company (2009).

[2] A. Das, 'Finite Temperature Field Theory', (W.Scientific, Singapour, 1977)

[3] M. Le Bellac, 'Thermal Field Theory (Cambridge University Press, Cambridge, 1996)

[4] J. I. Kapusta, 'Finite Temperature Field Theory' (Cambridge University Press, Cambridge, 1989).

[5] Y. Takahashi and H. Umezawa, 'Thermofield dynamics',Collective Phenomena, Vol. 2, pp.55-8 (1975).

[6] Y. Takahashi and H. Umezawa, 'Thermo field Dynamics', International journal of modern Physics B, Vol. 10, Nos. 1314 (1996) 1755-1805.

[7] Wanyun Zhao and F.C. Khanna, *"Screening lenght, dispersion relations and quark potentiel in thermofield* dynamics", hep-ph/0011182.

[8] H. Matsumoto, *"The causal function in many body problems"* ,Fortschritte der Physik , 25, 1-36(1977).

3

Thermal Oscillators: Bosons and Fermions

In this section, we present two elementary examples to consider how the thermal vaccum $\mid O(\beta)\rangle$ is constructed. We construct a complete set of orthonormal vectors showing that the doubling of the Hilbert space. The thermal states transform under BOGOLIOUBOV unitary transformations. A matrix notation is introduced for both bosonic and fermionic systems.

3.1 Real scalar field in TFD.

First, we consider the case of a real scalar field a1nd later, we will proceed to fields with spin, in particular the fermion field. This would establish a general procedure to study fields at finite temperature. The case of fields with interactions will be introduced later on.

3.1.1 case of a boson field with frequency ω.

The Hamiltonian of an ensemble of free bosons with frequency ω is

$$H = \omega a^{\dagger} a$$

with the operators a and $a^{\dagger}$ satisfy the relations

$$[a, a^{\dagger}] = 1$$

$$[a, a] = 0.$$

3. THERMAL OSCILLATORS: BOSONS AND FERMIONS

The Fock space is spanned by the set of orthonormal vectors

$$\mid O\rangle, a^{\dagger} \mid O\rangle, \ldots, \frac{1}{\sqrt{n!}}\left(a^{\dagger}\right)^{n} \mid O\rangle, \ldots.$$

where the vacuum state, $\mid O, \tilde{O}\rangle$, is simply denoted by $\mid O\rangle$.The eigenvalues and eigenstates of H are specified by

$$H \mid n\rangle = n\omega \mid n\rangle, n = 0, 1, 2, \ldots, \infty.$$

The system at finite temperature is defined by

$$\hat{H} = H - \tilde{H}$$

Let's introduce the tilde field

$$\tilde{H} = \omega \tilde{a}^{\dagger} \tilde{a}$$

with annihilation and creation operators obeying

$$[\tilde{a}, \tilde{a}^{\dagger}] = 1$$

$$[\tilde{a}, \tilde{a}] = 0$$

$$[a, \tilde{a}] = [a, \tilde{a}^{\dagger}] = 0.$$

It is clear that the scalar field in the usual and in the tilde space are distinct. This is clear from the commutation rules.

3.1.2 Thermal vacuum and Bogoliubov transformation.

Since the Hilbert space is doubled, any bosonic state is represented by the thermal vacuum and the states $\mid n, \tilde{n}\rangle$. The thermal vacuum is spanned by these states as follows

$$\begin{aligned} \mid O \quad (\beta)\rangle &= Z^{-1/2}(\beta) \sum_{n} e^{-\beta E_n/2} \mid n, \tilde{n}\rangle = Z^{-1/2}(\beta) \sum_{n} e^{-\beta n\omega/2} \frac{1}{n!}\left(a^{\dagger}\right)^{n}\left(\tilde{a}^{\dagger}\right)^{n} \mid O\rangle \\ &= \sqrt{(1 - e^{-\beta\omega})} \exp\left(e^{-\beta\omega/2} a^{\dagger} \tilde{a}^{\dagger}\right) \mid O\rangle. \end{aligned} \tag{3.1}$$

Then we use the Boson distribution function, $n_{\mathbf{B}}(\beta)$, to define

$$
\begin{aligned}
u(\beta) &= \left(1-e^{-\beta\omega}\right)^{-1/2} = \sqrt{(1+n_{\mathbf{B}}(\beta)} \\
G_{\mathbf{B}}(\beta) &= -i\theta(\beta)\left(\tilde{a}a - a^{\dagger}\tilde{a}^{\dagger}\right)
\end{aligned}
$$

where $n_{\mathbf{B}}(\beta) = \left(1+e^{\beta\omega}\right)^{-1}$ and the angle $\theta(\beta)$ is defined as

$$
\cos\theta(\beta) = u(\beta).
$$

Then, we write relation (3.1) as

$$
\mid O(\beta)\rangle = u^{-1}(\beta)\exp\left(\frac{v(\beta)}{u(\beta)}a^{\dagger}\tilde{a}^{\dagger}\right)\mid O\rangle = e^{-iG_{\mathbf{B}}(\beta)}\mid O\rangle. \tag{3.2}
$$

Hence, the unitary operator, transforming $\mid O,\tilde{O}\rangle$ into $\mid O(\beta)\rangle$ is given by

$$
U(\beta) = e^{-iG_{\mathbf{B}}(\beta)}. \tag{3.3}
$$

The operator $U(\beta)$ is called the *Bogoliubov transformation.*

3.1.3 Thermal operators for the system.

The temperature dependent operators are defined in terms of the zero temperature creation $\left(a^{\dagger},\tilde{a}^{\dagger}\right)$ and annihilation $(a,\tilde{a})$ as follows

$$
\begin{aligned}
a(\beta) &= e^{-iG_{\mathbf{B}}(\beta)}ae^{iG_{\mathbf{B}}(\beta)} = u(\beta)a - v(\beta)\tilde{a}^{\dagger} \\
\tilde{a}(\beta) &= e^{-iG_{\mathbf{B}}(\beta)}\tilde{a}e^{iG_{\mathbf{B}}(\beta)} = u(\beta)\tilde{a} - v(\beta)a^{\dagger} \\
a^{\dagger}(\beta) &= e^{iG_{\mathbf{B}}(\beta)}a^{\dagger}e^{-iG_{\mathbf{B}}(\beta)} = u(\beta)a^{\dagger} - v(\beta)\tilde{a} \\
\tilde{a}^{\dagger}(\beta) &= e^{-iG_{\mathbf{B}}(\beta)}\tilde{a}^{\dagger}e^{iG_{\mathbf{B}}(\beta)} = u(\beta)\tilde{a}^{\dagger} - v(\beta)a
\end{aligned} \tag{3.4}
$$

Then, the non-thermal operators a and $a^{\dagger}$ are written in terms of the thermal ones by inverting these relations to set

$$
\begin{aligned}
a &= u(\beta)a(\beta) + v(\beta)\tilde{a}^{\dagger}(\beta). \\
\tilde{a} &= u(\beta)\tilde{a}(\beta) + v(\beta)a^{\dagger}(\beta), \\
a^{\dagger} &= u(\beta)a^{\dagger}(\beta) + v(\beta)\tilde{a}(\beta), \\
\tilde{a}^{\dagger} &= u(\beta)\tilde{a}^{\dagger}(\beta) + v(\beta)a(\beta).
\end{aligned} \tag{3.5}
$$

Such relations are useful in expressing the zero temperature operators in terms of the temperature dependent operators. The physical observables are written in terms of a and $a^{\dagger}$ and it is then convenient to write these operators in terms of thermal ones to get temperature dependent expressions and relations.

The relations for annihilation operators at finite temperature to the vacuum state $\mid O(\beta)\rangle$ are

$$a(\beta) \mid O(\beta)\rangle = \tilde{a}(\beta) \mid O(\beta)\rangle = 0.$$

The number operator gives the statistical average

$$\langle O(\beta) \mid a^{\dagger} a \mid O(\beta)\rangle = v^{2}(\beta) = \frac{1}{(e^{\beta\omega}-1)} = n_{\mathbf{B}}(\omega)$$

which is the Bose-Einstein distribution. The Fock space is spanned by

$$\mid O(\beta)\rangle, a^{\dagger}(\beta) \mid O(\beta)\rangle, \tilde{a}^{\dagger}(\beta) \mid O(\beta)\rangle, \ldots, \frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}\left(a^{\dagger}(\beta)\right)^{n}\left(\tilde{a}^{\dagger}(\beta)\right)^{m} \mid O(\beta)\rangle, \ldots .$$

3.1.4 Matrix notation. Commutation relations.

A matrix notation can be introduced by writing the operators as

$$\begin{pmatrix} a(\beta) \\ \tilde{a}^{\dagger}(\beta) \end{pmatrix} = B(\beta)\begin{pmatrix} a \\ \tilde{a}^{\dagger} \end{pmatrix},$$

where

$$B(\beta) = \begin{pmatrix} u(\beta) & -v(\beta) \\ -v(\beta) & u(\beta) \end{pmatrix}$$

and

$$u(\beta) = \cosh\theta(\beta) = \frac{1}{\sqrt{(1-e^{-\beta\omega})}}, v(\beta) = \sinh\theta(\beta) = \frac{e^{-\beta\omega/2}}{\sqrt{(1-e^{-\beta\omega})}}$$

with

$$u^{2}(\beta) - v^{2}(\beta) = \cosh^{2}\theta(\beta) - \sinh^{2}\theta(\beta) = 1.$$

Since this transformation is unitary, the algebra of the original operators a and $\tilde{a}$ is kept invariant, i.e. the thermal operators $a(\beta)$ and $\tilde{a}(\beta)$ satisfy the following commutation relations

$$\left[a(\beta), a^{\dagger}(\beta)\right] = 1, \left[\tilde{a}(\beta), \tilde{a}^{\dagger}(\beta)\right] = 1, \tag{3.6}$$

with all other commutation relations being zero.

3.2 Dirac field in TFD.

Now we turn to a study of the Dirac field. The procedure to obtain temperature dependent operators and the Fock space are similar. The notion of spin has to be considered in order for the $T = 0$ and $T \neq 0$ theories are defined consistently.

3.2.1 Case of a fermion field with frequency ω.

The Hamiltonian of an ensemble of free fermions with frequency ω is defined by

$$H = \omega a^{\dagger} a$$

where the operators a and $a^{\dagger}$ satisfy the algebra

$$\left\{a, a^{\dagger}\right\} = 1, \left\{a^{\dagger}, a^{\dagger}\right\} = \{a, a\} = 0.$$

where $\{A, B\} = AB + BA$ is the anti-commutator. Using last equations and the fact that $\langle n \mid n\rangle = 1$, it can be shown that $n = 0, 1$. Further, if we introduce the tilde Hilbert space and the corresponding operators, we get

$$\tilde{H} = \omega \tilde{a}^{\dagger} \tilde{a}$$

with

$$\left\{\tilde{a}, \tilde{a}^{\dagger}\right\} = 1, \{\tilde{a}, \tilde{a}\} = \{a, \tilde{a}\} = \left\{a, \tilde{a}^{\dagger}\right\} = 0.$$

Again the creation and annihilation operators in the normal and tilde Hilbert space are distinct and they anticommute. The Fock space will be spanned by the set of orthonormal vectors

$$\mid O\rangle, a^{\dagger} \mid O\rangle, \tilde{a}^{\dagger} \mid O\rangle, a^{\dagger}\tilde{a}^{\dagger} \mid O\rangle,$$

where $\mid O\rangle$ denotes the state $\mid O, \tilde{O}\rangle$.The energy eigenvalues of H are then

$$H \mid n\rangle = \epsilon_n \mid n\rangle = n\omega \mid n\rangle, n = 0, 1.$$

The system at finite temperature is defined by the Hamiltonian, $\hat{H} = H - \tilde{H}$. The commutation relations will be useful to defining matrix elements at finite temperature.

3.2.2 Thermal vacuum and Bogoliubov transformation.

In order to construct a system at finite temperature, the Hilbert space is doubled. Any fermionic state is then represented by the basis

$$\mid O, \tilde{O}\rangle, \tilde{a}^{\dagger} \mid O\rangle =\mid O, \tilde{1}\rangle, a^{\dagger} \mid O\rangle =\mid 1, \tilde{O}\rangle, a^{\dagger}\tilde{a}^{\dagger} \mid O\rangle =\mid 1, \tilde{1}\rangle.$$

The thermal vaccum can be spanned by these states as follow

$$\begin{aligned} \mid \quad O\left(\beta\right)\rangle &= Z^{-1/2}\left(\beta\right)\sum_{n} e^{-\beta\epsilon_{n}/2} \mid n, \tilde{n}\rangle = \frac{1}{\sqrt{Z\left(\beta\right)}}\sum_{n}\left(\mid O, \tilde{O}\rangle + e^{-\beta\epsilon_{1}/2} \mid 1, \tilde{1}\rangle\right) \\ &= \frac{1}{\sqrt{Z\left(\beta\right)}}\left(1 + e^{-\beta\omega/2}a^{\dagger}\tilde{a}^{\dagger}\right) \mid O\rangle. \end{aligned} \tag{3.7}$$

We obtain $\mid O\left(\beta\right)\rangle$ from $\mid O\rangle$ by a *Bogoliuobov transformation*. Indeed, If we denote

$$\begin{aligned} \cosh\theta\left(\beta\right) &= u\left(\beta\right) = \left(1 + e^{-\beta\omega}\right)^{-1/2} = \sqrt{\left(1 - n_{\mathbf{F}}\left(\omega\right)\right)}. \\ \sinh\theta\left(\beta\right) &= v\left(\beta\right) = \left(1 + e^{\beta\omega}\right)^{-1/2} = \sqrt{n_{\mathbf{F}}\left(\omega\right)}, \\ G_{\mathbf{B}}\left(\beta\right) &= -i\theta\left(\beta\right)\left(\tilde{a}a - a^{\dagger}\tilde{a}^{\dagger}\right), \end{aligned}$$

where $n_{\mathbf{F}}\left(\omega\right)$ is the fermion distribution function, $n_{\mathbf{F}}\left(\omega\right) = \left[e^{\beta\omega} + 1\right]^{-1}$.Then we find that

$$u^{2}\left(\beta\right) + v^{2}\left(\beta\right) = 1$$

$$\mid O\left(\beta\right)\rangle = \left\{u\left(\beta\right) + v\left(\beta\right)a^{\dagger}\tilde{a}^{\dagger}\right\} \mid O\rangle$$

then, we can write the relation (3.7) in a canonical form as

$$\mid O\left(\beta\right)\rangle = e^{-iG_{\mathbf{F}}\left(\beta\right)} \mid O\rangle = \left(\cos\theta\left(\beta\right) + \sin\theta\left(\beta\right)a^{\dagger}\tilde{a}^{\dagger}\right) \mid O\rangle. \tag{3.8}$$

where

$$G_{\mathbf{F}}\left(\beta\right) = -i\theta\left(\beta\right)\left(\tilde{a}a - a^{\dagger}\tilde{a}^{\dagger}\right)$$

and $G_{\mathbf{F}}\left(\beta\right)$ defines the Bogoliubov transformation as

$$U_{\mathbf{F}}\left(\beta\right) = e^{G_{\mathbf{F}}\left(\beta\right)}.$$

3.2.3 Thermal operators for the system.

The thermal operators are defined in terms of the zero temperature operators, as in the boson case, by using the Bogoliubov transformation as

$$\begin{aligned}
a(\beta) &= e^{-iG_{\mathbf{F}}(\beta)} a e^{iG_{\mathbf{F}}(\beta)} = u(\beta)\, a - v(\beta)\, \tilde{a}^{\dagger} \\
\tilde{a}(\beta) &= e^{-iG_{\mathbf{F}}(\beta)} \tilde{a} e^{iG_{\mathbf{F}}(\beta)} = u(\beta)\, \tilde{a} + v(\beta)\, a^{\dagger} \\
a^{\dagger}(\beta) &= e^{iG_{\mathbf{F}}(\beta)} a^{\dagger} e^{-iG_{\mathbf{F}}(\beta)} = u(\beta)\, a^{\dagger} - v(\beta)\, \tilde{a} \\
\tilde{a}^{\dagger}(\beta) &= e^{-iG_{\mathbf{F}}(\beta)} \tilde{a}^{\dagger} e^{iG_{\mathbf{F}}(\beta)} = u(\beta)\, \tilde{a}^{\dagger} + v(\beta)\, a
\end{aligned} \tag{3.9}$$

Non-thermal operators a and $a^{\dagger}$ are obtained from the thermal ones by inverting these relations

$$\begin{aligned}
a &= u(\beta)\, a(\beta) + v(\beta)\, \tilde{a}^{\dagger}(\beta). \\
\tilde{a} &= u(\beta)\, \tilde{a}(\beta) - v(\beta)\, a^{\dagger}(\beta), \\
a^{\dagger} &= u(\beta)\, a^{\dagger}(\beta) + v(\beta)\, \tilde{a}(\beta), \\
\tilde{a}^{\dagger} &= u(\beta)\, \tilde{a}^{\dagger}(\beta) - v(\beta)\, a(\beta).
\end{aligned} \tag{3.10}$$

Often, as for the bosonic case, physical observables are written in terms of $T = 0$ operators a and $a^{\dagger}$, and it is then convenient to write them in terms of thermal ones to get the temperature dependent expressions.

The fermionic annihilation operators acting on the thermal vaccum give

$$a(\beta) \mid O(\beta)\rangle = \tilde{a}(\beta) \mid O(\beta)\rangle = 0.$$

The number operator gives the statistical average

$$\begin{aligned}
\langle O(\beta) &\mid N \mid O(\beta)\rangle = \langle O(\beta) \mid a^{\dagger} a \mid O(\beta)\rangle \\
&= \frac{1}{(1+e^{-\beta\omega})} \langle O, \tilde{O} \mid \left(1 + e^{-\beta\omega/2} \tilde{a} a\right) a^{\dagger} a \left(1 + e^{-\beta\omega/2} a^{\dagger} \tilde{a}^{\dagger}\right) \mid O, \tilde{O}\rangle \\
&= \frac{1}{(1+e^{\beta\omega})} = n_{\mathbf{F}}(\omega)
\end{aligned}$$

which is the Fermi-Dirac distribution. The Fock space is constructed from the vacuum $\mid O(\beta)\rangle$ and is spanned by the set of states

$$\left\{ \mid O(\beta)\rangle, a^{\dagger}(\beta) \mid O(\beta)\rangle, \tilde{a}^{\dagger}(\beta) \mid O(\beta)\rangle, a^{\dagger}(\beta)\, \tilde{a}^{\dagger}(\beta) \mid O(\beta)\rangle \right\}.$$

3.2.4 Matrix notation. Commutation relations.

A matrix notation is introduced from the equations (3.9) and (3.10) and is written as

$$\begin{pmatrix} a(\beta) \\ \tilde{a}^{\dagger}(\beta) \end{pmatrix} = B(\beta) \begin{pmatrix} a \\ \tilde{a}^{\dagger} \end{pmatrix},$$

where

$$B(\beta) = \begin{pmatrix} u(\beta) & -v(\beta) \\ v(\beta) & u(\beta) \end{pmatrix}$$

and

$$u(\beta) = \cos\theta(\beta) = \frac{1}{\sqrt{(1+e^{-\beta\omega})}}, v(\beta) = \sin\theta(\beta) = \frac{1}{\sqrt{(1+e^{\beta\omega})}}$$

with

$$u^2(\beta) + v^2(\beta) = \cos^2\theta(\beta) + \sin^2\theta(\beta) = 1.$$

Since this transformation is unitary, the algebra of the original operators a and $\tilde{a}$ is kept invariant, that is the thermal operators $a(\beta)$ and $\tilde{a}(\beta)$ satisfy the following anticommutation relations

$$\left\{a(\beta), a^{\dagger}(\beta)\right\} = 1, \left\{\tilde{a}(\beta), \tilde{a}^{\dagger}(\beta)\right\} = 1, \tag{3.11}$$

with all other commutation relations being zero.

3.3 Thermalization of a bosonic and fermionic oscillator using the Bogoliubov transformation

Let's recall that TFD is a real time formalism. To describe a statistical system at equilibrium, we need to introduce the partition function of the system given as

$$Z(\beta) = Tr\left(e^{-\beta\aleph}\right)$$

where, $\aleph$ is the hamiltonian of the system. MATSUBARA has shown that the statistical average of an operator is the same as the mean value of this operator in the vaccum at finite temperature If we define the mean value of an operator as

$$\langle A\rangle_{\beta} = \langle O(\beta) \mid A \mid O(\beta)\rangle = Z^{-1}(\beta)\sum_{n} e^{-\beta E_n}\langle n \mid A \mid n\rangle$$

where $\mid n\rangle$ is a specific state corresponding to the energy E_n.Then the finite temperature formalism will be completely parallel to the zero temperature field theory. The Lagrange formalism may be set up for the total system involving the physical and tilda fields [1],[2],[3],[4],[5],6]. If we introduce the free fields by

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} e^{-i\epsilon_k t} a_{\mathbf{k}}$$

$$\tilde{\psi}(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}} e^{i\epsilon_k t} \tilde{a}_{\mathbf{k}}, \left(\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m\right)$$

the Hamiltonian $\hat{H}$ becomes

$$\hat{H} = H - \tilde{H}$$

with

$$\begin{aligned} H &= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \\ \tilde{H} &= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}} \end{aligned}$$

and k is the the momentum of the oscillator. Then we observe that the total hamiltonian $\hat{H} = H - \tilde{H}$ is invariant under the *bogoliubov transformation*

$$\begin{aligned} a_k \longrightarrow a_k(\theta) &= a_k \cos\theta_k - \tilde{a}_k^\dagger \sin\theta_k \\ \tilde{a}_k \longrightarrow \tilde{a}_k(\theta) &= \tilde{a}_k \cos\theta_k + a_k^\dagger \sin\theta_k \end{aligned} \tag{3.12}$$

for fermions, and

$$\begin{aligned} a_k \longrightarrow a_k(\theta) &= a_k \cosh\theta_k - \tilde{a}_k^\dagger \sinh\theta_k \\ \tilde{a}_k \longrightarrow \tilde{a}_k(\theta) &= \tilde{a}_k \cosh\theta_k - a_k^\dagger \sinh\theta_k \end{aligned} \tag{3.13}$$

for bosons. These relations are written in a matrix form as

$$\begin{pmatrix} a_k(\beta) \\ \tilde{a}_k^\dagger(\beta) \end{pmatrix} = U(\theta_k) \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} = \begin{pmatrix} \cos\theta_k(\beta) & \text{-}\sin\theta_k(\beta) \\ \sin\theta_k(\beta) & \cos\theta_k(\beta) \end{pmatrix} \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} \tag{3.14}$$

for fermions, and

$$\begin{pmatrix} a_k(\beta) \\ \tilde{a}_k^\dagger(\beta) \end{pmatrix} = U(\theta_k) \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\theta_k(\beta) & \text{-}\sinh\theta_k(\beta) \\ \text{-}\sinh\theta_k(\beta) & \cosh\theta_k(\beta) \end{pmatrix} \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} \tag{3.15}$$

for bosons where $U(\theta_k)$ is a unitary transfomation and,

$$\sinh^2\theta_k = n_B(k) = \frac{1}{\left(e^{\beta|k_0|}-1\right)}\,,\; \sin^2\theta_k = n_F(k) = \frac{1}{\left(1+e^{\beta|k_0|}\right)}$$

being bose and fermion distributions, k_0 being the energy associated with the four-vector k.The generator of the transformation can easily be found to be

$$\hat{G} = \sum_{\mathbf{k}} i_{\mathbf{k}}\theta_{\mathbf{k}}\left\{a^\dagger_{\mathbf{k}}\tilde{a}^\dagger_{\mathbf{k}} - \tilde{a}_{\mathbf{k}}a_{\mathbf{k}}\right\},$$

where, as stated before for the one particle oscillator

$$\begin{aligned} a_{\mathbf{k}}(\theta) &= e^{-i\hat{G}}a_{\mathbf{k}}e^{i\hat{G}}, \\ \tilde{a}_{\mathbf{k}}(\theta) &= e^{-i\hat{G}}\tilde{a}_{\mathbf{k}}e^{i\hat{G}} \end{aligned} \tag{3.16}$$

the transformation (3.12) and (3.13) leave the total Hamiltonian invariant, the generator $\hat{G}$ is conserved

$$[\hat{G},\hat{H}] = [\hat{G}, H - \tilde{H}] = 0$$

The freedom of this transformation allows us to write the transformed Fock space for the vacuum as

$$\mid O(\theta)\rangle = e^{-i\hat{G}} \mid O\rangle$$

and the one particle states

$$\begin{aligned} a^\dagger_{\mathbf{k}}(\theta) &\mid O(\theta)\rangle = e^{-i\hat{G}}a^\dagger_{\mathbf{k}} \mid O\rangle \\ \tilde{a}^\dagger_{\mathbf{k}}(\theta) &\mid O(\theta)\rangle = e^{-i\hat{G}}\tilde{a}^\dagger_{\mathbf{k}} \mid O\rangle \end{aligned}$$

and so on. Using the explicit form of $\hat{G}$, and the property

$$a_{\mathbf{k}} \mid O\rangle = \tilde{a}_{\mathbf{k}} \mid O\rangle = 0$$

The temperature dependent vaccum is introduced using a bogoliubov transformation, then it can shown [1],[2],[3],[4],[5],[6] that

$$\mid O(\theta)\rangle = \prod_k \left(\cos\theta_k + \sin\theta_k a^\dagger_k\tilde{a}^\dagger_k\right) \mid O\rangle \tag{3.17}$$

for fermions and

$$\mid O(\theta)\rangle = \prod_k \left(\frac{1}{\cosh\theta_k}\right)\exp\left(\tanh\theta_k a^\dagger_k\tilde{a}^\dagger_k\right) \mid O\rangle \tag{3.18}$$

for bosons. As for the one particle system, from equation (3.16), we have

$$a_{\mathbf{k}}(\theta) \mid O(\theta)\rangle = \tilde{a}_{\mathbf{k}}(\theta) \mid O(\theta)\rangle = 0.$$

The new vacuum has many properties. After some algebra, we obtain

$$a_{\mathbf{k}}^{\dagger}(\theta) \mid O(\theta)\rangle = \left(\frac{1}{\cos\theta_{\mathbf{k}}}\right) a_{\mathbf{k}}^{\dagger} \mid O(\theta)\rangle = -\left(\frac{1}{\sin\theta_{\mathbf{k}}}\right) \tilde{a}_{\mathbf{k}} \mid O(\theta)\rangle$$

for fermions, and

$$a_{\mathbf{k}}^{\dagger}(\theta) \mid O(\theta)\rangle = \left(\frac{1}{\cosh\theta_{\mathbf{k}}}\right) a_{\mathbf{k}}^{\dagger} \mid O(\theta)\rangle = \left(\frac{1}{\sinh\theta_{\mathbf{k}}}\right) \tilde{a}_{\mathbf{k}} \mid O(\theta)\rangle$$

for bosons. Similarly, we have

$$\tilde{a}_{\mathbf{k}}^{\dagger}(\theta) \mid O(\theta)\rangle = \left(\frac{1}{\cos\theta_{\mathbf{k}}}\right) \tilde{a}_{\mathbf{k}}^{\dagger} \mid O(\theta)\rangle = \left(\frac{1}{\sin\theta_{\mathbf{k}}}\right) a_{\mathbf{k}} \mid O(\theta)\rangle$$

for fermions, and

$$\tilde{a}_{\mathbf{k}}^{\dagger}(\theta) \mid O(\theta)\rangle = \left(\frac{1}{\cosh\theta_{\mathbf{k}}}\right) \tilde{a}_{\mathbf{k}}^{\dagger} \mid O(\theta)\rangle = \left(\frac{1}{\sinh\theta_{\mathbf{k}}}\right) a_{\mathbf{k}} \mid O(\theta)\rangle$$

for bosons. As seen in the last equations, there are equal number of physical particles and tielda particles (fictitious particles) in the vacuum state $\mid O(\theta)\rangle$.The last four relations show that the addition of one particle without tielda (the physical state) to the vacuum is equivalent to the elimination of one particle with tielda. Therefore, the fictitious state may be interpreted as a hole for the physical state, a matter already discussed earlier in the introduction [1],[2],[3],[4],[5],[6]. Tield operation is introduced and to each zero temperature field $\phi(x)$ an vector column of fields is attached :

$$\begin{pmatrix} \phi(x) \\ \tilde{\phi}^{\dagger}(x) \end{pmatrix}$$

whose dynamics is described by the following thermal hamiltonian :

$$\hat{H} = H(\phi) - \tilde{H}(\phi) = H(\phi) - H^{*}\left(\tilde{\phi}\right)$$

where , $H(\phi) = H(\phi) - \mu N$ and N is the particle number operator.

The temperature dependent vacuum state described above is annihilated by the temperature dependent physical annihilation operators :

$$a_{\mathbf{k}}(\theta) \mid O(\theta)\rangle = \tilde{a}_{\mathbf{k}}(\theta) \mid O(\theta)\rangle = 0.$$

Also, the creation and annihilation operators satisfy ordinary commutation relations

$$\left[a_{\mathbf{k},\mathbf{s}}\left(\theta\right), a^{\dagger}_{\mathbf{k}',\mathbf{s}'}\left(\theta\right)\right]_{\pm} = \delta\left(\vec{k} - \vec{k}'\right)\delta_{ss'} \tag{3.19}$$

$$\left[\tilde{a}_{\mathbf{k},\mathbf{s}}\left(\theta\right), \tilde{a}^{\dagger}_{\mathbf{k}',\mathbf{s}'}\left(\theta\right)\right]_{\pm} = \delta\left(\vec{k} - \vec{k}'\right)\delta_{ss'} \tag{3.20}$$

all other commutators (anticommutators) vanish.

Bibliography

[1] F.C. khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, A. E. Santana, 'Thermal Quantum Field Theory, Algebraic Aspects and Applications',World Scientific Publishing Company (2009).

[2] A. Das, 'Finite Temperature Field Theory', (W.Scientific, Singapour, 1977)

[3] M. Le Bellac, 'Thermal Field Theory (Cambridge University Press, Cambridge, 1996)

[4] J. I. Kapusta, 'Finite Temperature Field Theory' (Cambridge University Press, Cambridge, 1989).

[5] Y. Takahashi and H. Umezawa, 'Thermofield dynamics',Collective Phenomena, Vol. 2, pp.55-8 (1975).

[6] Y. Takahashi and H. Umezawa, 'Thermo field Dynamics', International journal of modern Physics B, Vol. 10, Nos. 1314 (1996) 1755-1805

4

Wick's theorem and Renormalization.

4.1 Wick's theorem.

At non-zero temperature and at thermal equilibrium, we have to take a thermal average rather than an expectation value. In order to define Feynman's rules from the thermal average (see for example [1],[2],[3] for further reading), we need some equivalent of Wick's theorem. We should refer to it as an operator identity rather than a relation between expectation values. But fortunately one can derive the necessary result for thermal averages at equilibrium (see next chapter) : this result will be called the *" thermal Wick's theorem"*. It relates time ordered products of fields to normal ordered products and contractions. At zero temperature, normal ordered products of operators are defined such that annihilation operators are always placed to the right of creation operators. This ensures that vacuum expectation values of normal ordered products $N[\ldots]$ always vanish. However, in thermal field theory, thi is no longer true : ${}_n e^{-n\beta\omega} \langle n \left| a^\dagger a \right| n \rangle \neq 0$. But imposing some conditions on the fields as done by Thouless, one can prove that thermal expectation values of normal ordered products can be set to be zero. In the case of high energy, the split of fields which gives zero normal ordered products at finite temperature is tending to be the same as the traditional $T = 0$ one giving the same behaviour of Wick's theorem in both cases.

Normal ordering is defined in terms of arbitrary split, generalising the traditional $T = 0$ definition which is expressed in terms of annihilation and creation operators. For example, we define for two-point functions regardless of the nature of the fields and volume of the system :

4. WICK'S THEOREM AND RENORMALIZATION.

$$N\left[\psi_1\psi_2\right] := N_{1,2} = \psi_1^+\psi_2^+ + \psi_1^-\psi_2^+ + \sigma\psi_2^-\psi_1^+ + \psi_1^-\psi_2^-$$

for normal ordering and,

$$T\left[\psi_1\psi_2\right] := T_{1,2} = \psi_1\psi_2\theta\left(t_1 - t_2\right) + \sigma\psi_2\psi_1\theta\left(t_2 - t_1\right)$$

for time ordering and,

$$D\left[\psi_1\psi_2\right] := D_{1,2} := T_{1,2} - N_{1,2},$$

for the contraction of two fields where $\sigma = \pm 1$ for either bosonic or fermionic field.

To higher order, Wick's theorem can be obtained from the generating functional :

$$T\left[\exp\left\{-i\int d^4x j_i\left(x\right)\psi_i\left(x\right)\right\}\right] = N\left[\exp\left\{-i\int d^4x j_i\left(x\right)\psi_i\left(x\right)\right\}\right]\times$$

$$\times\exp\left\{-\frac{1}{2}\int d^4x d^4y j_i\left(x\right)D\left[\psi_i\left(x\right)\psi_j\left(y\right)\right]j_j\left(y\right)\right\},$$

where $j_i\left(x\right)$ are sources. Functional differentiation with respect to these sources gives Wick's theorem, and using symmetry properties of the time-ordered products, we obtain :

$$T_{1,2\ldots,2n} = \sum_{permutations}\left(-1\right)^p\sum_{m=0}^{n}\left[\frac{N_{a_1,a_2,\ldots,a_{2m}}}{(2m)!}\cdot\frac{1}{(n-m)!}\left(\prod_{j=m+1}^{n}\frac{1}{2}D_{a_{2j-1},a_{2j}}\right)\right]. \tag{4.1}$$

The sum is over all permutations of $1,\ldots,2n$, of $a_1,\ldots,a_{2n}$ and p is the number of pair interchanges of fermions in moving throught the permutations. For four bosonic fields, the equation (4.1) gives the well known result :

$$T_{1,2,3,4} = N_{1,2,3,4} + D_{1,2}N_{3,4} + D_{1,3}N_{2,4} + D_{1,4}N_{2,3} + D_{2,3}N_{1,4}+$$

$$+D_{2,4}N_{1,3} + D_{3,4}N_{1,2} + D_{1,2}D_{3,4} + D_{1,3}D_{2,4} + D_{1,4}D_{2,3}.$$

A recursive formula to $2n$ order gives :

$$\begin{aligned} T_{1,2\ldots,2n} &= N_{1,2,\ldots,2n} \\ &+D_{1,2}N_{3,4,\ldots,2n} + allothertermswithonecontraction \\ &+D_{1,2}D_{3,4}N_{5,6,\ldots,2n} + allothertermswithtwocontractions \\ &\vdots \\ &+D_{1,2}D_{3,4}\ldots D_{2n-1,2n} + allotherpossiblepairings\,. \end{aligned}$$

It holds whatever the nature of the fields.

4.2 Dimensional Regularization.

The infrared and UV singularities encountered in the last cha pter dealing with the analytic determination of the fragmentation function at finite temperature can be handled with in dimensional regularization. These divergences are represented by the terms $\beta^{\pm\epsilon}$ where $\epsilon = D - 4$.

A brief overview on how this works is given by the following [4],[5]. The divergent loop integrals of field theory are four dimensional in energy-momentum phase space. To make these integals finite, we modify the dimensionality of these integrals using Dimensional Regularization. We change from a four-dimensional space to a D-dimensional one where one should redefine the metric tensor as:

$$g^{00} = -g^{ii},\ i = 1, 2, \ldots, D-1,$$

$$g^{\alpha\beta} = 0, \alpha \neq \beta.$$

A four vector k^α is replaced by a vector with D components :

$$k^\alpha = \left(k^0, k^1, \ldots, k^{D-1}\right),$$

and,

$$k^2 = k^\alpha k_\alpha = \left(k^0\right)^2 - \sum_{i=1}^{D-1} \left(k^i\right)^2.$$

Loop integrals now become integrals in D dimensions with the volume element $d^D k = dk^0 dk^1 \ldots dk^{D-1}$.For example, the integral :

$$\int \frac{d^4k}{(k^2+m^2+i\varepsilon)^n}$$

becomes :

$$\int \frac{d^Dk}{(k^2+m^2+i\varepsilon)^n} = i\pi^{\frac{D}{2}} \frac{\Gamma\left(n-\frac{1}{2}D\right)}{\Gamma(n)} \frac{1}{(m^2)^{n-\frac{D}{2}}}$$

for integer values of $n \succ \frac{D}{2}$. For $n = \frac{D}{2}$, the left hand side of this equation is logarithmically divergent, and the right hand side is singular at $n = 0$. However, for non-integer values of D , the right hand side is perfectly finite. Therefore, we should use the last property to define the integral in D-dimensions for non-integer values of D and put :

$$D = 4 - \eta,$$

where, η is a small positive parameter. We should then take the limit $\eta \longrightarrow 0$ to restore the four-dimensinal space. That's what we used in chapter 8.

Bibliography

[1] A. Das, *"Finite temperature field theory"* , World Scientific , (1997).

[2] P. V. Landshoff, *"Introduction to equilibrium thermal field theory"* ,hep-ph/9808362.

[3] T.S. Evans, D.A.Steer, *"Wick's theorem at finite temperature"*, arXiv : hep-ph/9601268.

[4] Claude Itzykson and Jean-Bernard Zuber, *"Quantum field theory"*, McGraw-Hill International Editions, (1988).

[5] F. Mandl and G.shaw , "*Quantum field theory*" , John Wiley and Sons , (1991).

5

Scattering Process at Finite Temperature.

Over the years, scattering processes have been extensively used to study properties of matter throught collision of fragments of particles. The results obtained during the fragmentation of debries of matter during collision could give some information about the properpties of paricles. such properties could include the study of decay rates and transition amplitudes, and reaction rates at finite temperature [1],[2],[3],[4]. This leads us to the use of Finite temperature field theory called TFD [1],[5],[6],[7],[8],[9] described in the introductory chapter. Therefore, to treat many body systems at equilibrium and non-equilibrium, one uses the closed -time path method whereas the imaginary time approch could be used to investigate systems at equilibrium [1],[3],[5],[6],[7].

5.1 Scattering Matrix in TFD.

Let us consider a process for which we have particles that may be considered free at the beginning and end of the interaction, but that interact over some limited region of space and time and with a heat bath.We denote the state vector in the interaction representation by $\Phi(-\infty)$ for the remote past (in state) and $\Phi(\infty)$ for the remote future (out state), and look for an operator S such that

$$\Phi(\infty) = S\Phi(-\infty).$$

The operator S is the limit of the time evolution operator $U(t, t_0)$

$$S = \lim_{t \longrightarrow +\infty, t_0 \longrightarrow -\infty} U(t, t_0)$$

where the unitary operator $U(t,t_0)$ propagates the system from t_0 to t.Since in the interaction representation

$$i\frac{\partial \Phi(t)}{\partial t} = V(t)\Phi(t), \tag{5.1}$$

where

$$V(t) \equiv H_I(t) = e^{iH_0 t}\left(\int H_I^S d^3x\right) e^{-iH_0 t} \tag{5.2}$$

is the interaction Hamiltonian, we have by formal integration of (5.1)

$$\Phi(t) = U(t,t_0)\Phi(t_0) = \exp\left(-i\int_{t_0}^{t} V(t)\,dt\right)\Phi(t_0).$$

Now, the solution is to introduce a *chronological* or *time-ordering operator* T that has the property

$$T(V(t_1),\ldots,V(t_n)) = \delta_p V(t_\alpha)V(t_\beta)\ldots V(t_\delta)V(t_\sigma), \tag{5.3}$$

with $t_\alpha \geqslant t_\beta \geqslant \ldots t_\delta \geqslant t_\sigma$.The factor δ_p is 1 if only boson operators are involved. If fermion operators must be permuted $\delta_p = \pm 1$, depending on whether the number of permutations of fermion operators required for chronological ordering is even or odd. That is, T applied to a product of operators orders them according to their time arguments, with the earliest times standing to the right. This leads to the definition of S in terms of the *time-ordered exponential*

$$S = \lim_{t\longrightarrow+\infty, t_0\longrightarrow-\infty} U(t,t_0),$$

with

$$U(t,t_0) = T\left[\exp\left(-i\int_{t_0}^{t} V(t)\,dt\right)\right]$$

$$= T\left[\exp\left(-i\int_{t_0}^{t} dt\int d^3x H_I(\mathbf{x},t)\right)\right] = T\left[\exp\left(-i\int d^4x H_I(x)\right)\right]. \tag{5.4}$$

The formal expression (5.4) is to be interpreted as the expansion series

$$S = \sum_{n=0}^{\infty} S_n = \lim_{t\longrightarrow+\infty, t_0\longrightarrow-\infty} T\left[1 + \frac{1}{i}\int_{t_0}^{t} V(t)\,dt + \ldots + \frac{1}{i^n n!}\left(\int_{t_0}^{t} V(t)\,dt\right)^n + \ldots\right], \tag{5.5}$$

which will be the starting point for a perturbation series calculation of the transition probabilities. We note that for theories in which the interaction Lagrangian density $\mathcal{L}_I$

contains no derivative couplings the interaction Hamiltonian density differs from $\pounds_I$ by only a sign so that

$$V(t) = H_I(t) = -L_I(t) = -\int d^3x \pounds_I,$$

and (5.5) may be written

$$S = Te^{iA},$$

where A is the part of the action coming from the interaction Lagrangian density

$$A = \int d^4x \pounds_I.$$

The states Φ may be expanded in a complete basis $\Phi = \sum_{\alpha} \mid \alpha\rangle\langle\alpha \mid \Phi\rangle$, and the amplitude for the transition from $\Phi(-\infty)$ to some basis state$\mid \sigma\rangle$ at $t = +\infty$ is

$$F = \langle\sigma \mid \Phi(+\infty)\rangle = \langle\sigma \mid S \mid \Phi(-\infty)\rangle$$

$$= \sum_{\alpha}\langle\sigma \mid S \mid \alpha\rangle\langle\alpha \mid \Phi(-\infty)\rangle \tag{5.6}$$

Thus the operator S can be specified by its matrix elements

$$S_{\sigma\alpha} = \langle\sigma \mid S \mid \alpha\rangle \tag{5.7}$$

and is termed by the *S-matrix* or scattering matrix.

As is commonly the case in field theories, the S-matrix elements could be found by considering asymptotic states which are the major difficulty in these calculations. In Thermo-field dynamics, such in and out states are considered thermal states function of temperature β and from (5.3)-(5.7) the form of the nth-order contribution to the S-matrix element is

$$\langle b \mid S \mid a\rangle \simeq \int dt_1 \int dt_2 \ldots \int dt_n$$

$$\langle O(\beta) \mid b_1(\beta) b_2(\beta) \ldots b_l(\beta) T(V(t_1) V(t_2) \ldots V(t_n)) a_1^{\dagger}(\beta) a_2^{\dagger}(\beta) \ldots a_k^{\dagger}(\beta) \mid O(\beta)\rangle \tag{5.8}$$

where

$$\mid \psi_i\rangle = a_1^{\dagger}(\beta) a_2^{\dagger}(\beta) \ldots a_k^{\dagger}(\beta) \mid O(\beta)\rangle, \mid \psi_f\rangle = b_1^{\dagger}(\beta) b_2^{\dagger}(\beta) \ldots b_k^{\dagger}(\beta) \mid O(\beta)\rangle \tag{5.9}$$

represent initial and final $k-$ and $l-$particle thermal states generated by successive applications of appropriate creation operators to the thermal vacuum state $\mid O(\beta)\rangle, T$

is the chronological operator,and $V(t)$ is the interaction portion of the Hamiltonian in the interaction representation (5.1).

