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Présentée par :

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sujet :

# Contribution à la stabilisation des systèmes dynamiques " systèmes à commutations "

Devant le jury d'examen composé de :

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*I dedicate this thesis to my mother, to my father,*
*to my brother especially my little brother Rayane,*
*to my late grandfather "May God bless his soul and grant him paradise",*
*to my sisters, nephews : Amine, Annas, Baylasane*
*and all my friends.*

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# Notations and Acronyms

1. ***Acronyms :***
   - LTI : Linear Time Invariant .
   - LPV : Linear Parameter Varying .
   - LMI : Linear Matrix Inequality .
   - BMI : Bilinear matrix inequality .
   - SLF : Switching Lyapunov Function.
   - GAS : Global Asymptotically Stable.
   - CQLF : Commun Quadratic Lyapunov Function.

2. ***Notations :***
   - $\mathbb{R}$ is the set of real numbers ;
   - $\mathbb{R}^n$ is the vector space of dimension n built on the body of the real ;
   - $(\star)$ is used for blocks induced by symmetry in a matrix or in a matrix inequality ;
   - $A^T$ represents the transposed matrix of $A$ ;
   - $A^{-T}$ represents the inverse of matrix $A^T$ ;
   - $\mathbb{R}^{n\times m}$ is the set of all real matrices of $n$ rows and $m$ columns ;
   - $I$ is an identity matrix of appropriate size and $I_r$ represents a matrix of identity of dimension $r$
   - The value 0 represents a zero matrix of appropriate dimension, and $0_{(n,m)}$ designates null matrix with $n$ rows and $m$ columns ;
   - For a square matrix $S$, $S>0$ $(S<0)$ means that this matrix is positive definite (defined negative) ;
   - $A \leq B$ means that the matrix $B-A$ is semi-definite positive ;
   - $\mathrm{H}e(A)$ represents the sum $A+A^T$.
   - $\|.\|$ is the usual Euclidean norm ;
   - $l_2^s$ is the space of the summable square sequences, i.e.,

     $$l_2^s = \{x = (x(k))_k \subset \mathbb{R}^s, \sum_{k=0}^{\infty} \|x(k)\|^2 < +\infty\}$$

     with the norm $\|x\|_{l_2^s} = \left(\sum_{k=0}^{\infty} \|x(k)\|^2\right)^{\frac{1}{2}}$ ;
   - $\mathrm{diag}(A_1,\ldots,A_i)$ is the block diagonal matrix with $A_1,\ldots,A_i$ on its main diagonal ;
   - $e_s(i) = \underbrace{(0,\ldots,\overbrace{1}^{i\text{th}},\ldots,0)}_{s \text{ composantes}} \in \mathbb{R}^s$, $s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$.
   - $\mathscr{C}^1(\mathbb{R}^n)$ ) is the set of continuously differentiable functions,
   - $ImB$ is the subspace spanned by columns of matrix $B$,
   - $r_\sigma$ is the joint spectral radius.

# General Introduction

## Preamble

Dynamical systems that are described by an interaction between continuous and discrete dynamics are called switched systems. Such systems can be viewed as higher-level abstractions of hybrid systems, although they are of interest in their own right. A very simple example is the the model that describes the motion of an automobile [2] :

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) & (1)\\ \dot{x}_2(t) &= f(\alpha(t), q(t)) & (2) \end{aligned}$$

where $x_1(.)$ is the position, $x_2(.)$ is the velocity, $\alpha(.) \neq 0$ is the acceleration input, and $q(.) \in \{1,2,3,4,5,-1,0\}$ is the gear shift position. The function $f$ should be negative and decreasing in a when $q = -1$, negative and independent of $\alpha$ when $q = 0$, and increasing in $\alpha$, positive for sufficiently large $\alpha$, and decreasing in $q$ when $q > 0$. In this system, $x_1$ and $x_2$ are the continuous states and $q$ is the discrete state (called switching function). Hence, the switched system consists of seven distinct dynamic motions (seven distinct continuous state models) indexed by $q$ and concatenated in some way to produce an overall dynamic motion.
Clearly, the discrete transitions affect the continuous trajectory. In the case of an automatic transmission, the evolution of the continuous state $x_2$ is in turn used to determine the discrete transitions. In the case of a manual transmission, the discrete transitions are controlled by the driver.