The operator $V(t)$ will contain products of creation and destruction operators for the various fields interacting through each term and with the heat bath.

Symbolically Eq.(5.5), as an $S-$matrix, is written as

$$S=\sum_{n=0}^{\infty} S^n=\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 dx_2 \ldots dx_n T\left[H_I(x_1) H_I(x_2) \ldots H_I(x_n)\right].$$

Considering the doubling and the tilde-conjugation rules, the S-matrix for TFD is

$$\hat{S}=\sum_{n=0}^{\infty} \hat{S}^n=\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 dx_2 \ldots dx_n T\left[\hat{H}_I(x_1) \hat{H}_I(x_2) \ldots \hat{H}_I(x_n)\right],$$

where $\hat{H}_I(x)=H_I(x)-\tilde{H}_I(x)$. In particular at the tree level

$$\hat{S}=1-i \int d^4x\left[H_I(x)-\tilde{H}_I(x)\right].$$

In order to calculate amplitudes, expressions for $H_I(x)$ and $\tilde{H}_I(x)$ have to be introduced explicitly. Then Wick's theorem is used to obtain all the contributions in any order of perturbation theory.To find temperature dependence, the Bogoliubov transformations have to be used for the creation and annihilation operators. The cross section and decay rates depend closely on the $S-$matrix elements.

5.2 Reaction rates.

Consider the process

$$p_1+p_2+\ldots+p_l \longrightarrow p_1'+p_2'+\ldots+p_l'.$$

The amplitude at $T=0$ for this process is obtained by the usual Feynman rules by writing

$$\langle f \mid S \mid i\rangle=\sum_{n=0}^{\infty}\langle f \mid S^n \mid i\rangle$$

where $\mid i\rangle=a_{p_1}^{\dagger} a_{p_2}^{\dagger} \ldots a_{p_l}^{\dagger} \mid o\rangle$ and $\mid f\rangle=a_{p_1'}^{\dagger} a_{p_2'}^{\dagger} \ldots a_{p_l'}^{\dagger} \mid o\rangle$ with $\mid o\rangle$ being the vacuum state. For $T \neq 0$, similar procedure may be used. The amplitude for the process is given as

$$\langle f \mid \hat{S} \mid i\rangle=\sum_{n=0}^{\infty}\langle f \mid \hat{S}^n \mid i\rangle,$$

where

$$| i\rangle = a^{\dagger}_{p_1}(\beta)\, a^{\dagger}_{p_2}(\beta) \dots a^{\dagger}_{p_l}(\beta) \mid o(\beta)\rangle$$

$$| f\rangle = a^{\dagger}_{p'_1}(\beta)\, a^{\dagger}_{p'_2}(\beta) \dots a^{\dagger}_{p'_l}(\beta) \mid o(\beta)\rangle$$

the state $| o(\beta)\rangle$ is the thermal vacuum. The differential cross section for the process

$$p_1 + p_2 \longrightarrow p'_1 + p'_2 + \dots + p'_l.$$

is given as

$$d\sigma = (2\pi)^4\, \delta^4 \left(p'_1 + p'_2 + p'_3 \dots + p'_l - p_1 - p_2\right)$$
$$\times \frac{1}{4E_1 E_2 v_{rel}} \prod_j (2m_j) \prod_{j=1}^{l} \frac{d^3 p'_j}{(2\pi)^3\, 2E'_j} \mid M_{fi} \mid^2,$$

where $E_j = \sqrt{m_j^2 + \mathbf{p}_j^2}$ and v_{rel} is the relative velosity of the two initial particles with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. The amplitude M_{fi} is related to the $S-$matrix element by

$$\langle f \mid \hat{S} \mid i\rangle = i\,(2\pi)^4\, M_{f_i} \prod_{ext} \left(\frac{M}{VE}\right)^{\frac{1}{2}} \prod_{ext} \left[\frac{m}{V\omega}\right]^{\frac{1}{2}} \delta^4 (p_f - p_i)$$

here p_f and p_i are the total 4-momenta in the final and initial state. The product extends over all the external fermions and bosons, with $E(M)$ and $\omega(m)$ being the energy (mass) of fermions and bosons respectively, and V is the volume.

5. SCATTERING PROCESS AT FINITE TEMPERATURE.

Bibliography

[1] F.C. khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, A. E. Santana, 'Thermal Quantum Field Theory, Algebraic Aspects and Applications',World Scientific Publishing Company (2009).

[2] A. Rakhimov and F.C. Khanna,'Finite temperature amplitudes and reaction rates in thermofield dynamics', Phys. Rev. C, volume 64 064907.

[3] P. V. Landshoff, 'Introduction to equilibrium thermal field theory', hep-ph/9808362.

[4] M. H. Thoma,'New Developments and Applications of Thermal Field Theory', arXiv:hep-ph/0010164.

[5] A. Das, 'Finite Temperature Field Theory', (W.Scientific, Singapour, 1977)

[6] M. Le Bellac, 'Thermal Field Theory (Cambridge University Press, Cambridge, 1996)

[7] J. I. Kapusta, 'Finite Temperature Field Theory' (Cambridge University Press, Cambridge, 1989).

[8] Y. Takahashi and H. Umezawa, 'Thermofield dynamics',Collective Phenomena, Vol. 2, pp.55-8 (1975).

[9] Y. Takahashi and H. Umezawa, 'Thermo field Dynamics', International journal of modern Physics B, Vol. 10, Nos. 1314 (1996) 1755-1805

6

Total cross section for e^+e^- annihilation at finite temperature.

6.1 Introduction

The deconfined state of hadronic matter, a plasma of quarks and gluons, existed at the early times of our universe. However, the conversion to hadronic state was quite rapid. It is highly unlikely to find such a state in stars or galaxies. However such a state could be investigated in the laboratory with experiments at Relativistic Heavy Ion collider (RHIC) with formation of the quark gluon plasma (QGP). This is some sort of a soup of quarks and gluons moving at hight temperature [1],[2]. Usual predictions for such state of matter are made using perturbation theory with the quantum chromodynamics (QCD) and statistical mechanical models [3],[4],[5],[6]. Such experiments are for instance: (i) pp collisions [7]; (ii) Drell-Yang processes [7].(iii) The e^+e^- annihilation process [8][9][10][11][12][13][14]. The latter could be considered as a huge factory of information to investigate quark interactions. It provides us with a good test for QCD at hight energies. For other energy scales, some insight is obtained on the phenomenological models of quark interactions. Moreover, it is the cleanest arena in which we can study in detail since electroweak annihilation creates an almost asymptotically free initial quark-antiquark pair at very large Q^2, the most fundamental and elementary tests of the color field behaviour. Experimentally, LEP machine has provided a wealth of precision data on the electroweak interactions through a multitude of e^+e^- annihilation states recorded by ALEPH, DELPHI, L3 and OPAL [15][16]. The LEP colliders

6. TOTAL CROSS SECTION FOR E+E-ANNIHILATION AT FINITE TEMPERATURE.

operated at energies above the Z^0 resonance threshold :

$$\sqrt{s} = E_{cm} \succ M_{Z^0}.c^2$$

Mainly, the basic prediction of the standard model of electroweak interaction, for fermion-antifermion production annihilation around the Z^0 resonances are sketched in this process. The cross section for the W-pair production $e^+e^- \to W^+W^-(\gamma)$ is measured. Using QCD events in the e^+e^- annihilation, jet events have also been studied.e^+e^- annihilation process is a reliable and suitable process for the study of multiparticle production in jet physics. In accordance with QCD, it is realized through the production of γ or Z^0 boson into two quarks :

$$e^+e^- \to (Z^0/\gamma) \to q\bar{q}$$

Several hadronisation models were proposed [8]. In these models, hadronization occuring in the e^+e^- annihilation is proposed where the quark pair and diquark pair are produced. They then undergo Q^2 evolution via a virtual photon and fragment into hadrons. Light hadron physics at the B factories gives a good account of clusters of hadronic matter obtained in different channels [13]. Another process for e^+e^- annihihilation and QCD gives different fragmentation process (τ production for instance) in the Z^0 channel [11],[12]. A measurement of the total cross section for $e^+e^- \to hadrons$ at $\sqrt{s} = 10.25\ Gev$ has been made using the CLEO detector at the Cornell Electron Storage Ring [17]. Finally, new results on the low energy e^+e^- annihilation into hadrons from Novosibirsk and Beijing have been described [12].A brief review about e^+e^- annihilation process is also given in the books litterature by Le Bellac, Kapusta and Bailin and Love [5],[6],[18].

The role of temperature in the annihilation process arises in considering details of the production mechanisms and has been treated [19],[20]. It is extensively used in ultrarelativistic heavy ion collisions [21]. The hadron production in the e^+e^- annihilation process using real time formalism is proposed by Baier, Pire, Schiff [25],[34]. The first order corrections to the decay rate of dilepton pair produced in a thermalized medium are examined. A vanishing chemical potential has been assumed for simplicity. Our focus is not to describe the realm of QCD hadronic interaction using temperature. We do not wish describe temperature dependence for the cluster hadronic jets obtained during e^+e^- annihilation process of this deconfined phase. However, we do analyse the effect of temperature on the decay rate, particularly, for hadron production and $\pi^{\pm}$ production using real time formalism at the tree level. Indeed, the motivation is the following :

-our work, first of all, follows Rakhimov and Khanna original work [24] on calculating decay rates for different reactions using the real-time formalism. -a graphical representation of different decay rates as is function of temperature.

Numerous attempts have been made to study thermal effects in a quark-gluon plasma using Born Approximation to calculate dilepton production [25],[26]. This reinforces the need for the development of a finite temperature field theory which could provide answers about the transition from hadronic matterto QGP. Several methods have been proposed. One formalism using Wick rotation was proposed by Matsubara [27], the imaginary-time formalism. The extension of Matsubara work to quantum field theory was carried out by Ezawa,Tomozawa and Umezawa[28],[29]. Despite the lack of such a theory to fully describe the interaction at nonequilibrium,a real-time formalism was needed which is the closed-time path formulation due to Schwinger [30], Mahanthapa and Bakhshi [31], and Keldysh [32]. The approch uses a closed contour in the complex plane. This requires a doubling of the degrees of freedom of the theory in a sense that the green functions are represented by a 2*2 matrices. Umezawa and Takahashi introduced the method of Thermo-field Dynamics (TFD) that introduced a second Hilbert space. Bogoliubov transformation is used to mix states in the two Hilbert spaces. In equilibrium, all three methods are equivalent. Howeve, the objective of real-time theories was to extend the formalism to non-equilibrium processes.This has not been achieved as yet.

Our calculation is applied to total decay rates at finite temperature in laboratory framework in non-polarised e^+e^- annihilation process giving rise to hadronic matter within a heat bath (QGP). We obtain curves for different centre of mass energies and different chemical potentials. The nature of the nucleus is specified by the value of centre of mass energy $\sqrt{s}$.

6.2 Theoretical framework.

Field theory at finite temoerature has been studied long time ago with the pioneering work of Umezawa at al.[28],[29]. In order to calculate decay rates at finite temperature , one should apply real time formalism following Rakhimov and Khanna method [24]. An operator approach formalism has been elaborated by Umazawa et al.[28],[29], called THERMO-FIELD DYNAMICS to account for the thermal dynamic properties of statistical field theory using a thermal vaccum. The main features of the theory are presented before going to the main calculations.

6.3 General formalism.

6.3.1 Thermal fields, Thermal average and Bogoliubov Transformations.

Thermo-field dynamics (TFD) provided a real time formalism at finite temperature. To describe a statistical system at equilibrium, we need to introduce the partition function of the system given by

$$Z(\beta) = Tr\left(e^{-\beta H}\right)$$

where, H is the hamiltonian of the system. MATSUBARA has shown that the statistical average of an operator is the same as the mean value of this operator in vaccum at finite temperature If we define the mean value of an operator as :

$$\langle A\rangle_\beta = \langle O(\beta) \mid A \mid O(\beta)\rangle = Z^{-1}(\beta)\sum_n e^{-\beta E_n}\langle n \mid A \mid n\rangle \tag{6.1}$$

where $\mid n\rangle$ is a specific state corresponding to the energy E_n.Then the finite temperature formalism will be completely parallel to the zero temperature field theory. We observe that the total hamiltonian is invariant under the Bogoliubov transformation (relativistic case):

$$a_k \longrightarrow a_k(\theta) = a_k \cos\theta_k - \tilde{a}_k^\dagger \sin\theta_k$$

$$\tilde{a}_k \longrightarrow \tilde{a}_k(\theta) = \tilde{a}_k \cos\theta_k + a_k^\dagger \sin\theta_k$$

$$a_k^\dagger \longrightarrow a_k^\dagger(\theta) = a_k^\dagger \cos\theta_k - \tilde{a}_k \sin\theta_k$$

$$\tilde{a}_k^\dagger \longrightarrow \tilde{a}_k^\dagger(\theta) = \tilde{a}_k^\dagger \cos\theta_k + a_k \sin\theta_k$$

for fermions, and :

$$a_k \longrightarrow a_k(\theta) = a_k \cosh\theta_k - \tilde{a}_k^\dagger \sinh\theta_k$$

$$\tilde{a}_k \longrightarrow \tilde{a}_k(\theta) = \tilde{a}_k \cosh\theta_k - a_k^\dagger \sinh\theta_k$$

$$a_k^\dagger \longrightarrow a_k^\dagger(\theta) = a_k^\dagger \cosh\theta_k - \tilde{a}_k \sinh\theta_k$$

$$\tilde{a}_k^\dagger \longrightarrow \tilde{a}_k^\dagger(\theta) = \tilde{a}_k^\dagger \cosh\theta_k - a_k \sinh\theta_k$$

for bosons, Therefore doubling of degrees of freedom is required. These relations can be written in a matrix form as :

$$\begin{pmatrix} a_k(\beta) \\ \tilde{a}_k^\dagger(\beta) \end{pmatrix} = U(\theta_k) \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} = \begin{pmatrix} \cos\theta_k(\beta) & \text{-}\sin\theta_k(\beta) \\ \sin\theta_k(\beta) & \cos\theta_k(\beta) \end{pmatrix} \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} \tag{6.2}$$

for fermions, and :

$$\begin{pmatrix} a_k(\beta) \\ \tilde{a}_k^\dagger(\beta) \end{pmatrix} = U(\theta_k) \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\theta_k(\beta) & \text{-}\sinh\theta_k(\beta) \\ \text{-}\sinh\theta_k(\beta) & \cosh\theta_k(\beta) \end{pmatrix} \begin{pmatrix} a_k \\ \tilde{a}_k^\dagger \end{pmatrix} \tag{6.3}$$

for bosons where $U(\theta_k)$ is a unitary transfomation,

$$\sinh^2\theta_k = n_B(k) = \frac{1}{\left(e^{\beta|k_0|} - 1\right)}, \sin^2\theta_k = n_F(k) = \frac{1}{\left(1 + e^{\beta|k_0|}\right)} \tag{6.4}$$

being bose and fermion distributions. Here k_0 is the energy associated with the four-vector k.Introducing a temperature-dependent vaccum using bogoliubov transformation, we can show that :

$$| O(\beta)\rangle = \prod_k \left(\cos\theta_k + \sin\theta_k a_k^\dagger \tilde{a}_k^\dagger\right) | O\rangle \tag{6.5}$$

for fermions and,

$$| O(\beta)\rangle = \prod_k \left(\frac{1}{\cosh\theta_k}\right) \exp\left(\tanh\theta_k a_k^\dagger \tilde{a}_k^\dagger\right) | O\rangle \tag{6.6}$$

for bosons.Tilde operation is introduced and to each field $\phi(x)$ at zero temperature, a column vector of fields is introduced :

$$\begin{pmatrix} \phi(x) \\ \tilde{\phi}^\dagger(x) \end{pmatrix}$$

whose dynamics is described by the following thermal hamiltonian :

$$\hat{H} = H(\phi) - \tilde{H}(\phi) = H(\phi) - H^*\left(\tilde{\phi}\right)$$

where , $H(\phi) = H(\phi) - \mu N$ and N is the particle number operator. The temperature dependent vacuum state described above is annihilated by the temperature dependent physical annihilation operators :

$$a_k(\beta) \mid O(\beta)\rangle = \tilde{a}_k(\beta) \mid O(\beta)\rangle = 0. \tag{6.7}$$

Also, the creation and annihilation operators satisfy ordinary commutation relations :

$$\left[a_{k,s}(\beta), a^{\dagger}_{k',s'}(\beta)\right]_{\pm} = \delta\left(\vec{k} - \vec{k}'\right)\delta_{ss'} \tag{6.8}$$

$$\left[\tilde{a}_{k,s}(\beta), \tilde{a}^{\dagger}_{k',s'}(\beta)\right]_{\pm} = \delta\left(\vec{k} - \vec{k}'\right)\delta_{ss'} \tag{6.9}$$

all other commutators (anticommutators) vanish.

6.3.2 Thermal propagators in TFD.

In the following, we calculate the causal green function :

$$G_c^{\alpha\gamma}(x-y) = \theta(x_0 - y_0)\langle O(\beta) \mid \phi^{\alpha}(x)\phi^{\gamma\dagger}(y) \mid O(\beta)\rangle \mp \theta(x_0 - y_0)\langle O(\beta) \mid \phi^{\gamma\dagger}(y)\phi^{\alpha}(x) \mid O(\beta)\rangle \tag{6.10}$$

where $\theta(x)$ is the haveasead function and $\mid O(\beta)\rangle$ is the vaccum at finite temperature. The momentum representation of the green function is :

$$G^{\alpha\gamma}(x-y) = \frac{i}{(2\pi)^4}\int d^4p \exp(-ip(x-y))\, G^{\alpha\gamma}(p). \tag{6.11}$$

In QCD, It should be pointed out that it is convenient to rewrite the standard propagators in TFD, $\triangle(k)$ for bosons (gluons) and $S(p)$ for fermions (quarks) , as follows [28],[29]; for bosons

$$\triangle(k) = U_B(k_0, \beta)\,\bar{\triangle}(k)\,U_B^{\dagger}(k_0, \beta) =$$

$$= \begin{pmatrix} \frac{g_{\mu\nu}}{k^2+i\epsilon} - \frac{\left(1-\lambda^{-1}\right)k_\mu k_\nu}{\left(k^2+i\epsilon\right)^2} & 0 \\ 0 & -\frac{g_{\mu\nu}}{k^2-i\epsilon} + \frac{\left(1-\lambda^{-1}\right)k_\mu k_\nu}{\left(k^2-i\epsilon\right)^2} \end{pmatrix}$$

$$-2\pi i\delta\left(k^2\right)\frac{g_{\mu\nu}}{e^{\beta|k_0|}-1}\begin{pmatrix} 1 & e^{\frac{\beta|k_0|}{2}} \\ e^{\frac{\beta|k_0|}{2}} & 1 \end{pmatrix} \tag{6.12}$$

where

$$\bar{\triangle}(k) = \frac{\tau}{\left(k_0+i\delta\tau\right)^2-\vec{k}^2}\left[g_{\mu\nu} - \frac{\left(1-\lambda^{-1}\right)k_\mu k_\nu}{\left(k_0+i\delta\tau\right)^2-\vec{k}^2}\right],$$

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\lambda = 0, 1,$$

$$\sinh^2\theta\left(k_0,\beta\right) = \frac{1}{e^{\beta|k_0|}-1},$$

where $\beta = \frac{1}{k_B T}$, k_B and T are the Boltzmann constant and the temperature of the quark gluon system respectively;and for quarks

$$S(p) = U_F\left(\vec{p},\beta\right)\bar{S}(p)\,U_F^{\dagger}\left(\vec{p},\beta\right)$$

$$= \left(\bar{p}+m\right)\left[\begin{pmatrix} \frac{1}{p^2-m^2+i\epsilon} & 0 \\ 0 & \frac{1}{p^2-m^2-i\epsilon} \end{pmatrix}\right.$$

$$\left.+2\pi i\delta\left(p^2-m^2\right)\frac{1}{e^{\beta|p_0|}+1}\begin{pmatrix} 1 & -\epsilon\left(p_0\right)e^{\frac{\beta|p_0|}{2}} \\ -\epsilon\left(p_0\right)e^{\frac{\beta|p_0|}{2}} & 1 \end{pmatrix}\right] \tag{6.13}$$

where

$$\bar{S}(p) = \frac{I}{\bar{p}-m+i\delta\tau};\ \ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sinh^2\theta\left(\left|\vec{p}\right|,\beta\right) = \frac{1}{e^{\beta p_0}+1}$$

and

$$\epsilon\left(p_0\right) = \theta\left(p_0\right) - \theta\left(-p_0\right),\ \theta\left(p_0\right) = \left\{\begin{array}{ll} 1, & p_0 \succ 0, \\ 0, & p_0 \prec 0. \end{array}\right\}$$

The matrices U_B and U_F are the Bogoliubov transformation matrices [3],[28],[29] for bosons and fermions respectively.

6.4 Calculation.

We consider The e^+e^- annihilation process into hadrons through one photon annihilation within a color confined QGP medium at finite temperature. Here we use real-time formalism of TFD to calculate the total cross section of the annihilation process $e^+e^- \longrightarrow hX$ at finite temperature. Then, the cross section is calculated by taking thermal average of the product of thermal Green functions. The center of mass energies are fixed at $\sqrt{s} = 2\epsilon_0 = 10Gev, 100Gev$ .and chemical potentials : $\mu = 0Mev, 400Mev$. (throught γ and Z_0 channels).

In the following, we have to consider the quark-antiquark interactions without emission of a gluon in a hot non-abelian medium, a many body system at equilibrium. To deal with such a system, we use the approach of TFD. In this formalism, we introduce a temperature dependent vacuum state so that the average of any operator can be identified with the expectation value of the operator in this state. Doubling of degrees of freedom of all fields is required. That is to say, besides the normal operators, there exists "tild "operators : to each operator $\varphi(x)$ at finite temperature there exists a doublet of fields $(\varphi(x), \tilde{\varphi}(x))$ .The dynamics of such fields is controlled by a lagrangian :

$$\hat{L} = L(\varphi) - L^*(\tilde{\varphi})$$

It is important to remind that the advantage of the theory lies in the fact that all the machinery of the zero temperature field theory can be introduced at $T \neq 0$ and Feynman diagrams are well defined. Now, let us consider the process : $e^+e^- \longrightarrow q\bar{q} \longrightarrow hX$. We assume that we are working near the light-cone $x^2 \simeq 0$. The result of the calculation of the scattering amplitude at $T = 0$ is given by the optical theorem [33] :

$$\begin{aligned} \sigma_{tot} &= \frac{8\pi^2\alpha^2}{3(q^2)^2} \int d^{4x} e^{iqx} \langle 0 \mid [J_\mu(x), J^\mu(0)] \mid 0 \rangle \\ &= \frac{8\pi^2\alpha^2}{3(q^2)^2} \left[2Im \left(i \int d^{4x} e^{iqx} \langle 0 \mid T\left(J_\mu(x) J^\mu(0)\right) \mid 0 \rangle \right) \right] \end{aligned} \quad (6.14)$$

For a reminder, we write in a scattering process the scattering matrix formally in terms of the decay rate

$$S = 1 + iT$$

then,

$$S^{\dagger}S = \left(1 - iT^{\dagger}\right)(1 + iT)$$
$$= 1 - i\left(T^{\dagger} - T\right) + T^{\dagger}T$$

in the other hand, the scattering cross section is

$$\sigma \propto TT^{\dagger} = i\left(T^{\dagger} - T\right) = Im\left(T^{\dagger} - T\right).$$

The operator T is written in terms of a current, J_{μ}

$$T \to \langle 0 \mid J_{\mu} \mid 0\rangle$$

therefore, in our case

$$\sigma \propto \langle q\bar{q} \mid J_{\mu}^{q\bar{q}}(x)\, J_{\mu}^{e^{+}e^{-}}(0) \mid e^{+}e^{-}\rangle$$

To calculate σ_{tot} at $T \neq 0$, we follow the developments in ref. [3] , [24], using Wick's theorem [26] rather than the direct method. The theorem applies for fast quarks (hard photons) and for high temperatures as in the $T = 0$ case. We assume the following expression calculated in the vaccum at finite temperature, $\beta \neq 0$, and is :

$$\sigma_{tot} = \frac{8\pi^{2}\alpha^{2}}{3(q^{2})^{2}}\left[2Im\left(i\int d^{4}xe^{iqx}\langle O(\beta) \mid T\left(J_{\mu}(x)J^{\mu}(0)\right) \mid O(\beta)\rangle\right)\right] \tag{6.15}$$

The physical current of quarks fragmenting into hadrons is : $J_{\mu}(x) = \bar{q}(x)\gamma_{\mu}Qq(x)$, where $q(x)$ is the quark field and Q the fractional charge matrix. In the expression for the current, we assume that the sum over the quark color is implicit. This introduces a color factor of $N_c = 3$ in the expression whereas the flavors are introduced in the charge matrix. We assume the chemical potentiel $\mu = 0$ in the Fermi-Dirac distribution for the quarks. Then,

$$J_{\mu}(x)J^{\mu}(0) = Q^{2}\left[\bar{q}(x)\gamma_{\mu}q(x)\bar{q}(0)\gamma^{\mu}q(0)\right]$$

In the following, we are interested in taking the T- product on the vaccum at $\beta \neq 0$ and complete the calculation using the Wick's theorem :

$$\langle O(\beta) \mid T\left(J_{\mu}(x)J^{\mu}(0)\right) \mid O(\beta)\rangle = \langle O(\beta)Tr\left[T\left(q(0)\bar{q}(x)\right)\gamma_{\mu}T\left(q(x)\bar{q}(0)\right)\gamma_{\nu}Q^{2}\right] \mid O(\beta)\rangle. \tag{6.16}$$

6. TOTAL CROSS SECTION FOR E+E-ANNIHILATION AT FINITE TEMPERATURE.

The T-product is defined by :

$$\langle O(\beta) \mid T\left(q(0)\bar{q}(x)\right) \mid O(\beta)\rangle = iS_F(-x,m) = \frac{1}{(2\pi)^4}\int d^4k e^{+ikx} S_F(k,m) \qquad (6.17)$$

and,

$$\langle O(\beta) \mid T\left(q(x)\bar{q}(0)\right) \mid O(\beta)\rangle = iS_F(x,m) = \frac{1}{(2\pi)^4}\int d^4k^{'} e^{-ik^{'}x} S_F(k^{'},m) \qquad (6.18)$$

and the Feynman propagator for the physical fermion field is given by (we drop The subscript F for the moment) :

$$S(k,m) = S^{(0)}(k) + S^{(\beta)}(k) = \left(\bar{k}+m\right)\left(\frac{1}{k^2-m^2+i\epsilon} + 2i\pi n_F(k)\delta(k^2-m^2)\right)$$

Here $n_F(k)$ represents the fermionic distribution function defined by :

$$n_F(k) = \frac{1}{e^{\beta|k_0|}+1} \qquad (6.19)$$

Substituting fo the T-products (6.17) and (6.18) and taking the trace , expression (6.16) becomes :

$$\begin{aligned}\langle O(\beta) \quad &| \quad T\left(J_\mu(x)J_\nu(0)\right) \mid O(\beta)\rangle = Tr\left[iS_F(-x,m)\gamma_\mu iS_F(x,m)\gamma_\nu Q^2\right] = \\ &= \quad (-1)\,Tr\left(Q^2\right)Tr\left[\frac{1}{(2\pi)^4}\int d^4k e^{ikx}\frac{1}{(2\pi)^4}\int d^4k^{'} e^{-ik^{'}x} S(k,m)\gamma_\mu S(k^{'},m)\gamma_\nu\right]\end{aligned}$$

Substituting for the Feynman propagators, the latter becomes

$$\begin{aligned}\langle O(\beta) \quad &| \quad T\left(J_\mu(x)J_\nu(0)\right) \mid O(\beta)\rangle \quad = \\ &= \quad (-1)\,Tr\left(Q^2\right)Tr\left[\frac{1}{(2\pi)^4}\int d^4k e^{ikx}\frac{1}{(2\pi)^4}\int d^4k^{'} S(k,m)\gamma_\mu S(k^{'},m)\gamma_\nu\right] \\ &= \quad (-1)\,Tr\left(Q^2\right)Tr\frac{1}{(2\pi)^4}\int d^4k e^{ikx}\frac{1}{(2\pi)^4}\int d^4k' e^{-ik^{'}x} \\ &\quad \times[S^{(0)}(k,m)\gamma_\mu S^{(0)}(k^{'},m)\gamma_\nu + S^{(0)}(k,m)\gamma_\mu S^{(\beta)}(k^{'},m)\gamma_\nu \\ &\quad + S^{(\beta)}(k,m)\gamma_\mu S^{(0)}(k^{'},m)\gamma_\nu + S^{(\beta)}(k,m)\gamma_\mu S^{(\beta)}(k^{'},m)\gamma_\nu] \qquad (6.20)\end{aligned}$$

Therefore, the imaginery part of the T-product (6.20) is composed of a temperature independent and dependent contribution as follow :

$$
\begin{aligned}
& Im\left(i\int d^4xe^{iqx}\langle O(\beta)\mid T\left(J_\mu(x)J_\nu(0)\right)\mid O(\beta)\rangle\right)\\
= \; & Im\left((-i)\int d^4xe^{iqx}Tr\left(Q^2\right)Tr\left(\frac{1}{(2\pi)^4}\int d^4ke^{ikx}\frac{1}{(2\pi)^4}\int d^4k'e^{-ik'x}\right.\right.\\
& \times\; S^{(0)}(k,m)\gamma_\mu S^{(0)}(k',m)\gamma_\nu+S^{(\beta)}(k,m)\gamma_\mu S^{(\beta)}(k',m)\gamma_\nu\Big)\Big)\\
= \; & Im\left(i\int d^4xe^{iqx}\langle O\mid T\left(J_\mu(x)J_\nu(0)\right)\mid O\rangle\right)+\\
& +Im\left(i\int d^4xe^{iqx}Tr\left(Q^2\right)\frac{1}{(2\pi)^4}\int d^4ke^{ikx}\frac{1}{(2\pi)^4}\int d^4k'e^{-ik'x}Tr\left(iS^{(\beta)}(k,m)\gamma_\mu iS^{(\beta)}(k',m)\gamma_\nu\right)\right)
\end{aligned}
$$

Now, returning to our total cross section (6.15), we can write it as two contributions :

$$
\begin{aligned}
\sigma_{tot}(T \;\; \neq \;\; 0) &= \frac{8\pi^2\alpha^2}{3(q^2)^2}\left[2Im\left(i\int d^4xe^{iqx}\langle O(\beta)\mid T\left(J_\mu(x)J^\mu(0)\right)\mid O(\beta)\rangle\right)\right]=\\
&= \frac{8\pi^2\alpha^2}{3(q^2)^2}\left[2Imi\int d^4xe^{iqx}\langle O\mid T\left(J_\mu(x)J^\mu(0)\right)\mid O\rangle\right]-f\left((T)\right)\\
&= \sigma_{tot}(T=0)-f\left(T\right)
\end{aligned}
\tag{6.21}
$$

6.5 Investigation of the formula

Let's compute the temperature dependent contribution :

$$
\begin{aligned}
f(T) \;\; = \;\; & \frac{8\pi^2\alpha^2Tr\left(Q^2\right)}{3(q^2)^2}2Im(i\int d^4xe^{iqx}\frac{1}{(2\pi)^4}\int d^4ke^{ikx}\frac{1}{(2\pi)^4}\int d^4k'^{e^{-ik'x}}\left(4\pi^2\right)\\
& \times Tr\left\{\left(\bar{k}+m\right)\gamma_\mu\left(\bar{k}'+m\right)\gamma_\nu\right\}n_F\left(\left|k^0\right|\right)n_F\left(\left|k^0-q^0\right|\right)\delta\left(k^2-m^2\right)\delta\left((k-q)^2-m^2\right)
\end{aligned}
\tag{6.22}
$$

Try to calculate the trace. Upon a contraction with $g_{\mu\nu}$, we get

$$
\begin{aligned}
Tr\left\{\left(\bar{k}+m\right)\gamma_\mu\left(\bar{k}'+m\right)\gamma_\nu\right\}g^{\mu\nu} &= 4\left(k_\mu k'_\nu+k'_\mu k_\nu-g_{\mu\nu}k.k'\right)g^{\mu\nu}+4m^2g_{\mu\nu}g^{\mu\nu}\\
&= 4\left(k_\mu k'^\mu+k'_\mu k^\mu-4kk'\right)+4m^2\times 4\\
&= 4\left(kk'+kk'-4kk'\right)+16m^2\\
&= -8kk'+16m^2=4\left(q^2-2m^2\right)+16m^2=4q^2+8m^2.
\end{aligned}
$$

Therefore the expression (6.21) becomes (we apply the properties of Dirac Delta functions during different steps) :

$$\begin{aligned} f(T) &= \frac{8\pi^2\alpha^2 Tr\left(Q^2\right)}{3(q^2)^2}\int d^4x e^{iqx}\frac{1}{(2\pi)^4}\int d^4k e^{-ikx}\frac{1}{(2\pi)^4}\int d^4k' e^{+ik'x} \\ &\quad\times\left(8\pi^2\right)\left(4q^2+8m^2\right)n_F\left(\left|k^0\right|\right)n_F\left(\left|k^0-q^0\right|\right)\delta\left(k_0^2-\left(\vec{k}^2+m^2\right)\right)\delta\left(k_0'^2-\left(\vec{k}'^2+m^2\right)\right) \\ &= \frac{4\alpha^2 Tr\left(Q^2\right)}{3(q^2)^2}\int d^4k\int d^4k'\delta\left(q-k+k'\right) \\ &\quad\times\left(4q^2+8m^2\right)n_F\left(\left|k^0\right|\right)n_F\left(\left|k^0-q^0\right|\right)\left\{\delta\left(k_0^2-\left(\vec{k}^2+m^2\right)\right)\right\}^2 \\ &= \frac{4\alpha^2 Tr\left(Q^2\right)}{3(q^2)^2}\int d^4k\left(4q^2+8m^2\right)n_F\left(\left|k^0\right|\right)n_F\left(\left|k^0-q^0\right|\right)\left\{\frac{\delta\left(k_0-\sqrt{\left(\vec{k}^2+m^2\right)}\right)}{2\sqrt{\left(\vec{k}^2+m^2\right)}}\right\}^2 \\ &= \frac{32\alpha^2 Tr\left(Q^2\right)}{3(q^2)^2}\int d^3k\left(2\epsilon^2+m^2\right)n_F(\epsilon)\,n_F(\epsilon)\frac{\delta\left(k_0-\epsilon\right)}{4\epsilon^2} = \\ &= \frac{32\alpha^2 Tr\left(Q^2\right)}{3(q^2)^2}\int d^3k\left(2\epsilon^2+m^2\right)\left(\frac{1}{\left(1+e^{\beta\epsilon}\right)^2}\right)\frac{\delta\left(k_0-\epsilon\right)}{4\epsilon^2}. \end{aligned} \tag{6.23}$$

Now , we know that the Fermi-Dirac destribution for the quark is :

$$n_q\left(\left|k^0\right|\right)=\sin^2\theta_{k^0}=\frac{1}{\left(1+e^{\beta\epsilon}\right)}=\frac{1}{\left(1+e^{\beta\left|k^0\right|}\right)}$$

and for the antiquark is :

$$n_{\bar{q}}\left(\left|k^0-q^0\right|\right)=n_{\bar{q}}\left(k-q\right)=\sin^2\theta_{k'}=\frac{1}{\left(1+e^{\beta\epsilon}\right)}=\frac{1}{\left(1+e^{\beta\left|k^0\right|}\right)}$$

$$n_{\bar{q}}\left(\left|k^0-q^0\right|\right)=n_{\bar{q}}\left(k-q\right)=\sin^2\theta_{k'}=\frac{1}{\left(1+e^{\beta\epsilon}\right)}=\frac{1}{\left(1+e^{\beta\left|k^0\right|}\right)}$$

6.6 Result.

Let's try to investigate the final formula (6.23). Here the chemical potential is assumed to be : $\mu = 0$. Furthermore, $E_1 = E_2 = k^0$ is the energy of quark ,and antiquark , and $k, (m_1)$ and $k', (m_2)$ their 4-momentum (mass) respectively. We work in the center of mass of the photon of momentum q . We set

$$\begin{aligned} q &= k-k',\ \vec{k}+\vec{k}'=\vec{0}, m_1=m_2=m. \\ E_1 &= E_2=\sqrt{\vec{k}^2+m_1^2}=\sqrt{\vec{k}'^2+m_2^2}=\epsilon. \end{aligned}$$

The energy of the hard photon is

$$\begin{aligned} E_\gamma^2 &= s=q^2=\left(k-k'\right)^2=\left(\left(E_1,\ \vec{k}\right)-\left(-E_2,-\vec{k}'\right)\right)^2 \\ &= \left(\left(E_1+E_2\right)^2,\left(\vec{k}+\vec{k}'\right)^2\right)=\left(E_1+E_2\right)^2-\left(\vec{k}+\vec{k}'\right)^2=4\epsilon^2. \end{aligned}$$

and its momentum is

$$q^2=\left(k-k'\right)^2=k^2+k'^2-2kk'=4\epsilon^2$$

which implies

$$\left(-2kk'\right)=4\epsilon^2-\left(k^2+k'^2\right)=4\epsilon^2-\left(m_1^2+m_2^2\right)=4\epsilon^2-2m^2.$$

The integration over the phase space volume is :

$$d^3k=\left|\vec{k}\right|^2 d\left|\vec{k}\right| d\Omega_k=\left|\vec{k}\right|\left(\left|\vec{k}\right| d\left|\vec{k}\right|\right) d\Omega_k=\left|\vec{k}\right|\left(k_0 dk_0\right) d\Omega_k=\sqrt{k_0^2-m^2}k_0dk_0d\Omega_k$$

since

$$k=\left(k_0,\vec{k}\right)=\left(\epsilon,\vec{k}\right)\Longrightarrow\left|\vec{k}\right|^2=k_0^2-m^2$$

Moreover

$$\sigma_{tot}\left(T=0\right)=\left(\frac{4\pi\alpha^2}{3q^2}\right)Tr\left(Q^2\right).$$

Therefore the relation (6.23) becomes on substituting the above relations :

$$\begin{aligned} f(T) &= \frac{32\alpha^2 Tr\left(Q^2\right)}{3(q^2)^2}\int d^3k\left(2\epsilon^2+m^2\right)\left(\frac{1}{\left(1+e^{\beta\epsilon}\right)^2}\right)\frac{\delta\left(k_0-\epsilon\right)}{4\epsilon^2}= \\ &= \frac{32\pi\alpha^2 Tr\left(Q^2\right)}{3(q^2)^2}\int\sqrt{k_0^2-m^2}k_0dk_0\left(2\epsilon^2+m^2\right)\left(\frac{1}{\left(1+e^{\beta\epsilon}\right)^2}\right)\frac{\delta\left(k_0-\epsilon\right)}{\epsilon^2} \\ &= \frac{32\pi\alpha^2 Tr\left(Q^2\right)}{3(q^2)^2}\sqrt{1-\left(\frac{m}{\epsilon}\right)^2}\left(2\epsilon^2+m^2\right)\left(\frac{1}{\left(1+e^{\beta\epsilon}\right)^2}\right) \qquad (6.24) \end{aligned}$$

Now gathering all the results we get the final result :

$$
\begin{aligned}
R(\beta, s) &= \frac{\sigma_{tot}(T \neq 0)}{\sigma_{tot}(T=0)} = 1 - \left(\frac{8}{q^2}\right)\sqrt{1-\left(\frac{m}{\epsilon}\right)^2}\left(2\epsilon^2 + m^2\right)\left(\frac{1}{(1+e^{\beta\epsilon})^2}\right) \\
&= 1 - 4\sqrt{1-\left(\frac{m}{\epsilon}\right)^2}\left(1+\frac{1}{2}\left(\frac{m}{\epsilon}\right)^2\right)\left(\frac{1}{(1+e^{\beta\epsilon})^2}\right) \qquad (6.25)
\end{aligned}
$$

substituting for the centre mass energy we finally get :

$$
\mathbf{R}(\beta, \mathbf{s}) = \frac{\sigma_{tot}(T \neq 0)}{\sigma_{tot}(T=0)} = \mathbf{1} - 4\sqrt{1-\left(\frac{2m}{\sqrt{s}}\right)^2}\left(1+\frac{1}{2}\left(\frac{2m}{\sqrt{s}}\right)^2\right)\left(\frac{1}{\left(1+e^{\frac{\beta\sqrt{s}}{2}}\right)^2}\right) \qquad (6.26)
$$

Our result Eq.(6.25) confirms the result obtained in Ref.[34] using Born approximation.

6.7 Examining the case for quarks with chemical potential $\mu = 400\ Mev$.

Following the same steps as before, we consider The e^+e^- annihilation process into hadrons through one photon annihilation at tree level at finite temperature. We assume the chemical potential to be $\mu = 400\ Mev$. The center of mass energy is fixed at $\sqrt{s} = 2\epsilon_0 = 1Gev$. At the end, a graphical study of the total scattering cross section versus temperature is shown at a center of mass energy. Now , the Fermi-Dirac destributions for the quark are modified as

$$
n_q\left(\left|k^0\right|\right) = \sin^2\theta_{k^0} = \frac{1}{\left(1+e^{\beta(\epsilon-400)}\right)} = \frac{1}{\left(1+e^{\beta(|k^0|-400)}\right)}
$$

and for the antiquark as

$$
n_{\bar{q}}\left(\left|k^0 - q^0\right|\right) = n_{\bar{q}}(k-q) = \sin^2\theta_{k'} = \frac{1}{\left(1+e^{\beta(\epsilon-400)}\right)} = \frac{1}{\left(1+e^{\beta(|k^0|-400)}\right)}
$$

$$
n_{\bar{q}}\left(\left|k^0 - q^0\right|\right) = n_{\bar{q}}(k-q) = \sin^2\theta_{k'} = \frac{1}{\left(1+e^{\beta(\epsilon-400)}\right)} = \frac{1}{\left(1+e^{\beta(|k^0|-400)}\right)} \qquad (6.27)
$$

6.8 Result.

Let's try to investigate the final formula (6.23). For the same centre of mass energy, the chemical potential is assumed to be : $\mu = 400\ Mev$. Formula (6.26) will be modified as :

$$\mathbf{R}(\beta, \mathbf{s}) = \frac{\sigma_{tot}(T \neq 0)}{\sigma_{tot}(T = 0)} = \mathbf{1} - 4\sqrt{1 - \left(\frac{2m}{\sqrt{s}}\right)^2} \left(1 + \frac{1}{2}\left(\frac{2m}{\sqrt{s}}\right)^2\right) \left(\frac{1}{\left(1 + e^{\beta\left(\frac{\sqrt{s}}{2} - 400\right)}\right)^2}\right) \tag{6.28}$$

6.9 Annihilation Process throught $\mathbf{Z}_0$ channel.

The same process for the e^+e^- annihilation is seen throught Z_0 channel which dominates for energies above $\sqrt{s} = 100Gev$. The annihilation process takes place for high energies of $\sqrt{s} = 10Gev$ where the Z_0 dominate. The formula remaines the same as for the γ channel

$$\mathbf{R}(\beta, \mathbf{s}) == \mathbf{1} - 4\sqrt{1 - \left(\frac{2m}{\sqrt{s}}\right)^2} \left(1 + \frac{1}{2}\left(\frac{2m}{\sqrt{s}}\right)^2\right) \left(\frac{1}{\left(1 + e^{\beta\left(\frac{\sqrt{s}}{2} - \mu\right)}\right)^2}\right) \tag{6.29}$$

where μ is the chemical potential of the quark. All the curves are sketched in one graphical plot for different channels and chemical potentials.