Other phenomena that present a hybrid behavior are : a valve or a power switch opening and closing ; a thermostat that turns the heat on and off ; a server switching between the buffers in a queue ; network ; aircraft entering, crossing and leaving an air traffic control area ; the dynamics of a car changing abruptly because of the wheels that lock and unlock on the ice, ...etc. Hybrid systems became a very active area of current research. They present attractive theoretical challenges and are significant in many problems of the real world. Due to its interdisciplinary nature, the realm has attracted the attention of the scientists from diverse domains, mainly computer scientists, mathematicians, and engineers.

This thesis deals with the study of classes of switched systems from a control-theory point of view. More precisely, we are focused on the analysis of their stabilization (by a dynamic state feedback). The following example motivates the rich challenges and unexpected surprises intrinsic to the stability of switched systems.

**Example 0.0.1** ( [3]). *Consider the autonomous state dynamics $\dot{x}(t) = A_{p(t)}x(t)$, $t \geq 0$, where $x(t) = [x_1(t), x_2(t)] \in \mathbb{R}^2$ and $p(t) \in \{1,2\}$ and*

$$A_1 = \begin{bmatrix} -1 & -100 \\ 10 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}.$$

*Both $A_1$ and $A_2$ are stable, having identical eigenvalues $\lambda_{1,2} = -1 \pm i\sqrt{100}$. Define the switching function $p$ as follows :*

$$p(t^+) = \begin{cases} 1, & \text{if } p(t) = 2 \quad \text{and} \quad x_2(t) = -\frac{1}{k}x_1(t) \\ 2, & \text{if } p(t) = 1 \quad \text{and} \quad x_2(t) = kx_1(t) \end{cases} \tag{3}$$

*For any given initialization, this function specifies a rule with memory for switching the dynamic motion of the system between $A_1$ and $A_2$. For $k = -0.2$ and any initial condition $x(0) \neq 0$, the state trajectory diverges to $\infty$ (see Fig. 1, taken from [3]), illustrating that*

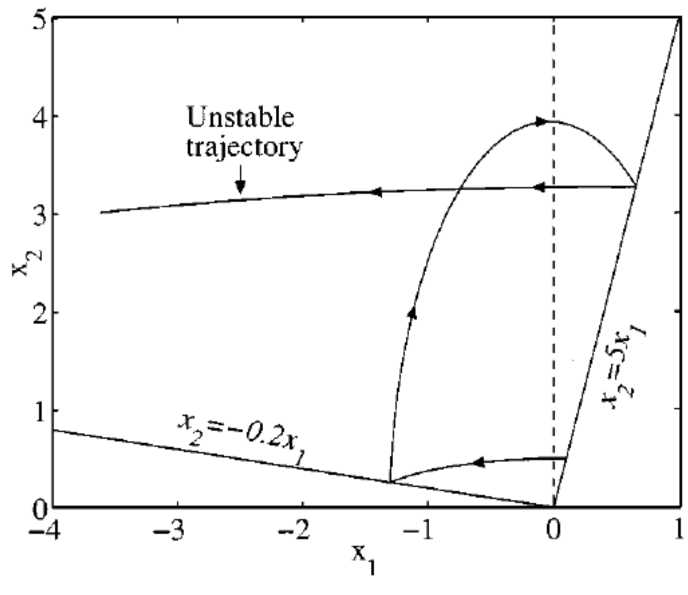



FIGURE 1 – Unstable state trajectory of switched stable systems.

*switching between two asymptotically stable systems can induce an unstable trajectory. This situation can occur in the control of the longitudinal dynamics of an aircraft with constrained angle of attack [4].*

Two main problems arise in the literature of switched systems :

1. What is the class of switching systems which are stable in the presence of any arbitrary switching rule ?
2. What are the switching rules which lead to stable switched systems even if the subsystems are unstable ?

In the context of this thesis, we focused on the study of the first issue.
Several methodologies have been developed in the literature for both continuous-time an discrete-time systems [5–9]. Among the existing methods, we have : dwell-time and average dwell-time approaches for stability analysis and stabilization problems [10,11] ; approaches based on a specific class of switching laws [12], [13] and under arbitrary switching sequences [14] ; sliding mode technique [15] ; algebraic approach [5] ; Lyapunov-Metzler approach [16, 17] ; input-output approach [18]. For an overview on stability analysis, we refer the reader to [2, 3, 19, 20], which summarize some contributions on the analysis and design of switching systems. New investigations on stabilization and control for both linear and nonlinear switched systems have been addressed in the monograph [21], see also [9].

In this thesis, we investigate the problem of robust observer-based stabilization for linear switched discrete-time systems in the presence of parameter uncertainties. The switching mode is assumed to be arbitrary, but its instantaneous values are available in real time.

Most of the existing control strategies of switched systems focus on full-state feedback ; see, e.g., [22] and [23]. However, in practice, full measurement of the states of a switched system may be expensive or unavailable at any cost. For this reason, considerable efforts have been paid to state estimation of linear and nonlinear switched systems [6, 12, 24]. On the other hand, it is always more suitable to design a control system which is not only stable, but also guarantees an adequate level of performance. This is why control systems design in the presence of model uncertainties has been a challenging topic and received considerable attention [9, 25–27].

One of hot topics in switched systems is to find non conservative conditions to guarantee the stabilization of the systems under arbitrary switching rules. A breakthrough regarding this issue is the switched quadratic Lyapunov functions (SLF) introduced in [7]. Within LMI framework, control techniques by switching among different controllers have been applied extensively in recent years, see in particular [16, 28–31]. Control synthesis techniques via static output feedback for switched systems under arbitrary switching rule have been first considered in [7]. Sufficient LMI conditions subject to an equality constraint that guarantees the asymptotic stability of the closed-loop system have been given. Using similar techniques, the issue has been reconsidered in [32], in the observer-based static output feedback context, in presence of parameter uncertainties. Relevant results and interesting improvements of [7] have been considered in [29]. As for the dynamic output feedback, it has been investigated in [16].

Finsler's lemma has been used previously in the control literature mainly in order to eliminate some unlike matrix terms, see e.g. [33]. Switched quadratic Lyapunov functions combined with Finsler's lemma have been used in [34] to get necessary and sufficient LMI conditions for the asymptotic stability issue. However, based on the pioneering work in [34], some attempts using Finsler's lemma-based approach have been presented in [1] and [27]. Unfortunately, the obtained LMI conditions still remain very conservative. Indeed, they are either subjected to strong equality constraints [27], or require particular choices of the decision variables [1]. From LMI point of view, the stabilization problem is far from being solved. Indeed, finding a systematic LMI technique for handling the bilinear matrix terms related to the controller gains and the Finsler inequality is a hard task. This is one of the main motivations of the work investigated in this thesis.

The main objective of this thesis consists in developing new and less conservative LMI synthesis conditions for the observer-based stabilization problem for switched discrete-time linear systems with parameter uncertainties. As mentioned previously, the addressed problem has been investigated in [1] and [27] in the LMI context by using switched Lyapunov functions combined with Finsler's lemma. However, the obtained LMI conditions are conservative because of the particular use of Finsler's lemma in its basic formulation. In addition, the switched Lyapunov matrices used in [1] are assumed to be diagonal, and the LMI synthesis conditions proposed in [27] require additional equality constraints, which turns out to be very conservative. To show the importance of the work we proposed in this thesis, we summarize the contributions in the following items :

— New linearization schemes and different scenarios of a convenient use of Finsler's Lemma are proposed to reduce the conservatism of some previous results in the literature. This novel use of Finsler's lemma may open some new directions to solve more complicated control design problems.

— As compared to the Young inequality based approach introduced recently in [35] and [36], the proposed technique in this thesis allows to eliminate some bilinear terms arising from the use of the Young relation-based approach, and then leads to less conservative LMI conditions.

— The proposed LMI methods are more general than those established in the literature for the same stabilization problem. We essentially demonstrated that the way to select the matrices inferred from Finsler's lemma plays an important role in the feasibility of the obtained LMIs.

— As an application, new enhanced LMI conditions are proposed to solve the problem of stabilization of Linear Parameter Varying (LPV) discrete-time systems with uncertain parameters. Indeed, asymptotic stability problem for switched linear systems with arbitrary switching is equivalent to the robust asymptotic stability problem for LPV systems, for which several strong stability conditions are available in the literature [8].

### Structure of thesis

This thesis is organized in five chapters that are structured as follows :

**Chapter 1** : The first chapter is a literature survey. The general stability problem is recalled in order to establish fundamental concepts that are necessary to the comprehension of manuscript. Then, we present several stability problems and stability criteria that are encountered in the domain of switched linear systems.

The rest of the chapters constitutes our contributions.
**Chapter 2** : In this chapter, we deal with observer-based control design for a class of switched discrete-time linear systems with parameter uncertainties. The main contributions of this chapter consists in a new and judicious use of the Finsler's Lemma to reduce the conservatism of some previous results in the literature. Especially, we correct the false application of Finsler's lemma in [1] and thanks to some new mathematical tools we introduced in this chapter, we will get new slack variables allowing more degree of freedom in the obtained LMI conditions. Two numerical examples followed by a Monte Carlo evaluation are proposed to show the superiority of the proposed design technique.