6.10 Conclusion and interpretation of the graph for $\mu = 0$ and $\mu \neq 0$.

1) for $T \simeq 0$ $(\beta \simeq \infty)$, $R(\beta, s) = 1$.

2) for $T \simeq \infty$ $(\beta \simeq 0)$, $R(\beta, s) = 0$.

Therefore, for high temperatures, the probability of formation of states of hadrons drops rapidely ((Fig. 1); (Fig. 2)). Now, we notice from the formula that it has a branch cut that runs from $q^2 = 4m^2$ to ∞ and another from $q^2 = 0$ to $-\infty$. This allows us to say that $R(\beta, s)$ is not defined for centre of mass energies such that : $s \geq 4m^2$.

6. TOTAL CROSS SECTION FOR E+E-ANNIHILATION AT FINITE TEMPERATURE.

This is obvious since the green functions show these discontinuities in the complex plane (see analytic properties of Green functions in the complex plane). Therefore we will work in the hard photon momentum region ($q^2 \gg 4m^2, 4\mu^2$).To allow for the production of hadron states at high energies, one should fix the energy of the center of mass (corresponding for short probing distances),say to $\sqrt{s} = \sqrt{q^2} = 10Gev, 100Gev$, and plot the corresponding curves for different chemical potentials $\mu = 0Mev, 400Mev$. In the present calculation, we a been interested in computing the total cross section oref the e^+e^- annihilation process into quark-antiquark pairs at finite temperature. At first sight, we can see that the production of hadrons at finite temperature is decreased compared to that at zero temperature (probability of formation of states of hadrons is diminished with increase of temperature). The curves have been plotted for different centre of mass energy : a) for low energies (through γ channel), the centre of mass energies are fixed at : $\sqrt{s} = 0.9Gev, 1Gev$ and chemical potential : $\mu = 0Mev, 400Mev$ ((Fig. 1);(Fig. 3a);(Fig. 3b)). b) for high energies (through Z_0 channel),The center of mass energies are fixed at $\sqrt{s} = 2\epsilon_0 = 10Gev, 100Gev$ .and chemical potentials : $\mu = 0Mev, 400Mev$ ((Fig. 2);(Fig. 3c);(Fig. 3d)).The decrease as the temperature increases implies that the probability of hadronisation into quark-antiquark bound state pairs is low. This accounts possibly for asymptotic freedom and deconfinement.

Bibliography

[1] J. W. Harris and B. Muller, 'The search for the Quark-Gluon Plasma',Xiv:hep-ph/9602235 v2.

[2] A.V. Smilga, 'Physics of Thermal QCD', arXiv:hep-ph/9612347 v3.

[3] F.C. khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, A. E. Santana, 'Thermal Quantum Field Theory, Algebraic Aspects and Applications',World Scientific Publishing Company (2009).

[4] A. Das, 'Finite Temperature Field Theory', (W.Scientific, Singapour, 1977)

[5] M. Le Bellac, 'Thermal Field Theory (Cambridge University Press, Cambridge, 1996)

[6] J. I. Kapusta, 'Finite Temperature Field Theory' (Cambridge University Press, Cambridge, 1989).

[7] WU-KI TUNG, 'Perturbative QCD and the Parton Structure of the Nucleon', based on a contribution to : At the frontier of Particle Physics-Handbook of QCD, vol. 2* 887-971, ed. M. Shifman, Pub. World Scientific Publisher, Singapore, New York, 2001.

[8] Chien-er LEE, Yeou-Wei YANG and Wen-Fa TSAI , 'Hadronization in the e+e- Annihilation: The Model',Chinese Journal of Physics (vol 22, N 03, 1984)

[9] 34th International Conference on light Energy Physics, Philadelphia, P.A, July 29th-August 5th, 2008.

[10] M.E.Peskin,'Introduction to the Standard Model', PiTP summer school, July 2005.

[11] J.D.Bjorken and B.L.Ioffe (Usp.Fiz.Nauk 116, september 15-144 (May 1975))

[12] S.I.Eidelman,'recent results from $e^+e^- \to$ hadrons 7th International workshop on Tau Lepton physics (Tau 02) Santa Cruz, CA, 10-13 september, 2002.

[13] Andreas Petzold representing the BABAR collaboration : the 2007 Europhysics Conference on Hight Energy Physics, Journal of Physics : Conference Series 110 (2008)022041.

[14] M.E.Peskin, 'Physics at e^+e^- linear colliders 1. e^+e^- annihilation into pairs., March, 2002.

[15] S. Bethke, 'Standard Model Physics at LEP', arXiv:hep-ex/00010223 v1.

[16] S. Bethke, 'Jet Physics at LEP and World Summary of α_s', arXiv:hep-ex/9812026.

[17] R. Ammar, K. Bloom, 'Mesurement of the total cross section for $e^+e^- \longrightarrow$Hadrons at $\sqrt{s} = 10.52\ Gev$ .CLEO collaboration, year 1998.

[18] D. Bailin and A. Love, 'Introduction to Gauge Field Theory' (IoP and Univ. of Sussex Press, Bristol, 1993).

[19] *E. L. Feinberg, Nuovo Cimento 34A, 391 (1976)
*K. Kajantie and H. Miehinen, Z. Phys. C-Particles and Fields 9, 341 (1981); ibid. 14, 357 (1982).
*R. C. Hwa and K. Kajantie, Phys. Rev. D32, 1109 (1985);
*E. V. Shuryak, Yad. Fiz. 28 796 (1978); and Phys. Rep. 61, 71(1980); Ī
*G. Domokos and J. I. Goldman, Phys. Rev. D23, 203 (1981);

[20] L. D. Mc Lerran and T. Toimela, Phys. Rev. D31, 545 (1985).

[21] See, for example, in Quark Matter, Proceedings of the Sixth International Conference on Ultra-relativistic Nucleus-Nucleus-Quark Matter 1987, edited by H. Satz, H. J. Specht, and R. Stock, in Z. Phys. C-Particles and Fields 38 (1988)

[22] P. V. Landshoff, 'Introduction to equilibrium thermal field theory', hep-ph/9808362.

[23] M. H. Thoma,'New Developments and Applications of Thermal Field Theory', arXiv:hep-ph/0010164.

[24] A.Rakhimov and F.C.Khanna,'Finite temperature amplitudes and reaction rates in thermofield dynamics', Phys. Rev. C, volume 64 064907.

[25] R. Bair, B. Pire and D. Schiff, 'Perturbative aspects of Dilepton Production at finite temperature', LPTHE Orsay 87/57, Septembre 1987.

[26] T.S. Evans, D.A.Steer, 'Wick's theorem at finite temperature', arXiv: hep-ph/9601268.

[27] T. Matsubara, Prog. Theor. Phys. **14,** 351 (1955).

[28] Y. Takahashi and H. Umezawa, 'Thermofield dynamics',Collective Phenomena, Vol. 2, pp.55-8 (1975).

[29] Y. Takahashi and H. Umezawa, 'Thermo field Dynamics', International journal of modern Physics B, Vol. 10, Nos. 1314 (1996) 1755-1805

[30] J. Schwinger, J. Math. Phys. 2, 407 (1961)

[31] *P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. 4, 1 and 12 (1963)
*K. T. Mahanthappa, Phys. Rev. 126, 329 (1962)

[32] L. V. Keldysh, Zh. Eksp. Toer. Fiz. 47, 1515 (1964) [Sov. Phys. JETP 20, 1018 (1965)].

[33] C. Itzykson and J. B. Zuber, 'Quantum Field Theory', McGraw-Hill International Editions, (1988).

[34] R. Bair, B. Pire and D. Schiff, 'Dilepton Production at finite temperature, Perturbative Treatment at Order s', BI-TP 88/20 July 1988.

7

Differential cross section at finite temperature for the q$\bar{\text{q}}$ bound state of a meson $\pi^{\pm}$. Comparison to the zero temperature case.

We consider The q$\bar{\text{q}}$ bound state of a meson π^{+}within a thermal bath that is a color confined QGP medium at finite temperature. We use the canonical formalism of TFD to conduct step by step the calculation process: We compute the differential cross section of the process $q\bar{q} \rightarrow \pi^{+}$ using Ŝ matrix expansion at first order at finite temperature and apply bogoliubov transformations on state operators concerned to evaluate the transition amplitude in the differential cross section. Finally, we compare with the result at zero temperature [1],[2].

7.1 Calculation.

We propose to study the probabilité of formation of a q$\bar{\text{q}}$ bound state of a meson π^{+} at finite temperature in QGP. Much of the work on calculation of reaction rates and decay of particles was given in references [3],[4], Where either Cutkosky rules or calculation of module of elements of the $\hat{S}$ matrix was used. in this paper we try to apply methods of calculation given in this article to QCD at finite temperature. We consider a QGP medium at finite temperature and pick up a quark and antiquark states, then calculate $\hat{S}$ matrix element at first order on the vaccum $\mid O(\beta)\rangle$, and finally evaluate the transition amplitude with the concept of T.F.D. For a reminder,

7. DIFFERENTIAL CROSS SECTION AT FINITE TEMPERATURE FOR THE $Q\bar{Q}$ BOUND STATE OF A MESON $\pi^{\pm}$. COMPARISON TO THE ZERO TEMPERATURE CASE.

we give useful relations of T.F.D in the appendix (see references therein).

let's consider a quark-antiquark bound state of a meson π^+: the quark q is of momentum k_1,energy $\varpi_1$and spin s_1. The antiquark $\bar{q}$ is of momentum k_2,energy $\varpi_2$and spin s_2.The minimal coupling in the interaction lagrangian (wihtout emission of a gluon) is :

$$L_{int} = -ig_s\pi^+(x)\,\bar{q}(x)\,\frac{1}{2}\tau_+\gamma_5 q(x) - ig_s\tilde{\pi}^+(x)\,\_\tilde{q}(x)\,\frac{1}{2}\tau_+\gamma_5\tilde{q}(x) \tag{7.1}$$

where $q(x)$ is either the up or down quark. The initial state is : $\mid i\rangle = c^+_{k_1}(\beta)\,d^+_{k_2}(\beta)\mid O(\beta)\rangle$ and the final state is : $\mid f\rangle = a^+_k(\beta)\mid O(\beta)\rangle$. The $\hat{S}$ matrix element is :

$$\langle f\mid\hat{S}\mid i\rangle = i(-ig_s)\int d^4x\langle f\mid\left[\pi^+(x)\,\bar{q}(x)\,\frac{1}{2}\tau_+\gamma_5 q(x) + \tilde{\pi}^+(x)\,\_\tilde{q}(x)\,\frac{1}{2}\tau_+\gamma_5\tilde{q}(x)\right]\mid i\rangle \tag{7.2}$$

In the center of mass frame of π^+ , we can write : $(\vec{k}_1 = -\vec{k}_2)$

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{c.o.m(i.e\pi^+)} &= \frac{1}{64\pi^2(\varpi_1+\varpi_2)^2}\frac{\mid\vec{k}\mid}{\mid\vec{k}_{1|}}\mid M_{fi}(T)\mid^2 \\ &= \frac{1}{32\pi^2(\varpi_1+\varpi_2)^2}\mid M_{fi}(T)\mid^2 \end{aligned}\tag{7.3}$$

in the rest mass frame of π^+ ,the meson has a mass M ,and each quark has a mass of $\frac{M}{2}$. The scattering amplitude is :

$$\begin{aligned}\langle f\mid\hat{S}\mid i\rangle = g_s\int d^4x\langle O(\beta)\mid a_k(\beta)\times\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3p}{\sqrt{2\varpi_p}}\left[a_pe^{-ipx}+a^+_pe^{ipx}\right]\right]\times \\ \times\left[\sum_s\int d^3p'N_{p'}\left[c^+_{p',s}\bar{u}(p',s)\,e^{ip'x}+d_{p',s}\bar{v}(p',s)\,e^{-ip'x}\right]\right]\times\frac{1}{2}\tau_+\gamma_5 \\ \times\left[\sum_{s'}\int d^3p"N_{p"}\left[c_{p",s'}u(p",s')\,e^{-ip"x}+d^+_{p",s'}v(p",s')\,e^{ip"x}\right]\right] \\ +\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3p}{\sqrt{2\varpi_p}}\left[\tilde{a}_pe^{ipx}+\tilde{a}^+_pe^{-ipx}\right]\right]\times \\ \times\left[\sum_s\int d^3p'N_{p'}\left[\tilde{c}^+_{p',s}\_\tilde{u}(p',s)\,e^{-ip'x}+\tilde{d}_{p',s}\_\tilde{v}(p',s)\,e^{-ip'x}\right]\right]\times\frac{1}{2}\tau_+\gamma_5 \\ \times\left[\sum_{s'}\int d^3p"N_{p"}\left[\tilde{c}_{p",s'}\tilde{u}(p",s')\,e^{-ip"x}+\tilde{d}^+_{p",s'}\tilde{v}(p",s')\,e^{ip"x}\right]\right]c^+_{k_1}(\beta)\,d^+_{k_2}(\beta)\mid O(\beta)\rangle\end{aligned}$$

7.1.1 Preliminary calculation of the first term of the sum.

Let's drop out the integral signs for the moment and consider the bits :

$$I_{kp}(x,\beta) = \langle O(\beta) \mid a_k(\beta)\left[a_p e^{-ipx} + a_p^+ e^{ipx}\right] \mid O(\beta)\rangle$$

and

$$I_{k_1k_2}(x,\beta) = \langle O(\beta) \mid \left[c_{p',s}^+ \bar{u}(p',s) e^{ip'x} + d_{p',s}\bar{v}(p',s) e^{-ip'x}\right] \times \frac{1}{2}\tau_+\gamma_5$$

$$\times \left[c_{p",s'} u(p",s') e^{-ip"x} + d_{p",s'}^+ v(p",s') e^{ip"x}\right] c_{k_1}^+(\beta) d_{k_2}^+(\beta) \mid O(\beta)\rangle$$

Applying the bogoliubov transformations (3.10) [5],[6] on bosons and fermions and the related properties of the action of the annihilation and creation operators over the thermal vaccum (see the appendix), we get

$$I_{kp}(x,\beta) = \langle O(\beta) \mid a_k(\beta)\left(\cosh\theta_p a_p(\beta) + \sinh\theta_p \tilde{a}_p^+(\beta)\right) e^{-ipx}$$

$$+ \left(\cosh\theta_p a_p^+(\beta) + \sinh\theta_p \tilde{a}_p(\beta)\right) e^{ipx} \mid O(\beta)\rangle$$

$$= \langle O(\beta) \mid a_k(\beta)\tilde{a}_p^+(\beta)\sinh\theta_p e^{-ipx} + a_k(\beta) a_p^+(\beta)\cosh\theta_p e^{ipx} \mid O(\beta)\rangle$$

$$= \cosh\theta_k e^{ikx} \tag{7.4}$$

Now the other bit :

$$I_{k_1k_2}(x,\beta) = \langle O(\beta) \mid d_{p',s}\bar{v}(p',s) e^{-ip'x}\frac{1}{2}\tau_+\gamma_5 c_{p",s'} u(p",s') e^{-ip"x} c_{k_1}^+(\beta) d_{k_2}^+(\beta) \mid O(\beta)\rangle$$

$$= \bar{v}(p',s)\frac{1}{2}\tau_+\gamma_5 u(p",s') e^{-i(p'+p")x}\langle O(\beta) \mid \left(\cos\theta_{-p'} d_{p',s}(\beta) + i\sin\theta_{-p'}\tilde{d}_{p',s}^+(\beta)\right) \times$$

$$\times \left(\cos\theta_{p"} c_{p",s'}(\beta) + i\sin\theta_{p"}\tilde{c}_{p",s'}^+(\beta)\right) c_{k_1}^+(\beta) d_{k_2}^+(\beta) \mid O(\beta)\rangle$$

$$= \bar{v}(p',s)\frac{1}{2}\tau_+\gamma_5 u(p",s') e^{-i(p'+p")x}\cos\theta_{-p'}\cos\theta_{p"}\delta(k_1 - p")\delta(k_2 - p')$$

$$= \bar{v}(k_2,s)\frac{1}{2}\tau_+\gamma_5 u(k_1,s') e^{-i(k_1+k_2)x}\cos\theta_{-k_2}\cos\theta_{k_1} \tag{7.5}$$

where we have used the properties and the action of the annihilation and creation operators over the thermal vaccum as well as different commutation relations over the fields (3.19) and (3.20).

7.1.2 Preliminary calculation of the second term of the sum.

let's compute the tilde ($\sim$) term :

$$\langle f \mid \hat{S} \mid i\rangle = \sum_{s,s'} g_s \frac{1}{(2\pi)^{\frac{3}{2}}} \left(\int d^4x \int \frac{d^3p}{\sqrt{2\varpi_p}} \int d^3p' N_{p'} \int d^3p" N_{p"} \right) \times$$

$$\langle O(\beta) \mid a_k(\beta) \left[\tilde{a}_p e^{ipx} + \tilde{a}_p^+ e^{-ipx}\right] * \left[\tilde{c}^+_{p',s} \bar{\tilde{u}}(p',s) e^{ip'x} + \tilde{d}_{p',s} \bar{\tilde{v}}(p',s) e^{-ip'x}\right] \times \frac{1}{2}\tau_+\gamma_5$$

$$\times \left[\tilde{c}_{p",s'} \tilde{u}(p",s') e^{-ip"x} + \tilde{d}^+_{p",s'} \tilde{v}(p",s') e^{ip"x}\right] c^+_{k_1}(\beta) d^+_{k_2}(\beta) \mid O(\beta)\rangle$$

we will consider the following two bits :

$$\tilde{I}_{kp}(x,\beta) = \langle O(\beta) \mid a_k(\beta) \left[\tilde{a}_p e^{ipx} + \tilde{a}_p^+ e^{-ipx}\right] \mid O(\beta)\rangle$$

and

$$\tilde{I}_{k_1k_2}(x,\beta) = \langle O(\beta) \mid \left[\tilde{c}^+_{p',s} \bar{\tilde{u}}(p',s) e^{ip'x} + \tilde{d}_{p',s} \bar{\tilde{v}}(p',s) e^{-ip'x}\right] \times \frac{1}{2}\tau_+\gamma_5$$

$$\times \left[\tilde{c}_{p",s'} \tilde{u}(p",s') e^{-ip"x} + \tilde{d}^+_{p",s'} \tilde{v}(p",s') e^{ip"x}\right] c^+_{k_1}(\beta) d^+_{k_2}(\beta) \mid O(\beta)\rangle.$$

Using Bogoliubov transformations [5],[6] the first bit becomes :

$$\tilde{I}_{kp}(x,\beta) = \langle O(\beta) \mid a_k(\beta) \left[\left(\sinh\theta_p a_p^+(\beta) + \cosh\theta_p \tilde{a}_p(\beta)\right) e^{ipx}\right.$$

$$\left. + \left(\sinh\theta_p a_p(\beta) + \cosh\theta_p \tilde{a}_p^+(\beta)\right) e^{-ipx}\right] \mid O(\beta)\rangle$$

$$= \langle O(\beta) \mid a_k(\beta) a_p^+(\beta) \sinh\theta_p e^{ipx} \mid O(\beta)\rangle = \sinh\theta_k e^{ikx} \tag{7.6}$$

The second bit on developing the calculation becomes :

$$\tilde{I}_{k_1k_2}(x,\beta) = \langle O(\beta) \mid \left[\left(i\sin\theta_{p'} c_{p',s}(\beta) + \cos\theta_{p'} \tilde{c}^+_{p',s}(\beta)\right) \bar{\tilde{u}}(p',s) e^{ip'x}\right.$$

$$\left. + \left(-i\sin\theta_{-p'} d^+_{p',s}(\beta) + \cos\theta_{-p'} \tilde{d}_{p',s}(\beta)\right) \bar{\tilde{v}}(p',s) e^{-ip'x}\right] \times \frac{1}{2}\tau_+\gamma_5$$

$$\times \left[\left(-i\sin\theta_{p"} c^+_{p",s}(\beta) + \cos\theta_{p"} \tilde{c}_{p",s}(\beta)\right) \tilde{u}(p",s') e^{-ip"x}\right.$$

$$\left. + \left(i\sin\theta_{-p"} d_{p",s'}(\beta) + \cos\theta_{-p"} \tilde{d}^+_{p",s'}(\beta)\right) \tilde{v}(p",s') e^{ip"x}\right] \times c^+_{k_1}(\beta) d^+_{k_2}(\beta) \mid O(\beta)\rangle$$

$$= -\sin\theta_{p'} \sin\theta_{-p"} \delta(k_2 - p") \delta(k_1 - p') \bar{\tilde{u}}(p',s) \frac{1}{2}\tau_+\gamma_5 \tilde{v}(p",s') e^{i(p'+p")x}$$

$$= -\sin\theta_{k_1}\sin\theta_{-k_2}e^{-i(k_1+k_2)x}\_\tilde{u}\left(k_1,s\right)\frac{1}{2}\tau_+\gamma_5\tilde{v}\left(k_2,s'\right) \tag{7.7}$$

We used the properties of the action of the thermal operators over the thermal vaccum, the Bogoliubov transformations (3.14) and (3.15) and the commutation relations (3.19) and (3.20). Gathering all the bits (7.4)-(7.7), we come to the following scattering amplitude

$$M_{fi} = \langle f \mid \hat{S} \mid i\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2\pi)^3}\left(\frac{1}{2\varpi_k}\right)\left(\frac{1}{2\varpi_{k_1}}\right)\left(\frac{1}{2\varpi_{k_2}}\right)\left(\frac{M}{2}\right)\times$$

$$\times\sum_{s,s'}g_s\left(\bar{v}\left(k_2,s\right)\frac{1}{2}\tau_+\gamma_5 u\left(k_1,s'\right)e^{-i(k_1+k_2)x}e^{ikx}\cos\theta_{-k_2}\cos\theta_{k_1}\cosh\theta_k\right.$$

$$\left.-\_\tilde{u}\left(k_1,s\right)\frac{1}{2}\tau_+\gamma_5\tilde{v}\left(k_2,s'\right)e^{-i(k_1+k_2)x}e^{ikx}\sin\theta_{k_1}\sin\theta_{-k_2}\sinh\theta_k\right) \tag{7.8}$$

Now, due to the conservation of the 4- momentum and after some algebra, expression (7.8) becomes :

$$\begin{aligned} M_{fi} \quad = \quad & \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2\pi)^3}\left(\frac{M}{2}\right)\left(\frac{1}{2\varpi_k}\right)\left(\frac{1}{2\varpi_{k_1}}\right)\left(\frac{1}{2\varpi_{k_2}}\right)\times \\ & \times\sum_{s,s'}g_s\left(\bar{v}\left(k_2,s\right)\frac{1}{2}\tau_+\gamma_5 u\left(k_1,s'\right)\cos\theta_{-k_2}\cos\theta_{k_1}\cosh\theta_k\right. \\ & \left.-\_\tilde{u}\left(k_1,s\right)\frac{1}{2}\tau_+\gamma_5\tilde{v}\left(k_2,s'\right)\sin\theta_{k_1}\sin\theta_{-k_2}\sinh\theta_k\right) \end{aligned}$$

Then, squaring the scattering amplitude and reorganizing the full expression of the amplitude : we use the tild conjugation rules over the Dirac spinors u and v and take the transpose of the product of these 4-vectors $\bar{v}^* \times u^*$ and similar expressions, we obtain a semi-final result

$$\mid M_{fi}\mid^2 = \frac{1}{(2\pi)^9}\left(\frac{1}{64}\right)\left(\frac{1}{\varpi_k\varpi_{k_1}\varpi_{k_2}}\right)\left(\frac{M}{2}\right)^2\times$$

$$\times g_s^2\sum_{s,s'}\cos^2\theta_{-k_2}\cos^2\theta_{k_1}\cosh^2\theta_k\left(\bar{v}\left(k_2,s\right)\frac{1}{2}\tau_+\gamma_5 u\left(k_1,s'\right)\bar{u}\left(k_1,s'\right)\frac{1}{2}\gamma_5\tau_- v\left(k_2,s\right)\right)$$

$$+\sin^2\theta_{k_1}\sin^2\theta_{-k_2}\sinh^2\theta_k\left(\_\tilde{u}\left(k_1,s\right)\frac{1}{2}\tau_+\gamma_5\tilde{v}\left(k_2,s'\right)\_\tilde{v}\left(k_2,s'\right)\frac{1}{2}\gamma_5\tau_-\tilde{u}\left(k_1,s\right)\right)$$

$$-\left(\cos\theta_{-k_2}\cos\theta_{k_1}\cosh\theta_k\sin\theta_{k_1}\sin\theta_{-k_2}\sinh\theta_k\right)\times$$

7. DIFFERENTIAL CROSS SECTION AT FINITE TEMPERATURE FOR THE $Q\bar{Q}$ BOUND STATE OF A MESON $\pi^{\pm}$. COMPARISON TO THE ZERO TEMPERATURE CASE.

$$\times\left(\bar{v}\left(k_2,s\right)\frac{1}{2}\tau_+\gamma_5 u\left(k_1,s'\right)\bar{u}\left(k_1,s\right)\frac{1}{2}\tau_-\gamma_5 v(k_2,s')\left(k_2,s'\right)+\_\tilde{u}\left(k_1,s\right)\frac{1}{2}\tau_+\gamma_5\tilde{v}\left(k_2,s'\right)\_\tilde{v}\left(k_2,s\right)\frac{1}{2}\tau_-\gamma_5\tilde{u}\left(k_1,s'\right)\right)$$

In the above, we use different normalization conditions given in the appendix for different Dirac spinors u and v, different thermal distributions Eq. (6.4) for the quarks, and the properties: $\frac{1}{2}\tau_+\tau_- = \left(1+\gamma^0\right), {\gamma_5}^2 = 1, {\gamma_5}^t = \gamma_5$. Therefore, the scattering amplitude becomes

$$\begin{aligned}
\mid M_{fi}\mid^2 &= \frac{1}{(2\pi)^9}\left(\frac{1}{64}\right)\left(\frac{1}{\varpi_k\varpi_{k_1}\varpi_{k_2}}\right)\left(\frac{M}{2}\right)^2 g_s^2\times\\
&\times\left[\left(n_q\left(\frac{M}{2}\right)-1\right)\left(n_{\bar{q}}\left(\frac{M}{2}\right)-1\right)\left(n_{\pi^+}(M)+1\right)+n_q\left(\frac{M}{2}\right)n_{\bar{q}}\left(\frac{M}{2}\right)n_{\pi^+}(M)\right.\\
&\left.-2\sqrt{\left(1-n_q\left(\frac{M}{2}\right)\right)}\sqrt{\left(1-n_{\bar{q}}\left(\frac{M}{2}\right)\right)}\sqrt{\left(n_{\pi^+}(M)+1\right)}\sqrt{n_q\left(\frac{M}{2}\right)}\sqrt{n_{\bar{q}}\left(\frac{M}{2}\right)}\sqrt{n_{\pi^+}(M)}\right]
\end{aligned}$$

Therefore the differential cross section can be written as :

$$\left(\frac{d\sigma}{d\Omega}\right)_{c.o.m(\pi^+)} = 2g_s^2\frac{1}{(4\pi)^{11}}\left(\frac{1}{\varpi_k\varpi_{k_1}\varpi_{k_2}}\right)\left(\frac{1}{\left(\varpi_{k_1}+\varpi_{k_2}\right)^2}\right)\left(\frac{M}{2}\right)^2\times$$

$$\times\left[\left(n_q\left(\frac{M}{2}\right)-1\right)\left(n_{\bar{q}}\left(\frac{M}{2}\right)-1\right)\left(n_{\pi^+}(M)+1\right)+n_q\left(\frac{M}{2}\right)n_{\bar{q}}\left(\frac{M}{2}\right)n_{\pi^+}(M)\right.$$
$$\left.-2\sqrt{\left(1-n_q\left(\frac{M}{2}\right)\right)}\sqrt{\left(1-n_{\bar{q}}\left(\frac{M}{2}\right)\right)}\sqrt{\left(n_{\pi^+}(M)+1\right)}\sqrt{n_q\left(\frac{M}{2}\right)}\sqrt{n_{\bar{q}}\left(\frac{M}{2}\right)}\sqrt{n_{\pi^+}(M)}\right]$$

To compare with the zero temperature case, we can write the final result in the center of mass frame of π^+:

$$\begin{aligned}
\frac{\left(\frac{d\sigma}{d\Omega}\right)_{(T\neq 0)}}{\left(\frac{d\sigma}{d\Omega}\right)_{(T=0)}} &= \left[\left(n_q\left(\frac{M}{2}\right)-1\right)^2\left(n_{\pi^+}(M)+1\right)+n_q^2\left(\frac{M}{2}\right)n_{\pi^+}(M)\right.\\
&\left.-2\left(1-n_q\left(\frac{M}{2}\right)\right)n_q\left(\frac{M}{2}\right)\sqrt{n_{\pi^+}(M)}\sqrt{\left(n_{\pi^+}(M)+1\right)}\right]\\
&= \left(\left(n_q\left(\frac{M}{2}\right)-1\right)\sqrt{\left(n_{\pi^+}(M)+1\right)}+n_q\left(\frac{M}{2}\right)\sqrt{n_{\pi^+}(M)}\right)^2 \quad (7.9)
\end{aligned}$$

Substituting for the bosonic and fermionic distributions , we get

$$\frac{\left(\frac{d\sigma}{d\Omega}\right)_{(T\neq 0)}}{\left(\frac{d\sigma}{d\Omega}\right)_{(T=0)}} = \left[\left(\frac{1}{\left(e^{\beta\frac{M}{2}}+1\right)}-1\right)^2\left(\frac{1}{(e^{\beta M}-1)}+1\right)+\frac{1}{\left(e^{\beta\frac{M}{2}}+1\right)^2}\frac{1}{(e^{\beta M}-1)}\right.$$
$$\left.-2\left(1-\frac{1}{\left(e^{\beta\frac{M}{2}}+1\right)}\right)\frac{1}{\left(e^{\beta\frac{M}{2}}+1\right)}\frac{1}{(e^{\beta M}-1)^{\frac{1}{2}}}\frac{e^{\beta\frac{M}{2}}}{(e^{\beta M}-1)^{\frac{1}{2}}}\right]$$
$$= \tanh\left(\beta\frac{M}{4}\right) = \tanh\left(\frac{35}{T}\right). \tag{7.10}$$

for $M = m_{\pi^+} \simeq 140\ Mev$.

7.2 Conclusion and interpretation of the graph.

We notice from the curve (Fig. 4) that the formation of π^+ bound states build up does diminish when temperature goes up. Beyond a certain temperature not high enough, the probability of formation of π^+ bound states is not decreasing rapidely. For high temperatures , the plotting drops at a null value . We consider the centre of mass energy of the pion.

7.3 Parity conserving part. Parity violating part.

For pseudo-vectors, the full current will be written as

$$J_\mu(x)J^\mu(0) = Q^2\left[\bar{q}(x)\left(\gamma_\mu + a\gamma_\mu\gamma_5\right)q(x)\bar{q}(0)\left(\gamma^\mu + a\gamma^\mu\gamma^5\right)q(0)\right]$$

then

$$J_\mu(x)J^\mu(0) = Q^2\left[\bar{q}(x)\gamma_\mu q(x)\bar{q}(0)\gamma^\mu q(0)\right] + a^2Q^2\left[\bar{q}(x)\left(\gamma_\mu\gamma_5\right)q(x)\bar{q}(0)\left(\gamma^\mu\gamma^5\right)q(0)\right] +$$
$$+ aQ^2\left[\bar{q}(x)\gamma_\mu q(x)\bar{q}(0)\gamma^\mu\gamma^5 q(0)\right] + aQ^2\left[\bar{q}(x)\gamma_\mu\gamma_5 q(x)\bar{q}(0)\gamma^\mu q(0)\right] \tag{7.11}$$

the second line is the usual part already calculated, and the second line is the parity-violating part. As for Eq.(6.24) for the total cross section of the parity-violating part could be written as

7. DIFFERENTIAL CROSS SECTION AT FINITE TEMPERATURE FOR THE $Q\bar{Q}$ BOUND STATE OF A MESON $\pi^{\pm}$. COMPARISON TO THE ZERO TEMPERATURE CASE.

$$\begin{aligned}
\sigma_{tot}^{NC}(T \neq 0) &= \frac{8\pi^2\alpha^2}{3(q^2)^2}\left[2Im\left(i\int d^4xe^{iqx}\langle O(\beta)\mid T\left(J_\mu(x)J^\mu(0)\right)\mid O(\beta)\rangle\right)\right] = \\
&= \frac{8\pi^2\alpha^2}{3(q^2)^2}\left[2Imi\int d^4xe^{iqx}\langle O\mid T\left(J_\mu(x)J^\mu(0)\right)\mid O\rangle\right] - Thermaldependentcontribution \\
&= \sigma_{tot}^{NC}(T=0) - Thermaldependentcontribution
\end{aligned}$$

Let's compute the temperature dependent contribution :

$$\begin{aligned}
f(T) &= \frac{16\pi^2a\alpha^2Tr\left(Q^2\right)}{3(q^2)^2}2Im(i\int d^4xe^{iqx}\frac{1}{(2\pi)^4}\int d^4ke^{ikx}\frac{1}{(2\pi)^4}\int d^4k'^{e^{-ik'x}} \\
&\left(4\pi^2\right)Tr\left\{\left(\bar{k}+m\right)\gamma_\mu\left(\bar{k}'+m\right)\gamma_\nu\gamma_5\right\}n_F\left(\left|k^0\right|\right)n_F\left(\left|k^0-q^0\right|\right)\delta\left(k^2-m^2\right)\delta\left((k-q)^2-m^2\right))
\end{aligned} \tag{7.12}$$

the latter contains a factor of two pieces of two equal terms of violating-part in the third line of Eq.(7.11). Let's calculate the trace

$$\begin{aligned}
Tr\left\{\left(\bar{k}+m\right)\gamma_\mu\left(\bar{k}'+m\right)\gamma_\nu\gamma_5\right\} &= Tr\left\{\left(\bar{k}\gamma_\mu+m\gamma_\mu\right)\left(\bar{k}'\gamma_\nu\gamma_5+m\gamma_\nu\gamma_5\right)\right\} \\
&= Tr\left\{\bar{k}\gamma_\mu\bar{k}'\gamma_\nu\gamma_5+m\bar{k}\gamma_\mu\gamma_\nu\gamma_5+m\gamma_\mu\bar{k}'\gamma_\nu\gamma_5+m^2\gamma_\mu\gamma_\nu\gamma_5\right\} \\
&= Tr\left(\bar{k}\gamma_\mu\bar{k}'\gamma_\nu\gamma_5\right) = k_\rho k'_\sigma Tr\left(\gamma_5\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\right) = 4i\varepsilon_{\rho\mu\sigma\nu}k_\rho k'_\sigma.
\end{aligned} \tag{7.13}$$

Upon replacing Eq.(7.13) in Eq.(7.12), the imaginery part of the temperature-dependent parity-violating part is (in centre of mass frame)

$$\begin{aligned}
f(T) &= \frac{16\pi^2a\alpha^2Tr\left(Q^2\right)}{3(q^2)^2}\int d^4xe^{iqx}\frac{1}{(2\pi)^4}\int d^4ke^{-ikx}\frac{1}{(2\pi)^4}\int d^4k'e^{+ik'x}\left(8\pi^2\right)\left(4g^{\mu\nu}\varepsilon_{\rho\mu\sigma\nu}k^\rho k'^\sigma\right) \\
&n_F\left(\left|k^0\right|\right)n_F\left(\left|k^0-q^0\right|\right)\delta\left(k_0^2-\left(\vec{k}^2+m^2\right)\right)\delta\left(k_0'^2-\left(\vec{k}'^2+m^2\right)\right) \\
&= \frac{8a\alpha^2Tr\left(Q^2\right)}{3(q^2)^2}\int d^4k\left(4\varepsilon^\mu_{\rho\mu\sigma}k^\rho k'^\sigma\right)n_F\left(\left|k^0\right|\right)n_F\left(\left|k^0-q^0\right|\right)\left\{\frac{\delta\left(k_0-\sqrt{\left(\vec{k}^2+m^2\right)}\right)}{2\sqrt{\left(\vec{k}^2+m^2\right)}}\right\}^2 \\
&= \frac{8a\alpha^2Tr\left(Q^2\right)}{3(q^2)^2}\int d^3k\int dk_0\left(4\varepsilon^\mu_{\rho\mu\sigma}\left(\frac{g^{\rho\sigma}}{4}\right)kk'\right)n_F\left(\left|k^0\right|\right)n_F\left(\left|k^0-q^0\right|\right)\left\{\frac{\delta\left(k_0-\epsilon\right)}{2\epsilon}\right\}^2 \\
&= \frac{8a\alpha^2Tr\left(Q^2\right)}{3(q^2)^2}\int d^3k\left(4kk'\right)n_F\left(\epsilon\right)n_F\left(\epsilon\right)\frac{\delta\left(k_0-\epsilon\right)}{4\epsilon^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{32\pi a\alpha^2 Tr\left(Q^2\right)}{3\left(q^2\right)^2}\int\sqrt{k_0^2-m^2}k_0dk_0\frac{\left(m^2-2\epsilon^2\right)}{\left(1+e^{\beta\epsilon}\right)^2}\frac{\delta\left(k_0-\epsilon\right)}{\epsilon^2}\\
&= \frac{32\pi a\alpha^2 Tr\left(Q^2\right)}{3\left(q^2\right)^2}\left(1-\left(\frac{m}{\epsilon}\right)^2\right)^{1/2}\left(2\epsilon^2\right)\left(1-\frac{1}{2}\left(\frac{m}{\epsilon}\right)^2\right)\left(\frac{1}{\left(1+e^{\beta\epsilon}\right)^2}\right) \qquad (7.14)
\end{aligned}
$$

putting $\epsilon=\frac{\sqrt{s}}{2}, q^2=s=4\epsilon^2$ we get

$$
f(T)=\frac{32\pi a\alpha^2 Tr\left(Q^2\right)}{3\left(q^2\right)^2}\left(1-\left(\frac{2m}{\sqrt{s}}\right)^2\right)^{1/2}\frac{q^2}{2}\left(1-\frac{1}{2}\left(\frac{2m}{\sqrt{s}}\right)^2\right)\left(\frac{1}{\left(1+e^{\beta\frac{\sqrt{s}}{2}}\right)^2}\right)
$$

on the other hand

$$
\sigma_{tot}^{NC}\left(T=0\right)=\frac{2a\alpha^2}{3\left(q^2\right)^2}\mu^{\varepsilon}Tr\left(Q^2\right)\left(m^2-\frac{1}{6}q^2\right)
$$

Now gathering all the results we get the final result :

$$
\begin{aligned}
R_{NC}(\beta,s) &= \frac{\sigma_{tot}^{NC}(T\neq 0)}{\sigma_{tot}^{NC}(T=0)}\\
&= 1-\frac{\left[\frac{32\pi a\alpha^2 Tr\left(Q^2\right)}{3\left(q^2\right)^2}\right]}{\left[\frac{2a\alpha^2 Tr\left(Q^2\right)\mu^{\varepsilon}}{3\left(q^2\right)^2}\right]}\frac{\frac{q^2}{2}}{\left(\frac{1}{6}q^2-m^2\right)}\left(1-\left(\frac{2m}{\sqrt{s}}\right)^2\right)^{1/2}\left(1-\frac{1}{2}\left(\frac{2m}{\sqrt{s}}\right)^2\right)\left(\frac{1}{\left(1+e^{\beta\epsilon}\right)^2}\right)\\
&= 1-\frac{48\pi}{\mu^{\varepsilon}}\frac{1}{\left(1-6\left(\frac{m}{\sqrt{s}}\right)^2\right)}\left(1-\left(\frac{2m}{\sqrt{s}}\right)^2\right)^{1/2}\left(1-\frac{1}{2}\left(\frac{2m}{\sqrt{s}}\right)^2\right)\left(\frac{1}{\left(1+e^{\beta\epsilon}\right)^2}\right) \qquad (7.15)
\end{aligned}
$$

substituting for the centre mass energy we finally get :

$$
\mathbf{R}_{NC}(\beta,\mathbf{s}) == \mathbf{1}-\frac{\mathbf{48}\pi}{\mu^{\varepsilon}}\frac{\mathbf{1}}{\left(\mathbf{1}-\mathbf{6}\left(\frac{\mathbf{m}}{\sqrt{\mathbf{s}}}\right)^{\mathbf{2}}\right)}\left(\mathbf{1}-\left(\frac{\mathbf{2m}}{\sqrt{\mathbf{s}}}\right)^{\mathbf{2}}\right)^{\mathbf{1/2}}\left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\mathbf{2m}}{\sqrt{\mathbf{s}}}\right)^{\mathbf{2}}\right)\left(\frac{\mathbf{1}}{\left(\mathbf{1}+\mathbf{e}^{\beta\frac{\sqrt{\mathbf{s}}}{\mathbf{2}}}\right)^{\mathbf{2}}}\right) \qquad (7.16)
$$

7. DIFFERENTIAL CROSS SECTION AT FINITE TEMPERATURE FOR THE $Q\bar{Q}$ BOUND STATE OF A MESON $\pi^{\pm}$. COMPARISON TO THE ZERO TEMPERATURE CASE.

Bibliography

[1] Claude Itzykson and Jean-Bernard Zuber, *"Quantum field theory"*, McGraw-Hill International Editions, (1988).

[2] F. Mandl and G.shaw , "*Quantum field theory*" , John Wiley and Sons , (1991).

[3] A. Rakhimov and F.C. Khanna, *"Finite temperature amplitudes and reaction rates in thermofield dynamics"*, Phys. Rev. C, volume 64 064907.