**Chapter 3** : This chapter is an extension of results in **chapter 2**. We provide different scenarios of the use of Finsler's Lemma to reduce the conservatism of some previous results in the literature. The validity and effectiveness of the proposed design methodologies are shown through numerical evaluations.

**Chapter 4** : Our objective in this chapter is to extend and improve the study in the previous chapter, by taking into account the presence of $l_2$-bounded disturbances in state equations, and by introducing a more general structure of the slack variable coming from Finsler's lemma.

**Chapter 5** : In this chapter, we investigate the observer-based stabilization problem for a class of LPV discrete-time systems. The problem is motivated by the work proposed by Heemels et al [37], Jetto and Orsini [38] and recently by Zemouche et al [39]. One of the contributions of this chapter consists in the use of a relaxed reformulation for the parameter uncertainty. This relaxation is inspired from recent design methods for observer design of nonlinear Lipschitz systems Zemouche and Boutayeb [40], Phanomchoeng et al. [41]. Using this reformulation, a new LMI synthesis method is developed to design observer-based controllers for the studied LPV systems. The approach used a new congruence principle by pre- and post multiplying the basic BMI condition by new and ingenious slack matrices. We propose relaxations of some approaches in the literature, especially the Young inequality based approach.

# Personal Publications

Below is a list of publications related to the work of this thesis :

- **Journals**

  1. ***H. BIBI**, F. BEDOUHENE, A. ZEMOUCHE, H.R. KARIMI & H. KHELOUFI, "Output feedback stabilization of switching discrete-time linear systems with parameter uncertainties " Journal of the Franklin Institute – (**354**) : 5895-5918 (2017).*

- **International Conferences**

  1. ***H. BIBI**, F. BEDOUHENE, A. ZEMOUCHE, H. KHELOUFI & H. TRINH " Observer-Based Control Design via LMIs for a Class of Switched Discrete-Time Linear Systems with Parameter Uncertainties " In Proceeding $55^{th}$ IEEE Conference on Decision and Control (pp. 5321 - 5326) Las Vegas, USA.*
  2. ***H. BIBI**, F. BEDOUHENE, A. ZEMOUCHE & A. AITOUCHE " $\mathscr{H}_\infty$ Observer-Based Stabilization of Switched Discrete-Time Linear Systems " In Proceeding IEEE $6^{th}$ International Conference on Systems and Control. (pp. 285 - 290 ) University of Batna 2, Batna, Algeria.*
  3. *C. BENNANI , F. BEDOUHENE, A. ZEMOUCHE, **H. BIBI** & A. AITOUCHE " A Modified Two-Steps LMI Method to Design Observer-based controller for Linear Discrete-Time Systems with Parameter Uncertainties " In Proceeding IEEE $6^{th}$ International Conference on Systems and Control. (pp. 279 - 284 ) University of Batna 2, Batna, Algeria.*
  4. ***H. BIBI**, F. BEDOUHENE, A. ZEMOUCHE & H.R. KARIMI " A New LMI-Based Output Feedback Controller Design Method for Discrete-Time LPV Systems with Uncertain Parameters " In Proceeding $20^{th}$ IFAC World Congress. (pp.11847 - 11852) Toulouse Convention Center, Toulouse, France.*
  5. ***H. BIBI**, A. ZEMOUCHE, F. BEDOUHENE, K.C DRAA & A. AITOUCHE " Robust Observer-Based $\mathscr{H}_\infty$ Stabilization for a Class of Switched Discrete-Time Linear Systems with Parameter Uncertainties " In Proceeding $56^{th}$ IEEE Conference on Decision and Control. (Accepted)*

- **National Conference**

  1. The First National Doctoriales of Mathematics, Constantine, Algeria, OCTOBER 28-31, 2017

CHAPITRE

1

# Generalities and Previous Literature

## Sommaire



## 1.1 Introduction

The aim of this chapter is to represent both the necessary and crucial reminders for the comprehension of this script, and previous literature of the different methods LMIs synthesis by observer-based of switching systems discrete-time with parameters uncertainties. First, a reminder of the stability (Lyapunov's direction) is given. We also state briefly the criteria of controllability, observability and stabilizability of systems to explore. We move to present the LMI approach, depicting the rationale of using it. Finally, we close this chapter exposing the results of some recent studies tackling the same class of systems which are parts of the comparison in the following chapters.

## 1.2 Switched systems

Switched systems provide an introduced framework for mathematical modeling of many physical or man-made systems displaying switching features such as power electronics, flight control systems, and network control systems. More specifically, a switched dynamical system is a two-level hybrid system with the lower level governed by a set of modes described by differential and/or difference equations and the upper level a coordinator that orchestrates the switching among the modes. Switching systems are particular case hybrid dynamic systems [2]. They combine the elegance of the simplicity and the richness of the described phenomena. A switched-system is composed of a finite set $N \in \mathbb{N}^*$ dynamics, among one and only one is activated in each time by the switching mode. A switched-system is said in a discrete time if, more precisely, consists of dynamics in differences. Such system can be defined under the following form :

$$x_{t+1} = f_{\sigma(t)}(t, x_t, u_t), \qquad \forall t \in \mathbb{N} \tag{1.1}$$

$$y_t = g_{\sigma(t)}(t, x_t, u_t) \tag{1.2}$$

where $x_t \in \mathbb{R}^n$ is vector state, $u_t \in \mathbb{R}^m$ is the control input vector and $y_t \in \mathbb{R}^p$ is the output measurement vector, and $\sigma : \mathbb{N} \to \Lambda \triangleq \{1, 2, \ldots, N\}$ $t \to \sigma_t$ is a switching rule. If there is no ambiguity, we simply write $\sigma$ instead of $\sigma_t$, for $\sigma \in \Lambda$ is said to be a subsystem of the switched system.
The N couples of functions $f_i(.,.,.)$ and $g_i(.,.,.)$, $i \in \Lambda$ is subsystem dynamic or mode.
The correspondent model of the switched systems (1.1)-(1.2) is too broad. In this script, the class of the considered systems will be a restrain for the switched-linear systems, among the modes are linear time invariant. Moreover, without losing a generality, we can consider the output $y_t$ without the controller input $u_t$.

$$x_{t+1} = A_{\sigma(t)} x_t + B_{\sigma(t)} u_t, \qquad \forall t \in \mathbb{N} \tag{1.3}$$

$$y_t = C_{\sigma(t)} x_t \tag{1.4}$$

If no control input is imposed on the system, then, the system is said to be a *switched autonomous system*, or an unforced switched system. The unforced switched linear system is described by

$$x_{t+1} = A_{\sigma(t)} x_t, \qquad \forall t \in \mathbb{N} \tag{1.5}$$

$$y_t = C_{\sigma(t)} x_t \tag{1.6}$$

### 1.2.1 Switching linear system

A switching linear system is a dynamic system composed of a finite number subsystems and a so-called switching law to manage the switching between these different subsystems. In discrete time, a Switching linear system has the form

$$x_{t+1} = A_{\sigma(t)} x_t \tag{1.7}$$

In continuous time,

$$\dot{x} = A_{\sigma(t)} x \tag{1.8}$$

where $x \in \mathbb{R}^n$ is the state vector, $t \in \mathbb{N}$ the time, $\sigma : \mathbb{N} \longrightarrow \Lambda$ the switching arbitrary, $A_{\sigma(t)} \in \mathbb{R}^{n\times n}$ the dynamic matrix of the system, and $\Lambda$ the set of indices $\{1,\ldots,N\}$ representing the modes of the linear switching systems (1.7) and (1.8).
The stability analysis in the case of linear switched systems is different from the LTI case in so far as some interesting phenomena are observed. We can cite the case where even if all modes (subsystems) are asymptotically stable.

### 1.2.2 Solutions of Switched Linear Systems

For a nonlinear system, usually it is very hard (if not impossible) to explicitly express the solution in terms of the system parameters in an analytic way. On the other hand, for well-posed switched linear systems, this is always possible. For clarity, let $\phi(t;t_0,x_0,u,\sigma)$ denote the state trajectory at time $t$ of a continuous-time switched linear system

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t) \tag{1.9}$$

initialized at $x(t_0) = x_0$ with input $u$ and switching signal $\sigma$. Suppose that the switching signal is well-defined and its switching sequence

$$x_0, (t_0,i_0), (t_1,i_1), \ldots, (t_l,i_l).$$

As the $i_0^{th}$ subsystem is active during $[t_0,t_1[$, we have

$$\dot{x}(t) = A_{i_0} + B_{i_0} u(t), \qquad x(t_0) = x_0, \quad t \in [t_0,t_1[$$

This is a linear differential equation with an initial condition, so its solution can be given explicitly by

$$\phi(t,t_0,x_0,u,\sigma) = e^{A_{i_0}(t-t_0)} x_0 + \int_{t_0}^{t_1} e^{A_{i_0}(t-\tau)} B_{i_0} u(\tau) d\tau, \qquad t \in [t_0,t_1[$$

and (by the continuity of the state trajectory)

$$x_1 = x(t_1) = \phi(t_1,t_0,x_0,u,\sigma) = e^{A_{i_0}(t_1-t_0)} x_0 + \int_{t_0}^{t_1} e^{A_{i_0}(t_1-\tau)} B_{i_0} u(\tau) d\tau,$$