[4] P. V. Landshoff, *"Introduction to equilibrium thermal field theory"* ,hep-ph/9808362.

[5] Y.Takahashi and H.Umezawa , *"Thermo Field Dynamics"* , Collective Phenomena, Vol. 2. pp 55-80 (1975).

[6] A. Das, *"Finite temperature field theory"* , World Scientific , (1997).

8

Fragmentation Functions.

8.1 Preliminaries: Fragmentation Functions, an overview.

To account for different processes which take place during quark or gluon fragmentation into hadrons , we require a definition for a function analougous to structure function of a quark inside a hadron. It should be pointed out that these functions have been defined in a general framework by many authors namely Collins , Stermann and Soper[1], [2],[3],[4]. Their definitions are universal and extracted from specific processes (e.g. e^+e^- annihilation, Drel-Yang processes, etc...). We give here an overview or a sketch of the fragmentation functions in QCD. Imagine a fast moving proton between x^+ planes in the infinit momentum frame. Such a case needs to be worked in field theory near the light cone. We use light cone coordinates defined by [5]

$$x^{\pm} = \frac{\left(x^0 \pm x^3\right)}{\sqrt{2}}, P^{\pm} = \frac{\left(P^0 \pm P^3\right)}{\sqrt{2}}$$

where :$P^+ \gg 1, P^-$ is small enough and $\vec{P}_T = \vec{0}$, for the hadron and $x^{\pm}$ are the light cone coordinates of the hadron.The quark field moving on the x^+ plane may be defined as

$$\Lambda_+ Q\left(x^+, x^-, x_T\right) = \frac{1}{(2\pi)^3} \int_0^{\infty} \frac{dk^+}{2k^+} \int d\vec{k}_T \sum_s \left(\sqrt{2}k^+\right)^{\frac{1}{2}} \left\{c_{k,s} u\left(k,s\right) e^{-\left(ik^+ x_- - \vec{k}_T.\vec{x}_T\right)} + d^+_{k,s} v\left(k,s\right) e^{+\left(ik^+ x_- - \vec{k}_T.\vec{x}_T\right)}\right\} \quad (8.1)$$

where $\Lambda_+ = \frac{1}{2}\gamma^-\gamma^+$ is the projection operator on the x^+ plane and $Q\left(x^+, x^-, x_T\right)$ defines the quark field at x^+, x^-,and x_T . Now, the hadron state is defined as $\mid H\rangle =\mid$

$P^{+}, \vec{P}_T\rangle$. Then we construct the unrenormalized fragmentation function for a quark of flavor j inside a hadron as

$$D_{H/Q}(z) = \frac{1}{2z(2\pi)^3} \int d\vec{k}_T \sum_s \langle P'^{+}, \vec{P}'_T \mid N\left(zk^{+}, k_T, s, x^{+}\right) \mid P^{+}, \vec{P}_T\rangle \tag{8.2}$$

where

$$N\left(zk^{+}, k_T, s, x^{+}\right) = c_j^{+}\left(zk^{+}, k_T, s, x^{+}\right) c_j\left(zk^{+}, k_T, s, x^{+}\right)$$

is the quark number operator for flavor j and z is defined as the boost-invariant longitudinal momentum :

$$z = \frac{P^{+}}{k_j^{+}}, with 0 \prec z \prec 1.$$

With these definitions, we find

$$D_{H/Q}(z) = \frac{1}{4\pi} \int dy_{-} e^{-\frac{iP^{+}y^{-}}{z}} \langle P^{+}, \vec{0}_T \mid \bar{Q}_j\left(0, y^{-}, \vec{0}_T\right) \gamma^{+} Q_j\left(0, 0, \vec{0}_T\right) \mid P^{+}, \vec{0}_T\rangle \tag{8.3}$$

To make the formule gauge invariant , we work in the gauge $A^{+} = 0$. Therefore, the gauge invariant definition is

$$D_{H/Q}(z) = \frac{1}{4\pi} \int dy_{-} e^{-\frac{iP^{+}y^{-}}{z}} \langle P^{+}, \vec{0}_T \mid \bar{Q}_j\left(0, y^{-}, \vec{0}_T\right) \gamma^{+} \mathbf{O}_0 Q_j\left(0, 0, \vec{0}_T\right) \mid P^{+}, \vec{0}_T\rangle \tag{8.4}$$

where

$$\mathbf{O}_0\left(y^{-}\right) = P \exp\left(-ig_s \int_0^{y^{-}} dx_\mu A^\mu(x)\right)$$

is a path ordered product called the Wilson line and represents a phase factor. Another approch to fragmentation functions can be found through their evolutions with the probing scale. The resultant Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [6],[7] have been tested againt experiment in $e^{+}e^{-}$ annihilation processes and can even be tested to measure the scale dependence of the running strong coupling constant (see the case of heavy flavoured hadron production in $e^{+}e^{-}$ collisions [7]).The fragmentation functions, as well as the structure functions, must be measured at some energy scale, and their value at any other scale can be obtained using the Altarelli-Parisi evolution equation. What is measured experimentally in $e^{+}e^{-}$ annihilations is the probability of observing a hadron from the process $ee \longrightarrow \gamma^{*} \longrightarrow hadrons$ carrying a certain fraction z of the centre of mass energy ,

$$z = \frac{2E_h}{Q}$$

where E_h is the hadron enegy and Q is the photon or the centre of mass energy. Parton production is followed by hadronisation and therefore formation of collimated jets of hadrons after collision within a heat bath. The distribution of hadrons inside a jet is known as the jet fragmentation function. We can approch this definition by using the following approach. The inclusive hadron cross section is related to the parton differential cross sections according to

$$d\sigma^h(z,Q^2) = \left(\frac{d\sigma^q}{dy}\right) dyD_q^h(x)dx + \left(\frac{d\sigma^{\bar{q}}}{dy}\right) dyD_{\bar{q}}^h(x)dx + \left(\frac{d\sigma^g}{dy}\right) dyD_g^h(x)dx, \quad (8.5)$$

this formula is understood as a product of probabilities where $(d\sigma^q/dy)dy$ is the probability of finding a quark with energy

$$E_q = \frac{1}{2}yQ,$$

and $D_q^h(x)dx$ is the probability that a quark of energy E_q fragments into a hadron carrying fractional energy,

$$x = \frac{E_h}{E_q}.$$

Similarly $(d\sigma^g/dy)dy$ is the probability of finding a gluon with energy $E_g = yQ/2$ and $D_g^h(x)$ is the gluon fragmentation function. The experimental fractional variable z is related to the parton variables x and y by

$$x = \frac{z}{y},$$

and $0 \leq x \leq 1$ implies that $z \leq y \leq 1$.Thus,

$$\frac{d\sigma^h}{dz}(Q^2) = 3\sigma(\mu\mu)\int_z^1 \frac{dy}{y}\{_{i=1}^{n_f} e_{q_i}^2 \left[D_{q_i}^h\left(\frac{z}{y}\right) + D_{\bar{q}_i}^h\left(\frac{z}{y}\right)\right]$$

$$\left[\left(1+\frac{\alpha_s}{\pi}\right)\delta\left(1-y\right) + \frac{\alpha_s}{2\pi}P_{q\longrightarrow qg}(y)\log\left(\frac{Q^2}{m^2}\right) + \alpha_s f_q^{e^+e^-}(y)\right]$$

$$+ 2_{i=1}^{n_f} e_{q_i}^2 D_g^h\left(\frac{z}{y}\right)\left[\frac{\alpha_s}{2\pi}P_{q\longrightarrow gq}(y)\log\left(\frac{Q^2}{m^2}\right) + \alpha_s f_g^{e^+e^-}(y)\right]\}, \quad (8.6)$$

where the sum over n_f quark flavors is assumed and where ,

$$P_{q\longrightarrow gq}(x) = P_{q\longrightarrow qg}(1-x) = \frac{4}{3}\left\{\frac{1+(1-x)^2}{(x)_+} + \frac{3}{2}\delta\left(x\right)\right\},$$

is the splitting function, and,

$$\alpha_s f_g^{e^+e^-}(x) = \frac{2\alpha_s}{3\pi}\left\{2\left(1+(1-x)^2\right)\left(\frac{\log(x)}{x}\right)_+ - 2x - \frac{7}{4}\delta(x)\right\},$$

$$\int_0^1 \left[f_q^{e^+e^-}(x) + f_g^{e^+e^-}(x)\right] dx = 0,$$

are the "little f" functions. $\sigma(\mu\mu)$ is the $\gamma^* \longrightarrow \mu^+\mu^-$ rate in μ production in e^+e^- collisions. The fragmentation functions provide information on how a parton (quark or gluon) decays into a hadron. This is identically applied to a QGP in a hot medium. Nevertheless, at zero temperature, the absence of an energy scale means that the fragmentation functions depend only on the scaling variable z. At finite temperature, the thermal average introduces dependence on the temperature as well. Therefore, the fragmentation function will depend on two parameters : the temperature T, and the energy scale, z, the absolute momenta of the initial partons. The energy conservation implies the sum rule

$$\int_0^1 xD_{q_i}^h(x)\,dx = 1, \tag{8.7}$$

and,

$$\int_0^1 xD_g^h(x)\,dx = 1. \tag{8.8}$$

We now define the fragmentation function to order α_s as

$$D_q^h(z,Q^2) = \int_z^1 \frac{dy}{y}\{D_{q_i}^h\left(\frac{z}{y}\right)$$

$$\left[\delta(1-y) + \frac{\alpha_s}{\pi}P_{q\longrightarrow qg}(y)\log\left(\frac{Q^2}{m^2}\right) + \alpha_s f_q^{e^+e^-}(y)\right]$$

$$+ D_g^h\left(\frac{z}{y}\right)\left[\frac{\alpha_s}{2\pi}P_{q\longrightarrow gq}(y)\log\left(\frac{Q^2}{m^2}\right) + \alpha_s f_g^{e^+e^-}(y)\right]\}. \tag{8.9}$$

In the following, we try to give an approximation of the fragmentation function at $T = 0$ and compare our results (see curves : (Fig. 6);(Fig. 7)) with J. P. Ma's [8]. In the next section, we try to exhibit the dependence on temperature of these fragmentation functions since much is known about the scale energy z between the hadron and the initial parton. Variation with temperature of the fragmentation is an important information about the formation of quark-gluon plasma in a heavy-ion collision.

8.2 Applications:

8.3 Attempt of a direct calculation of parton fragmentation functions at zero temperature using definitions : Case of a fast moving quark or a gluon fragmenting into a spinless HADRON $q\bar{q}$.

In this section, we use the calculation of parton fragmentation functions from définitions at $T = 0$. The fragmentation functions computed in this work is extracted from specific processes (i.e. Drell-Yang process, ee$^+$annihilation ...etc). It should be noted that these fragmentation functions are already given by Collins and Soper (see references [3],[4]) and their definitions are universal and independent of the process. Some authors (e.g. J.P.Ma [8]) use Feynman diagram technique to derive the fragmentation function for heavy quarks inside heavy hadrons in effective theories such as the HQET. In this section we try to give an attempt of an expression of a given fragmentation function at finite temperature for a hadron as a bound state of two heavy quarks. We examine the case of a fast moving quark or a gluon fragmenting into a hadron via the exchange of a virtual photon in the above processes.

8.3.1 Calculation.

In the infinite momentum frame the momentum of the incoming hadron is almost only in the P^+ direction $P_\mu \approx (P^+, 0, 0)$ .It is convenient to work in the light cone coordinate system, defined as

$$a_\mu = (a_+, a_-, a_{\mathbf{T}})\, where, a_\pm = \frac{(a_0 \pm a_3)}{\sqrt{2}} and a_{\mathbf{T}} = (a_1, a_2)$$

the inner product of two vactors is given by

$$a.b = a_+ b_- + a_- b_+ - a_{\mathbf{T}} b_{\mathbf{T}}$$

The fragmentation function for a spinless hadron is defined as

$$D_{H/Q}(z) = \frac{z}{4\pi} \int dx^- \exp\left(-iP^+ x^-/z\right) \times \left(\frac{1}{3}\right) Tr_{color} \left(\frac{1}{2}\right) Tr_{Dirac}$$

$$\left[n.\gamma\langle 0\mid Q\left(0\right)P\exp\left(-ig_s\int_0^{x^-}dx_\mu A^\mu\left(x\right)\right)a_H^+\left(P^+,\mathbf{0_T}\right)a_H\left(P^+,\mathbf{0_T}\right)\bar{Q}\left(0,x^-,\mathbf{0_T}\right)\mid 0\rangle\right] \tag{8.10}$$

where

$$P\exp\left(-ig_s\int_0^{x^-}dx_\mu A^\mu\left(x\right)\right) \tag{8.11}$$

is called the Wilson line, which describes a path ordered phase factor of the parton (quark or gluon) moving from 0 to x^-. A field operator creates a parton at position 0 coming from infinity (∞) and another field annihilates it at position x^-. Therefore we should parametrize the world line of the fragmenting parton as the following path

$$x_\mu = n_\mu t, for - \infty\langle t\langle 0 and x_\mu = x^- t for 0\langle t\langle 1 \tag{8.12}$$

where we choose x_μ light-like

$$x_\mu = \left(0,x^-,\mathbf{0}\right), and n_\mu = \frac{P_\mu}{M} = \left(P^+,P^-,\mathbf{0}\right)/M time - like, such that n^2 = \frac{P^2}{M^2} = 1.$$

where,

$$A_\mu(x) = A_\mu^a(x)T^a, A_\mu^a(x)$$

is the gluon field and T^a is the generators of the $SU\left(3\right)$ algebra of the color field of the gluons $\left(a = 8 gluons\right)$.

8.3.2 Developing the calculations.

On substituting the quark fields in Dirac representation and writing out explicitely the hadronique states, equation (8.10) becomes :

$$D_{H/Q}(z) = \frac{z}{4\pi}\int dx^-\exp\left(-iP^+x^-/z\right)\times\left(\frac{1}{N_c}\right)Tr_{color}\left(\frac{1}{2}\right)Tr_{Dirac}$$

$$\left[\left(n.\gamma\right)\langle 0\mid Q\left(0\right)P\exp\left(-ig_s\int_0^{x^-}dx_\mu A^\mu\left(x\right)\right)a_{\mathbf{H}}\left(P^+,\mathbf{0_T}\right)a_{\mathbf{H}}^\dagger\left(P^+,\mathbf{0_T}\right)\bar{Q}\left(0,x^-,\mathbf{0_T}\right)\mid 0\rangle\right]$$

$$= \frac{z}{4\pi}\int dx^-\exp\left(-iP^+x^-/z\right)\times\left(\frac{1}{N_c}\right)Tr_{color}\left(\frac{1}{2}\right)Tr_{Dirac}$$

$$\left(n.\gamma\right)\langle 0\mid\left[\sum_s\int d^3\vec{p}N_p\left(c_{p,s}u\left(p,s\right)+d_{p,s}^\dagger v\left(p,s\right)\right)\right]P\exp\left(-ig_s\int_0^{x^-}dx_\mu A^\mu\left(x\right)\right)\times$$

$$\times\chi_{s_1s_2}^2 d\left(p_2^+,s_2\right)c\left(p_1^+,s_1\right)c^\dagger\left(p_1^+,s_1\right)d^\dagger\left(p_2^+,s_2\right)\times$$

$$\times \left[\sum_{s'} \int d^3\vec{p}' N_{p'} \left(c^{\dagger}_{p',s'} \bar{u}\left(p',s'\right) e^{ip'^{+}.x^{-}} + d_{p',s'} \bar{v}\left(p',s'\right) e^{-ip'^{+}.x^{-}} \right)\right] \mid 0\rangle$$

substituting for the wilson line to second order, we come to

$$D_{H/Q}(z) = \frac{z}{4\pi} \int dx^{-} \exp\left(-iP^{+}x^{-}/z\right) \times \left(\frac{1}{N_c}\right) Tr_{color} \left(\frac{1}{2}\right) Tr_{Dirac} \times$$

$$\times (n.\gamma) \langle 0 \mid \left[\sum_{s} \int d^3\vec{p} N_p \left(c_{p,s} u\left(p,s\right)\right)\right] \times \left(1 + ig_s \int_0^{x^-} dx_\mu A^\mu\left(x\right) + \left(ig_s\right)^2 \int_0^{x^-} dx_\mu \int_0^{x_\mu} dy_\upsilon A^\mu\left(x\right) A^\upsilon$$

$$\times \chi^2_{s_1 s_2} d\left(p_2^+, s_2\right) c\left(p_1^+, s_1\right) c^\dagger\left(p_1^+, s_1\right) d^\dagger\left(p_2^+, s_2\right) \times \left[\sum_{s'} \int d^3\vec{p}' N_{p'} \left(c^{\dagger}_{p',s'} \bar{u}\left(p',s'\right) e^{ip'^{+}.x^{-}} \right)\right] \mid 0\rangle \tag{8.13}$$

Gathering all the terms, the last equation (8.13) becomes

$$\Longrightarrow D_{H/Q}(z) = \frac{z}{4\pi} \int dx^{-} \exp\left(-iP^{+}x^{-}/z\right) \times \left(\frac{1}{N_c}\right) Tr_{color} \left(\frac{1}{2}\right) Tr_{Dirac} \times$$

$$\times (n.\gamma) \langle 0 \mid \left(1 - g_s^2 \int_0^{x^-} dx_\mu \int_0^{x_\mu} dy_\upsilon A^\mu\left(x\right) A^\upsilon\left(y\right)\right)$$

$$\times \sum_{s,s'} \int d^3\vec{p} \int d^3\vec{p}' N_p N_{p'} \chi^2_{s_1 s_2} c_{p,s} d\left(p_2^+, s_2\right) c\left(p_1^+, s_1\right) c^\dagger\left(p_1^+, s_1\right) d^\dagger\left(p_2^+, s_2\right) c^\dagger_{p',s'} u\left(p,s\right) \bar{u}\left(p',s'\right) e^{ip'^{+}.x^{-}} \tag{8.14}$$

When sandwhished between the vaccum, the first order term of Wilson line cancels.

8.3.3 Investigation of the formula.

let's try to investigate this formula (8.14). Using commutation relations (3.19) and (3.20), we get

$$D_{H/Q}(z) = \frac{z}{4\pi} \int dx^{-} \exp\left(-iP^{+}x^{-}/z\right) \times \left(\frac{1}{2}\right) Tr_{Dirac} \times$$

$$\times (n.\gamma) \langle 0 \mid \left(1 - g_s^2 \int_0^{x^-} dx_\mu \int_0^{x_\mu} dy_\upsilon A_a^\mu\left(x\right) A_b^\upsilon\left(y\right)\right) \left(\frac{1}{N_c}\right) Tr_{color} \left\{T^a T^b\right\}$$

$$\times \sum_{s,s'} \int d^3\vec{p} \int d^3\vec{p}' N_p N_{p'} \chi^2_{s_1 s_2} \left\{ c_{p,s} c^\dagger_{p',s'} - c_{p,s} c^\dagger\left(p_1^+, s_1\right) c\left(p_1^+, s_1\right) c^\dagger_{p',s'} \right\} u\left(p,s\right) \bar{u}\left(p',s'\right) e^{ip'^{+}.x^{-}} \mid 0\rangle$$

using relations (3.19) and (3.20) again

$$D_{H/Q}(z) = \frac{z}{4\pi} \int dx^{-} \exp\left(-iP^{+}x^{-}/z\right) \times \left(\frac{1}{2}\right) Tr_{Dirac} \times$$

$$\times(n.\gamma)\langle 0\mid\left(1-g_s^2\int_0^{x^-}dx_\mu\int_0^{x_\mu}dy_v A_a^\mu(x)\,A_b^v(y)\right)\left(\frac{3}{2}\right)\left(\frac{N_c^2-1}{N_c}\right)\times\sum_{s,s'}d^3\vec{p}d^3\vec{p}'N_pN_{p'}\chi_{s_1s_2}^2\times$$

$$\times\left[\delta\left(\vec{p}-\vec{p}'\right)\delta_{s,s'}\left\{\delta\left(\vec{p}-\vec{p}_1^+\right)\delta_{s,s_1}-c^\dagger\left(p_1^+,s_1\right)c_{p,s}\right\}\left\{\delta\left(\vec{p}'-\vec{p}_1^+\right)\delta_{s',s_1}-c_{p',s'}^\dagger c\left(p_1^+,s_1\right)\right\}\right]\times$$

$$\times u(p,s)\,\bar{u}\left(p',s'\right)e^{ip'^+.x^-}\mid 0\rangle$$

some terms cancel when the annihilation operators act on the vaccum; then one term is left, the Dirac delta function term

$$D_{H/Q}(z)=\frac{z}{4\pi}\int dx^-\exp\left(-iP^+x^-/z\right)\times\left(\frac{1}{2}\right)Tr_{Dirac}\times$$

$$\times(n.\gamma)\langle 0\mid\left(1-g_s^2\int_0^{x^-}dx_\mu\int_0^{x_\mu}dy_v A_a^\mu(x)\,A_b^v(y)\right)\left(\frac{3}{2}\right)\left(\frac{N_c^2-1}{N_c}\right)\times\sum_{s,s'}\int d^3\vec{p}\int d^3\vec{p}'N_pN_{p'}\chi_{s_1s_2}^2\times$$

$$\times\left[\delta\left(\vec{p}-\vec{p}'\right)\delta_{s,s'}\delta\left(\vec{p}-\vec{p}_1^+\right)\delta_{s,s_1}\delta\left(\vec{p}'-\vec{p}_1^+\right)\delta_{s',s_1}\right]\times$$

$$\times u(p,s)\,\bar{u}\left(p',s'\right)e^{ip'^+.x^-}\mid 0\rangle$$

$$=\frac{z}{4\pi}\int dx^-\exp\left(-iP^+x^-/z\right)\times\left(\frac{1}{2}\right)Tr_{Dirac}\times$$

$$\times(n.\gamma)\langle 0\mid\left(1-g_s^2\int_0^{x^-}dx_\mu\int_0^{x_\mu}dy_v A_a^\mu(x)\,A_b^v(y)\right)\left(\frac{3}{2}\right)\left(\frac{N_c^2-1}{N_c}\right)N_{p_1^+}^2Tr\left(\chi_{s_1s_2}^2\right)\times$$

$$\times\sum_s u\left(p_1^+,s\right)\bar{u}\left(p_1^+,s\right)e^{ip_1^+.x^-}\mid 0\rangle \qquad (8.15)$$

where we put $\vec{p}=\vec{p}'=\vec{p}_1^+\simeq\frac{m_1}{m_1+m_2}\vec{P}^+\simeq\frac{m_2}{m_1+m_2}\vec{P}^+$ the quark, antiquark momentum, and $(m_1=m_2,q\simeq 0)\,,y_i=\frac{m_i}{M}=\frac{m_i}{m_1+m_2}\ (i=1,2)\,,\ m_1,m_2$ being the quark and antiquark masses, M, the mass of the hadron, and where $s=s'=s_1$, $\sum_s u\left(p_1^+,s\right)\bar{u}\left(p_1^+,s\right)=\frac{\left(\bar{p}_1^++m_1\right)}{2m_1}$, $\chi_{s_1s_2}=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, s,s',s_1 are the quark spins and spin matrix respectively.

8.3.4 Evaluation of the Wilson line.

After applying the preceding closed relation for Dirac spinors given in the appendix, we get a compact form where we need evaluate the left Wilson line only in the equation (8.15)

$$D_{H/Q}(z)=\frac{3z}{16\pi}\left(\frac{N_c^2-1}{N_c}\right)Tr\left(\chi_{s_1s_2}^2\right)\int dx^-\exp\left(-iP^+x^-/z\right)\times$$

$$\times\langle 0 \mid \left(1 - g_s^2 \int_0^{x^-} dx_\mu \int_0^{x_\mu} dy_v A_a^\mu (x) A_b^v (y)\right) \mid 0\rangle \times Tr_{Dirac} \left\{(n.\gamma) \frac{\left(\bar{p}_1^+ + m_1\right)}{2m_1}\right\} N_{p_1^+}^2 e^{ip_1^+ .x^-}$$

$$= \frac{8z}{32\pi} \frac{1}{(2\pi)^3} \left(\frac{1}{E_{p_1^+}}\right) \times Tr_{Dirac} \left\{(n.\gamma) y_1 \bar{P}^+\right\} \int dx^- \exp\left(-iP^+ x^-/z\right) \times \exp\left(iy_1 P^+ .x^-\right)$$

$$\times \langle 0 \mid \left(1 - g_s^2 \int_0^{x^-} dx_\mu \int_0^{x_\mu} dy_v A_a^\mu (x) A_b^v (y)\right) \mid 0\rangle \tag{8.16}$$

Finally, we arrive at a compact forme

$$D_{H/Q}(z) = \frac{8z}{32\pi} \frac{1}{(2\pi)^3} \left(\frac{1}{E_{p_1^+}}\right) \times Tr_{Dirac} \left\{(n.\gamma) y_1 \bar{P}^+\right\} \int dx^- \exp\left(iP^+ x^- \left(\frac{y_1 z - 1}{z}\right)\right) \times$$

$$\times \langle 0 \mid \left(1 - g_s^2 \int_0^{x^-} dx_\mu \int_0^{x_\mu} dy_v A_a^\mu (x) A_a^v (y)\right) \mid 0\rangle \tag{8.17}$$

Let $D_{ab}^{\mu v}(x - y) = -g^{\mu v} \delta_{ab} D(x - y)$ be the virtual propagator from x to y and using the world line parametrisation (8.12) equation (8.17) becomes

$$D_{H/Q}(z) = \frac{8z}{32\pi} \frac{1}{(2\pi)^3} \left(\frac{1}{E_{p_1^+}}\right) \times Tr_{Dirac} \left\{(n.\gamma) y_1 \bar{P}^+\right\} \int dx^- \exp\left(iP^+ x^- \left(\frac{y_1 z - 1}{z}\right)\right) \times$$

$$\times \left(1 - g_s^2 \int_{-\infty}^0 dt_1 n_\mu \int_0^1 dt_2 x_v^- D_{ab}^{\mu v} \left(x_v^- t_2 - n_\mu t_1\right)\right)$$

$$= \frac{8z}{32\pi} \frac{1}{(2\pi)^3} \left(\frac{1}{E_{p_1^+}}\right) \times Tr_{Dirac} \left\{(n.\gamma) y_1 \left(\gamma.P^+\right)\right\} \int dx^- \exp\left(iP^+ x^- \left(\frac{y_1 z - 1}{z}\right)\right) \times$$

$$\times \left(1 - g_s^2 \left(n.x^-\right) \int_{-\infty}^0 dt_1 \int_0^1 dt_2 D\left(x^- t_2 - n t_1\right)\right) \tag{8.18}$$

Using the Feynman rules for the propagator in 4 dimensions

$$D(z) = \frac{\Gamma(1)}{4\pi^2} \frac{1}{-z^2 + i\epsilon} = \frac{\Gamma(1)}{4\pi^2} \frac{1}{-\left(x^- t_2 - n t_1\right)^2} = \frac{\Gamma(1)}{4\pi^2} \frac{1}{\left(-t_1^2 + 2n.x^- t_1 t_2\right)^2} \tag{8.19}$$

where $n^2 = 1$ and $\left(x^-\right)^2 = 0$.Try to evaluate the integral of the Wilson line. On substituting (8.19) into (8.18) Wilson line piece, we get

$$\begin{aligned} I\left(x^-\right) &= -\left(n.x^-\right) \left(\int_{-\infty}^0 dt_1 \int_0^1 dt_2 D\left(x^- t_2 - n t_1\right)\right) \\ &= \left(n.x^-\right) \left(\int_{-\infty}^0 dt_1 \int_0^1 dt_2 \frac{\Gamma(1)}{4\pi^2} \frac{1}{\left(t_1^2 - 2n.x^- t_1 t_2\right)^2}\right) \end{aligned}$$

perform the change of variables : $u = t_1^2 - 2n.x^- t_1 t_2$, then , $du = -2n.x^- t_1$; we obtain

$$I\left(x^-\right) = \left(n.x^-\right)\frac{\Gamma\left(1\right)}{4\pi^2}\left(\int_{-\infty}^{0} dt_1 \int_0^1 \frac{du}{u^2}\frac{1}{\left(2n.x^- t_1\right)}\right) = -\frac{\Gamma\left(1\right)}{4\pi^2}\int_{+\infty}^{0} dt_1 \left\{\frac{n.x^-}{t_1^2\left(2n.x^- t_1 + t_1^2\right)}\right\}$$

then,we put the following change of variables : $t_1 = (2n.x^-)t'$. Whence

$$\begin{aligned} I\left(x^-\right) &= \frac{\Gamma\left(1\right)}{4\pi^2}\int_0^{+\infty} dt_1 \left\{\frac{n.x^-}{t_1^3\left(2n.x^- + t_1\right)}\right\} \\ &= \frac{\Gamma\left(1\right)}{4\pi^2}\left(\frac{n.x^-}{2}\right)\int_0^{+\infty}\left\{\frac{\left(2n.x^-\right)^2 dt'}{\left(2n.x^-\right)^3 t'^3\left(2n.x^- + \left(2n.x^-\right)t'\right)}\right\} \\ &= \frac{\Gamma\left(1\right)}{32\pi^2}\frac{1}{\left(n.x^-\right)}\int_0^{+\infty} t'^{-3}\left(1+t'\right)^{-1} dt' = \frac{\Gamma\left(3\right)\Gamma\left(-2\right)}{32\pi^2}\frac{1}{\left(n.x^-\right)} \end{aligned}$$

8.3.5 The Result and Application of the sum rule.

Finally , equation (8.18) becomes

$$D_{H/Q}\left(z\right) = \frac{8z}{32\pi}\frac{1}{\left(2\pi\right)^3}\left(\frac{1}{E_{p_1^+}}\right) \times Tr_{Dirac}\left\{\left(n.\gamma\right)y_1\left(\gamma.P^+\right)\right\} \times$$
$$\times \int dx^- \exp\left(iP^+x^-\left(\frac{y_1 z - 1}{z}\right)\right) \times \left(1 + \frac{g_s^2}{\left(n.x^-\right)}\frac{\Gamma\left(3\right)\Gamma\left(-2\right)}{32\pi^2}\right) \qquad (8.20)$$

Or, we know that : $n.x^- = n^-.x^- = x^-$.Therefore the fragmentation function (8.10) for a quark fragmenting into a spinless hadron is :

$$D_{H/Q}\left(z\right) =$$
$$= \frac{z}{32\pi^4}\left(\frac{1}{E_{p_1^+}}\right)\times Tr_{Dirac}\left\{\left(\gamma^+\right)y_1\left(\gamma^+.P^+\right)\right\}\int dx^- \exp\left(iP^+x^-\left(\frac{y_1 z - 1}{z}\right)\right)\left(1 + \frac{g_s^2}{\left(n.x^-\right)}\frac{\Gamma\left(3\right)\Gamma\left(-2\right)}{32\pi^2}\right)$$

For fast quarks , $E_{p_1^+} \simeq p_1^+ \simeq y_1 P^+$. Hence, the last equation becomes

$$D_{H/Q}\left(z\right) = \frac{z}{32\pi^4}\times Tr_{Dirac}\left\{\gamma^+\gamma^+\right\}\int dx^- \exp\left(iP^+x^-\left(\frac{y_1 z - 1}{z}\right)\right)\left(1 + \frac{g_s^2}{\left(n.x^-\right)}\frac{\Gamma\left(3\right)\Gamma\left(-2\right)}{32\pi^2}\right)$$

Whence,

$$D_{H/Q}\left(z\right) = \frac{z}{32\pi^4}\int_1^{\infty} dx^-\left(1 + \frac{g_s^2}{\left(n.x^-\right)}\frac{1}{32\pi^2}\right)\exp\left(iP^+x^-\left(\frac{y_1 z - 1}{z}\right)\right) =$$

$$= \frac{z}{32\pi^4}\int_1^{\infty} dx^- \left(1+\frac{g_s^2}{(n.x^-)}\frac{1}{32\pi^2}\right)\exp\left(-iP^+x^-\left(\frac{1-y_1z}{z}\right)\right) \tag{8.21}$$

Put

$$\bar{z} = iP^+\left(\frac{1-y_1z}{z}\right)$$

Then, equation (8.21) becomes

$$D_{H/Q}(z) = \frac{z}{32\pi^4}\int_1^{\infty} dx^- \left(1+\frac{g_s^2}{(n.x^-)}\frac{1}{32\pi^2}\right)\exp\left(-iP^+x^-\left(\frac{1-y_1z}{z}\right)\right)$$

$$= \frac{z}{32\pi^4}\int_1^{\infty} dx^- \left(1+\frac{g_s^2}{(n.x^-)}\frac{1}{32\pi^2}\right)\exp\left(-\bar{z}x^-\right)$$

$$= \frac{z}{32\pi^4}\left[\frac{\exp(-\bar{z})}{\bar{z}}+\frac{g_s^2}{32\pi^2}E_1(\bar{z})\right] = \frac{z}{32\pi^4}\left[\frac{\exp\left(-i\sqrt{2s}\left(\frac{1-y_1z}{z}\right)\right)}{i\sqrt{2s}\left(\frac{1-y_1z}{z}\right)}+\frac{g_s^2}{32\pi^2}E_1\left(i\sqrt{2s}\left(\frac{1-y_1z}{z}\right)\right)\right] \tag{8.22}$$

where , $E_1(\bar{z})$ is the exponential integral which is given by

$$E_1(\bar{z}) = -\gamma - \ln\bar{z} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\frac{\bar{z}^k}{k}$$

Where γ is the Euler constante. We should note that this is a complex number. The quantity which has a physical meaning of a probability density which is the fragmentation function in our case is the norme of the complex number weighted by a normalisation constant N obeying the following sum rule

$$\int_0^1 zD_{H/Q}(z)dz = 1 \tag{8.23}$$

Therefore, we should have

$$\int_0^1 \frac{Nz^2}{32\pi^4}\left[\frac{\exp(-\bar{z})}{\bar{z}}+\frac{g_s^2}{32\pi^2}E_1\left(iP^+\left(\frac{1-y_1z}{z}\right)\right)\right]dz = 1$$

where,

$$P^+ = \frac{P^0+P^3}{\sqrt{2}} = \sqrt{2}P^0 = \sqrt{2}E_{Hadron} = \sqrt{2}\sqrt{s} = \sqrt{2s} \simeq 10Gev.$$

We find,

$$\frac{N}{32\pi^4} = \frac{1}{0.021185} = 47.203.$$

Finally the absolute value of the complex number is

$$D_{H/Q}\left(z\right)=47.203z\left|\frac{\exp\left(-i\sqrt{2s}\left(\frac{1-y_1z}{z}\right)\right)}{i\sqrt{2s}\left(\frac{1-y_1z}{z}\right)}+\frac{g_s^2}{32\pi^2}E_1\left(i\sqrt{2s}\left(\frac{1-y_1z}{z}\right)\right)\right| \tag{8.24}$$

Further, the fragmentation function should have the following properties

$$forz=0,1then\,D_{H/Q}\left(z\right)=0,$$

physically $z=0$ means that there is no fragmentation and $z=1$ means that the hadron is in its initial state. therefore, Eq.(8.22) should be written as

$$D_{H/Q}\left(z\right)=47.203\times z\times\frac{\left(1-z\right)^2}{\left(1-y_1z\right)^6}\times$$

$$\times\left|\frac{\left(-i\right)}{\sqrt{2s}}\left[\frac{z\left(1-y_1z\right)^5}{\left(1-z\right)^2}\right]\exp\left(-i\sqrt{2s}\left(\frac{1-y_1z}{z}\right)\right)+\frac{g_s^2}{32\pi^2}\left[\frac{\left(1-y_1z\right)^5}{\left(1-z\right)^2}\right]E_1\left(i\sqrt{2s}\left(\frac{1-y_1z}{z}\right)\right)\right|$$

where the absolute value should be finite. After develloping in a power series with MATHEMATICA, we obtain the following approximation for a comparative study with J.P.Ma result [8]

$$\begin{aligned}D_{H/Q}\left(z\right) &= 47.203\times z\frac{\left(1-z\right)^2}{\left(1-y_1z\right)^6}\times\left[1-2\left(1-3y_1\right)z+\left(1-12y_1+21y_1^2\right)z^2\right.\\ &\quad\left.+2y_1\left(3-21y_1+28y_1^2\right)z^3+7y_1^2\left(3-16y_1+18y_1^2\right)z^4+o\left(z^5\right)\right]\end{aligned}$$

If we choose $y_1=\frac{1}{2}$ for a pion for example, and put

$$\sqrt{2s}=10Gev,\,and\,g_s=0.3.$$

we arrive at the following final expression

$$D_{H/Q}\left(z\right)=3020.992\times z\times\frac{\left(1-z\right)^2}{\left(2-z\right)^6}\times\left(1+z+0.25\times z^2-0.5\times z^3-0.875\times z^4\right)$$

8.4 Parton fragmentation functions at finite temperature.

The parton fragmentation functions at finite temperature (i.e. at $T\neq 0$) is calculated using the real-time formalism of Thermofield dynamics (TFD). The motivation for this work is due to the lack of such calculations, in particular for processes likee^+e^- annihilation. It should be noted that these fragmentation functions at T=0 are already given by Collins and Soper [1],[2],[3],[4]. These definitions are universal and independent of the process. Their study exhausted nearly all the analytical and phenomenologecal properties of such functions. Feynman and Field have given a phenomenologecal model [6]. The European Muon Collaboration gave an exhaustive study of a type of fragmentation function obtained for hadronic matter [11]. To account for the theory, many experiments have been carried out in order to understand the underlying mechanism for the fragmenting matter [12],[13]. Many attempts have been made in order to describe the fragmentation function for light as well as for heavy quarks. Numerous models give formulas for these functions. Results of next-to-leading order QCD calculation for heavy quarks have been given [7],14]. However, the study for non-zero temperature have been infrequent [14]. In the present study, we attempt to get an expression of the fragmentation function at finite temperature for hadron collision on a bound state of two heavy quarks. We examine the case of a fast moving quark or a gluon fragmenting into a hadron via the exchange of a virtual photon. Our conclusions at $T \neq 0$ will stress comparison with $T = 0$ case. We should point out that our study relies on the work of J. P. Ma who gave perturbative predictions for parton fragmentation function for heavy quarks at one loop level using Feynman diagrams [8],[15].

The motivation for this presentation is two folds. The first is the lack of literature for the treatment of such functions at $T \neq 0$.Secondly, the behaviour of the fragmentation function at finite temperature is not known. Such a result could find an experimental confirmation, a task that can be carried out during e^+e^- annihilation experiments.

The fact that a quantum formalism is strongly founded on the basis of representation of linear algebras suggests that a $T \neq 0$ field theory needs a real-time *operator* structure. Such a theory, based on a vacuum state at finite temperature ($\beta = 1/k_B T, k_B$, is a Bolttzmann constant), $|0(\beta)\rangle$, was presented by Takahashi and Umezawa [17],[18] and they labelled it *Thermofield Dynamics*(TFD). As a consequence of the real-time requirement, a doubling of the original Hilbert space is defined for the system, through the algebraic structure called *tilde conjugation rules*, the temperature is introduced by introducing a Bogoliubov transformation; that is, a rotation in the doubled algebra space.

The Takahashi and Umezawa approach has been developed for practical purposes and some results should be mentioned, such as the proof of the Goldstone theorem within this formalism with a quite amazing physical and mathematical appeal, and the perturbative scheme with Feynman rules established to carry out calculations completely in parallel with the zero-temperature quantum field theory [19]. Thus it has been successfully applied to study superconductivity [20], magnetic systems like ferromagnets and paramagnets [20], quantum optics and transport phenomena [21], d-branes [22],[23], among others. Furthermore, the propagators are 2×2 matrices. An association of the Matsubara and Schwinger-Keldysh methods has been established [19],[24]. The TFD formalism provides results that agree with other methods for systems in equilibrium.

Formally the thermal theory has been studied using TFD, within c^* algebras and symmetry groups [25], with the notion of thermo-Lie groups [26], opening a broad spectrum of possibilities for the study of thermal effects. For instance, the kinectic theory has been formulated for the first time from the analysis of representations of kinematical group and elements of the q-group have been considered, where the effect of temperature is related to a deformation in the Weyl-Heisenberg algebra [20]. The analysis of thermal theories via c^* algebras was carried out earlier [27],[28], resulting in the doubled structure of the Tomita-Takesaki (standard) representation that can be immediately used to construct several aspects of TFD. It is worth mentioning that some elements of TFD were pointed out in a paper by Laplae, Mancini, Umezawa [28] studying superconductors; but much earlier the doubling of Hilbert space explored by Verboven [29],[30], studying thermal oscillators. Despite the interest and importance, the c^*-aproach does not provide a complete physical understanding for the doubling, that is present in any thermal theory, and this is one aspect to be addressed here from a physical and algebraic point of view.

8.4.1 Fragmentation function in TFD.

In the infinite momentum frame, the momentum of the incoming hadron is almost entirely in the P^+ direction $P_\mu \approx (P^+, 0, 0)$ .It is convenient to work in the light cone coordinate system defined in [5]. The fragmentation function for a spinless hadron at finite temperature β is modified and will be defined as :

$$D_{H/Q}(z, \beta) = \frac{z}{4\pi} \int dx^- \exp\left(-iP^+ x^-/z\right) \times \left(\frac{1}{3}\right) Tr_{color} \left(\frac{1}{2}\right) Tr_{Dirac} \times (n.\gamma)$$

$$\times \left[\langle O\left(\beta\right) \mid Q\left(0\right) L\left(\vec{x}\right) \times a_H^+\left(P^+, \mathbf{0_T}\right) a_H\left(P^+, \mathbf{0_T}\right) \bar{Q}\left(0, x^-, \mathbf{0_T}\right) \mid O\left(\beta\right)\rangle\right] \quad (8.25)$$

where

$$L(\vec{x}) = P\exp\left(-ig_s\int_0^{x^-} dx_0 A^0(\vec{x},\tau)\right)$$

is the Wilson line [32],[33],[34] which describes a path ordered phase factor of the parton (quark or gluon) moving from 0 to x^-. A field operator creates a parton at position 0 coming from infinity (∞) and another field annihilates it at position x^-. Therefore we parametrize the world line of the fragmenting parton along the following path [34] :

$$x_\mu = n_\mu t, for - \infty\langle t\langle 0 and x_\mu = x^- t, for 0\langle t\langle 1 \tag{8.26}$$

where we choose x_μ to be light-like :$x_\mu = (0, x^-, \mathbf{0})$, and $n_\mu = \frac{P_\mu}{M} = (P^+, P^-, \mathbf{0})/M$ time-like , such that $n^2 = \frac{P^2}{M^2} = 1$. The field $A_\mu(\vec{x},\tau)$ is defined as

$$A_\mu(\vec{x},\tau) = A^a_\mu(\vec{x},\tau)T^a,$$

where $A^a_\mu(\vec{x},\tau)$ is the time component of the gluon field and T^a are the generators of the fundamental representation of the $SU(3)$ algebra of the color field of the gluons ($a = 8$ gluons . Expand the quark and hadron fields into plane waves as

$$Q(0) = \sum_s \int d^3p N_p \left[c_{p,s}u(p,s) + d^\dagger_{p,s}v(p,s)\right],$$

$$\bar{Q}(0, x^-, p_{\mathbf{T}}) = \sum_{s'} \int d^3p' N_{p'} \left[c^\dagger_{p',s'}u(p',s')e^{ip^+x^-} + d_{p',s'}\bar{v}(p',s')e^{-ip^+x^-}\right],$$

$$a_{\mathbf{H}}(p^+.0_{\mathbf{T}}) = d(p_2^+, s_2)\,c(p_1^+, s_1),$$

$$a^\dagger_{\mathbf{H}}(p^+.0_{\mathbf{T}}) = c^\dagger(p_1^+, s_1)\,d^\dagger(p_2^+, s_2).$$

Furthermore, we expand the Wilson loop to second order

$$Pe^{-ig_s\int_0^{x^-} dx_0 A^0(\vec{x},\tau)} = P\left[1 - ig_s\int_0^{x^-} dx_0 A^0(\vec{x},\tau) + (ig_s)^2\int_0^{x^-} dx_0\int_0^{x^0} dy_0 A^0(\vec{x},\tau)A^0(\vec{y},\tau')\right].$$

We find that the linear term in $A^0(\vec{x},\tau)$ does not contribute. Now, we find the operators are defined at $T = 0$ while the vacuum state is at finite temperature. Now we must apply the Bogoliubov transformation to the operators, thus defining finite temperature creation and annihilation operators. For fermions, we use

$$
\begin{aligned}
c_{p,s} &= \cos\theta_p c_{p,s}(\beta) + i\sin\theta_p \tilde{c}^{\dagger}_{p,s}(\beta) \\
d_{p,s} &= \cos\theta_{-p} d_{p,s}(\beta) + i\sin\theta_{-p} \tilde{d}^{\dagger}_{p,s}(\beta) \\
c^{\dagger}_{p,s} &= \cos\theta_p c^{\dagger}_{p,s}(\beta) - i\sin\theta_p \tilde{c}_{p,s}(\beta) \\
d^{\dagger}_{p,s} &= \cos\theta_{-p} d^{\dagger}_{p,s}(\beta) - i\sin\theta_{-p} \tilde{d}_{p,s}(\beta)
\end{aligned}
$$

We apply the Bogoliubov transformations over the fields [16],[17],[18],[35],[36] thus the creation and annihilation operators are changed to finite temperature operators

$$G\left(u\left(\beta\right),v\left(\beta\right)\right)=$$

$$=\left[\left(\cos\theta_p c_{p,s}(\beta)+i\sin\theta_p\tilde{c}^{\dagger}_{p,s}(\beta)\right)u(p,s)+\left(\cos\theta_{-p}d^{\dagger}_{p,s}(\beta)-i\sin\theta_{-p}\tilde{d}_{p,s}(\beta)\right)v(p,s)\right]$$

$$d_{p_2^+,s_2}(\beta)\,c_{p_1^+,s_1}(\beta)\,c^{\dagger}_{p_1^+,s_1}(\beta)\,d^{\dagger}_{p_2^+,s_2}(\beta)$$

$$\times\left[\left(\cos\theta_{p'}c^{\dagger}_{p',s'}(\beta)+i\sin\theta_{p'}\tilde{c}_{p',s'}(\beta)\right)\bar{u}\left(p',s'\right)e^{ip'^{+}.x^{-}}\right.$$

$$\left.+\left(\cos\theta_{-p'}d_{p',s'}(\beta)+i\sin\theta_{-p'}\tilde{d}^{\dagger}_{p',s'}(\beta)\right)\bar{v}\left(p',s'\right)e^{-ip'^{+}.x^{-}}\right]$$

Using the relations for the annihilation operators,

$$c_{p,s}(\beta)\mid O(\beta)\rangle=d_{p,s}(\beta)\mid O(\beta)\rangle=0\quad and\quad \tilde{c}_{p,s}(\beta)=\tilde{d}_{p,s}(\beta)\mid O(\beta)\rangle=0$$

we get

$$\langle O(\beta)\mid G\left(u(\beta),v(\beta)\right)\mid O(\beta)\rangle=$$

$$=\langle O(\beta)\mid\left[\cos\theta_p c_{p,s}(\beta)\,u(p,s)-i\sin\theta_{-p}\tilde{d}_{p,s}(\beta)\,v(p,s)\right]\times d_{p_2^+,s_2}(\beta)\,c_{p_1^+,s_1}(\beta)\,c^{\dagger}_{p_1^+,s_1}(\beta)\,d^{\dagger}_{p_2^+,s_2}(\beta)$$

$$\times\left[\cos\theta_{p'}c^{\dagger}_{p',s'}(\beta)\,\bar{u}\left(p',s'\right)e^{ip'^{+}.x^{-}}+i\sin\theta_{-p'}\tilde{d}^{\dagger}_{p',s'}(\beta)\,\bar{v}\left(p',s'\right)e^{-ip'^{+}.x^{-}}\right]\mid O(\beta)\rangle$$

Thus the expectation value between vaccum state at finite temperature becomes

$$\langle O(\beta)\mid G\left(u(\beta),v(\beta)\right)\mid O(\beta)\rangle=$$

$$=\langle O(\beta)\mid\cos\theta_p\cos\theta_{p'}u(p,s)\,\bar{u}\left(p',s'\right)e^{ip'^{+}.x^{-}}$$

$$\times c_{p,s}(\beta)\,d_{p_2^+,s_2}(\beta)\,c_{p_1^+,s_1}(\beta)\,c^{\dagger}_{p_1^+,s_1}(\beta)\,d^{\dagger}_{p_2^+,s_2}(\beta)\,c^{\dagger}_{p',s'}(\beta)$$

$$+\sin\theta_{-p'}\sin\theta_{-p}v(p,s)\,\bar{v}\left(p',s'\right)e^{-ip'^{+}.x^{-}}$$

$$\times \tilde{d}_{p,s}(\beta)\, d_{p_2^+,s_2}(\beta)\, c_{p_1^+,s_1}(\beta)\, c^{\dagger}_{p_1^+,s_1}(\beta)\, d^{\dagger}_{p_2^+,s_2}(\beta)\, \tilde{d}^{\dagger}_{p',s'}(\beta) \mid O(\beta)\rangle$$

Now consider the following expression

$$I(\beta) = \left(\frac{1}{N_c} Tr_{color}\left\{T^a T^b\right\}\right) \sum_{s,s'} d^3\vec{p} d^3\vec{p}' N_p N_{p'} \chi^2_{s_1 s_2} \langle O(\beta) \mid G(u(\beta), v(\beta)) \mid O(\beta)\rangle$$

$$= \left(\frac{3}{2}\right)\left(\frac{N_c^2-1}{N_c}\right) \times \sum_{s,s'} d^3\vec{p} d^3\vec{p}' N_p N_{p'} \chi^2_{s_1 s_2}$$

$$\times \langle O(\beta) \mid \left[\delta\left(\vec{p}' - \vec{p}\right)\delta_{s,s'} \cos\theta_p \cos\theta_{p'} u(p,s)\, \bar{u}\left(p', s'\right) e^{ip'^+ . x^-}\right.$$

$$\left. + \delta\left(\vec{p}' - \vec{p}\right)\delta_{s,s'} \sin\theta_{-p'} \sin\theta_{-p} v(p,s)\, \bar{v}\left(p', s'\right) e^{-ip'^+ . x^-}\right] \mid O(\beta)\rangle$$

$$= \left(\frac{3}{2}\right)\left(\frac{N_c^2-1}{N_c}\right) \times \sum_{s} \int d^3\vec{p} N_p^2 \chi^2_{s_1 s_2}$$

$$\times \left[\cos^2\theta_p u(p,s)\, \bar{u}(p,s)\, e^{ip^+ . x^-} + \sin^2(\theta_{-p})\, v(p,s)\, \bar{v}(p,s)\, e^{-ip^+ . x^-}\right]$$

Here the canonical commutation relations between T-dependent operators are used. The fragmentation function becomes

$$D_{H/Q}(z,\beta) =$$

$$= \frac{z}{4\pi} \int dx^- \exp\left(-iP^+ x^-/z\right) \times \left(\frac{1}{2}\right) Tr_{Dirac} \times (n.\gamma) \langle O(\beta) \mid L(\vec{x}) \times \left(\frac{3}{2}\right)\left(\frac{N_c^2-1}{N_c}\right) Tr\left(\chi^2_{s_1 s_2}\right) \times$$

$$\left[\sum_s \int d^3\vec{p} N_p^2 \cos^2\theta_p \left(\sum_s u(p,s)\, \bar{u}(p,s)\, e^{ip^+ . x^-}\right) + N_p^2 \sin^2(\theta_{-p}) \left(\sum_s v(p,s)\, \bar{v}(p,s)\, e^{-ip^+ . x^-}\right)\right] \mid O \tag{8.27}$$

where we use in the centre of mass of the photon

$$\vec{p} = \vec{p}' = \vec{p}_1^+ \simeq \frac{m_1}{m_1+m_2}\vec{P}^+ \simeq \frac{m_2}{m_1+m_2}\vec{P}^+ \;\; (m_1 = m_2, q \simeq 0), y_i = \frac{m_i}{M} = \frac{m_i}{m_1+m_2} \; (i=1,2)$$

$$s = s' = s_1 \;,\; \sum_s u\left(p_1^+, s\right) \bar{u}\left(p_1^+, s\right) = \frac{\left(\bar{p}_1^+ + m_1\right)}{2m_1} \;,\; \chi_{s_1 s_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\vec{p}$, $\vec{p}'$, and $\vec{P}^+$ being respectively the quark, antiquark and the photon momentum. $m_1$and m_2 are the quark, antiquark masses, s, s' their spin respectively. $M = m_1 + m_2$ the mass of the hadron. $\chi_{s_1 s_2}$ the spin matrix. $N_c = 3$ the color number for the quarks.

8.4.2 Presentation of the result.

Finally, the expression for the fragmentation function has been reduced to a compact form. Now we evaluate the Wilson line only. The temperature dependence of the fragmentation function Eq.(8.25) is

$$D_{H/Q}(z,\beta) = \frac{3z}{16\pi}\left(\frac{N_c^2-1}{N_c}\right) Tr\left(\chi^2_{s_1 s_2}\right)\int dx^- \exp\left(-iP^+x^-/z\right)\times\langle 0\left(\beta\right) \mid L\left(\vec{x}\right) \mid 0\left(\beta\right)\rangle$$

$$\times\int d^3\vec{p}N_p^2\left[\cos^2\theta_p - \sin^2\left(\theta_{-p}\right)\right] Tr_{Dirac}\left\{(n.\gamma)\frac{(\bar{p}+m_1)}{2m_1}\right\} e^{-ip^+.x^-}]$$

hence,

$$D_{H/Q}(z,\beta) = -\frac{8z}{32\pi}\frac{1}{(2\pi)^3}\times Tr_{Dirac}\left\{(n.\gamma)\,\bar{p}\right\}\int dx^- \exp\left(-iP^+x^-/z\right)\times\int\left(\frac{1}{E_p}\right) d^3\vec{p}\exp\left(-ip^+.x^-\right)$$

$$\times\left(2\cos^2\theta_p - 1\right)\times\langle O\left(\beta\right) \mid L\left(\vec{x}\right) \mid O\left(\beta\right)\rangle$$

Finally, after going onto light cone coordinates for $d^3\vec{p} = |p_{\mathbf{T}}|\, d\,|p_{\mathbf{T}}|\, dp^+$, and changing $dp^+ \longrightarrow -dp^+$ we arrive at

$$D_{H/Q}(z,\beta) = i[\frac{8z}{32\pi}\frac{1}{(2\pi)^3}\left(\frac{1}{E_{p_1^+}}\right)\times Tr_{Dirac}\left\{(n.\gamma)\,y_1\bar{P}^+\right\}\int\left(\frac{dx^-}{x^-}\right)\exp\left(iP^+x^-\left(\frac{y_1 z-1}{z}\right)\right)$$

$$\times\int d\,|p_{\mathbf{T}}|^2\left(2\cos^2\theta_p - 1\right)]\times$$

$$\times\langle O\left(\beta\right) \mid \left(1 - g_s^2\int_0^{x^-} dx_0\int_0^{x_0} dy_0 A_a^0\left(\vec{x},\tau\right) A_b^0\left(\vec{y},\tau'\right)\right) \mid O\left(\beta\right)\rangle$$

Now

$$p^+p^- + p^-p^+ - \mid p_T\mid^2 = 2p^+p^- - \mid p_T\mid^2 = m^2 \Rightarrow \mid p_T\mid^2 = -m^2$$

We know that $\langle O\left(\beta\right) \mid A_a^0\left(\vec{x},\tau\right) A_b^0\left(\vec{y},\tau'\right) \mid O\left(\beta\right)\rangle = G_0\left(x-y\right)$ and using the world line parametrisation Eq.(8.26) we arrive at

$$D_{H/Q}(z,\beta) = \frac{8z}{32\pi}\frac{1}{(2\pi)^3}\left(\frac{m^2}{E_{p_1^+}}\right)\times Tr_{Dirac}\left\{(n.\gamma)\,y_1\left(\gamma.P^+\right)\right\}\int\left(\frac{dx^-}{x^-}\right)\exp\left(iP^+x^-\left(\frac{y_1 z-1}{z}\right)\right)$$

$$\times\left(2\cos^2\theta_p - 1\right)\times\left(1 - g_s^2\int_0^{x^-} dx_0\int_0^{x_0} dy_0 G_0\left(\left(\vec{x},\tau\right) - \left(\vec{y},\tau'\right)\right)\right)$$

8.4.3 Solving for the Wilson line.

The Feynman propagator in 4 dimensions is given as

$$G_0(k) = \frac{1}{k^2 + i\epsilon} + 2\pi i n_{\mathbf{B}}(k)\,\delta\left(k^2\right).$$

The Wilson loop to second order is written as

$$I\left(x^-, \beta, z\right) = \int_0^{x^-} d\tau \int_0^{\tau} d\tau' G_0\left((\vec{x}, \tau) - (\vec{y}, \tau')\right)$$

Then we have

$$I\left(x^-, \beta, z\right) = \int_0^{x^-} d\tau \int_0^{\tau} d\tau' G_0\left((\vec{x}, \tau) - \left(\vec{0}, \tau'\right)\right)$$

$$= \int_0^{x^-} d\tau \int_0^{\tau} d\tau' \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} e^{-ik.(x-y)} +$$

$$+ \int_0^{x^-} d\tau \int_0^{\tau} d\tau' \frac{1}{(2\pi)^4} \int dk_0 \int dk_3 \int d^2k_{\mathbf{T}} \left(\frac{2\pi i}{e^{\beta k_0} - 1}\delta\left(k_0^2 - k_3^2 - k_{\mathbf{T}}^2\right)\right) e^{-ik_0.(\tau-\tau') + ik_3.\tau} \tag{8.28}$$

integrating over the phase space in $D-$ dimensions ,

$$d^Dk = dk_0 dk_3 d^{D-2}k_{\mathbf{T}} = (2\pi)^{D-3} dk_0 dk_3 \left|k_{\mathbf{T}}\right|^{D-3} d\left|k_{\mathbf{T}}\right| = (2\pi)^{D-3} dk_0 dk_3 d\left(\left|k_{\mathbf{T}}\right|^{(D-2)}\right) \times \frac{1}{(D-2)}$$

substituting the latter into Eq.(8.30) we get :

$$I\left(x^-, \beta, z\right) = -\frac{1}{(2\pi)^2} \int_0^{x^-} d\tau \int_0^{\tau} d\tau' \left(\frac{1}{(x-y)^2 + i\varepsilon}\right) +$$

$$+ \int_0^{x^-} d\tau \int_0^{\tau} d\tau' \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} dk_0 \int_{-k_0}^{k_0} dk_3 \int_0^{\infty} d\left(\left|k_{\mathbf{T}}\right|^2\right) \left(\frac{\pi i}{e^{\beta k_0} - 1}\delta\left(k_0^2 - k_3^2 - k_{\mathbf{T}}^2\right)\right) e^{-ik_0.(\tau-\tau') + ik_3.\tau}$$

$$= -\frac{1}{(2\pi)^2} \int_0^{x^-} d\tau \int_0^{\tau} d\tau' \left(\frac{1}{(\tau-\tau')^2 + i\varepsilon}\right)$$

$$+ \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \frac{dk_0}{k_0} \left(\frac{1}{e^{\beta k_0} - 1}\right) \int_0^{x^-} d(k_0\tau) \left\{\left(\frac{\sin(k_0\tau)}{k_0\tau}\right)\left(1 - e^{-ik_0.\tau}\right)\right\}$$

Therefore, we get

$$I\left(x^-, \beta, z\right) = I\left(x^-\right) + \left(\frac{1}{8\pi^3}\right) \int_{-\infty}^{\infty} \frac{dk_0}{k_0} \left(\frac{1}{e^{\beta k_0} - 1}\right) \int_0^{x^-} dx \left\{\frac{4\sin^3\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{x} + i\frac{\sin^2 x}{x}\right\} =$$

$$= I\left(x^{-}\right)+\left(\frac{1}{8\pi^{3}}\right)\int_{-\infty}^{\infty}\frac{dk_{0}}{k_{0}}\left(\frac{1}{e^{\beta k_{0}}-1}\right)\times\left\{\int_{0}^{x^{-}}dx\frac{4\sin^{3}\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{x}+i\left(\frac{1}{2}\right)\left(\lg\left(x^{-}\right)-Ci\left(2x^{-}\right)\right)\right\}$$

where we use

$$I\left(x^{-}\right)=-\frac{1}{\left(2\pi\right)^{2}}\int_{0}^{x^{-}}d\tau\int_{0}^{\tau}d\tau'\left(\frac{1}{\left(\tau-\tau'\right)^{2}+i\varepsilon}\right).$$

and $Ci\left(2\beta\right)$ is the Cosine integral and $x=k_{0}\tau$ [37],[38].

8.4.4 The result.

Combining different factors (e.g. equation (8.8)), the final result for the fragmentation function at β is

$$D_{H/Q}\left(z,\beta\right)=\frac{8z}{32\pi}\frac{1}{\left(2\pi\right)^{3}}\left(\frac{m^{2}}{E_{p_{1}^{+}}}\right)\times Tr_{Dirac}\left\{\left(n.\gamma\right)y_{1}\left(\gamma.P^{+}\right)\right\}\int\left(\frac{dx^{-}}{x^{-}}\right)\exp\left(iP^{+}x^{-}\left(\frac{y_{1}z-1}{z}\right)\right)\times$$

$$\times\left(2\left(\frac{e^{\frac{\beta\left|p^{0}\right|}{2}}}{\sqrt{\left(e^{\beta\left|p^{0}\right|}+1\right)}}\right)^{2}-1\right)\times\left[1-g_{s}^{2}I\left(x^{-},\beta,z\right)\right].\qquad(8.29)$$

Also we have : $n.x^{-}=n^{+}.x^{-}=x^{-}$ and for fast quarks ,$E_{p_{1}^{+}}\simeq p_{1}^{+}\simeq y_{1}P^{+}$Therefore the fragmentation function Eq.(8.25) for a quark fragmenting into a spinless hadron at temperature β is

$$D_{H/Q}\left(z,\beta\right)=\frac{z}{32\pi^{4}}m^{2}\int_{1}^{\infty}\left(\frac{dx^{-}}{x^{-}}\right)\exp\left(-iP^{+}x^{-}\left(\frac{1-y_{1}z}{z}\right)\right)\times\left(2\left(\frac{e^{\frac{\beta\left|p^{0}\right|}{2}}}{\sqrt{\left(e^{\beta\left|p^{0}\right|}+1\right)}}\right)^{2}-1\right)$$

$$\times\left[1-g_{s}^{2}I\left(x\right)-g_{s}^{2}\left(\frac{1}{8\pi^{3}}\right)\int_{-\infty}^{\infty}\frac{dk_{0}}{k_{0}}\left(\frac{1}{e^{\beta k_{0}}-1}\right)\right.$$

$$\left.\times\left\{\int_{0}^{x^{-}}dx\frac{4\sin^{3}\left(x\right)\cos\left(x\right)}{x}+i\left(\frac{1}{2}\right)\left(\lg\left(x^{-}\right)-Ci\left(2x^{-}\right)\right)\right\}\right]$$

8.4.5 Dimensional regularization.

Dimensional regularization gives :

$$D_{H/Q}\left(z,\beta\right)=$$

$$= \frac{z}{32\pi^4} m^2 \int_1^\infty \left(\frac{dx^-}{x^-}\right) \exp\left(-iP^+ x^- \left(\frac{1-y_1 z}{z}\right)\right) \times \left(2\left(\frac{e^{\frac{\beta\left|p^0\right|}{2}}}{\sqrt{\left(e^{\beta|p^0|}+1\right)}}\right)^2 - 1\right)$$

$$\times \left[1 - g_s^2 I\left(x^-\right) - g_s^2 \left(\frac{1}{8\pi^3}\right) \mu^\varepsilon \int_{-\infty}^{\infty} \frac{d^{(D-3)}k_0}{k_0} \left(\frac{1}{e^{\beta k_0}-1}\right)\right.$$

$$\left.\times \left\{\int_0^{x^-} dx \frac{4\sin^3(x)\cos(x)}{x} + i\left(\frac{1}{2}\right)\left(\lg\left(x^-\right) - Ci\left(2x^-\right)\right)\right\}\right]$$

For $T=0$,the fragmentation function is

$$D_{H/Q}(z) = \frac{z}{32\pi^4} m^2 \int_1^\infty dx^- \left(\frac{1-g_s^2 I\left(x^-\right)}{x^-}\right) \exp\left(-iP^+ x^- \left(\frac{1-y_1 z}{z}\right)\right) \tag{8.30}$$

and $\left|p_2^0\right| = \left|p_1^0\right| = \left|p^0\right| = \frac{\sqrt{s}}{2}$, then the ratio of finite and zero temperature fragmentation function gives us the net effect of finite temperature to be

$$\frac{D_{H/Q}(z,\beta)}{D_{H/Q}(z)} = \left(2\left(\frac{e^{\frac{\beta\sqrt{s}}{4}}}{\sqrt{\left(e^{\beta\frac{\sqrt{s}}{2}}+1\right)}}\right)^2 - 1\right) \times$$

$$\times \left(1 - \left(\frac{1}{8\pi^3}\right)\left(\frac{g_s^2}{1-g_s^2 I\left(x^-\right)}\right) \mu^\varepsilon \int_{-\infty}^{\infty} \frac{d^{(D-3)}k_0}{k_0} \left(\frac{1}{e^{\beta k_0}-1}\right)\right.$$

$$\left.\times \left\{Si\left(2x^-\right) - \left(\frac{1}{2}\right) Si\left(4x^-\right) + i\left(\frac{1}{2}\right)\left(\lg\left(x^-\right) - Ci\left(2x^-\right)\right)\right\}\right) \tag{8.31}$$

where $Si\left(2x^-\right) - \left(\frac{1}{2}\right) Si\left(4x^-\right) = \int_0^{x^-} dx \frac{4\sin^3(x)\cos(x)}{x}$ [37],[38].

Now, going over to the dimensional regularization we put ,

$$\int d^{(D-3)}x = \int x^{(D-4)} dx \int d\Omega_{(D-4)}, \quad with \int d\Omega_{(D-4)} = \frac{2\pi^{\frac{(D-3)}{2}}}{\Gamma\left(\frac{(D-3)}{2}\right)} = 2^{(D-4)} \pi^{\left(\frac{(D-4)}{2}\right)} \frac{\Gamma\left(\frac{(D-4)}{2}\right)}{\Gamma(D-4)}.$$

substituting the three-space D-volume, we can get for $I\left(x^-, \beta, z\right)$

$$I(\beta) = \mu^\varepsilon \int_{-\infty}^{\infty} \frac{d^{(D-3)}k_0}{k_0} \left(\frac{1}{e^{\beta k_0}-1}\right) = 2^{(D-4)} \pi^{\left(\frac{(D-4)}{2}\right)} \frac{\Gamma\left(\frac{(D-4)}{2}\right)}{\Gamma(D-4)} \mu^\varepsilon \int \frac{k_0^{(D-4)} dk_0}{k_0} \left(\frac{1}{e^{\beta k_0}-1}\right)$$

$$= 2^{(D-4)} \pi^{\left(\frac{(D-4)}{2}\right)} \Gamma\left(\frac{(D-4)}{2}\right) \mu^\varepsilon \beta^{(4-D)} \zeta(D-4),$$

where

$$\mu^{\varepsilon}\int dk_0 \frac{k_0^{(D-5)}}{e^{\beta k_0}-1} = \mu^{\varepsilon}\beta^{(4-D)}\Gamma\left(D-4\right)\zeta\left(D-4\right).$$

Here $\zeta\left(x\right)$ is the Riemann zeta function given as

$$\zeta\left(x\right) = \frac{1}{\Gamma\left(x\right)}\int_0^{\infty} du \frac{u^{(x-1)}}{e^{u}-1}.$$

Then the ratio of the fragmentation function at $T \neq 0$ and $T = 0$ is given as

$$\frac{D_{H/Q}\left(z,\beta\right)}{D_{H/Q}\left(z\right)} = \left(2\left(\frac{e^{\frac{\beta\sqrt{s}}{4}}}{\sqrt{\left(e^{\beta\frac{\sqrt{s}}{2}}+1\right)}}\right)^2 - 1\right)\times$$

$$\times\left(1-\left(\frac{g_s^2}{1-g_s^2 I\left(x^-\right)}\right)\times 2^{(D-7)}\pi^{\left(\frac{(D-10)}{2}\right)}\Gamma\left(\frac{(D-4)}{2}\right)\zeta\left(D-4\right)\times\mu^{\varepsilon}\beta^{(4-D)}\right.$$

$$\left.\times\left\{Si\left(2x^-\right)-\left(\frac{1}{2}\right)Si\left(4x^-\right)+i\left(\frac{1}{2}\right)\left(\lg\left(x^-\right)-Ci\left(2x^-\right)\right)\right\}\right). \tag{8.32}$$

We put $D = 4+\varepsilon$, and expanding in the power of ε,

$$\frac{\Gamma\left(\varepsilon/2\right)}{\left(4\pi\right)^{-\varepsilon/2}} = \frac{2}{\varepsilon} - \gamma + \ln\left(4\pi\right) + O\left(\varepsilon\right),$$

where $\gamma = 0.5772...$ is the Euler constant,

$$\mu^{\varepsilon}\left(\beta^2\right)^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2}\left(\ln\beta^2 - \ln\mu^2\right).$$

Write

$$\zeta\left(D-4\right) = \zeta\left(0\right) + \left(D-4\right)\zeta'\left(D-4\right) = -\frac{1}{2} + a\varepsilon$$

where $a = \zeta'\left(D-4\right)$.Then, we have

$$\frac{D_{H/Q}\left(z,\beta\right)}{D_{H/Q}\left(z\right)} = \left(2\left(\frac{e^{\frac{\beta\sqrt{s}}{4}}}{\sqrt{\left(e^{\beta\frac{\sqrt{s}}{2}}+1\right)}}\right)^2 - 1\right)\times$$

$$\times\left(1-\left(\frac{8\pi^3 g_s^2}{1-g_s^2 I\left(x^-\right)}\right)\times\frac{\Gamma\left(\varepsilon/2\right)}{\left(4\pi\right)^{-\varepsilon/2}}\times\zeta\left(\varepsilon\right)\times\mu^{\varepsilon}\left(\beta^2\right)^{-\varepsilon/2}\right.$$

$$\left.\times\left\{Si\left(2x^-\right)-\left(\frac{1}{2}\right)Si\left(4x^-\right)+i\left(\frac{1}{2}\right)\left(\ln\left(x^-\right)-Ci\left(2x^-\right)\right)\right\}\right)$$

$$= \left(2 \left(\frac{e^{\frac{\beta\sqrt{s}}{4}}}{\sqrt{\left(e^{\beta\frac{\sqrt{s}}{2}}+1\right)}} \right)^2 - 1 \right) \times$$

$$\times \left(1 + \left(\frac{4\pi^3 g_s^2}{1 - g_s^2 I(x^-)} \right) \times \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) - 4a - \ln\left[\frac{\beta^2}{\mu^2}\right] \right) \right.$$

$$\left. \times \left\{ Si(2x^-) - \left(\frac{1}{2}\right) Si(4x^-) + i \left(\frac{1}{2}\right) \left(\ln(x^-) - Ci(2x^-) \right) \right\} \right).$$

8.4.6 Minimal subtraction scheme (MS).

Here we subtract out the pole in ε to get

$$\frac{D_{H/Q}(z,\beta)}{D_{H/Q}(z)} = \left(2 \left(\frac{e^{\frac{\beta\sqrt{s}}{4}}}{\sqrt{\left(e^{\beta\frac{\sqrt{s}}{2}}+1\right)}} \right)^2 - 1 \right) \times$$

$$\times \left(1 + \left(\frac{4\pi^3 g_s^2}{1 - g_s^2 I(x^-)} \right) \times \left(-\gamma + \ln(4\pi) - \ln\left[\frac{\beta^2}{\mu^2}\right] \right) \right.$$

$$\left. \times \left\{ Si(2x^-) - \left(\frac{1}{2}\right) Si(4x^-) + i \left(\frac{1}{2}\right) \left(\ln(x^-) - Ci(2x^-) \right) \right\} \right).$$

8.4.7 Discussion.

1) for low temperatures $(\beta \longrightarrow \infty)$, taking the limit,$\beta \longrightarrow \infty$, we have

$$\frac{D_{H/Q}(z,\beta)}{D_{H/Q}(z)} \longrightarrow 1. \tag{8.33}$$

We recover the $T = 0$ case.

2) for high temperatures $(\beta \longrightarrow 0)$.

We will get results for very high temperatures. Taking the limit, of $\beta \longrightarrow 0$, we get

$$\frac{D_{H/Q}(z,\beta)}{D_{H/Q}(z)} \longrightarrow 0. \tag{8.34}$$

8.4.8 Conclusion.

In order to calculate the production of hadron states at high energies, we fix the energies of the centre of masses (corresponding to short probing distances), say to $\sqrt{s}/2$ $= \sqrt{q^2}/2 = 1TeV, 5TeV, 10TeV$, and plot the corresponding curves (Fig. 8). In our calculation, we interested in calculating the fragmentation function of hadron production process into quark-antiquark pairs at finite temperature. At first sight, we find that the production of hadrons at finite temperature decreases compared to that at zero temperature (probability of formation of states of hadrons is diminished with increase of temperature). The curve (Fig. 8) is plotted in the light-cone gauge for centre of mass energies : $\sqrt{s}/2 = 1TeV, 5TeV, 10TeV$. The dropping of the curves as we go up in temperature means that the probability of hadronisation into quark-antiquark bound state paires is low as T grows up which leads to weak quark-antiquark couplings for high temperatures in the hot QGP, therefore not allowing for quark-antiquark paires to form easily. This accounts possibly for asymptotic frydoom and deconfinement.

It should be mentioned that in the finite temperature case, we can see that Bose-Einstein or Fermi-Dirac thermal ditribution weights arise naturally without needing to define them in the formula of the hadronic tensor as stated in reference [14]. It should be pointed out that the result of the fragmentation function takes a null value as temperature goes up . A result to be confirmed experimentaly. We should remark that we have extracted from $T \neq 0$ calculations of the fragmentation function, a result (8.30) for the $T = 0$ case.

Bibliography

[1] D.E. Soper,*"Parton Distribution functions"*, arXiv:hep-lat/9609018.

[2] J.C. Collins, D.E.Soper and G.Sterman , *"Factorization of Hard Processes in QCD"*, arXiv:hep-ph/0409313.

[3] J.C. Collins, D.E. Soper, *"Parton Distribution and Decay Functions"*, OITS-166, June 1981.

[4] J. C. Collins, D. E. Soper, 'Parton Distribution and Decay Functions', Nuclear Physics **B 194** (1982) 445-492

[5] S.J.Brodsky , H.C. Pauli, and S.S.Pinsky, *"Quantum Chromodynamics and Other Field Theories on the Light Cone"*, arXiv: hep-ph/9705477

[6] Richard D. Field, *"Application of Perturbative QCD"* , Addison-Wesley publishing company, (1989).

[7] B. Mele and P. Nason, *"Next-to-leading QCD calculation of the heavy quark fragmentation function"*, Nucl.Phys. **B 361** (1991) 626; Phys.lett. **B 245** (1990) 635.

[8] J. P. Ma , *"Calculating fragmentation functions from definitions"* , Physics Letters **B 332** (1994).

[9] J. D. Whitehead, H. Matsumoto and H. Umezawa, phys.lett.103, 408 (1984).

[10] 'Measurements of the u Valence Quark Distribution Function in the Proton and u Quark Fragmentation Functions', CERN-EP/89-25 January 31st, 1989.

[11] TASSO Collab.,W.Branschweig et al., Z.Phys. C47 (1990)187;TASSO collab., W. Braun Schweig et al., Z.Phys. C22 (1984) 307.

[12] OPAL collab., M. AKrway et al., Phys.lett. B247 (1990) 617. TOPAZ collab., R. Itoh et al., Phys. lett.B 345 (1995) 335.

[13] B. Mele and P. Nason ,*"Next-to-leading QCD Calculation of the Heavy Quark Fragmentation Function"* , CERN-TH 5707/90 , April 1990.

[14] Jonathan A.Osborn , Enke Wang and Xin-Nian Wang , *"Evolution of Parton Fragmentation Functions at Finite Temperature"*, arXiv : hep-ph/0212131.

[15] J.P.Ma , *"Perturbative Prediction for Parton Fragmentation into Heavy Hadron"* , Nucl. Phys. **B 506** (1997) 329-347.

[16] Y. Takahashi, H. Umezawa, 'Thermo Field Dynamics', International Journal of Modern Physics, **B,** Vol.**10**. **Nos. 13 & 14** (1996) 1755-1805.

[17] Y.Takahashi and H.Umezawa , *"Thermo Field Dynamics"* , Collective Phenomena, 1975, Vol. 2. pp 55-80.

[18] H. Umezawa, '*Advanced Field theory, Micro, Macro and Thermal physics*' (AIP, New-York, 1993)

[19] T. Kopf, A. E. Santana and F. C. Khanna, J. Math. Phys. 38, 4971 (1997).

[20] L. M. Silva, A. E. Santana and J. D. M. Vianna, Braz. J. phys. 27, 619 (1996).

[21] M. C. B. Abdala, A. L. Gadelha and I. V. Vancea, Phys. Rev D.64 086005 (2001)

[22] M. C. B. Abdala, A. L. Gadelha and I. V. Vancea, Int. J. Mod. Phys, A.18, 2109 (2003)

[23] H. Chu and H. Umezawa, *Int. J. Mod. Phys.* **A9**, 2363 (1994).

[24] A. E. Santana and F. C. Khanna, Phys. lett. A 203, 68 (1995)

[25] F. C. Knanna, A. P. C. Malbouisson, J. M. C. Malbouisson, A. E. Santana, 'Thermal Quantum Field Theory, Algebraic Aspects and Applications', World Scientific Publishing Company (2009).

[26] L. E. Segal, Ann. Math. **48,** 930 (1947).

[27] G. C. Emch, *Algebraic Methods in statistical and quantum field theory* (John Wiley, N. York,1972).

[28] L. Laplae, F. Mancini and H. Umezawa., *Phys. Rep.* **102**. 151 (1974).

[29] E. J. Verboven, *Phys. Lett.* **21** , 391 (1966).

[30] E. J.Verboven, *Physica* **32**, 2081 (1966).

[31] J. Wirstam, *"One-Loop QCD Corrections to the Thermal Wilson Line Model"*, arXiv: hep-ph/0106141.

[32] R. D. Pisarsky , *"Effective Theory of Wilson Lines and Deconfinement"*,arXiv: hep-ph/0608242.

[33] R. D. Pisarsky , *"Quark-Gluon Plasma as a Condensate of* $SU(3)$ *Wilson Lines"*, arXiv: hep-ph/0000.

[34] R. Frederix , *"Wilson lines in QCD"*, Ph.d Thesis, Universiteit Utrecht, October 2005.

[35] M. Le Bellac, 'Thermal Field Theory (Cambridge University Press, Cambridge, 1996).

[36] J. L. Kapusta, 'Finite Temperature Field Theory' (Cambridge University Press, Cambridge,1989)

[37] I.S. Gradshteyn and I.M. Ryzhik , *"Tables of Integrals, Series, and Products"* , Academic Press , (1980).

[38] A.Nikiforov, V.Ouvarov, *"Fonctions Spéciales de la Physique Mathématique"*, Edition MIR.Moscou,1983.

9

Non-equilibrium evolution of Green functions in TFD: a canonical approach.

9.1 Introduction.

Thermofield dynamics (TFD) studied by Takahashi and Umezawa is an *operator formalism* which describes isolated systems at equilibrium [1],[2],[3],[4],[5],[6]. On the other hand, there are many systems exhibiting nonequilibrium properties. These are called open systems. they interact with a heat bath and show spatial distribution of temperature which is not homogeneous, it will naturally initiate a temporal change in the thermal state trying to reach a homogeneous situation. This is commonly known as heat conduction. Since this requires a quantum field theory with spatially inhomogeneous vaccum, it requires an extension of thermal quantum field theories to deal with such a nonequilibrium system. This kind of formalism for space-time dependent thermal physics will find many applications in condensed matter physics and phase transitions [7]. Time-dependent thermal quantum theories are also useful in the study of dissipatif systems and the evolution of the universe. Since this evolution is supposed to be controlled by all of the fundamental quantum fields and since the evolution is supposed to involve thermal effects, the need for a time-dependent thermal quantum field theories is obvious [3],[6].

The idea presented here, i.e the use of a new approach which is **LvN** formalism is the start to a new point of view of nonequilibrium TFD. We consider the theory of explicitly time-dependent invariants for a quantum system whose Hamiltonian operator $H(t)$ is explicitly time-dependent. The simplicity of the rules for constructing these invariants and the instructive relation of the invariants to the asymptotic expansion of the adiabatic invariant theory have stimulated an interest in using the invariants for solving some explicit quantum-mechanical problems. We apply the approach to systems undergoing time-dependent evolution in a time-dependent heat bath of time-varying temperature $\beta(t)$. The **LvN** approach is a canonical method that unifies the functional Schrodinger equation for quantum evolution of pure states and the **LvN** equation for the quantum description of mixed states of either equilibrium or nonequilibrium. One of the advantages is that one mixes the well-known techniques of quantum mechanics and quantum many-particle systems. It is based on the observation by Lewis and Riesenfeld [9],[10] that the quantum **LvN** equation, which was originally used to define the density operator for a mixed state, can also be used to find exact pure states of a time-dependent harmonic oscillator.This observation makes it possible to find not only the mixed state but also the pure state of a time-dependent system. The usefulness and simplicity of the **LvN** method in obtaining two-point correlators were illustrated using the quantum-mechanical anharmonic oscillator. The formalism was then extended to a scalar field and to Φ^4 field theory.

In sec.2 we present an introduction of three pictures to describe a quantum system. Sec.3 we use the thermofield dynamics (TFD) introduced by Takahashi and Umezawa as a canonical formalism to describe nonequilibrium systems. We treat the example of a harmonic oscillator. Sec.4 we treat time-dependent anharmonic oscillator. Sec.5 we study correlation function for anharmonic oscillator. Sec.6 we develop further the latter study and extend it to calculate the time-dependent Green functions for a scalar field. Sec.7 the same sudy for Φ^4 theory. Sec.8 is devoted to time-dependent green functions for fermions.

9.2 Three dependent pictures for Quantum theory : The Schrödinger, the Heisenberg and Liouville-von Neumann (LvN) representations.

For stationary (time-independent) systems or isolated system, two pictures have been used : the Schrödinger and the Heisenberg pictures. in this paper we consider the Schrodinger picture to solve the secular equation encountered in quantum theory or what is called the eigenvalue problem in perturbation theory for $\lambda\Phi^4(x)$ theory. In the

9.2 Three dependent pictures for Quantum theory : The Schrödinger, the Heisenberg and Liouville-von Neumann (LvN) representations.

Heisenberg picture, operators evolve in time whereas states do not change in time. So we have the prescription

$$O_H(t), \mid \Psi\rangle$$

for open systems (out of equilibrium) or systems changing in time, it is possible to show that another description, the LvN equation, applies besides the Heisenberg picture. In the Schrödinger picture, the quantum states evolve in time according to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t} \mid \Psi(t)\rangle = H(t) \mid \Psi(t)\rangle$$

where $H(t)$ is the Hamiltonian of the system. All the quantum information is encoded in the state $\mid \Psi(t)\rangle$, whereas the operators do not change. The mean value of operators O_S is given by

$$\langle O_S(t)\rangle = \langle \Psi(t) \mid O_S \mid \Psi(t)\rangle.$$

The operator describing the evolution of quantum states and operators satisfies the equation

$$i\hbar\frac{\partial}{\partial t}U(t) = H(t)U(t),$$

that has the time-ordered solution

$$U(t) = T\exp\left(-\frac{i}{\hbar}\int_0^t H(t')\,dt'\right).$$

Then the state vector is then written as

$$\mid \Psi(t)\rangle = U(t) \mid \Psi\rangle$$

where $\mid \Psi\rangle$ is any state independent of time. In the Heisenberg picture, opertaors are given by

$$O_H(t) = U^{\dagger}(t)\,OU(t)$$

where O is any operator independent of time. The Heisenberg operator satisfies the equation

$$i\hbar\frac{\partial}{\partial t}O_H(t) + [H_H(t), O_H(t)] = 0.$$

where $H_H(t)$ is the Hamiltonian in the Heisenberg representation. For stationary systems (isolated) $U(t)$ commutes with H_0, so $H_H(t)$=H_0. Therefore the expectation value for any operator in the two representations is given by

$$\langle O_H(t)\rangle = \langle \Psi \mid O_H(t) \mid \Psi\rangle = O_H(t) = \langle \Psi \mid U^{\dagger}(t)\,OU(t) \mid \Psi\rangle = \langle \Psi(t) \mid O \mid \Psi(t)\rangle = \langle O_S(t)\rangle$$

9. NON-EQUILIBRIUM EVOLUTION OF GREEN FUNCTIONS IN TFD: A CANONICAL APPROACH.

We introduce another invariant operator called the LvN operator

$$O_L(t) = U(t)\,OU^{\dagger}(t)\,,$$

The invariant operator $O_L(t)$ evolves backward in time, whereas the Heisenberg operator $O_H(t)$ changes forward in time. The invariant operators satisfy the quantum LvN equation

$$i\hbar\frac{\partial}{\partial t}O_L(t) + [O_L(t)\,, H(t)] = 0.$$

We wish to use the LvN equation to consider systems that change with time. Therefore, we develop in later sections a time-dependent approach which is LvN method in that: i) its invariant operators would be solutions of the Schroedinger equation whose eigenvalues are time-independent [9],[10] ii) LvN formulation is in the Schrodinger picture. That leads to higher order effects beyond the Hartree approximation for TFD in non-equilibrium [8].