Continuing with the above procedure, the solution for the switched system can be computed to be

$$\phi(t,t_0,x_0,u,\sigma) = e^{A_{i_k}(t_k-t_0)} e^{A_{i_{k-1}}(t_{k-1}-t_0)} \ldots e^{A_{i_0}(t_1-t_0)} x_0$$

$$+ e^{A_{i_k}(t-t_k)} \dots e^{A_{i_1}(t2-t_1)} \int_{t_0}^{t_1} e^{A_{i_0}(t_1-\tau)} B_{i_0} u(\tau) d\tau,$$
$$+ \dots + e^{A_{i_k}(t-t_k)} \int_{t_{k-1}}^{t_k} e^{A_{i_{k-1}}(t_k-\tau)} B_{i_{k-1}} u(\tau) d\tau,$$
$$+ \int_{t_k}^{t} e^{A_{i_k}(t-\tau)} B_{i_k} u(\tau) d\tau, \qquad t \in [t_k, t_{k+1}[.$$

Let

$$\Psi(t, \sigma, x_0) = \exp^{A_{i_k}(t-t_k)} \dots e^{A_{i_0}(t1-t_0)} \qquad t \in [t_k, t_{k+1}[$$

It is clear that the state transition matrix is given by

$$\Phi(t_1, t_2, \sigma, x_0) = \Psi(t_1, \sigma, x_0)(\Psi(t_2, \sigma, x_0))^{-1}$$

The solution of the system can be re-written, in terms of the transition matrix, as

$$\phi(t, t_0, x_0, u, \sigma) = \Phi(t, t_0, \sigma, x_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau, \sigma, x_0)u(\tau)d\tau.$$

From this expression, we can draw a few useful conclusions as follows :

— For a switched linear system, if the switching signal is well-defined and the input is globally integrable, then the system always permits a unique solution for the forward time space.

— The solution is usually not continuously differentiable at the switching instants, even if the input is smooth.

— The state transition matrix is a multiple multiplication of matrix function of the form $e^{At}$. Accordingly, properties of functions in this form play an important role in the analysis of switched linear systems.

Similarly, for a discrete-time switched linear system

$$x_{t+1} = A_\sigma x_t + B_\sigma u_t \qquad (1.10)$$

the solution is

$$x_t = A_{\sigma(t-1)} \dots A_{\sigma(1)} A_{\sigma(0)} x_0 + A_{\sigma(t-1)} \dots A_{\sigma(1)} B_{\sigma(0)} u_0 + \dots A_{\sigma(t-1)} B_{\sigma(t-2)} u_{t-2} + B_{\sigma(t-1)} u_{t-1}.$$

The state transition matrix is

$$\Phi(t_1, t_2, \sigma) = A_{\sigma(t_1-1)} \dots A_{\sigma(t_2)} \qquad t_1 > t_2.$$

In terms of the transition matrix, the solution can be rewritten as

$$x_t = \Phi(t, 0, \sigma)x_0 + \sum_{j=0}^{t-1} \Phi(t, j, \sigma)u_j.$$

From the solution, we have :

— For a switched linear system, the system permits a unique solution for the forward time space. Hence, any discrete-time switched system is wellposed.
— The state transition matrix is a multiple multiplication of matrices. Accordingly, properties of matrix multiplication play an important role in analyzing the switched system.

Finally, for both continuous-time and discrete-time switched linear systems, the state trajectory possesses several nice properties under mild conditions.

## 1.3 Lyapunov Stability Analysis

The notion of stability has a broad range of definition in the literature. In our case, we will focus on the stability in the sense of Lyapunov, who constitutes a powerful tool in studying the stability of an equilibria point.
Considering now a nonlinear discrete-time system, we write

$$x_{t+1} = f(x(t)). \tag{1.11}$$

where $f(.,.)$ is function of $\mathbb{N} \times \mathbb{R}^n$ in $\mathbb{R}^n$, locally Lipschitz with second variable.
We note $x_{eq} = 0$ equilibria point of (1.11).The theory of Lyapunov allows also to provide sufficient conditions of stability in the general case.