9.3 TFD for thermal harmonic oscillators: Bosons and Fermions in Non-equilibrium state.

In this section, we presente two elemantary examples to see how the time-dependent thermal vaccum $\mid O(\beta)\,, t\rangle$ can be constructed out of time dependent creation and annihilation operators. We construct a complete set of orthonormal vectors showing the doubling ofdegrees of freedom of the Hilbert space, and to complete the theory. The time-dependent thermal states are obtained by applying time-dependent BOGOLIOUBOV unitary transformations. A matrix notation will be introduced for both bosonic and fermionic systems. It is possible that time-dependence of an open system may be considered by using time-dependence of the mass $m(t)$ of the system.

9.3.1 Real scalar field in TFD.

First, we consider the case of a boson field with time-dependent frequency, $\omega(t)$. Then, the Hamiltonian of an ensemble of free bosons with frequency $\omega(t)$ is

$$H(t) = \omega(t)\,a^{\dagger}(t)\,a(t)$$

the usual commutation relations hold

$$[a(t), a^{\dagger}(t)] = 1$$

$$[a(t), a(t)] = 0.$$

The Fock space is spanned by a set of orthonormal vectors

$$\mid O, t\rangle, a^{\dagger}(t) \mid O, t\rangle, \ldots, \frac{1}{\sqrt{n!}} \left(a^{\dagger}(t)\right)^{n} \mid O, t\rangle, \ldots.$$

where $\mid O, \tilde{O}, t\rangle$ is simply denoted by $\mid O, t\rangle$.The eigenvalues and eigenstates of $H(t)$ are specified by

$$H(t) \mid n, \tilde{n}, t\rangle = n\omega(t) \mid n, \tilde{n}, t\rangle, n = 0, 1, 2, \ldots, \infty.$$

Let's introduce the tilda fields

$$\tilde{H}(t) = \omega(t)\, \tilde{a}^{\dagger}(t)\, \tilde{a}(t)$$

with annihilation and creation operators obeying

$$[\tilde{a}(t), \tilde{a}^{\dagger}(t)] = 1$$

$$[\tilde{a}(t), \tilde{a}(t)] = 0$$

$$[a(t), \tilde{a}(t)] = [a(t), \tilde{a}^{\dagger}(t)] = 0.$$

The total Hamiltonian is

$$\hat{H}(t) = H(t) - \tilde{H}(t) = \omega(t) \left[a^{\dagger}(t)\, a(t) - \tilde{a}^{\dagger}(t)\, \tilde{a}(t)\right]. \tag{9.1}$$

The eigenvalue solutions to the Schrödinger equation is

$$H(t) \mid \Psi(t)\rangle = E_n \mid \Psi(t)\rangle. \tag{9.2}$$

Let's remark that the eigenvalues (spectrum of energy) of this Hamiltonian is time-independent [9],[10].

9.3.1.1 Thermal vacuum and time-dependent Bogoliubov transformation.

Since the Hilbert space is doubled, any bosonic state is represented by $\mid n, \tilde{n}, t\rangle$ and the thermal vaccum is spanned by these states as follows [1],[2],[3],[4],[5],[11]

$$\begin{aligned} \mid \quad O(\beta), t\rangle &= Z^{-1/2}(\beta, t)_n e^{-\beta E_n/2} \mid n, \tilde{n}, t\rangle = Z^{-1/2}(\beta, t)_n e^{-\beta n\omega/2} \frac{1}{n!} \left(a^{\dagger}(t)\right)^n \left(\tilde{a}^{\dagger}(t)\right)^n \mid O, t\rangle \\ &= \sqrt{\left(1-e^{-\beta\omega}\right)} \exp\left(e^{-\beta\omega/2} a^{\dagger}(t)\, \tilde{a}^{\dagger}(t)\right) \mid O, t\rangle. \end{aligned} \tag{9.3}$$

If we denote

$$u(\beta, t) = \left(1 - e^{-\beta\omega(t)}\right)^{-1/2} = \sqrt{\left(1 + n_{\mathbf{B}}(t)\right)} \tag{9.4}$$

with

$$\cos\theta(\beta) = u(\beta, t).$$

Then, we find

$$\mid O(\beta), t\rangle = u^{-1}(\beta) \exp\left(\frac{v(\beta, t)}{u(\beta, t)} a^{\dagger}(t)\, \tilde{a}^{\dagger}(t)\right) \mid O, t\rangle = e^{-iG_{\mathbf{B}}(\beta, t)} \mid O, t\rangle. \tag{9.5}$$

Hence, the unitary operator, transforming $\mid O, \tilde{O}, t\rangle$ into $\mid O(\beta), t\rangle$ is given by

$$U(\beta, t) = e^{-iG_{\mathbf{B}}(\beta, t)}. \tag{9.6}$$

where

$$G_{\mathbf{B}}(\beta, t) = -i\theta(\beta)\left(\tilde{a}(t)\, a((t)) - a^{\dagger}(t)\, \tilde{a}^{\dagger}(t)\right)$$

The operator $U(\beta)$ is called the *Bogoliubov transformation.* The temperature dependent operators are defined as follow

$$\begin{aligned} a(\beta, t) &= e^{-iG_{\mathbf{B}}(\beta, t)} a(t)\, e^{iG_{\mathbf{B}}(\beta, t)} = u(\beta, t)\, a(t) - v(\beta, t)\, \tilde{a}^{\dagger}(t) \\ \tilde{a}(\beta, t) &= e^{-iG_{\mathbf{B}}(\beta, t)} \tilde{a}(t)\, e^{iG_{\mathbf{B}}(\beta, t)} = u(\beta, t)\, \tilde{a}(t) - v(\beta, t)\, a^{\dagger}(t). \end{aligned} \tag{9.7}$$

The physical observables for zero temperature systems are written in terms of $a(t)$ and $a^{\dagger}(t)$ and for non-thermal operators, it is appropriate to work with operators at finite temperature. Bogoliubov transformations are appropriate for such a change.

Then, the annihilation operators satisfy

$$a(\beta, t) \mid O(\beta), t\rangle = \tilde{a}(\beta, t) \mid O(\beta), t\rangle = 0. \tag{9.8}$$

The number operator gives the statistical average

$$\langle O(\beta), t \mid a^{\dagger}(t) a(t) \mid O(\beta), t\rangle = v^2(\beta) = \frac{1}{(e^{\beta\omega} - 1)} = n_{\mathbf{B}}(\omega) \tag{9.9}$$

which is the Bose-Einstein distribution. The Fock space is spanned by the states as follows

$$\mid O(\beta), t\rangle, a^{\dagger}(\beta, t) \mid O(\beta), t\rangle, \tilde{a}^{\dagger}(\beta, t) \mid O(\beta), t\rangle, \ldots, \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{m!}} \left(a^{\dagger}(\beta, t)\right)^n \left(\tilde{a}^{\dagger}(\beta, t)\right)^m \mid O(\beta), t\rangle, \ldots$$

Since this transformation is unitary, the algebra of the original time-dependent operators $a(t)$ and $\tilde{a}(t)$ is kept invariant, that is the thermal operators $a(\beta, t)$ and $\tilde{a}(\beta, t)$ satisfy the following commutation relations

$$[a(\beta, t), a^{\dagger}(\beta, t)] = 1, [\tilde{a}(\beta, t), \tilde{a}^{\dagger}(\beta, t)] = 1, \tag{9.10}$$

with all other commutation relations being zero.

9.3.2 Dirac field in TFD.

Next, we consider a system of free fermions with frequency $\omega(t)$. Then, the Hamiltonian of an ensemble of free fermions with frequency $\omega(t)$ is defined by

$$H(t) = \omega(t) a^{\dagger}(t) a(t)$$

where the operators $a(t)$ and $a^{\dagger}(t)$ satisfy the relations

$$\left\{a(t), a^{\dagger}(t)\right\} = 1, \left\{a^{\dagger}(t), a^{\dagger}(t)\right\} = \{a(t), a(t)\} = 0.$$

where $\{A, B\} = AB + BA$ is the anti-commutator. Using the last equation and the fact that $\langle n \mid n\rangle = 1$, we get $n = 0, 1$. Further, if we introduce the fictitious time-dependent system characterized by

$$\tilde{H}(t) = \omega(t) \tilde{a}^{\dagger}(t) \tilde{a}(t)$$

with

$$\left\{\tilde{a}(t), \tilde{a}^{\dagger}(t)\right\} = 1, \{\tilde{a}(t), \tilde{a}(t)\} = \{a(t), \tilde{a}(t)\} = \left\{a(t), \tilde{a}^{\dagger}(t)\right\} = 0.$$

The Fock space will be spanned by the set of time-dependent orthonormal vectors

$$\mid O, t\rangle, a^{\dagger}(t) \mid O, t\rangle, \tilde{a}^{\dagger}(t) \mid O, t\rangle, a^{\dagger}(t) \tilde{a}^{\dagger}(t) \mid O, t\rangle,$$

where $\mid O, t\rangle$ denotes $\mid O, \tilde{O}, t\rangle$.The energy eigenvalues of $H(t)$ are independent of time [9],[10]

$$H(t) \mid n, \tilde{n}, t\rangle = \epsilon_n \mid n, \tilde{n}, t\rangle = n\omega \mid n, \tilde{n}, t\rangle, n = 0, 1.$$

9.3.2.1 Thermal vacuum and time-dependent Bogoliubov transformation.

In order to construct time-dependent TFD for this system, the Hilbert space is doubled, any fermionic state is represented by the basis

$$\mid O,\tilde{O},t\rangle,\tilde{a}^{\dagger}(t)\mid O,t\rangle=\mid O,\tilde{1},t\rangle,a^{\dagger}(t)\mid O,t\rangle=\mid 1,\tilde{O},t\rangle,a^{\dagger}(t)\,\tilde{a}^{\dagger}(t)\mid O,t\rangle=\mid 1,\tilde{1}\rangle. \tag{9.11}$$

The thermal vaccum is spanned by these states as follows [1],[2],[3],[4],[5],[6],[11],[12],[13]

$$\begin{aligned}\mid\quad O(\beta),t\rangle &= Z^{-1/2}(\beta,t)_{n}\,e^{-\beta\epsilon_{n}/2}\mid n,\tilde{n},t\rangle=\frac{1}{\sqrt{Z(\beta,t)}_{n}}\left(\mid O,\tilde{O},t\rangle+e^{-\beta\epsilon_{1}/2}\mid 1,\tilde{1},t\rangle\right)\\ &= \frac{1}{\sqrt{Z(\beta,t)}}\left(1+e^{-\beta\omega/2}a^{\dagger}(t)\,\tilde{a}^{\dagger}(t)\right)\mid O,t\rangle.\end{aligned} \tag{9.12}$$

We obtain $\mid O(\beta),t\rangle$ from $\mid O\rangle$ by a *Bogoliuobov transformation* defined by

$$U_{\mathbf{F}}(\beta,t)=e^{-iG_{\mathbf{F}}(\beta,t)}$$

where

$$G_{\mathbf{F}}(\beta,t)=-i\theta(\beta)\left(\tilde{a}(t)\,a(t)-a^{\dagger}(t)\,\tilde{a}^{\dagger}(t)\right).$$

Indeed, If we denote

$$\begin{aligned}\cosh\theta(\beta) &= u(\beta,t)=\left(1+e^{-\beta\omega(t)}\right)^{-1/2}=\sqrt{(1-n_{\mathbf{F}}(\omega(t)))}\\ \sinh\theta(\beta) &= v(\beta,t)=\left(1+e^{\beta\omega(t)}\right)^{-1/2}=\sqrt{n_{\mathbf{F}}(\omega(t))}\end{aligned} \tag{9.13}$$

then we have

$$u^{2}(\beta)-v^{2}(\beta)=1 \tag{9.14}$$

$$\mid O(\beta),t\rangle=\left\{u(\beta,t)+v(\beta,t)\,a^{\dagger}(t)\,\tilde{a}^{\dagger}(t)\right\}\mid O,t\rangle \tag{9.15}$$

Now the relation (9.12) in a canonical form is given as

$$\mid O(\beta),t\rangle=e^{-iG_{\mathbf{F}}(\beta,t)}\mid O,t\rangle=\left(\cos\theta(\beta)+\sin\theta(\beta)\,a^{\dagger}(t)\,\tilde{a}^{\dagger}(t)\right)\mid O,t\rangle. \tag{9.16}$$

The thermal operators are defined in a manner similar to the bosonic case

$$\begin{aligned}a(\beta,t)&=e^{-iG_{\mathbf{F}}(\beta,t)}a(t)\,e^{iG_{\mathbf{F}}(\beta,t)}=u(\beta,t)\,a(t)-v(\beta,t)\,\tilde{a}^{\dagger}(t)\\ \tilde{a}(\beta,t)&=e^{-iG_{\mathbf{F}}(\beta,t)}\tilde{a}(t)\,e^{iG_{\mathbf{F}}(\beta,t)}=u(\beta,t)\,\tilde{a}(t)+v(\beta,t)\,a^{\dagger}(t)\end{aligned} \tag{9.17}$$

Non-thermal operators a and $a^{\dagger}$ are derived from the thermal ones by inverting these relations [3]

$$a(t) = u(\beta,t)\,a(\beta,t) + v(\beta,t)\,\tilde{a}^{\dagger}(\beta,t)\,.$$
$$\tilde{a}(t) = u(\beta,t)\,\tilde{a}(\beta,t) - v(\beta,t)\,a^{\dagger}(\beta,t)\,, \tag{9.18}$$

Such relations are obtained if we assume that, without loss of generality, $\tilde{a}(t) = -a(t)$ for fermions. The fermionic annihilation operators satisfy

$$a(\beta,t) \mid O(\beta)\,,t\rangle = \tilde{a}(\beta,t) \mid O(\beta)\,,t\rangle = 0. \tag{9.19}$$

The number operator gives the time-independent statistical average

$$\begin{aligned} \langle O(\beta)\,,t \quad &| \quad N(t) \mid O(\beta)\,,t\rangle = \langle O(\beta)\,,t \mid a^{\dagger}(t)\,a(t) \mid O(\beta)\,,t\rangle \\ &= \frac{1}{(1+e^{-\beta\omega})}\langle O,\tilde{O},t \mid \left(1+e^{-\beta\omega/2}\tilde{a}(t)\,a(t)\right)a^{\dagger}(t)\,a(t)\left(1+e^{-\beta\omega/2}a^{\dagger}(t)\,\tilde{a}^{\dagger}(t)\right) \mid O,\tilde{O},t\rangle \\ &= \frac{1}{\left(1+e^{\beta\omega(t)}\right)} = n_{\mathbf{F}}(\omega(t)) \end{aligned} \tag{9.20}$$

which is the Fermi-Dirac distribution. The Fock space is constructed from the time-dependent vacuum $\mid O(\beta)\,,t\rangle$ and is spanned by the set of states

$$\left\{\mid O(\beta)\,,t\rangle, a^{\dagger}(\beta,t) \mid O(\beta)\,,t\rangle, \tilde{a}^{\dagger}(\beta,t) \mid O(\beta)\,,t\rangle, a^{\dagger}(\beta,t)\,\tilde{a}^{\dagger}(\beta,t) \mid O(\beta)\,,t\rangle\right\}. \tag{9.21}$$

Since this transformation is unitary, the algebra of the original operators a and $\tilde{a}$ is kept invariant, that is the thermal operators $a(\beta)$ and $\tilde{a}(\beta)$ satisfy the following anticommutation relations

$$\left\{a(\beta,t)\,,a^{\dagger}(\beta,t)\right\} = 1, \left\{\tilde{a}(\beta,t)\,,\tilde{a}^{\dagger}(\beta,t)\right\} = 1, \tag{9.22}$$

with all other commutation relations being zero.

9.3.3 Thermalization of a bosonic and fermionic oscillator with infinite degree of freedom via time-dependent Bogoliubov transformation.

Let's recall that TFD is a real-time finite temperature formalism. To describe a statistical system out of equilibrium, we need to introduce the partition function of the system depending on time for an open system and given by

9. NON-EQUILIBRIUM EVOLUTION OF GREEN FUNCTIONS IN TFD: A CANONICAL APPROACH.

$$Z(\beta, t) = Tr\left(e^{-\beta \hat{H}(t)}\right) \tag{9.23}$$

where, $H(t)$ is the total hamiltonian of the system. Matsubara has shown that the statistical average of an operator is the same as the mean value of this operator on the vaccum at finite temperature If we can define the mean value of an operator as

$$\langle A(t)\rangle_\beta = \langle O(\beta), t \mid A(t) \mid O(\beta), t\rangle = Z^{-1}(\beta, t) \sum_n e^{-\beta E_n} \langle n, \tilde{n}, t \mid A(t) \mid n, \tilde{n}, t\rangle \tag{9.24}$$

where $\mid n, \tilde{n}, t\rangle$ is a specific state corresponding to the energy E_n.Then the finite temperature formalism will be completely parallel to the zero temperature field theory. The Lagrange formalism may be set up for the total system involving the physical and the tilde field [1],[2],[3],[4],[5],[11]. If we introduce the free fields by

$$\psi(x) = \frac{1}{\sqrt{V}}{}_{\mathbf{k}} e^{i\mathbf{kx}} e^{-i\epsilon_k t} a_{\mathbf{k}}(t)$$

$$\tilde{\psi}(x) = \frac{1}{\sqrt{V}}{}_{\mathbf{k}} e^{-i\mathbf{kx}} e^{i\epsilon_k t} \tilde{a}_{\mathbf{k}}(t), \left(\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m\right)$$

the Hamiltonian $\hat{H}$ becomes

$$\hat{H}(t) = H(t) - \tilde{H}(t)$$

with

$$\begin{aligned} H(t) &= {}_{\mathbf{k}}\epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}}(t)\, a_{\mathbf{k}}(t) \\ \tilde{H}(t) &= {}_{\mathbf{k}}\epsilon_{\mathbf{k}} \tilde{a}^\dagger_{\mathbf{k}}(t)\, \tilde{a}_{\mathbf{k}}(t) \end{aligned}$$

and If k , and s are respectively the momentum and spin of the oscillator , then we observe that the total hamiltonian $\hat{H}(t) = H(t) - \tilde{H}(t)$ is invariant under the *Bogoliubov transformation*

$$a_k(t) \longrightarrow a_k(\theta, t) = a_k(t)\cos\theta_k - \tilde{a}^\dagger_k(t)\sin\theta_k$$

$$\tilde{a}_k(t) \longrightarrow \tilde{a}_k(\theta, t) = \tilde{a}_k(t)\cos\theta_k + a^\dagger_k(t)\sin\theta_k \tag{9.25}$$

for fermions, and

$$a_k(t) \longrightarrow a_k(\theta,t) = a_k(t)\cosh\theta_k - \tilde{a}_k^{\dagger}(t)\sinh\theta_k$$

$$\tilde{a}_k(t) \longrightarrow \tilde{a}_k(\theta,t) = \tilde{a}_k(t)\cosh\theta_k - a_k^{\dagger}(t)\sinh\theta_k \tag{9.26}$$

for bosons. These relations are written in a matrix form as

$$\begin{pmatrix} a_k(\beta,t) \\ \tilde{a}_k^{\dagger}(\beta,t) \end{pmatrix} = U_{\mathbf{F}}(\theta_k)\begin{pmatrix} a_k(t) \\ \tilde{a}_k^{\dagger}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta_k(\beta) & \text{-}\sin\theta_k(\beta) \\ \sin\theta_k(\beta) & \cos\theta_k(\beta) \end{pmatrix}\begin{pmatrix} a_k(t) \\ \tilde{a}_k^{\dagger}(t) \end{pmatrix} \tag{9.27}$$

for fermions, and

$$\begin{pmatrix} a_k(\beta,t) \\ \tilde{a}_k^{\dagger}(\beta,t) \end{pmatrix} = U_{\mathbf{B}}(\theta_k)\begin{pmatrix} a_k(t) \\ \tilde{a}_k^{\dagger}(t) \end{pmatrix} = \begin{pmatrix} \cosh\theta_k(\beta) & \text{-}\sinh\theta_k(\beta) \\ \text{-}\sinh\theta_k(\beta) & \cosh\theta_k(\beta) \end{pmatrix}\begin{pmatrix} a_k(t) \\ \tilde{a}_k^{\dagger}(t) \end{pmatrix} \tag{9.28}$$

for bosons where $U_{\mathbf{F}}(\theta_k)$ and $U_{\mathbf{B}}(\theta_k)$ are Bogoliubov unitary transfomations. Then we write

$$\sinh^2\theta_k = n_B(k,t) = \frac{1}{\left(e^{\beta|k_0(t)|}-1\right)}\,,\ \sin^2\theta_k = n_F(k,t) = \frac{1}{\left(1+e^{\beta|k_0(t)|}\right)} \tag{9.29}$$

as bose and fermion distribution functions with time-dependence, $k_0(t)$ being the energy associated with the four-vector k. The generator of the transformation is easily be found to be

$$\hat{G}(t) = i_{\mathbf{k}}\theta_{\mathbf{k}}\left\{a_{\mathbf{k}}^{\dagger}(t)\,\tilde{a}_{\mathbf{k}}^{\dagger}(t) - \tilde{a}_{\mathbf{k}}(t)\,a_{\mathbf{k}}(t)\right\}, \tag{9.30}$$

where the one particle operators are

$$\begin{aligned} a_{\mathbf{k}}(\theta,t) &= e^{-i\hat{G}(t)}a_{\mathbf{k}}(t)\,e^{i\hat{G}(t)}, \\ \tilde{a}_{\mathbf{k}}(\theta,t) &= e^{-i\hat{G}(t)}\tilde{a}_{\mathbf{k}}(t)\,e^{i\hat{G}(t)}. \end{aligned} \tag{9.31}$$

The transformation given by (9.31) leaves the total Hamiltonian invariant (the generator $\hat{G}(t)$ is conserved) i.e.

$$[\hat{G}(t), \hat{H}(t)] = [\hat{G}(t), H(t) - \tilde{H}(t)] = 0 \tag{9.32}$$

The freedom of this transformation allows us to write the transformed Fock space for the vacuum as

$$\mid O(\theta), t\rangle = e^{-i\hat{G}(t)} \mid O, t\rangle \tag{9.33}$$

and the one particle states as

$$\begin{aligned} a_{\mathbf{k}}^{\dagger}(\theta,t) &\mid O(\theta),t\rangle = e^{-i\hat{G}(t)} a_{\mathbf{k}}^{\dagger}(t) \mid O,t\rangle \\ \tilde{a}_{\mathbf{k}}^{\dagger}(\theta,t) &\mid O(\theta),t\rangle = e^{-i\hat{G}(t)} \tilde{a}_{\mathbf{k}}^{\dagger}(t) \mid O,t\rangle \end{aligned} \tag{9.34}$$

and so on. Using the explicit form of (41), and the property

$$a_{\mathbf{k}}(t) \mid O,t\rangle = \tilde{a}_{\mathbf{k}}(t) \mid O,t\rangle = 0 \tag{9.35}$$

A temperature dependent vaccum is introduced under a time-dependent bogoliubov transformation and we can show [1],[2],[3],[4],[5],[6],[11],[12],[13] as for the time-independent case that

$$\mid O(\theta),t\rangle = \prod_k \left(\cos\theta_k + \sin\theta_k a_k^{\dagger}(t)\, \tilde{a}_k^{\dagger}(t)\right) \mid O,t\rangle \tag{9.36}$$

for fermions and

$$\mid O(\theta),t\rangle = \prod_k \left(\frac{1}{\cosh\theta_k}\right) \exp\left(\tanh\theta_k a_k^{\dagger}(t)\, \tilde{a}_k^{\dagger}(t)\right) \mid O,t\rangle \tag{9.37}$$

for bosons. As for the one particle field state, from equations (9.35), we have

$$a_{\mathbf{k}}(\theta,t) \mid O(\theta),t\rangle = \tilde{a}_{\mathbf{k}}(\theta,t) \mid O(\theta),t\rangle = 0. \tag{9.38}$$

Also, the creation and annihilation operators satisfy ordinary commutation relations

$$\left[a_{\mathbf{k},\mathbf{s}}(\theta,t)\,, a_{\mathbf{k}',\mathbf{s}'}^{\dagger}(\theta,t')\right]_{\pm} = \delta\left(k-k'\right)\delta_{ss'}$$

$$\left[\tilde{a}_{\mathbf{k},\mathbf{s}}(\theta,t)\,, \tilde{a}_{\mathbf{k}',\mathbf{s}'}^{\dagger}(\theta,t')\right]_{\pm} = \delta\left(k-k'\right)\delta_{ss'} \tag{9.39}$$

all other commutators (anticommutators) vanish.

9.4 Time-dependent Anharmonic Oscillator in TFD.

In this section we extend the formalism developed above to a time-dependent anharmonic oscillator with the Hamiltonian

$$\hat{H}(t) = \frac{1}{2}\frac{p^2}{m(t)} + m(t)\, V(q) = \frac{1}{2}\frac{p^2}{m(t)} + \frac{1}{2} m(t)\, \omega^2(t)\, q^2 + \frac{\lambda}{4!} q^4.$$

Here the potential $V(q)$ may be written in the form

$$V(q) = \frac{\lambda_{2n}(t)}{(2n)!} q^{2n}$$

where $\lambda_{2n}(t)$ are constants depending on time. The LvN equations for the creation and annihilation operators are

$$i\frac{\partial a(t)}{\partial t} + \left[a(t), \hat{H}(t)\right] = 0, i\frac{\partial a^{\dagger}(t)}{\partial t} + \left[a^{\dagger}(t), \hat{H}(t)\right] = 0$$

and we have the following relations

$$q = \hbar\left[u(t)\, a(t) + u^{*}(t)\, a^{\dagger}(t)\right]$$

$$p = \hbar m(t)\left[\dot{u}(t)\, a(t) + \dot{u}^{*}(t)\, a^{\dagger}(t)\right]$$

where $a(t)$ and $u(t)$ satisfy respectively the usual commutation relations and the Wronskian conditions

$$\left[a(t), a^{\dagger}(t)\right] = 1$$

$$\hbar m(t)\left[\dot{u}^{*}(t)\, u(t) - \dot{u}(t)\, u^{*}(t)\right] = i.$$

Now the following relations follow

$$\bar{q} = \langle O(\beta), t \mid q \mid O(\beta), t\rangle = 0$$

$$\bar{p} = \langle O(\beta), t \mid p \mid O(\beta), t\rangle = 0$$

$$\langle q^{2}\rangle = \langle O(\beta), t \mid q^{2} \mid O(\beta), t\rangle = \hbar^{2} u(t)\, u^{*}(t)$$

$$\langle p^{2}\rangle = \langle O(\beta), t \mid p^{2} \mid O(\beta), t\rangle = \hbar^{2} m^{2}(t)\, \dot{u}(t)\, \dot{u}^{*}(t)$$

9.4.1 Calculation.

We substitute the Bogoliubov transformations Eq.(9.18) for the boson fields in the above relations, then

$$\bar{q} = \langle O(\beta), t \mid \hbar\left[u(t)\, a(t) + u^{*}(t)\, a^{\dagger}(t)\right] \mid O(\beta), t\rangle =$$

$$= \langle O(\beta), t \mid \hbar\left[u(t)\left(u(\beta,t)\, a(\beta,t) + v(\beta,t)\, \tilde{a}^{\dagger}(\beta,t)\right) + \right.$$

$$\left. + u^{*}(t)\left(u(\beta,t)\, a^{\dagger}(\beta,t) + v(\beta,t)\, \tilde{a}(\beta,t)\right)\right] \mid O(\beta), t\rangle = 0$$

and

$$\bar{p} = \langle O(\beta), t \mid p \mid O(\beta), t\rangle = \langle O(\beta), t \mid \hbar m(t) \left[\dot{u}(t) a(t) + \dot{u}^*(t) a^\dagger(t)\right] \mid O(\beta), t\rangle$$

$$= \hbar m(t) \langle O(\beta), t \mid \dot{u}(t) \left[u(\beta,t) a(\beta,t) + v(\beta,t) \tilde{a}^\dagger(\beta,t)\right] +$$

$$+\dot{u}^*(t) \left[u(\beta,t) a^\dagger(\beta,t) + v(\beta,t) \tilde{a}(\beta,t)\right] \mid O(\beta), t\rangle = 0.$$

The other relations

$$\langle O(\beta), t \mid p^2 \mid O(\beta), t\rangle = \langle O(\beta), t \mid \hbar^2 m^2(t) \left[\dot{u}(t) a(t) + \dot{u}^*(t) a^\dagger(t)\right]$$

$$\times \left[\dot{u}(t) a(t) + \dot{u}^*(t) a^\dagger(t)\right] \mid O(\beta), t\rangle =$$

$$= \hbar^2 m^2(t) \dot{u}(t) \dot{u}^*(t).$$

This gives the same results as in the $T = 0$ case. We used the following relations :

$$a^2(t) = \left[u(\beta,t) a(\beta,t) + v(\beta,t) \tilde{a}^\dagger(\beta,t)\right]^2$$

$$= u^2(\beta,t) a^2(\beta,t) + v^2(\beta,t) \tilde{a}^{\dagger 2}(\beta,t) + u(\beta,t) v(\beta,t) a(\beta,t) \tilde{a}^\dagger(\beta,t) + v(\beta,t) u(\beta,t) \tilde{a}^\dagger(\beta,t) a(\beta,t),$$

the second relation

$$a^{\dagger 2}(t) = a^\dagger(t) = \left[u(\beta,t) a^\dagger(\beta,t) + v(\beta,t) \tilde{a}(\beta,t)\right]^2$$

$$= u^2(\beta,t) a^{\dagger 2}(\beta,t) + v^2(\beta,t) \tilde{a}^2(\beta,t) + u(\beta,t) v(\beta,t) \tilde{a}(\beta,t) a^\dagger(\beta,t) + u(\beta,t) v(\beta,t) a^\dagger(\beta,t) \tilde{a}(\beta,t),$$

and the third relation

$$a(t) a^\dagger(t) + a^\dagger(t) a(t) = \left[u(\beta,t) a(\beta,t) + v(\beta,t) \tilde{a}^\dagger(\beta,t)\right] \times \left[u(\beta,t) a^\dagger(\beta,t) + v(\beta,t) \tilde{a}(\beta,t)\right]$$

$$+ \left[u(\beta,t) a^\dagger(\beta,t) + v(\beta,t) \tilde{a}(\beta,t)\right] \times \left[u(\beta,t) a(\beta,t) + v(\beta,t) \tilde{a}^\dagger(\beta,t)\right]$$

$$= u^2(\beta,t) a(\beta,t) a^\dagger(\beta,t) + u(\beta,t) v(\beta,t) a(\beta,t) \tilde{a}(\beta,t)$$

$$+v(\beta,t) u(\beta,t) \tilde{a}^\dagger(\beta,t) a^\dagger(\beta,t) + v^2(\beta,t) \tilde{a}^\dagger(\beta,t) \tilde{a}(\beta,t)$$

$$+u^2(\beta,t) a^\dagger(\beta,t) a(\beta,t) + u(\beta,t) v(\beta,t) a^\dagger(\beta,t) \tilde{a}^\dagger(\beta,t)$$

$$+v(\beta,t) u(\beta,t) \tilde{a}(\beta,t) a(\beta,t) + v^2(\beta,t) \tilde{a}(\beta,t) \tilde{a}^\dagger(\beta,t).$$

The other relation is

$$\langle O(\beta), t \mid q^2 \mid O(\beta), t\rangle = \langle O(\beta), t \mid \hbar^2 \left(u(t) a(t) + u^*(t) a^\dagger(t)\right)^2 \mid O(\beta), t\rangle$$

$$= \hbar^2 \times u(t)\,u^*(t)\,\langle O(\beta), t \mid \left(u^2(\beta,t)\,a(\beta,t)\,a^\dagger(\beta,t) + v^2(\beta,t)\,\tilde{a}(\beta,t)\,\tilde{a}^\dagger(\beta,t)\right) \mid O(\beta), t\rangle = \hbar^2 u(t)\,u^*$$

We obtain the same results as in the $T = 0$ case. The last relation

$$\langle qp\rangle = \langle O(\beta), t \mid \hbar^2 \left[u(t)\,a(t) + u^*(t)\,a^\dagger(t)\right] \times \hbar m(t)\left[\dot{u}(t)\,a(t) + \dot{u}^*(t)\,a^\dagger(t)\right] \mid O(\beta), t\rangle,$$

Hence

$$\begin{aligned}\langle qp\rangle &= \langle O(\beta), t \mid \hbar^2 m(t)\,[u^*(t)\,\dot{u}(t)\left(v^2(\beta,t)\,\tilde{a}(\beta,t)\,\tilde{a}^\dagger(\beta,t)\right) \\ &\quad + u(t)\,\dot{u}^*(t)\left(u^2(\beta,t)\,a(\beta,t)\,a^\dagger(\beta,t)\right)] \mid O(\beta), t\rangle \\ &= \hbar^2 m(t)\,[u^*(t)\,\dot{u}(t)\,v^2(\beta,t) + u(t)\,\dot{u}^*(t)\,u^2(\beta,t)],\end{aligned}$$

for bosons, we have

$$v^2(\beta,t) = 1 - u^2(\beta,t) = n_{\mathbf{B}}(\omega(t)), and v^2(\beta,t) = \frac{1}{e^{\beta\omega(t)} - 1} = n_{\mathbf{B}}(\omega(t)).$$

Therefore

$$\begin{aligned}\langle qp\rangle &= \hbar^2 m(t)\,[u^*(t)\,\dot{u}(t)\,n_{\mathbf{B}}(\omega(t)) + u(t)\,\dot{u}^*(t)\,(1 - n_{\mathbf{B}}(\omega(t)))] \\ &= \hbar^2 m(t)\,[u(t)\,\dot{u}^*(t) + n_{\mathbf{B}}(\omega(t))\,(u^*(t)\,\dot{u}(t) - u(t)\,\dot{u}^*(t))] \\ &= \hbar^2 m(t)\,[u(t)\,\dot{u}^*(t) - \frac{i}{\hbar m(t)} n_{\mathbf{B}}(\omega(t))] = \hbar^2 m(t)\,u(t)\,\dot{u}^*(t) - i\hbar n_{\mathbf{B}}(\omega(t)).\end{aligned}$$

Therefore, the two-point correlators are

$$g_{qq}(\beta,t) = \langle q^2\rangle - \bar{q}^2 = \hbar^2 u(t)\,u^*(t).$$

$$g_{pp}(\beta,t) = \langle p^2\rangle - \bar{p}^2 = \hbar^2 m^2(t)\,\dot{u}(t)\,\dot{u}^*(t).$$

$$g_{qp}(\beta,t) = g_{pq}(\beta,t) = \langle qp\rangle - \bar{q}\bar{p} = \hbar^2 m(t)\,u(t)\,\dot{u}^*(t) - i\hbar n_{\mathbf{B}}(\omega(t)).$$

9.5 Correlation functions in TFD.

9.5.1 Free scalar field.

We consider a free complex scalar field, the mass of which changes in time. The system is described by the lagrangian density

$$\begin{aligned}\pounds(x) &= \left[\frac{1}{2}\partial_\mu \Phi(x)\,\partial^\mu \Phi(x) - \frac{1}{2} m^2(t)\,\Phi^2(x)\right] \\ &\quad - \left[\frac{1}{2}\partial_\mu \tilde{\Phi}(x)\,\partial^\mu \tilde{\Phi}(x) - \frac{1}{2} m^2(t)\,\tilde{\Phi}^2(x)\right]\end{aligned}$$

the Hamiltonian is given by

$$H(t) = \int d^4x \left[\frac{1}{2}\Pi^2(x) + \frac{1}{2}(\nabla\Phi(x))^2 + \frac{1}{2}m^2(t)\Phi^2(x)\right] - \left[\frac{1}{2}\tilde{\Pi}^2(x) + \frac{1}{2}\left(\nabla\tilde{\Phi}(x)\right)^2 + \frac{1}{2}m^2(t)\tilde{\Phi}^2(x)\right],$$

where

$$\Pi(x) = \frac{\delta\pounds(\Phi(x))}{\delta\dot{\Phi}(x)} = \dot{\Phi}(x)$$

are conjugate momenta. The same equation for the tilda fields.

The field and momentum are Fourier decomposed as

$$\Phi(x) = \int [dk]\,\phi_k(t)\,e^{ikx}, \Pi(x) = \int [dk]\,\pi_k(t)\,e^{-ikx}$$

where

$$[dk] = \frac{d^4k}{(2\pi)^4}.$$

The Hermiticity of $\Phi(x)$ and $\Pi(x)$ implies that: $\phi_k^\dagger(t) = \phi_{-k}(t)$ and $\pi_k^\dagger(t) = \pi_{-k}(t)$.

So space integrals of quadratic fields and momenta result in momentum integrals for the decoupled modes

$$\int d^4x\Phi^2(x) = \int [dk]\,\phi_k(t)\,\phi_{-k}(t), \tag{9.40}$$

$$\int d^4x\Pi^2(x) = \int [dk]\,\pi_k(t)\,\pi_{-k}(t), \tag{9.41}$$

$$\int d^4x\nabla\Phi(x).\nabla\Phi(x) = \int [dk]\,\mathbf{k}^2\phi_k(t)\,\phi_k(t). \tag{9.42}$$

Then the Hamiltonian is written as

$$H(t) = \Pi(x)\dot{\Phi}(x) - \tilde{\Pi}(x)\dot{\tilde{\Phi}}(x) - \pounds\left(\Phi(x), \tilde{\Phi}(x), \partial_\mu\Phi(x), \partial_\mu\tilde{\Phi}(x)\right) =$$

$$= \Pi^2(x) - \tilde{\Pi}^2(x) - \pounds\left(\Phi(x), \tilde{\Phi}(x), \partial_\mu\Phi(x), \partial_\mu\tilde{\Phi}(x)\right)$$

$$= \int [dk]\,\frac{1}{2}\left\{\pi_k^2(t) + \left[\mathbf{k}^2 + m^2(t)\right]\phi_k^2(t)\right\} - \frac{1}{2}\left\{\tilde{\pi}_k^2(t) + \left[\mathbf{k}^2 + m^2(t)\right]\tilde{\phi}_k^2(t)\right\}.$$

Canonical quantization is prescribed by imposing the commutation relations at equal times,

$$[\Phi(x), \Pi(x)] = i\delta(\mathbf{x} - \mathbf{y})$$

$$\left[\tilde{\Phi}\left(x\right),\tilde{\Pi}\left(x\right)\right]=i\delta\left(\mathbf{x}-\mathbf{y}\right)$$

which lead to the relations

$$\left[\phi_{k'}\left(t\right),\pi_{k}\left(t\right)\right]=i\delta\left(k'+k\right),\tag{9.43}$$

$$\left[\tilde{\phi}_{k'}\left(t\right),\tilde{\pi}_{k}\left(t\right)\right]=i\delta\left(k'+k\right).\tag{9.44}$$

All other commutation relations vanish. The annihilation and creation operators satisfy, for each mode

$$a_{k}\left(t\right)=i\left[\varphi_{k}^{*}\left(t\right)\pi_{k}\left(t\right)-\dot{\varphi}_{k}^{*}\left(t\right)\phi_{k}\left(t\right)\right]\tag{9.45}$$

$$a_{k}^{\dagger}\left(t\right)=-i\left[\varphi_{-k}\left(t\right)\pi_{-k}\left(t\right)-\dot{\varphi}_{-k}\left(t\right)\phi_{-k}\left(t\right)\right]\tag{9.46}$$

where φ_k satisfy the classical equation of motion

$$\ddot{\varphi}_{k}\left(t\right)+\left[\mathbf{k}^{2}+m^{2}\left(t\right)\right]\varphi_{k}\left(t\right)=0.$$

It is assumed that $\varphi_{-k}\left(t\right)=\varphi_{k}\left(t\right)$ and the Wronskian condition

$$\dot{\varphi}_{k}^{*}\left(t\right)\varphi_{k}\left(t\right)-\varphi_{k}^{*}\left(t\right)\dot{\varphi}_{k}\left(t\right)=i$$

The creation and annihilation operators satisfy the standard commutation relations

$$\left[a_{k'}\left(t\right),a_{k}^{\dagger}\left(t\right)\right]=\delta\left(k'-k\right),\tag{9.47}$$

by inverting equation (9.45) and (9.46), one expresses the fields as

$$\phi_{k}\left(t\right)=\left[\varphi_{k}\left(t\right)a_{k}\left(t\right)+\varphi_{-k}^{*}\left(t\right)a_{-k}^{\dagger}\left(t\right)\right],$$

$$\pi_{k}\left(t\right)=\dot{\phi}_{k}\left(t\right)=\left[\dot{\varphi}_{k}\left(t\right)a_{k}\left(t\right)+\dot{\varphi}_{-k}^{*}\left(t\right)a_{-k}^{\dagger}\left(t\right)\right].$$

Now the thermal expectation value of the correlation functions is given as

$$g_{11}\left(x-y\right)=\left\langle O\left(\beta\right),t\mid\Phi\left(x\right)\Phi\left(y\right)\mid O\left(\beta\right),t\right\rangle=$$

$$=\left\langle O\left(\beta\right),t\mid\int\left[dk\right]\phi_{k}\left(t\right)e^{ikx}\int\left[dk'\right]\phi_{k'}\left(t\right)e^{ik'y}\mid O\left(\beta\right),t\right\rangle$$

$$=\int\left[dk\right]\int\left[dk'\right]e^{i\left(kx+k'y\right)}\left\langle O\left(\beta\right),t\mid\phi_{k}\left(t\right)\phi_{k'}\left(t\right)\mid O\left(\beta\right),t\right\rangle$$

9. NON-EQUILIBRIUM EVOLUTION OF GREEN FUNCTIONS IN TFD: A CANONICAL APPROACH.

For a real scalar field, then $\phi_k^*(t) = \phi_k(t)$ and we have

$$\langle O(\beta), t \mid \phi_k(t)\phi_{k'}(t) \mid O(\beta), t\rangle = \langle O(\beta), t \mid \left[\varphi_k(t)a_k(t) + \varphi_{-k}^*(t)a_{-k}^\dagger(t)\right] \times$$

$$\times \left[\varphi_{k'}(t)a_{k'}(t) + \varphi_{-k'}^*(t)a_{-k'}^\dagger(t)\right] \mid O(\beta), t\rangle.$$

Using the Bogoliubov transformations Eq.(9.18) for the boson fields, the correlation function becomes

$$\langle O(\beta), t \mid \phi_k(t)\phi_{k'}(t) \mid O(\beta), t\rangle =$$

$$= \langle O(\beta), t \mid \left[\varphi_k(t)\left(u(\beta,t)a_k(\beta,t) + v(\beta,t)\tilde{a}_k^\dagger(\beta,t)\right) + \varphi_{-k}^*(t)\left(u(\beta,t)a_{-k}^\dagger(\beta,t) + v(\beta,t)\tilde{a}_{-k}(\beta,t)\right)\right] \times$$

$$\times \left[\varphi_{k'}(t)\left(u(\beta,t)a_{k'}(\beta,t) + v(\beta,t)\tilde{a}_{k'}^\dagger(\beta,t)\right) + \varphi_{-k'}^*(t)\left(u(\beta,t)a_{-k'}^\dagger(\beta,t) + v(\beta,t)\tilde{a}_{-k'}(\beta,t)\right)\right] \mid O(\beta), t\rangle$$

$$= \langle O(\beta), t \mid [\varphi_k(t)\varphi_k^*(t)u^2(\beta,t) + \varphi_{-k}^*(t)\varphi_{-k}(t)v^2(\beta,t)] \mid O(\beta), t\rangle.$$

Leading to

$$\langle O(\beta), t \mid \phi_k(t)\phi_{k'}(t) \mid O(\beta), t\rangle = \varphi_k(t)\varphi_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right)\delta\left(k+k'\right).$$

Therefore

$$g_{11}(x-y) = \int [dk]\int \left[dk'\right] e^{i(kx+k'y)}\langle O(\beta), t \mid \phi_k(t)\phi_{k'}(t) \mid O(\beta), t\rangle$$

$$= \int [dk]\int \left[dk'\right] e^{i(kx+k'y)}\varphi_k(t)\varphi_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right)\delta\left(k+k'\right)$$

$$= \int [dk]\, e^{ik(x-y)}\varphi_k(t)\varphi_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right)$$

Thus, the Fourier transform of $g_{11}(x-y)$ is

$$g_{11}(k,t) = \varphi_k(t)\varphi_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right).$$

in momentum space, the Fourier transform of the equal time two point correlation function is

$$g_{ij}(x-y) = \int [dk]\, g_{ij}(k,t)\, e^{ik(x-y)}$$

where $i, j = 1, 2$.Now we compute $g_{22}(k,t), g_{12}(k,t), g_{21}(k,t)$.We have

$$g_{22}(k,t) = \langle O(\beta), t \mid \pi_k(t)\pi_{k'}(t) \mid O(\beta), t\rangle$$

$$= \langle O(\beta), t \mid \left[\dot{\varphi}_k(t)a_k(t) + \dot{\varphi}_{-k}^*(t)a_{-k}^\dagger(t)\right] \times \left[\dot{\varphi}_{k'}(t)a_{k'}(t) + \dot{\varphi}_{-k'}^*(t)a_{-k'}^\dagger(t)\right] \mid O(\beta), t\rangle$$

$$= \dot{\varphi}_k(t)\,\dot{\varphi}_k^*(t)\,u^2(\beta,t) + \dot{\varphi}_{-k}^*(t)\,\dot{\varphi}_{-k}(t)\,v^2(\beta,t) = \dot{\varphi}_k(t)\,\dot{\varphi}_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right),$$

$$g_{12}(k,t) = \langle O(\beta),t \mid \phi_k(t)\,\pi_{k'}(t) \mid O(\beta),t\rangle =$$

$$= \langle O(\beta),t \mid \left[\varphi_k(t)\,a_k(t) + \varphi_{-k}^*(t)\,a_{-k}^\dagger(t)\right]\times\left[\dot{\varphi}_{k'}(t)\,a_{k'}(t) + \dot{\varphi}_{-k'}^*(t)\,a_{-k'}^\dagger(t)\right] \mid O(\beta),t\rangle$$

$$= \varphi_k(t)\,\dot{\varphi}_k^*(t)\,u^2(\beta,t) + (\varphi_k(t)\,\dot{\varphi}_k^*(t) - i)\,v^2(\beta,t) =$$

$$= \varphi_k(t)\,\dot{\varphi}_k^*(t)\left(\frac{e^{\beta\omega_k}}{e^{\beta\omega_k}-1}\right) + (\varphi_k(t)\,\dot{\varphi}_k^*(t) - i)\left(\frac{1}{e^{\beta\omega_k}-1}\right) = \varphi_k(t)\,\dot{\varphi}_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right) - i\sinh^2(\beta\omega_k).$$

then

$$g_{12}(k,t) = \langle O(\beta),t \mid \phi_k(t)\,\pi_k(t) \mid O(\beta),t\rangle = \varphi_k(t)\,\dot{\varphi}_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right) - i\sinh^2(\beta\omega_k).$$

And finally $g_{21}(k,t)$ is given as

$$g_{21}(k,t) = \langle O(\beta),t \mid \pi_k(t)\,\phi_{k'}(t) \mid O(\beta),t\rangle = \langle O(\beta),t \mid \left[\phi_{k'}(t)\,\pi_k(t) - i\delta\left(k+k'\right)\right] \mid O(\beta),t\rangle$$

$$= \langle O(\beta),t \mid \phi_{k'}(t)\,\pi_k(t) \mid O(\beta),t\rangle - i\delta\left(k+k'\right) = \varphi_k(t)\,\dot{\varphi}_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right) - i\sinh^2(\beta\omega_k) - i$$

$$= \varphi_k(t)\,\dot{\varphi}_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right) - i\cosh^2(\beta\omega_k).$$

The Fourier transforms of the two point correlators are

$$g_{11}(k,t) = \varphi_k(t)\,\varphi_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right) \tag{9.48}$$

$$g_{22}(k,t) = \dot{\varphi}_k(t)\,\dot{\varphi}_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right) \tag{9.49}$$

$$g_{12}(k,t) = \varphi_k(t)\,\dot{\varphi}_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right) - i\sinh^2(\beta\omega_k) \tag{9.50}$$

$$g_{21}(k,t) = \varphi_k(t)\,\dot{\varphi}_k^*(t)\coth\left(\frac{\beta\omega_k}{2}\right) - i\cosh^2(\beta\omega_k). \tag{9.51}$$

9.5.2 Φ^4 field theory.