**Definition 1.3.1.** *The equilibrium points $x_{eq}$ the real solutions of equation*

$$f(t, x_{eq}) = x_{eq}, \qquad \forall t \in \mathbb{N}. \tag{1.12}$$

*An equilibrium point $x_{eq}$ of the system* (1.11) *is said*

— *Stable at time $t_0$, if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, t_0) > 0$ such that*

$$||x_{t_0} - x_{eq}|| < \delta \Rightarrow ||x_t - x_{eq}|| < \varepsilon, \quad \forall t > t_0; \tag{1.13}$$

— *asymptotically stable at time $t_0$, si $x_{eq}$ stable and $\delta = \delta(\varepsilon, t_0) > 0$ can be chosen as then*

$$||x_{t_0} - x_{eq}|| < \delta \Rightarrow \lim_{t \to +\infty} x_t = x_{eq}; \tag{1.14}$$

— *exponentially stable if there are three positive reels $c, K$ and $\rho (0 \leq \rho \leq 1)$ such that*

$$||x_t - x_{eq}|| < K||x_{t_0} - x_{eq}||\rho^{t-t_0}, \forall ||x_{t_0} - x_{eq}|| < c; \tag{1.15}$$

**Definition 1.3.2.** *[42]*

— *A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to class $\mathscr{K}$ if it is continuous, strictly increasing and $\phi(0) = 0$ and to class $\mathscr{K}_\infty$ if additionally $\phi(s) \to \infty$ as $s \to \infty$.*
— *A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ belongs to class $\mathscr{K}\mathscr{L}$ if for each fixed $k \in \mathbb{R}^+$, $\beta(., k) \in \mathscr{K}$ and for each fixed $s \in \mathbb{R}^+$, $\beta(s, .)$ is decreasing and $\lim_{k \to \infty} \beta(s, k) = 0$.*

**Definition 1.3.3.** *[42] A continuous function $V : \mathbb{R}^n \to \mathbb{R}^+$ is called an ISS-Lyapunov function for system* (1.11) *if the following holds :*

1. *There exist $\mathscr{K}_\infty$ functions $\alpha_1, \alpha_2$ such that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \forall x \in \mathbb{R}^n. \tag{1.16}$$

2. *There exist $\mathscr{K}_\infty$ function $\alpha_3$ and $\mathscr{K}_\infty$ function $\alpha_4$ such that*

$$V(f(x,\mu)) - V(x) \leq -\alpha_3(|x|) + \alpha_4(|\mu|), \qquad \forall x \in \mathbb{R}^n, \forall \mu \in \mathbb{R}^m. \tag{1.17}$$

*A smooth ISS-Lyapunov function is one which is smooth.*

**Theorem 1.3.4.** *Equilibria point $x_{eq} = 0$ is said*

— ***locally stable*** *if a function exists $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a neighbourhood $\Omega \subset \mathbb{R}^n$ of origin as :*

- $V(x) > 0, \forall x \in \Omega$ *et* $V(0) = 0,$
- $\Delta V(x(t)) \leq 0, \forall x \in \Omega, x \neq 0,$

— ***locally asymptotically stable*** *if a function exists $V(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a neighbourhood $\Omega \subset \mathbb{R}^n$ of origin as :*

- $V(x) > 0, \forall x \in \Omega$ *et* $V(0) = 0,$
- $\Delta V(x(t)) < 0, \forall x \in \Omega, x \neq 0,$

— ***globally asymptotically stable*** *if a function exists $V(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ as in*

- $V(x) > 0, \forall x \in \mathbb{R}^n$ *and* $V(0) = 0,$
- $\Delta V(x(t)) < 0, \forall x \in \mathbb{R}^n, x \neq 0,$
- $V(x) \longrightarrow \infty$ *when* $\|x\| \longrightarrow \infty$

*with $\Delta V(x(t)) = V(x(t+1)) - V(x(t))$.*

Consider the discrete-time switched system

$$x_{t+1} = A_{\sigma(t)} x_t \tag{1.18}$$

The following definitions are needed to precisely state what is "stable" in the context of switched systems

**Definition 1.3.5.** *[43] The switched system* (1.18) *is globally uniformly asymptotically stable (GUAS) if there exists a class $\mathscr{K}\mathscr{L}$ function $\beta(\|.\|,.)$ such that for all switching signals $\sigma$ and all initial conditions $x_{t_0}$, the solutions of* (1.18) *satisfy the inequality $\|x_t\| \leq \beta(\|x_{t_0}\|,\ t - t_0),\ \forall t \geq t_0$.*

**Definition 1.3.6.** *[43] The switched system* (1.18) *is globally uniformly exponentially stable (GUES) if for constants $c > 0, 0 < \lambda < 1$, all switching signals $\sigma$ and all initial conditions $x_{t_0}$, the solutions of* (1.18) *satisfy the inequality $\|x_t\| \leq c\lambda^{t-t_0}\|x_{t_0}\|,\ \forall t \geq t_0$.*

For discrete-time switched system (1.18), it should be pointed out that the reason why the switched system is not stable under arbitrary switching is due to the fact that some (at least one) of the subsystems satisfy $\|A_i\| \geq 1, i \in \{1,2,\ldots,N\}$.