For a system out of equilibrium depending in time (open system), the mean value of an operator evolving in time is

$$\langle A(t)\rangle = \langle A(t)\rangle_{(\beta,t)} = \langle O(\beta), t \mid A(t) \mid O(\beta), t\rangle = \frac{1}{Z(\beta,t)} Tr\left(e^{-\beta H(t)} A(t)\right) =$$
$$= Z^{-1}(\beta,t) \sum_n e^{-\beta E_n} \langle n, \tilde{n}, t \mid A(t) \mid n, \tilde{n}, t\rangle \tag{9.52}$$

assuming that $H(t) \mid n, \tilde{n}, t\rangle = E_n \mid n, \tilde{n}, t\rangle$ with $\langle n \mid m\rangle = \delta_{nm}$.

We next consider a scalar Φ^4 field theory, with the mass changing in time. The system is described by the lagrangian density

$$\begin{aligned} \pounds(x) &= \left[\frac{1}{2}\partial_\mu \Phi(x)\partial^\mu \Phi(x) - \frac{1}{2}m^2(t)\Phi^2(x) - \frac{\lambda}{4!}\Phi^4(x)\right] \\ &- \left[\frac{1}{2}\partial_\mu \tilde{\Phi}(x)\partial^\mu \tilde{\Phi}(x) - \frac{1}{2}m^2(t)\tilde{\Phi}^2(x) - \frac{\lambda}{4!}\tilde{\Phi}^4(x)\right] \end{aligned}$$

The Hamiltonian is written as

$$\begin{aligned} \tilde{H}(t) = H(t) - \tilde{H}(t) &= \int d^4x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\bigtriangledown\Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4\right] \\ &- \int d^4x \left[\frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\left(\bigtriangledown\tilde{\Phi}\right)^2 + \frac{m^2}{2}\tilde{\Phi}^2 + \frac{\lambda}{4!}\tilde{\Phi}^4\right] \\ &= \hat{H}_0 + \int d^4x \frac{\lambda}{4!}\left(\Phi^4 - \tilde{\Phi}^4\right) \end{aligned} \tag{9.53}$$

where

$$\hat{H}_0(\Phi) = \int d^4x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\bigtriangledown\Phi)^2 + \frac{m^2}{2}\Phi^2\right) - \int d^4x \left(\frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\left(\bigtriangledown\tilde{\Phi}\right)^2 + \frac{m^2}{2}\tilde{\Phi}^2\right) \tag{9.54}$$

is the Hamiltonian satisfying the Klein-Gordon equation. Now, we write the hamiltonian as

$$\hat{H}(t) = \hat{H}_0(t) + \lambda\hat{W}(t) \tag{9.55}$$

where

$$\hat{W}(t) = \int d^4x \frac{1}{4!}\left(\Phi^4 - \tilde{\Phi}^4\right). \tag{9.56}$$

The *Euler-Lagrange field equations* of motion are given as

$$\partial_\mu \frac{\partial \pounds}{\partial(\partial_\mu \Phi)} - \frac{\partial \pounds}{\partial \Phi} = 0,$$

which leads to

$$\partial_\mu \partial^\mu \Phi(x) + m^2(t)\,\Phi(x) + \frac{\lambda}{6}\Phi^3(x) = 0. \tag{9.57}$$

Now, the two-point correlators for Φ^4 field theory follow from Eq.(9.48-9.51). However, the coefficients $\varphi_k(t)$ are different for the two theories. We pick out the physical term without the tilda term. We know that the **LvN** equations are

$$i\dot{a}_k^\dagger(t) + \left[a_k^\dagger(t), H(t)\right] = 0$$

$$i\dot{a}_k(t) + \left[a_k(t), H(t)\right] = 0.$$

The relations for the creation and annihilation operators are

$$a_k(t) = i\left[\varphi_k^*(t)\,\pi_k - \dot{\varphi}_k^*(t)\,\phi_k\right], \tag{9.58}$$

$$a_k^\dagger(t) = -i\left[\varphi_{-k}(t)\,\pi_{-k} - \dot{\varphi}_{-k}(t)\,\phi_{-k}\right]. \tag{9.59}$$

Using relations in Eqs. (9.41), (9.42), (9.43) the Hamiltonian is written as

$$H(t) = \int d^4x\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\bigtriangledown\Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4\right],$$

$$= \frac{1}{2}\int [dk]\left[\pi_k\pi_{-k} + \left(k_0^2 - m^2\right)\phi_k\phi_{-k} + m^2\phi_k\phi_{-k}\right] + \frac{\lambda}{4!}\int [dk]\,\phi_k^2\phi_{-k}^2$$

$$= \frac{1}{2}\int [dk]\left[\pi_k\pi_{-k} + k_0^2\right] + \frac{\lambda}{4!}\int [dk]\,\phi_k^2\phi_{-k}^2.$$

Now derivation with respect of time of equations (9.58) and (9.59) gives

$$\dot{a}_k(t) = i\left[\dot{\varphi}_k^*(t)\,\pi_k + \varphi_k^*(t)\,\dot{\pi}_k - \ddot{\varphi}_k^*(t)\,\phi_k - \dot{\varphi}_k^*(t)\,\dot{\phi}_k\right]$$

$$\dot{a}_k^\dagger(t) = (-i)\left[\dot{\varphi}_{-k}(t)\,\pi_{-k} + \varphi_{-k}(t)\,\dot{\pi}_{-k} - \ddot{\varphi}_{-k}(t)\,\phi_{-k} - \dot{\varphi}_{-k}(t)\,\dot{\phi}_{-k}\right].$$

Then, we get

$$\dot{a}_k(t) = i\left[\dot{\varphi}_k^*(t)\,\pi_k + \varphi_k^*(t)\,\dot{\pi}_k - \ddot{\varphi}_k^*(t)\,\phi_k - \dot{\varphi}_k^*(t)\,\pi_k\right],$$

$$\dot{a}_k^\dagger(t) = (-i)\left[\dot{\varphi}_{-k}(t)\,\pi_{-k} + \varphi_{-k}(t)\,\dot{\pi}_{-k} - \ddot{\varphi}_{-k}(t)\,\phi_{-k} - \dot{\varphi}_{-k}(t)\,\pi_{-k}\right].$$

Finally

$$\dot{a}_k(t) = i\left[\varphi_k^*(t)\,\dot{\pi}_k - \ddot{\varphi}_k^*(t)\,\phi_k\right]$$

$$\dot{a}_k^\dagger(t) = (-i)\left[\varphi_{-k}(t)\,\dot{\pi}_{-k} - \ddot{\varphi}_{-k}(t)\,\phi_{-k}\right].$$

The first **LvN** equation becomes

$$i\dot{a}_k(t) + [a_k(t), H(t)] = -\left[\varphi_k^*(t)\,\dot{\pi}_k - \ddot{\varphi}_k^*(t)\,\phi_k\right] +$$

$$+\left[i\left(\varphi_k^*(t)\,\pi_k - \dot{\varphi}_k^*(t)\,\phi_k\right), \frac{1}{2}\int [dk']\,\pi_{k'}\pi_{-k'} + \frac{1}{2}\int [dk']\,k_0'^2\phi_{k'}\phi_{-k'} + \frac{\lambda}{4!}\int [dk']\,\phi_{k'}^2\phi_{-k'}^2\right].$$

The latter gives

$$i\dot{a}_k(t) + [a_k(t), H(t)] = -\left[\varphi_k^*(t)\,\dot{\pi}_k - \ddot{\varphi}_k^*(t)\,\phi_k\right] +$$

$$+\left(\frac{1}{2}i\right)\int [dk'] \times \left[\left(\varphi_k^*(t)\,\pi_k - \dot{\varphi}_k^*(t)\,\phi_k\right), \pi_{k'}\pi_{-k'} + k_0'^2\phi_{k'}\phi_{-k'} + \frac{\lambda}{12}\phi_{k'}^2\phi_{-k'}^2\right]$$

$$= -\left[\varphi_k^*(t)\,\dot{\pi}_k - \ddot{\varphi}_k^*(t)\,\phi_k\right] + k_0'^2\varphi_k^*(t)\,\phi_k + \frac{\lambda}{6}\varphi_k^*(t)\,\phi_{-k}\phi_k^2 + \dot{\varphi}_k^*(t)\,\pi_k = 0.$$

Therefore, we obtain

$$\varphi_k^*(t)\,\dot{\pi}_k - \dot{\varphi}_k^*(t)\,\pi_k = \ddot{\varphi}_k^*(t)\,\phi_k + k_0^2\varphi_k^*(t)\,\phi_k + \frac{\lambda}{6}\varphi_k^*(t)\,\phi_{-k}\phi_k^2.$$

The same steps are followed for the second **LvN** equation

$$i\dot{a}_k^\dagger(t) + \left[a_k^\dagger(t), H(t)\right] = \left[\varphi_{-k}(t)\,\dot{\pi}_{-k} - \ddot{\varphi}_{-k}(t)\,\phi_{-k}\right] +$$

$$+\left[(-i)\left[\varphi_{-k}(t)\,\pi_{-k} - \dot{\varphi}_{-k}(t)\,\phi_{-k}\right], \frac{1}{2}\int [dk']\,\pi_{k'}\pi_{-k'} + \frac{1}{2}\int [dk']\,k_0'^2\phi_{k'}\phi_{-k'} + \frac{\lambda}{4!}\int [dk']\,\phi_{k'}^2\phi_{-k'}^2\right],$$

that leads to

$$i\dot{a}_k^\dagger(t) + \left[a_k^\dagger(t), H(t)\right] = \left[\varphi_{-k}(t)\,\dot{\pi}_{-k} - \ddot{\varphi}_{-k}(t)\,\phi_{-k}\right] +$$

$$+\left((-i)\,\frac{1}{2}\right)\int [dk'] \times \left[\left[\varphi_{-k}(t)\,\pi_{-k} - \dot{\varphi}_{-k}(t)\,\phi_{-k}\right], \pi_{k'}\pi_{-k'} + k_0'^2\phi_{k'}\phi_{-k'} + \frac{\lambda}{12}\phi_{k'}^2\phi_{-k'}^2\right] =$$

$$= \left[\varphi_{-k}(t)\,\dot{\pi}_{-k} - \ddot{\varphi}_{-k}(t)\,\phi_{-k}\right] - k_0^2\varphi_{-k}(t)\,\phi_{-k} - \frac{\lambda}{6}\varphi_{-k}(t)\,\phi_k\phi_{-k}^2 - \dot{\varphi}_{-k}(t)\,\pi_{-k} = 0.$$

Therefore, we have

$$-\varphi_{-k}(t)\,\dot{\pi}_{-k} + \dot{\varphi}_{-k}(t)\,\pi_{-k} = -\ddot{\varphi}_{-k}(t)\,\phi_{-k} - k_0^2\varphi_{-k}(t)\,\phi_{-k} - \frac{\lambda}{6}\varphi_{-k}(t)\,\phi_k\phi_{-k}^2.$$

With the sign change $k \longrightarrow -k$ in the latter, we get

$$\varphi_k(t)\,\dot{\pi}_k - \dot{\varphi}_k(t)\,\pi_k = \ddot{\varphi}_k(t)\,\phi_k + k_0^2\varphi_k(t)\,\phi_k + \frac{\lambda}{6}\varphi_k(t)\,\phi_{-k}\phi_k^2.$$

To summarize the results, we have

$$\varphi_k^*(t)\,\dot{\pi}_k-\dot{\varphi}_k^*(t)\,\pi_k=\ddot{\varphi}_k^*(t)\,\phi_k+\mathbf{k}_0^2\varphi_k^*(t)\,\phi_k+\frac{\lambda}{6}\varphi_k^*(t)\,\phi_{-k}\phi_k^2, \tag{9.60}$$

$$\varphi_k(t)\,\dot{\pi}_k-\dot{\varphi}_k(t)\,\pi_k=\ddot{\varphi}_k(t)\,\phi_k+\mathbf{k}_0^2\varphi_k(t)\,\phi_k+\frac{\lambda}{6}\varphi_k(t)\,\phi_{-k}\phi_k^2 \tag{9.61}$$

multiply Eq. (9.60) by $\dot{\varphi}_k(t)$and Eq. (9.61) by $\dot{\varphi}_k^*(t)$, we get

$$\dot{\varphi}_k(t)\,\varphi_k^*(t)\,\dot{\pi}_k-\dot{\varphi}_k(t)\,\dot{\varphi}_k^*(t)\,\pi_k=\dot{\varphi}_k(t)\,\ddot{\varphi}_k^*(t)\,\phi_k+k_0^2\dot{\varphi}_k(t)\,\varphi_k^*(t)\,\phi_k+\frac{\lambda}{6}\dot{\varphi}_k(t)\,\varphi_k^*(t)\,\phi_{-k}\phi_k^2, \tag{9.62}$$

$$\dot{\varphi}_k^*(t)\,\varphi_k(t)\,\dot{\pi}_k-\dot{\varphi}_k^*(t)\,\dot{\varphi}_k(t)\,\pi_k=\dot{\varphi}_k^*(t)\,\ddot{\varphi}_k(t)\,\phi_k+k_0^2\dot{\varphi}_k^*(t)\,\varphi_k(t)\,\phi_k+\frac{\lambda}{6}\dot{\varphi}_k^*(t)\,\varphi_k(t)\,\phi_{-k}\phi_k^2. \tag{9.63}$$

Now, substract Eq. (9.63) from Eq. (9.62) to get

$$\begin{aligned}&\dot{\varphi}_k^*(t)\,\varphi_k(t)\,\dot{\pi}_k-\dot{\varphi}_k(t)\,\varphi_k^*(t)\,\dot{\pi}_k=\\ &=\dot{\varphi}_k^*(t)\,\ddot{\varphi}_k(t)\,\phi_k+k_0^2\dot{\varphi}_k^*(t)\,\varphi_k(t)\,\phi_k+\frac{\lambda}{6}\dot{\varphi}_k^*(t)\,\varphi_k(t)\,\phi_{-k}\phi_k^2-\\ &-\left(\dot{\varphi}_k(t)\,\ddot{\varphi}_k^*(t)\,\phi_k+k_0^2\dot{\varphi}_k(t)\,\varphi_k^*(t)\,\phi_k+\frac{\lambda}{6}\dot{\varphi}_k(t)\,\varphi_k^*(t)\,\phi_{-k}\phi_k^2\right),\end{aligned}$$

which becomes

$$\begin{aligned}&[\dot{\varphi}_k^*(t)\,\varphi_k(t)-\dot{\varphi}_k(t)\,\varphi_k^*(t)]\,\dot{\pi}_k=\\ &=[\dot{\varphi}_k^*(t)\,\ddot{\varphi}_k(t)-\dot{\varphi}_k(t)\,\ddot{\varphi}_k^*(t)]\,\phi_k+k_0^2\,[\dot{\varphi}_k^*(t)\,\varphi_k(t)-\dot{\varphi}_k(t)\,\varphi_k^*(t)]\,\phi_k+\\ &+\frac{\lambda}{6}\,[\dot{\varphi}_k^*(t)\,\varphi_k(t)-\dot{\varphi}_k(t)\,\varphi_k^*(t)]\,\phi_{-k}\phi_k^2.\end{aligned}$$

Now, using the Wronskian condition, the last relation reads

$$i\dot{\pi}_k=[\dot{\varphi}_k^*(t)\,\ddot{\varphi}_k(t)-\dot{\varphi}_k(t)\,\ddot{\varphi}_k^*(t)]\,\phi_k+ik_0^2\phi_k+i\frac{\lambda}{6}\phi_{-k}\phi_k^2.$$

Thus, we obtain a useful relation

$$\dot{\pi}_k=(-i)\,[\dot{\varphi}_k^*(t)\,\ddot{\varphi}_k(t)-\dot{\varphi}_k(t)\,\ddot{\varphi}_k^*(t)]\,\phi_k+k_0^2\phi_k+\frac{\lambda}{6}\phi_{-k}\phi_k^2. \tag{9.64}$$

Let's go back to Eq. (9.63).π_k and ϕ_k being independent variables, Eq. (9.63) reads

$$\ddot{\varphi}_k(t)\,\phi_k+k_0^2\varphi_k(t)\,\phi_k+\frac{\lambda}{6}\varphi_k(t)\,\phi_{-k}\phi_k^2=0\Longrightarrow\ddot{\varphi}_k(t)+k_0^2\varphi_k(t)+\frac{\lambda}{6}\varphi_k(t)\,\phi_{-k}\phi_k=0$$

which becomes a second order differential equation in $\varphi_k(t)$

$$\ddot{\varphi}_k(t)+\left(\mathbf{k}^2+\mathbf{m}^2(t)+\frac{\lambda}{6}\phi_{-k}\phi_k\right)\varphi_k(t)=\mathbf{0}.$$

9.5.3 Finite smooth quench.

We implement the finite smooth quench to phase transitions. Systems do become out of equilibrium in general during phase transitions because their coupling parameters depend explicitly on time through interaction with an environment (heat bath). Hence the nonequilibrium dynamics of phase transitions should differ significantly from the equilibrium dynamics. As the phase transition proceeds, the fluctuations grow and stability is lost as we will see in the example. In the Hartree approximation, the last equation reads

$$\ddot{\varphi}_k(t) + \left(\mathbf{k}^2 + \mathbf{m}^2(t) + \frac{\lambda}{6}\langle\phi_{-k}\phi_k\rangle\right)\varphi_k(t) = \mathbf{0}$$

We suppose that the duration of the transition quench is t_{sp} defined from $\langle\phi^2\rangle_{t=t_{sp}} \sim \eta^2$. We suppose that the time-interval quench is $[0, 2\tau_q]$, with $\tau_q \gg t_r \sim \mu^{-1}$. the differential equation related to the following varying mass

$$m^2(t) = \begin{cases} \mu^2 & \text{for} \quad t \leq 0 \\ \mu^2 - \frac{t\mu^2}{\tau_q} & \text{for} \quad 0 \prec t \leq 2\tau_q \\ -\mu^2 & \text{for} \quad t \geq 2\tau_q \end{cases}$$

the field behaves as a free field in an interval of t_{sp},where

$$\langle\phi^2\rangle \sim \eta^2$$

the equation of motion for the mode $\varphi_k(t)$ with wavenumber k is, in a quench period

$$\left[\frac{d^2}{dt^2} + k^2 + \mu^2 + \frac{\lambda}{6}\langle\phi_{-k}\phi_k\rangle - \frac{\mu^2 t}{\tau_q}\right]\varphi_k(t) = \mathbf{0}.$$

put

$$\omega_k^2 = k^2 + \mu^2 + \frac{\lambda}{6}\langle\phi_{-k}\phi_k\rangle$$

then, the last differential equation is given as

$$\ddot{\varphi}_k(t) + \left(\omega_k^2 - \left(\frac{\mu^2}{\tau_q}\right)t\right)\varphi_k(t) = 0.$$

Then, we put the following change of variables

$$-t' = \omega_k^2 - \left(\frac{\mu^2}{\tau_q}\right) tanddt = \left(\frac{\tau_q}{\mu^2}\right) dt'$$

Then

$$\frac{d^2\varphi_k}{dt^2} = \frac{d}{dt}\left(\frac{d\varphi_k}{dt}\right) = \left(\frac{\mu^2}{\tau_q}\right)^2 \frac{d^2\varphi_k}{dt'^2}$$

we obtain

$$\left(\frac{\mu^2}{\tau_q}\right)^2 \ddot{\varphi}_k\left(t'\right) - t'\varphi_k\left(t'\right) = 0.$$

Now, we put the following change of variable

$$t" = t'\left(\frac{\mu^2}{\tau_q}\right)^2,$$

then

$$dt" = dt'\left(\frac{\mu^2}{\tau_q}\right)^2 and \frac{d}{dt"}\left(\frac{d}{dt"}\right) = \left(\frac{\mu^2}{\tau_q}\right)^{-4}\frac{d}{dt'}\left(\frac{d}{dt'}\right)$$

therefore, the differential equations reads

$$\ddot{\varphi}_k\left(t"\right) - t"\varphi_k\left(t"\right) = 0.$$

the total change of variable is

$$t" = \left(\frac{\mu^2}{\tau_q}\right)^2\left(\left(\frac{\mu^2}{\tau_q}\right)t - \omega_k^2\right).$$

we distinguish two cases

a) before and after the quench, the solution of the differential equation is

$$\varphi_k\left(t\right) = C_1 e^{-i\omega_\pm(k)t} + C_2 e^{i\omega_\pm(k)t}$$

where

$$\omega_\pm^2\left(k\right) = \mathbf{k}^2 \pm \mu^2 + \frac{\lambda}{6}\langle\phi_{-k}\phi_k\rangle and C_1 and C_2 are complex constants.$$

If the frequency is negative, the solution will suffer from spinodal instability. The solutions to the differential equation are found separately for the stable modes and unstable modes. The regime of the stable modes are oscillatory.

b) the solution for the differential equation during the quench is

$$\varphi_k\left(t\right) = Ai\left[\left(\frac{\mu^2}{\tau_q}\right)^2\left(\left(\frac{\mu^2}{\tau_q}\right)t - \omega_k^2\right)\right] + Bi\left[\left(\frac{\mu^2}{\tau_q}\right)^2\left(\left(\frac{\mu^2}{\tau_q}\right)t - \omega_k^2\right)\right]$$

where $Ai\left[z\right]$ and $Bi\left[z\right]$ are the Airy functions. Due to the nature of the Airy functions, the unstable modes dominate the correlation function over the stable modes. The second term grows exponentially due to the AiryBi(z) function.

9.6 Time-dependent Green functions for Klein-Gordon field theory.

The thermal average of an observable, A , has been defined as :$\langle A\rangle_{(\beta,t)} = \langle O(\beta) \mid A \mid O(\beta)\rangle$. Let's compute the time-dependent thermal propagator, using the thermal vaccum. The thermal Feynmann propagator for the real scalar field is defined by :

$$G_{K.G}(x-y,\beta,t) = \langle O(\beta),t \mid T[\Phi(x)\Phi(y)] \mid O(\beta),t\rangle \tag{9.65}$$

or

$$iG_{K.G}(x-y,\beta,t) = \theta\left(x^0-y^0\right) g(x-y,\beta,t) + \theta\left(y^0-x^0\right) g(y-x,\beta,t) \tag{9.66}$$

where $\theta(x)$ is the step function, such that $\theta(x) = 1$,for $x \succ 1, \theta(x) = 0$,for $x \prec 1$ and $g(x-y,\beta,t) = \langle O(\beta),t \mid \Phi(x)\Phi(y) \mid O(\beta),t\rangle$.On using the explicit expressions for Klein-Gordon fields, we can write :

$$g(x-y,\beta,t) = \langle O(\beta),t \mid \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left[a(k,t)e^{-ikx} + a^\dagger(k,t)e^{ikx}\right]$$

$$\times \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[a(p,t)e^{-ipy} + a^\dagger(p,t)e^{ipy}\right] \mid O(\beta),t\rangle.$$

which reads :

$$g(x-y,\beta,t) = \langle O(\beta),t \mid \int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_k}\frac{1}{2\omega_p}\left[a(k,t)e^{-ikx} + a^\dagger(k,t)e^{ikx}\right]$$

$$\times\left[a(p,t)e^{-ipy} + a^\dagger(p,t)e^{ipy}\right] \mid O(\beta),t\rangle$$

$$= \langle O(\beta),t \mid \int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_k}\frac{1}{2\omega_p}\times$$

$$[a(k,t)a(p,t)e^{-(ikx+py)} + a(k,t)a^\dagger(p,t)e^{-(ikx-py)}$$

$$a^\dagger(k,t)a(p,t)e^{(ikx-py)} + a^\dagger(k,t)a^\dagger(p,t)e^{(ikx+py)}] \mid O(\beta),t\rangle. \tag{9.67}$$

Using the BOGOLIUBOV transformation, we get for the operators in thermal states as

$$\langle O(\beta),t \mid a(k,t)a(p,t) \mid O(\beta),t\rangle = \langle O(\beta),t \mid \left[u(k,\beta,t)a(k,\beta,t) + v(k,\beta,t)\tilde{a}^\dagger(k,\beta,t)\right]$$

$$\times\left[u(p,\beta,t)a(p,\beta,t) + v(p,\beta,t)\tilde{a}^\dagger(p,\beta,t)\right] \mid O(\beta),t\rangle =$$

$$= \langle O(\beta), t \mid u(k,\beta,t)\, v(p,\beta,t)\, a(k,\beta,t)\, \tilde{a}^{\dagger}(p,\beta,t) \mid O(\beta), t\rangle = 0. \tag{9.68}$$

the second term

$$\langle O(\beta), t \mid a(k,t)\, a^{\dagger}(p,t) \mid O(\beta), t\rangle = \langle O(\beta), t \mid \Big[u(k,\beta,t)\, a(k,\beta,t) + v(k,\beta,t)\, \tilde{a}^{\dagger}(k,\beta,t)\Big]$$

$$\times \Big[u(p,\beta,t)\, a^{\dagger}(p,\beta,t) + v(p,\beta,t)\, \tilde{a}(p,\beta,t)\Big] \mid O(\beta), t\rangle =$$

$$= \langle O(\beta), t \mid u(k,\beta,t)\, u(p,\beta,t)\, a(k,\beta,t)\, a^{\dagger}(p,\beta,t) \mid O(\beta), t\rangle$$

$$= u(k,\beta,t)\, u(p,\beta,t)\, (2\pi)^3\, 2k_0 \delta(\mathbf{k}-\mathbf{p}) = (2\pi)^3\, 2k_0 u^2(k,\beta,t). \tag{9.69}$$

the same procedure is used for the third term

$$\langle O(\beta), t \mid a^{\dagger}(k,t)\, a(p,t) \mid O(\beta), t\rangle = \langle O(\beta), t \mid \Big[u(k,\beta,t)\, a^{\dagger}(k,\beta,t) + v(k,\beta,t)\, \tilde{a}(k,\beta,t)\Big]$$

$$\times \Big[u(p,\beta,t)\, a(p,\beta,t) + v(p,\beta,t)\, \tilde{a}^{\dagger}(p,\beta,t)\Big] \mid O(\beta), t\rangle$$

$$= \langle O(\beta), t \mid v(k,\beta,t)\, v(p,\beta,t)\, \tilde{a}(k,\beta,t)\, \tilde{a}^{\dagger}(p,\beta,t) \mid O(\beta), t\rangle =$$

$$= v(k,\beta,t)\, v(p,\beta,t)\, (2\pi)^3\, 2k_0 \delta(\mathbf{k}-\mathbf{p}) = (2\pi)^3\, 2k_0 v^2(k,\beta,t). \tag{9.70}$$

the last term is zero i.e.

$$\langle O(\beta), t \mid a^{\dagger}(k,t)\, a^{\dagger}(p,t) \mid O(\beta), t\rangle = 0. \tag{9.71}$$

Substituting these equations into Eq.(9.67), we get

$$g(x-y,\beta,t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big[u^2(k,\beta,t)\, e^{-ik(x-y)} + v^2(k,\beta,t)\, e^{ik(x-y)}\Big]$$

Hence, Eq.(9.67) reads

$$iG_{K.G}(x-y,\beta,t) = \theta\left(x^0-y^0\right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big[u^2(k,\beta,t)\, e^{-ik(x-y)} + v^2(k,\beta,t)\, e^{ik(x-y)}\Big]$$

$$+\theta\left(y^0-x^0\right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big[u^2(k,\beta,t)\, e^{-ik(y-x)} + v^2(k,\beta,t)\, e^{ik(y-x)}\Big]$$

using Eq.(9.4) and (9.9), then

$$iG_{K.G}(x-y,\beta,t) = \theta\left(x^0-y^0\right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k(t)} \Big[\left(v^2(k,\beta,t)+1\right) e^{-ik(x-y)} + v^2(k,\beta,t)\, e^{ik(x-y)}\Big]$$

$$+\theta\left(y^0-x^0\right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k(t)} \Big[\left(v^2(k,\beta,t)+1\right) e^{-ik(y-x)} + v^2(k,\beta,t)\, e^{ik(y-x)}\Big]$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k(t)} \left[\theta\left(x^0 - y^0\right) e^{-ik(x-y)} + \theta\left(y^0 - x^0\right) e^{-ik(y-x)}\right]$$

$$+ \int \frac{d^3k}{(2\pi)^3} v^2(k,\beta,t) \frac{1}{2\omega_k(t)} \left[\theta\left(x^0 - y^0\right) e^{-ik(x-y)} + \theta\left(y^0 - x^0\right) e^{-ik(y-x)}\right]$$

$$+ \int \frac{d^3k}{(2\pi)^3} v^2(k,\beta,t) \frac{1}{2\omega_k(t)} \left[\theta\left(x^0 - y^0\right) e^{ik(x-y)} + \theta\left(y^0 - x^0\right) e^{ik(y-x)}\right]. \quad (9.72)$$

Using the Fourier representation of $\theta(x)$, i.e.

$$I = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\left(x^0-y^0\right)}}{k^0 - \omega_k + i\eta} = -i\theta\left(x^0 - y^0\right) e^{-i\omega_k\left(x^0-y^0\right)}.$$

we get for the first term of the last expression when changing $\mathbf{k}$ to $-\mathbf{k}$

$$G_{K.G}(x-y,t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k(t)} \left[\theta\left(x^0 - y^0\right) e^{-ik(x-y)} + \theta\left(y^0 - x^0\right) e^{-ik(y-x)}\right]$$

$$= \int \frac{d\mathbf{k}}{(2\pi^3)\, 2\omega_k} e^{i\mathbf{k}.(x-y)} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \left(\frac{e^{-ik^0\left(x^0-y^0\right)}}{k^0 - \omega_k + i\eta} + \frac{e^{ik^0\left(x^0-y^0\right)}}{k^0 - \omega_k + i\eta} \right)$$

$$= \int \frac{d\mathbf{k}}{(2\pi^3)\, 2\omega_k} e^{i\mathbf{k}.(x-y)} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \left(\frac{e^{-ik^0\left(x^0-y^0\right)}}{k^0 - \omega_k + i\eta} - \frac{e^{-ik^0\left(x^0-y^0\right)}}{k^0 + \omega_k - i\eta} \right)$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{!}{k^2 - m^2 + i\eta}.$$

as for the second term of Eq.(9.72), we get

$$G_{K.G}(x-y,\beta,t) = \int \frac{d^3k}{(2\pi)^3} v^2(k,\beta,t) \frac{1}{2\omega_k(t)} \left[\theta\left(x^0 - y^0\right) e^{-ik(x-y)} + \theta\left(y^0 - x^0\right) e^{-ik(y-x)}\right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{n_{\mathbf{B}}(\omega_k(t))}{2\omega_k(t)} \left[e^{-i\omega_k\left(x^0-y^0\right)} + e^{i\omega_k\left(x^0-y^0\right)}\right] e^{i\mathbf{k}(x-y)}$$

$$= \int \frac{d^3k}{(2\pi)^3} dk_0 n_{\mathbf{B}}(|k_0(t)|) \frac{1}{2\omega_k(t)} \left[\delta\left(k_0(t) - \omega_k(t)\right) + \delta\left(k_0((t)) + \omega_k(t)\right)\right] e^{-ik^0\left(x^0-y^0\right)} e^{i\mathbf{k}(x-y)}$$

$$= \int \frac{d^4k}{(2\pi)^4} 2\pi n_{\mathbf{B}}(|k_0(t)|)\, \delta\left(k^2 - m^2(t)\right) e^{-ik(x-y)}.$$

Therefore, taking the Fouries representation of the Green function, we obtain :

$$G_{K.G}(k,\beta,t) = G_{K.G}(k,t) + 2\pi i n_{\mathbf{B}}(k,\beta,t)\, \delta\left(k^2 - m^2(t)\right).$$

9.7 Thermal Green Functions for $\lambda\Phi^4(x)$ theory.

For a system out of equilibrium depending in time (open system), the mean value of an operator evolving in time is

$$\langle A(t)\rangle = \langle A(t)\rangle_{(\beta,t)} = \langle O(\beta), t \mid A(t) \mid O(\beta), t\rangle = \frac{1}{Z(\beta,t)} Tr\left(e^{-\beta H(t)} A(t)\right) =$$

$$= Z^{-1}(\beta,t) \sum_n e^{-\beta E_n} \langle n, \tilde{n}, t \mid A(t) \mid n, \tilde{n}, t\rangle \tag{9.73}$$

assuming that $H(t) \mid n, \tilde{n}, t\rangle = E_n \mid n, \tilde{n}, t\rangle$ with $\langle n \mid m\rangle = \delta_{nm}$. For $\lambda\Phi^4(x)$ theory, the Hamiltonian is

$$\hat{H}(t) = H(t) - \tilde{H}(t) = \int d^4x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4\right]$$

$$-\int d^4x \left[\frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\left(\nabla\tilde{\Phi}\right)^2 + \frac{m^2}{2}\tilde{\Phi}^2 + \frac{\lambda}{4!}\tilde{\Phi}^4\right]$$

Then, we get

$$\hat{H}(t) = \hat{H}_0 + \int d^4x \frac{\lambda}{4!}\left(\Phi^4 - \tilde{\Phi}^4\right) \tag{9.74}$$

where

$$\hat{H}_0(\Phi) = \int d^4x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2\right) - \int d^4x \left(\frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\left(\nabla\tilde{\Phi}\right)^2 + \frac{m^2}{2}\tilde{\Phi}^2\right) \tag{9.75}$$

is the Hamiltonian satisfying the Klein-Gordon equation. The momentum modes are given by the Fourier transform of a field operator and its inverse transform,

$$F(x) = \int d^4k F(k) e^{ikx},$$

$$F(k) = \int d^4x F(x) e^{-ikx},$$

Then, we have the relation

$$\int d^4x F^2(x) = \int [dk] F(k) F(-k). \tag{9.76}$$

where $[dk] = \frac{d^4k}{(2\pi)^4}$. The Fouries modes of $\Phi(x)$ and $\Pi(x)$ are

$$\Phi_k = \int d^4x \Phi(x) e^{-ikx},$$

$$\Pi_k = \int d^4x \Pi(x) e^{-ikx},$$

The operators $\Phi(x)$ and $\Pi(x)$ are Hermitian, therefore

$$\Phi_k^\dagger = \Phi_{-k} and \Pi_k^\dagger = \Pi_{-k}.$$

the commutation relations of the fields

$$[\Phi(x), \Pi(y)] = i\delta(\mathbf{x} - \mathbf{y}) \tag{9.77}$$

leads to commutation relations of modes in momentum space

$$[\Phi_k, \Pi_{k'}] = i(2\pi)^4 \delta(k + k'). \tag{9.78}$$

in terms of the annihilation and creation operators, then commutation relations for creation and annihilation operators are

$$\left[a_k(t), a_{k'}^\dagger(t)\right] = (2\pi)^4 \delta(k - k'). \tag{9.79}$$

The momentum space operators of the fields Φ_k and $\Pi_{k'}$may be expressed as

$$\Phi_k = \varphi_k(t) a_k(t) + \varphi_{-k}^*(t) a_{-k}^\dagger(t),$$

$$\Pi_k = \dot{\varphi}_k(t) a_k(t) + \dot{\varphi}_{-k}^*(t) a_{-k}^\dagger(t). \tag{9.80}$$

It is assumed that

$$\varphi_{-k}(t) = \varphi_k(t) and \dot{\varphi}_k^*(t) \varphi_k(t) - \varphi_k^*(t) \dot{\varphi}_k(t) = i. \tag{9.81}$$

the inverse relations are

$$a_k(t) = i\left[\varphi_k^*(t) \Pi_k - \dot{\varphi}_k^* \Phi_k\right],$$

$$a_k^\dagger(t) = (-i)\left[\varphi_{-k}(t) \Pi_{-k} - \dot{\varphi}_{-k} \Phi_{-k}\right].$$

Here $a_k(t)$ and $a_k^\dagger(t)$ are time dependent operators satisfying the LvN equation. Now, we write the hamiltonian as

$$\hat{H}(t) = \hat{H}_0(t) + \lambda \hat{W}(t) \tag{9.82}$$

where

$$\hat{W}(t) = \int d^4x \frac{1}{4!}\left(\Phi^4 - \tilde{\Phi}^4\right). \tag{9.83}$$

Now, we compute the green function for the theory

$$G(x-y,\beta,t) = \langle O(\beta), t \mid T[\Phi(x)\Phi(y)] \mid O(\beta), t\rangle, \tag{9.84}$$

then,

$$iG(x-y,\beta,t) = \theta\left(x^0-y^0\right) g(x-y,\beta,t) + \theta\left(y^0-x^0\right) g(y-x,\beta,t) \tag{9.85}$$

where,

$$\begin{aligned}
g(x-y,\beta,t) = \langle O(\beta), t \mid \Phi(x)\Phi(y) \mid O(\beta), t\rangle &= \frac{1}{Z(\beta,t)} Tr\left(e^{-\beta H(t)}\Phi(x)\Phi(y)\right) \\
&= \frac{1}{Z(\beta,t)} Tr\left(e^{-\beta\left(\hat{H}_0(t)+\lambda\hat{W}(t)\right)}\Phi(x)\Phi(y)\right) \\
&= \frac{1}{Z(\beta,t)} Tr\left(e^{-\beta\hat{H}_0(t)} e^{-\beta\lambda\hat{W}(t)} e^{-\beta\lambda\left[\hat{H}_0(t),\hat{W}(t)\right]}\Phi(x)\Phi(y)\right) \\
&= \frac{1}{Z(\beta,t)} \sum_n \langle n,\tilde{n},t \mid e^{-\beta\hat{H}_0(t)} e^{-\beta\lambda\hat{W}(t)} e^{-\beta\lambda\left[\hat{H}_0(t),\hat{W}(t)\right]}\Phi(x)\Phi(y) \mid n,\tilde{n},t\rangle.
\end{aligned} \tag{9.86}$$

Upon using the completeness relation, $\sum_n \mid n,\tilde{n},t\rangle\langle n,\tilde{n},t \mid = 1$, this gives :

$$g(x-y,\beta,t) = \frac{1}{Z(\beta,t)} \times$$

$$\begin{aligned}
&\times \sum_n \langle n,\tilde{n},t \mid e^{-\beta\hat{H}_0(t)} \sum_n \mid n,\tilde{n},t\rangle\langle n,\tilde{n},t \mid e^{-\beta\lambda\hat{W}(t)} e^{-\beta\lambda\left[\hat{H}_0(t),\hat{W}(t)\right]}\Phi(x)\Phi(y) \mid n,\tilde{n},t\rangle \\
&= \frac{1}{Z(\beta,t)} \sum_n \langle n,\tilde{n},t \mid e^{-\beta\hat{H}_0(t)}\Phi(x)\Phi(y) \mid n,\tilde{n},t\rangle \times \left[\sum_n \langle n,\tilde{n},t \mid e^{-\beta\lambda\hat{W}(t)} e^{-\beta\lambda\left[\hat{H}_0(t),\hat{W}(t)\right]} \mid n,\tilde{n},t\rangle\right] \\
&= g_{K.G}(x-y) \times \left[\sum_n \langle n,\tilde{n},t \mid e^{-\beta\lambda\hat{W}(t)} e^{-\beta\lambda\left[\hat{H}_0(t),\hat{W}(t)\right]} \mid n,\tilde{n},t\rangle\right].
\end{aligned} \tag{9.87}$$

The first order expansion of Eq.(9.87) gives

$$g(x-y,\beta,t) \simeq g_{K.G}(x-y) \times \left[\sum_n \langle n,\tilde{n},t \mid \left(1-\beta\lambda\hat{W}(t)\right)\left(1-\beta\lambda\left[\hat{H}_0(t),\hat{W}(t)\right]\right) \mid n,\tilde{n},t\rangle\right]$$

$$\begin{aligned}
&= g_{K.G}(x-y)\Bigg(1-(\beta\lambda)\sum_n \langle n,\tilde{n},t \mid \hat{W}(t) \mid n,\tilde{n},t\rangle - (\beta\lambda)\sum_n \langle n,\tilde{n},t \mid \left[\hat{H}_0(t),\hat{W}(t)\right] \mid n,\tilde{n},t\rangle \\
&\quad + (\lambda\beta)^2 \times \left[\sum_n \langle n,\tilde{n},t \mid \hat{W}(t)\left[\hat{H}_0(t),\hat{W}(t)\right] \mid n,\tilde{n},t\rangle\right]\Bigg)
\end{aligned}$$

to first order

$$g(x-y,\beta,t) \simeq g_{K.G}(x-y)\left(1-(\beta\lambda)\sum_n \langle n,\tilde{n},t \mid \hat{W}(t) \mid n,\tilde{n},t\rangle - (\beta\lambda)\sum_n \langle n,\tilde{n},t \mid \left[\hat{H}_0(t),\hat{W}(t)\right] \mid n,\tilde{n},t\rangle\right)$$

The propagator for $\lambda\Phi^4$ theory to first order is

$$iG(x-y,\beta,t) \simeq iG_{K.G}(x-y,\beta,t)\times$$
$$\times\left(1-(\beta\lambda)\sum_n \langle n,\tilde{n},t \mid \hat{W}(t) \mid n,\tilde{n},t\rangle - (\beta\lambda)\sum_n \langle n,\tilde{n},t \mid \left[\hat{H}_0(t),\hat{W}(t)\right] \mid n,\tilde{n},t\rangle\right). \tag{9.88}$$

If λ goes to zero, we simply find the free Klein-Gordon propagator. Now, we write the states $\mid n,\tilde{n},t\rangle = \frac{1}{n!}\left(a_k^\dagger(t)\right)^n\left(\tilde{a}_k^\dagger(t)\right)^{\tilde{n}} \mid O,t\rangle$; Note that $\mid O,t\rangle = \mid O,t\rangle_G$ is the Gaussian vacuum annihilated by all $a_k(t)$ and $\tilde{a}_k(t)$:

$$a_k(t) \mid O,t\rangle = \tilde{a}_k(t) \mid O,t\rangle = 0 \tag{9.89}$$

or the product of Gaussian vacuum state for each $a_k(t)$:

$$\mid O,t\rangle = \mid O,t\rangle_G =_k \mid O_k,t\rangle_G. \tag{9.90}$$

We drop the subscript G for the rest of the calculation. Let's calculate the second term explicitly. We substitute LvN equations (9.80) for Fouriers modes. But first, we have :

$$\hat{W}(t) = W(t) + \tilde{W}(t) = \int d^4x \frac{1}{4!}\left(\Phi^4 - \tilde{\Phi}^4\right). \tag{9.91}$$

We are interested in the term

$$W(t) = \int d^4x \frac{1}{4!}\Phi^4(x) = \int d^4x \frac{1}{4!}\left(\Phi^2(x)\right)^2, \tag{9.92}$$

which is written as

$$W(t) = \frac{1}{4!}\int d^4x \left(\Phi^2(x)\right)^2 = \int [dk]\, \Phi^2(k)\, \Phi^2(-k). \tag{9.93}$$

Also we have the following equations :

$$\hat{H}_0(t) = \int d^4x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2\right) - \int d^4x \left(\frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\left(\nabla\tilde{\Phi}\right)^2 + \frac{m^2}{2}\tilde{\Phi}^2\right),$$
$$\frac{1}{2}\int d^4x \pi^2(x) = \int [dk]\pi(k)\,\pi(-k)\,, \frac{1}{2}\int d^4x \tilde{\pi}^2(x) = \int [dk']\tilde{\pi}(k')\,\tilde{\pi}(-k')\,,$$

$$\frac{1}{4!}\int d^4x\Phi^4(x)=\int[dk'']\Phi^2(k'')\Phi^2(-k''),\frac{1}{4!}\int d^4x\tilde{\Phi}^4(x)=\int[dk''']\tilde{\Phi}^2(k''')\tilde{\Phi}^2(-k''').$$

For the following commutator

$$\left[\hat{H}_0(t),\hat{W}(t)\right]=\int[dk'']\int[dk]\left[\pi(k)\pi(-k),\Phi^2(k'')\Phi^2(-k'')\right]$$

$$+\int[dk']\int[dk''']\left[\tilde{\pi}(k')\tilde{\pi}(-k'),\tilde{\Phi}^2(k''')\tilde{\Phi}^2(-k''')\right],$$

we use the identity

$$[AB,CD]=AC[B,D]+A[B,C]D+[A,C]DB+C[A,D]B$$

where

$$A=\pi(k),\tilde{\pi}(k'),B=\pi(-k),\tilde{\pi}(-k'),C=\Phi^2(k''),\tilde{\Phi}^2(k'''),D=\Phi^2(-k''),\tilde{\Phi}^2(-k''').$$

Therefore the commutator for the non-tilde terms in Eq.(9.88) simplifies to

$$[H_0(t),W(t)]=(-i)\int d^4k\pi(k)\left\{\Phi^2(-k)\Phi(k)+\Phi^2(-k)\Phi(k)+\Phi(k)\Phi^2(-k)+\Phi(k)\Phi^2(-k)\right\}$$

$$+\{\Phi(-k)\Phi^2(k)+\Phi(-k)\Phi^2(k)+\Phi^2(k)\Phi(-k)+\Phi^2(k)\Phi(-k)\}\pi(-k).$$

The full commutator including the tilde term is

$$\left[\hat{H}_0(t),\hat{W}(t)\right]=$$

$$=(-i)\int d^4k\times 2\pi(k)\{\Phi^2(-k)\Phi(k)+\Phi(k)\Phi^2(-k)\}+2\{\Phi(-k)\Phi^2(k)+\Phi^2(k)\Phi(-k)\}\pi(-k)$$

$$+2\tilde{\pi}(k)\{\tilde{\Phi}^2(-k)\tilde{\Phi}(k)+\tilde{\Phi}(k)\tilde{\Phi}^2(-k)\}+2\{\tilde{\Phi}(-k)\tilde{\Phi}^2(k)+\tilde{\Phi}^2(k)\tilde{\Phi}(-k)\}\tilde{\pi}(-k).$$

Therefore, the Green function is

$$iG(x-y,\beta,t)\simeq iG_{K.G}(x-y,\beta,t)\times$$

$$\times\left(1-(\beta\lambda)\sum_n\langle n,\tilde{n},t\mid\int[dk]\left[\Phi^2(k)\Phi^2(-k)-\tilde{\Phi}^2(k)\tilde{\Phi}^2(-k)\right]\mid n,\tilde{n},t\rangle\right.$$

$$-(\beta\lambda)\sum_n\langle n,\tilde{n},t\mid(-i)\int d^4k\times 2\pi(k)\{\Phi^2(-k)\Phi(k)+\Phi(k)\Phi^2(-k)\}+$$

$$+2\{\Phi(-k)\Phi^2(k)+\Phi^2(k)\Phi(-k)\}\pi(-k)\mid n,\tilde{n},t\rangle$$

$$+\sum_n\langle n,\tilde{n},t\mid(-i)\int d^4k\times 2\tilde{\pi}(k)\{\tilde{\Phi}^2(-k)\tilde{\Phi}(k)+\tilde{\Phi}(k)\tilde{\Phi}^2(-k)\}+$$

$$+2\{\tilde{\Phi}(-k)\,\tilde{\Phi}^2(k)+\tilde{\Phi}^2(k)\,\tilde{\Phi}(-k)\}\tilde{\pi}(-k)\mid n,\tilde{n},t\rangle\Big)$$

in the Hartree approximation [8], we get

$$iG(x-y,\beta,t)\simeq iG_{K.G}(x-y,\beta,t)\times$$

$$\times\Bigg(1-(\beta\lambda)\sum_n\langle n,\tilde{n},t\mid\int[dk]\,\langle\Phi^2(k)\rangle[\Phi^2(-k)]-\langle\tilde{\Phi}^2(k)\rangle[\tilde{\Phi}^2(-k)]\mid n,\tilde{n},t\rangle$$

$$-(\beta\lambda)\sum_n\langle n,\tilde{n},t\mid(-i)\int d^4k\times\langle\Phi^2(-k)\rangle\times 4\pi(k)\,\Phi(k)+$$

$$+4\langle\Phi^2(k)\rangle\times\Phi(-k)\,\pi(-k)\mid n,\tilde{n},t\rangle$$

$$+\sum_n\langle n,\tilde{n},t\mid(-i)\int d^4k\times 4\langle\tilde{\Phi}^2(-k)\rangle\times\tilde{\pi}(k)\,\tilde{\Phi}(k)+$$

$$+4\langle\tilde{\Phi}^2(k)\rangle\times\tilde{\Phi}(-k)\,\tilde{\pi}(-k)\mid n,\tilde{n},t\rangle\Bigg)$$

Let's try to work out

$$\langle n,\tilde{n},t\mid\pi(k)\,\Phi(k)\mid n,\tilde{n},t\rangle=$$

$$=\langle n,\tilde{n},t\mid\left[\dot{\varphi}_k(t)\,a_k(t)+\dot{\varphi}^*_{-k}(t)\,a^\dagger_{-k}(t)\right]\times\left[\varphi_k(t)\,a_k(t)+\varphi^*_{-k}(t)\,a^\dagger_{-k}(t)\right]\mid n,\tilde{n},t\rangle$$

$$=\langle n,\tilde{n},t\mid\dot{\varphi}_k(t)\,\varphi_k(t)\,a_k(t)\,a_k(t)+\dot{\varphi}_k(t)\,\varphi^*_{-k}(t)\,a_k(t)\,a^\dagger_{-k}(t)+$$

$$+\dot{\varphi}^*_{-k}(t)\,\varphi_k(t)\,a^\dagger_{-k}(t)\,a_k(t)+\dot{\varphi}^*_{-k}(t)\,\varphi^*_{-k}(t)\,a^\dagger_{-k}(t)\,a^\dagger_{-k}(t)\mid n,\tilde{n},t\rangle=$$

$$\langle n,\tilde{n},t\mid\left[\dot{\varphi}^*_{-k}(t)\,\varphi_k(t)-\dot{\varphi}_k(t)\,\varphi^*_{-k}(t)\right]a^\dagger_{-k}(t)\,a_k(t)\mid n,\tilde{n},t\rangle=\langle N\rangle\times i$$

where N is the number operator and where we have used the Wronskian condition and the fact that

$$\langle n,\tilde{n},t\mid a_k(t)\,a_k(t)\mid n,\tilde{n},t\rangle=\langle n,\tilde{n},t\mid a^\dagger_{-k}(t)\,a^\dagger_{-k}(t)\mid n,\tilde{n},t\rangle=0.$$

Also, we have

$$\langle n,\tilde{n},t\mid\Phi^2(-k)\mid n,\tilde{n},t\rangle=\langle n,\tilde{n},t\mid\Phi^2(k)\mid n,\tilde{n},t\rangle=$$

$$=\langle n,\tilde{n},t\mid\left[\varphi_k(t)\,a_k(t)+\varphi^*_{-k}(t)\,a^\dagger_{-k}(t)\right]\times\left[\varphi_k(t)\,a_k(t)+\varphi^*_{-k}(t)\,a^\dagger_{-k}(t)\right]\mid n,\tilde{n},t\rangle=$$

$$=\langle n,\tilde{n},t\mid\left[\varphi_k(t)\,\varphi^*_{-k}(t)-\varphi^*_{-k}(t)\,\varphi_k(t)\right]a^\dagger_{-k}(t)\,a_k(t)\mid n,\tilde{n},t\rangle=0.$$

Therefore the time-dependent green function is

$$iG\left(x-y,\beta,t\right)\simeq iG_{K.G}\left(x-y,\beta,t\right)\times$$

$$\left(1-4\left(\beta\lambda\right)\int d^{4}k\times\left(\langle\Phi^{2}\left(-k\right)\rangle-\langle\Phi^{2}\left(k\right)\rangle\right)\times\langle N\rangle+\left(\langle\tilde{\Phi}^{2}\left(-k\right)\rangle-\langle\tilde{\Phi}^{2}\left(k\right)\rangle\right)\times\langle\tilde{N}\rangle\right)$$

where $\langle N\rangle$ and $\langle\tilde{N}\rangle$ are the mean value of the number operator for the non-tilde and tilde operator respectively. If we set

$$\Phi_{\pm}=\langle\Phi\left(k\right)\rangle\mp i\langle\Phi\left(-k\right)\rangle, and\tilde{\Phi}_{\pm}=\langle\tilde{\Phi}\left(k\right)\rangle\mp i\langle\tilde{\Phi}\left(-k\right)\rangle$$

then, the green function will be given by

$$iG\left(x-y,\beta,t\right)\simeq iG_{K.G}\left(x-y,\beta,t\right)\times\left(1-4\left(\beta\lambda\right)\int d^{4}k\times\left(\Phi_{+}\Phi_{-}\right)\times\langle N\rangle+\left(\tilde{\Phi}_{+}\tilde{\Phi}_{-}\right)\times\langle\tilde{N}\rangle\right)$$

using the above equations and in order to examin the higher order effects, the green function, to second order in λ, is given as

$$iG\left(x-y,\beta,t\right)\simeq iG_{K.G}\left(x-y,\beta,t\right)\times\left(1-4\left(\beta\lambda\right)\int d^{4}k\times\left(\Phi_{+}\Phi_{-}\right)\times\langle N\rangle+\left(\tilde{\Phi}_{+}\tilde{\Phi}_{-}\right)\times\langle\tilde{N}\rangle\right.$$

$$\left.+16\left(\beta\lambda\right)^{2}\left(\int d^{4}k\times\left(\Phi_{+}\Phi_{-}\right)\times\langle N\rangle+\left(\tilde{\Phi}_{+}\tilde{\Phi}_{-}\right)\times\langle\tilde{N}\rangle\right)^{2}\right).$$

9.8 Time-dependent Green functions for fermions.

Now, let's look at the time-dependent green functions for fermions ψ interacting with scalar bosons ϕ.Let be the following lagrangian for the interaction

$$\pounds=\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi-\frac{1}{2}m_{\phi}^{2}\phi^{2}+i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-m_{\psi}\bar{\psi}\psi+g\phi\bar{\psi}\psi.$$

The full Hamiltonian for both interaction fields is given as

$$\hat{H}\left(t\right)=\pi_{\phi}\left(x\right)\dot{\phi}\left(x\right)-\tilde{\pi}_{\phi}\left(x\right).\dot{\tilde{\phi}}\left(x\right)+\pi_{\psi}\left(x\right)\dot{\psi}\left(x\right)-\tilde{\pi}_{\psi}\left(x\right).\dot{\tilde{\psi}}\left(x\right)-\hat{\pounds}\left(\phi\left(x\right),\tilde{\phi}\left(x\right),\partial_{\mu}\phi\left(x\right),\partial_{\mu}\tilde{\phi}\left(x\right)\right)$$

$$-\hat{\pounds}\left(\psi\left(x\right),\tilde{\psi}\left(x\right),\partial_{\mu}\psi\left(x\right),\partial_{\mu}\tilde{\psi}\left(x\right)\right)=\pi_{\phi}^{2}\left(x\right)-\tilde{\pi}_{\phi}^{2}\left(x\right)-\pounds\left(\phi\left(x\right),\tilde{\phi}\left(x\right),\partial_{\mu}\phi\left(x\right),\partial_{\mu}\tilde{\phi}\left(x\right)\right)+$$

$$+i\bar{\psi}\left(x\right)\gamma^{0}\dot{\psi}\left(x\right)-i\overline{\tilde{\psi}}\left(x\right)\gamma^{0}\left(x\right).\dot{\tilde{\psi}}\left(x\right)-\hat{\pounds}\left(\psi\left(x\right),\tilde{\psi}\left(x\right),\partial_{\mu}\psi\left(x\right),\partial_{\mu}\tilde{\psi}\left(x\right)\right)$$

$$=\hat{H}_{0}\left(\phi,\psi\right)-g\left(\phi\bar{\psi}\psi+\tilde{\phi}\overline{\tilde{\psi}}\tilde{\psi}\right)$$

9. NON-EQUILIBRIUM EVOLUTION OF GREEN FUNCTIONS IN TFD: A CANONICAL APPROACH.

We can split the Hamiltonian as two pieces. For this theory, the Hamiltonian is

$$\tilde{H}(t) = H(t) - \tilde{H}(t) = \int d^4x \left[\frac{1}{2}\pi_\phi^2 + \frac{1}{2}(\bigtriangledown\phi)^2 + \frac{m_\phi^2}{2}\phi^2 + \right] - \int d^4x \left[\frac{1}{2}\tilde{\pi}_\phi^2 + \frac{1}{2}\left(\bigtriangledown\tilde{\phi}\right)^2 + \frac{m_\phi^2}{2}\tilde{\phi}^2 + \right]$$

$$+\int d^4x \left[\bar{\psi}\vec{\gamma}.\vec{\nabla}\psi + m_\psi\bar{\psi}\psi - g\phi\bar{\psi}\psi\right] - \int d^4x \left[\overline{\tilde{\psi}}\vec{\gamma}.\vec{\nabla}\tilde{\psi} + m_\psi\overline{\tilde{\psi}}\tilde{\psi} - g\tilde{\phi}\overline{\tilde{\psi}}\tilde{\psi}\right]$$

$$= \hat{H}_0(\phi,\psi) + \hat{W}(\phi,\psi). \tag{9.94}$$

where

$$\hat{H}_0(\phi,\psi) = \int d^4x \left(\frac{1}{2}\pi_\phi^2 + \frac{1}{2}(\bigtriangledown\phi)^2 + \frac{m^2}{2}\phi^2\right) - \int d^4x \left(\frac{1}{2}\tilde{\pi}_\phi^2 + \frac{1}{2}\left(\bigtriangledown\tilde{\phi}\right)^2 + \frac{m^2}{2}\tilde{\phi}^2\right)$$
$$+i\int d^4x \left[\left(\bar{\psi}\vec{\gamma}.\vec{\nabla}\psi + m_\psi\bar{\psi}\psi\right) - \left(\overline{\tilde{\psi}}\vec{\gamma}.\vec{\nabla}\tilde{\psi} + m_\psi\overline{\tilde{\psi}}\tilde{\psi}\right)\right] \tag{9.95}$$

and

$$\hat{W}(\phi,\psi) = -\int d^4x g\left(\phi\bar{\psi}\psi + \tilde{\phi}\overline{\tilde{\psi}}\tilde{\psi}\right)$$

in momentum space

$$\hat{H}_0(\phi,\psi) = \int [dk]\frac{1}{2}\left\{\pi_k^2(t) + \left[\mathbf{k}^2 + m^2(t)\right]\phi_k^2(t)\right\} - \frac{1}{2}\left\{\tilde{\pi}_k^2(t) + \left[\mathbf{k}^2 + m^2(t)\right]\tilde{\phi}_k^2(t)\right\}.$$

$$+i\int [dk]\left[\left(\bar{\psi}_k\vec{\gamma}.\vec{\mathbf{k}}\psi_k + m_\psi\bar{\psi}_k\psi_k\right) - \left(\overline{\tilde{\psi}}_k\vec{\gamma}.\vec{\mathbf{k}}\tilde{\psi}_k + m_\psi\overline{\tilde{\psi}}_k\tilde{\psi}_k\right)\right]$$

and

$$\hat{W}(\phi,\psi) = -\int [dk]\,[dk']g\left(\phi_{k'}\bar{\psi}_k\psi_k + \tilde{\phi}_{k'}\overline{\tilde{\psi}}_k\tilde{\psi}_k\right).$$

For g small, the green function for the theory, to first order, is,

$$iG(x-y,\beta,t) \simeq iG_0(x-y,\beta,t)\times$$
$$\left(1 - (\beta)\sum_{n,m}\langle n,\tilde{n},m,\tilde{m},t \mid \hat{W}(\phi,\psi) \mid n,\tilde{n},m,\tilde{m},t\rangle - (\beta)\sum_{n,m}\langle n,\tilde{n},m,\tilde{m},t \mid \left[\hat{H}_0(\phi,\psi),\hat{W}(\phi,\psi)\right] \mid n,\tilde{n},m,\tilde{m},t\rangle\right)$$
$$\simeq iG_0(x-y,\beta,t)\times\left(1 - (\beta)\sum_{n,m}\langle n,\tilde{n},m,\tilde{m},t \mid \left[\hat{H}_0(\phi,\psi),\hat{W}(\phi,\psi)\right] \mid n,\tilde{n},m,\tilde{m},t\rangle\right) \tag{9.96}$$

where the state $\mid n,\tilde{n},m,\tilde{m},t\rangle$ contains n bosons with m fermions. The only non-vanishing commutator is

$$[H_0(\phi,\psi), W(\phi,\psi)] = \left[\int [dk]\frac{1}{2}\left\{\pi_k^2(t) + \left[\mathbf{k}^2 + m^2(t)\right]\phi_k^2(t)\right\}, -\int [dk]\,[dk']g\phi_{k'}\bar{\psi}_k\psi_k\right]$$

$$= -\frac{1}{2}g \int [dk] [dk'] \left[\pi_k^2 (t), \phi_{k'}\right] \bar{\psi}_k \psi_k = 2ig (2\pi)^4 \int [dk][dk']\delta (k-k') \bar{\psi}_k \psi_k = 2ig \int [dk]\bar{\psi}_k \psi_k.$$

with its tilde countrepart. Therefore, the full green function to first order is

$$iG (x-y, \beta, t) \simeq iG_0 (x-y, \beta, t) \times \left(1 - (2i\beta g) \int [dk]\langle \bar{\psi}_k \psi_k + \bar{\tilde{\psi}}_k \tilde{\psi}_k \rangle \right).$$

To second order, the last equation becomes

$$iG (x-y, \beta, t) \simeq iG_0 (x-y, \beta, t) \times \left(1 - (2i\beta g) \int [dk]\langle \bar{\psi}_k \psi_k + \bar{\tilde{\psi}}_k \tilde{\psi}_k \rangle - 2\beta^2 g^2 \left(\int [dk]\langle \bar{\psi}_k \psi_k + \bar{\tilde{\psi}}_k \tilde{\psi}_k \rangle \right)\right.$$

9.9 Conclusion.

In this chapter, we have used the LvN formalism, a canonical method, to study the nonequilibrium evolution of equal-time, two-point functions for a symmetric ϕ^4 field theory. The usefulness and simplicity of the LvN method in obtaining the two-point correlation functions were first illustrated using the quantum mechanical anharmonic oscillator model. The formalism was then extended to a scalar field and a ϕ^4 field theory and used to obtain the evolution equations for the connected two-point function in the Hartree approximation, for nonequilibrium thermal case. This provides an alternative and nonperturbative approach to investigate systematically the nonequilibrium evolution of quantum fields. It would be straightforward to generalize this technique for studying the nonequilibrium evolution of spontaneously broken field theories. Symmetry breaking would result in the generation of linear and cubic terms in the potential, as a consequense of which even connected odd n-point functions would contribute to the dynamical evolution. Higher order terms describing the evolution of the system out of equilibrium using Hartree approximation have been obtained to second order. The study includes, also, the evolution of time-dependent green functions for fermions.

9. NON-EQUILIBRIUM EVOLUTION OF GREEN FUNCTIONS IN TFD: A CANONICAL APPROACH.

Bibliography

[1] Y. Takahashi and H. Umezawa , *"Thermo field dynamics"* , Collective Phenomena, Vol. 2, pp.55-80 (1975).

[2] A. Das, *"Finite temperature field theory"* , World Scientific, Singapour , (1997).

[3] F.C. khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, A. E. Santana, 'Thermal Quantum Field Theory, Algebraic Aspects and Applications',World Scientific Publishing Company (2009).

[4] M. Le Bellac, 'Thermal Field Theory (Cambridge University Press, Cambridge, 1996)

[5] J. I. Kapusta, 'Finite Temperature Field Theory' (Cambridge University Press, Cambridge, 1989).

[6] M.H.Thoma, *" New Developments and Applications of Thermal Field Theory"*, arXiv:hep-ph/0010164.

[7] S.P. Kim and Chul.H. Lee, 'Nonequilibrium quantum dynamics of second order phase transtions', Phys.Rev. D, Vol.**62**, 125020.

[8] S. Sengupta, F.C. Khanna and S.P Kim, *'Nonequilibrium evolution of correlation functions: A canonical approach'*, Phys.Rev D**68**.105014 (2003).

[9] H.R. Lewis, JR., and W.B. Riesenfeld, *'An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field'*, Journal of Mathematical Physics Vol.10. Number 8, August 1969.

[10] H.R. Lewis, JR, *'Class of Exact Invariants for Classical and Quantum Time-dependent Harmonic Oscillators'*, Journal of Mathematical Physics Vol.9. Number11, November 1968.

[11] H. Umezawa, H. Matsumoto, and M. Tachiki, *'Thermofield Dynamics and Condensed States'* (North-Holland, Amsterdam, 1982); H. Umezawa, *Advanced Field Theory: Micro, Macro and Thermal Physics'* (AIP, New York, 1993).

[12] H. Matsumoto, *"The causal function in many body problems"* ,Fortschritte der Physik , 25, 1-36(1977).

[13] P. V. Landshoff, *"Introduction to equilibrium thermal field theory"* ,hep-ph/9808362.

10

Appendix

10.1 Appendix A.

The quark and antiquark fields may be written as :

$$q\left(x\right)=\sum_{s}\int d^{3}pN_{p}\left[c_{p,s}u\left(p,s\right)e^{-ipx}+d_{p,s}^{+}\upsilon\left(p,s\right)e^{ipx}\right]$$

$$\bar{q}\left(x\right)=\sum_{s}\int d^{3}pN_{p}\left[c_{p,s}^{+}\bar{u}\left(p,s\right)e^{ipx}+d_{p,s}\bar{\upsilon}\left(p,s\right)e^{-ipx}\right]$$

where we have used the normalization conditions :

$$N_{p}=\frac{1}{\left(2\pi\right)^{\frac{3}{2}}}\sqrt{\left(\frac{m}{E_{p}}\right)},E_{p}=\left(m^{2}+\vec{p}^{2}\right)^{\frac{1}{2}}$$

The meson fields (bosons) π^{+} may be written as :

$$\pi^{+}\left(x\right)=\frac{1}{\left(2\pi\right)^{\frac{3}{2}}}\int\frac{d^{3}p}{\sqrt{2\varpi_{p}}}\left[a_{p}e^{-ipx}+a_{p}^{+}e^{ipx}\right]$$

$$\tilde{\pi}^{+}\left(x\right)=\frac{1}{\left(2\pi\right)^{\frac{3}{2}}}\int\frac{d^{3}p}{\sqrt{2\varpi_{p}}}\left[\tilde{a}_{p}e^{ipx}+\tilde{a}_{p}^{+}e^{-ipx}\right]$$

For fermions, with the creation $c_{p,s}^{+}$ and $d_{p,s}^{+}$ and annihilation $c_{p,s}$ and $d_{p,s}$ operators, the bogoliubov transformations lead to the relations :

10. APPENDIX

$$
\begin{aligned}
c_{p,s} &= \cos\theta_p c_{p,s}(\beta) + i\sin\theta_p \tilde{c}^{+}_{p,s}(\beta) \\
d_{p,s} &= \cos\theta_{-p} d_{p,s}(\beta) + i\sin\theta_{-p} \tilde{d}^{+}_{p,s}(\beta) \\
c^{+}_{p,s} &= \cos\theta_p c^{+}_{p,s}(\beta) - i\sin\theta_p \tilde{c}_{p,s}(\beta) \\
d^{+}_{p,s} &= \cos\theta_{-p} d^{+}_{p,s}(\beta) - i\sin\theta_{-p} \tilde{d}_{p,s}(\beta)
\end{aligned}
$$

For the tild ($\sim$) fields :

$$
\begin{aligned}
\tilde{c}_{p,s} &= -i\sin\theta_p c^{+}_{p,s}(\beta) + \cos\theta_p \tilde{c}_{p,s}(\beta) \\
\tilde{d}_{p,s} &= -i\sin\theta_{-p} d^{+}_{p,s}(\beta) + \cos\theta_{-p} \tilde{d}_{p,s}(\beta)
\end{aligned}
$$

For the bose fields, the bogoliubov transformations are :

$$
\begin{aligned}
a_p &= \cosh\theta_p a_p(\beta) + \sinh\theta_p \tilde{a}^{+}_p(\beta) \\
a^{+}_p &= \cosh\theta_p a^{+}_p(\beta) + \sinh\theta_p \tilde{a}_p(\beta)
\end{aligned}
$$

as for the tild fields :

$$
\begin{aligned}
\tilde{a}_p &= \sinh\theta_p a^{+}_p(\beta) + \cosh\theta_p \tilde{a}_p(\beta) \\
\tilde{a}^{+}_p &= \sinh\theta_p a_p(\beta) + \cosh\theta_p \tilde{a}^{+}_p(\beta)
\end{aligned}
$$

where :

$$
\sin^2\theta_p = n_F(p) = \frac{1}{e^{\beta p_0}+1}, \sin^2\theta_{-p} = n_F(p)
$$

and :

$$
\sinh^2\theta_p = n_B(p) = \frac{1}{e^{\beta p_0}-1}, \sinh^2\theta_{-p} = n_B(p)
$$

where p_0 is either the bosonic or fermionic energy.

The anticommutation relations for creation and annihilation operators are similar to those at zero temperature. for example for fermions, we have :

$$
\left[c_{p',s'}(\beta), c^{+}_{p,s}(\beta)\right]_{+} = \left[d_{p',s'}(\beta), d^{+}_{p,s}(\beta)\right]_{+} = \delta\left(\vec{p}-\vec{p}'\right)\delta_{s,s'}
$$

the same relations hold for the tild (∼)operators. All other anticommutators are zero.

Also, the temperature dependent vaccum state $\mid O(\beta)\rangle$ is annihilated by the temperature dependent physical annihilation operators :

$$c_k(\beta) \mid O(\beta)\rangle = d_k(\beta) \mid O(\beta)\rangle = c_k(\beta) \mid O(\beta)\rangle = d_k(\beta) \mid O(\beta)\rangle = 0$$

We also used the normalization conditions :

$$\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s) = \left(\frac{\bar{p}+M}{2M}\right)_{\alpha\beta}$$

$$\sum_s \tilde{v}_\rho(p,s)\,\bar{\tilde{v}}_\sigma(p,s) = \left(\frac{-\bar{p}+M}{2M}\right)_{\rho\sigma}$$

$$\bar{u}^{(\alpha)}(p,s)\,u^{(\beta)}(p,s) = u^{\dagger(\alpha)}(p,s)\,u^{(\beta)}(p,s) = \delta^{\alpha\beta}$$

$$\bar{v}^{(\alpha)}(p,s)\,v^{(\beta)}(p,s) = v^{\dagger(\alpha)}(p,s)\,v^{(\beta)}(p,s) = -\delta^{\alpha\beta}$$

$$\bar{v}^{(\alpha)}(p,s)\,u^{(\beta)}(p,s) = \bar{u}^{(\alpha)}(p,s)\,v^{(\beta)}(p,s) = 0.$$

Gordon identities :

$$u^{(\alpha)}(p)\,\gamma^\mu u^{(\beta)}(q) = \frac{1}{2m}u^{(\alpha)}(p)\left[(p+q)^\mu + i\sigma^{\mu\nu}(p-q)_\nu\right]u^{(\beta)}(q)$$

10.2 Appendix B

We know that at finite temperature bosonic and fermionic Green functions are written in the matrix form as :

$$S(p,m) = (\bar{p}+m)\left[\begin{array}{c}\left(\begin{array}{cc}\frac{1}{p^2-m^2+i\varepsilon} & -2i\pi\theta\left(-p^0\right)\delta\left(p^2-m^2\right)\\ -2i\pi\theta\left(p^0\right)\delta\left(p^2-m^2\right) & -\frac{1}{p^2-m^2-i\varepsilon}\end{array}\right)\\ +2i\pi n_F\left(\left|p^0\right|\right)\delta\left(p^2-m^2\right)\left(\begin{array}{cc}1 & 1\\ 1 & 1\end{array}\right)\end{array}\right]$$

and for the bosonic representation in momentum space :

$$D(p,m)=\left[\begin{array}{c}\left(\begin{array}{cc}\frac{1}{p^2-m^2+i\varepsilon} & -2i\pi\theta\left(-p^0\right)\delta\left(p^2-m^2\right)\\ -2i\pi\theta\left(p^0\right)\delta\left(p^2-m^2\right) & -\frac{1}{p^2-m^2-i\varepsilon}\end{array}\right)\\ +2i\pi n_B\left(\left|p^0\right|\right)\delta\left(p^2-m^2\right)\left(\begin{array}{cc}1 & 1\\ 1 & 1\end{array}\right)\end{array}\right]$$

Here n_F and n_B represent the fermionic and the bosonic distribution functions respectively , namely,

$$n_B\left(\left|p^0\right|\right)=\frac{1}{e^{\beta\left|p^0\right|}-1},n_F\left(\left|p^0\right|\right)=\frac{1}{e^{\beta\left|p^0\right|}+1}$$

10.3 Appendix C.

10.3.1 Identities, Conventions & Useful Formulas.

We work in natural units $\hbar=c=k_{\mathbf{B}}=1$. In this system,

$$[length]=[time]=[energy]^{-1}=[mass]^{-1}$$

and the light cone metric is :

$$g^{\mu\upsilon}=\left(\begin{array}{cccc}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{array}\right)$$

From this metric, we can define directly the light cone variables and their inner product and integration measure :

$$a_\mu=(a_+,a_-,\mathbf{a_T})=\left(\frac{a_0+a_3}{\sqrt{2}},\frac{a_0-a_3}{\sqrt{2}},a_1,a_2\right)$$

$$a.b=a_+.b_-+a_-.b_+-\mathbf{a_T}.\mathbf{b_T}$$

$$d^4a=da_+da_-d^2\mathbf{a_T}$$

We define the gluon propagator in the feynman gauge in the coordinate represqentation. We do this by making a fourier transformation in the momentum representation :

$$\triangle(x) = \int \frac{d^4q}{(2\pi)^4} e^{-iq.x} \bar{\triangle}(q),$$

where,

$$\bar{\triangle}^{\mu\upsilon}_{ab}(q) = -g^{\mu\upsilon}\delta_{ab}\bar{\triangle}(q) = \frac{-ig^{\mu\upsilon}\delta_{ab}}{q^2+i\epsilon}$$

is the free gluon propagator in the feynman gauge.

Wilson lines to order $O\left(g^2\right)$, is given by,

$$P\exp\left(ig\int_a^b dx_\mu A^\mu(x)\right) = \left(1 + ig\int_a^b dx_\mu A^\mu(x) + (ig)^2\int_a^b dx_\mu \int_a^x dy_\upsilon A^\mu(x) A^\upsilon(y)\right)$$

For further development see reference [2]. Now the quark and antiquark fields and their tild counterparts are given by :

$$Q(x) = \sum_s \int d^3\vec{p} N_p \left[c_{p,s} u(p,s) e^{-ipx} + d^\dagger_{p,s} v(p,s) e^{ipx}\right]$$

$$\tilde{Q}(x) = \sum_s \int d^3\vec{p} N_p \left[\tilde{c}_{p,s} \tilde{u}(p,s) e^{ipx} + \tilde{d}^\dagger_{p,s} \tilde{v}(p,s) e^{-ipx}\right]$$

$$\bar{Q}(x) = \sum_s \int d^3\vec{p} N_p \left[c^\dagger_{p,s} \bar{u}(p,s) e^{ipx} + d_{p,s} \bar{v}(p,s) e^{-ipx}\right]$$

$$\overline{\tilde{Q}}(x) = \sum_s \int d^3\vec{p} N_p \left[\tilde{c}^\dagger_{p,s} \overline{\tilde{u}}(p,s) e^{-ipx} + \tilde{d}_{p,s} \overline{\tilde{v}}(p,s) e^{ipx}\right]$$

where,

$$N_p = \frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{E_p}}, E_p = \sqrt{m^2 + (\vec{p})^2}, \tilde{u}(p,s) = u^*(p,s).$$

As for the spinless hadron field we take with omission of the radial function for simplicity :

$$a^\dagger_{\mathbf{H}}\left(P^+, \mathbf{0_T}\right) = \chi_{s_1 s_2} c^\dagger(p_2, s_2) d^\dagger(p_1, s_1) \mid 0\rangle$$

where ,

10. APPENDIX

$$p_1 = \frac{m_1}{m_1+m_2}P+q, s_1\, , and p_2 = \frac{m_1}{m_1+m_2}P-q, s_2\, are respectively the quark$$
$$and antiquark momentum and spin; P is the total momentum of the hadron;$$
$$q\ the relative coordinate$$

For cut and uncut propagators the expressions

$$S\left(k\right) = \left[\frac{i}{k^2+i\epsilon} - 2\pi f\left(k^+\right)\delta\left(k^2\right)\right]\bar{k},$$

$$\triangle_{\mu\upsilon}\left(k\right) = \left[\frac{i}{k^2+i\epsilon} + 2\pi n_{\mathbf{B}}\left(k^+\right)\delta\left(k^2\right)\right]d_{\mu\upsilon}\left(k\right)$$

replace the ordinary zero-temperature propagators. The distribution functions :

$$f\left(x\right) = \frac{1}{e^{\beta|x|}+1}, n\left(x\right) = \frac{1}{e^{\beta|x|}-1}$$

represent the effect of thermal medium on free propagation. Cut propagators of thermal partons are represented by

$$S^{cut}\left(k\right) = \left[\Theta\left(k^+\right) - f\left(k^+\right)\right]\bar{k}2\pi\delta\left(k^2\right)$$

$$\triangle^{cut}_{\mu\upsilon}\left(k\right) = \left[\Theta\left(k^+\right) + f\left(k^+\right)\right]d_{\mu\upsilon}2\pi\delta\left(k^2\right)$$

where $\Theta\left(x\right)$ is the step operator.

10.3.2 Some useful formulas.

Exponential integral :

$$E_m\left(z\right) = \int_1^\infty \frac{e^{-zs}}{s^m}ds \Longrightarrow E_1\left(z\right) = -\gamma - \lg z + \sum_{k=1}^{\infty}\frac{\left(-1\right)^{k-1}}{k}\frac{z^k}{k\,!}$$

Asymptotic representation :

$$E_m\left(z\right) = \frac{e^{-z}}{z}\left[\sum_{k=0}^{n}\frac{\left(-1\right)^k m\left(m+1\right)...\left(m+k-1\right)}{z^k} + O\left(\frac{1}{z^{n+1}}\right)\right] for z \gg 1.$$

The relation between the exponential integral and the exponential integral function :

$$E_1(z) = -Ei(-z).$$

Furthermore we have :

$$\int_0^\infty \frac{\cos\theta \sin^3\theta}{\theta} d\theta = \frac{\pi}{16}.$$

Cosinus integral :

$$cosintegral(z) = Ci(z) = \int_\infty^z \frac{\cos s}{s} ds \Longrightarrow Ci(z) = \gamma + \lg z - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^{2k}}{(2k)(2k)!`}$$

Asymptotic representation :

$$Ci(z) = \frac{\sin z}{z} P(z) - \frac{\cos z}{z} Q(z), with, P(z) = \sum_{k=0}^{n} \frac{(-1)^k (2k)!`}{z^{2k}} + O\left(z^{-2n-2}\right),$$

$$Q(z) = \sum_{k=0}^{n} \frac{(-1)^k (2k+1)!`}{z^{2k+1}} + O\left(z^{-2n-3}\right)$$

Angular volume :

$$\Omega_m = \int d\Omega_m = \frac{2\pi^{\frac{(m+1)}{2}}}{\Gamma\left(\frac{(m+1)}{2}\right)} = 2^m \pi^{\left(\frac{m}{2}\right)} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma(m)}$$

Riemann zeta function :

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty du \frac{u^{(x-1)}}{e^u - 1}.$$

Relation between **the exponential integral** and **the exponential integral function** :

$$E_1(z) = -Ei(-z).$$

10. APPENDIX

[2] NAME. **Title**. *Journal of Sth*, 2006.

References

[1] LASTNAME. **Title**. *Journal of Sth*, 2007.

Declaration

I herewith declare that I have produced this paper without the prohibited assistance of third parties and without making use of aids other than those specified; notions taken over directly or indirectly from other sources have been identified as such. This paper has not previously been presented in identical or similar form to any other German or foreign examination board.

The thesis work was conducted from 2001 to 2011 under the supervision of Professor M.Ladrem at 'L'ECOLE NORMALE SUPERIEURE' (ENS de Kouba, Algeria), with the collaboration of Professor F. C. Khanna and Professor A.D. Santana at the University of Alberta (CANADA) .