In the particular case of the discrete-time switched-system, the existence condition of Lyanpunov function proves to be necessary and sufficient.
Indeed, considering the system

$$x_{t+1} = Ax_t. \tag{1.19}$$

The following theorem recalls the asymptotic stability of the LTI discrete-time systems.

**Theorem 1.3.7.** *The point $x_{eq} = 0$ is asymptotically stable if and only if, for all $Q = Q^T > 0$, a matrix exists $P = P^T > 0$ verifying Lyapunov's equation*

$$A^T PA - P + Q = 0. \tag{1.20}$$

To consider stability for the discrete-time switched system (1.19),

**Remark 1.3.8.** *Consider the linear time-invariant symmetric system* (1.19) *and $A^T = A$. The system* (1.19) *is Schur stable if and only if*

$$A^2 < I.$$

**Remark 1.3.9.** *The system* (1.19) *is asymptotically stable if and only if the eigenvalues of $A$ satisfy $|\lambda_i| < 1$ ; i.e if they lie strictly within the unit disc of the complex plane.*

**Theorem 1.3.1.** *[44] For discrete-time linear time-invariant system* (1.19)*, the following statements are equivalent :*

1. *the system is asymptotically stable ;*
2. *the system is exponential stable ;*
3. *matrix $A$ is Schur ;*
4. *the Lyapunov difference equation*

$$P - A^T PA = Q \tag{1.21}$$

*has a unique solution $P > 0$ for any $Q > 0$ ; and equation* (1.21) *has a unique solution $P > 0$ for some $Q > 0$.*

Consider the linear time-invariant system given by

$$\dot{x}(t) = Ax(t) \qquad t \geq 0 \tag{1.22}$$

In the same way as if in the LTI systems of a discrete-time, in this special case, Lyapunov theory is very complete, and we have the following theorem.

**Theorem 1.3.2.** *[44]For linear time-invariant system* (1.22)*, the following statements are equivalent :*

1. *the system is asymptotically stable ;*

2. *the system is exponential stable ;*
3. *matrix A is Hurwitz ;*
4. *the Lyapunov equation*
$$A^T P + PA = -Q \tag{1.23}$$
*has a unique solution $P > 0$ for any $Q > 0$ ; and*
5. *Equation* (1.23) *has a unique solution $P > 0$ for some $Q > 0$.*

### 1.3.1 Common Quadratic Lyapunov Function

From a practical point of view, it is very difficult to verify the criteria proposed by the previous theorem. This difficulty has led many researchers to limit their search to a quadratic Lyapunov function in the form

$$V(x) = x^T Px.$$

The existence of such a function is a sufficient condition for stability. The latter may be expressed in terms of linear matrix inequalities [33] whose solution can be found by convex optimization algorithms. In discrete time, we have the following result

**Theorem 1.3.10.** *If there exists a positive definite symmetric matrix P satisfying*

$$A_i^T PA_i - P < 0, \;\; pour \;\; i = 1, \ldots, N \tag{1.24}$$

*then the function $V(x) = x^T Px$ is a common quadratic Lyapunov function (CQLF) for the system* (1.7). *The switching system* (1.7) *is then GAS.*

The system of linear matrix inequalities (LMIs) (1.24) is said to be feasible if there exists a solution $P$, otherwise it is said to be unfeasible. For example, the existence of a CQLF is equivalent to checking the feasibility of the LMIs (1.24). In recent decades, convex optimization algorithms have been to solve this type of LMIs [33]. In [45], the authors proposed an algorithm to converge to a CQLF in a finite step number. But, until now, the conditions necessary and sufficient for the existence of a CQLF stay an open problem in the neral. In [46], the authors used Lie algebra to express the conditions of existence of a CQLF in terms of solvable Lie algebra.

**Example 1.3.11.** *Consider the switched linear system composed of two subsystems with the following system matrices,*

$$A_1 = \begin{bmatrix} -0.9 & 3.2 \\ 0 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.3 & 0 \\ -1.9 & -0.9 \end{bmatrix}.$$

*It is clear that both subsystems are stable. However, it can be seen in Figure.*(1.1) *that the system is not stable under the switching shown in the figure.*

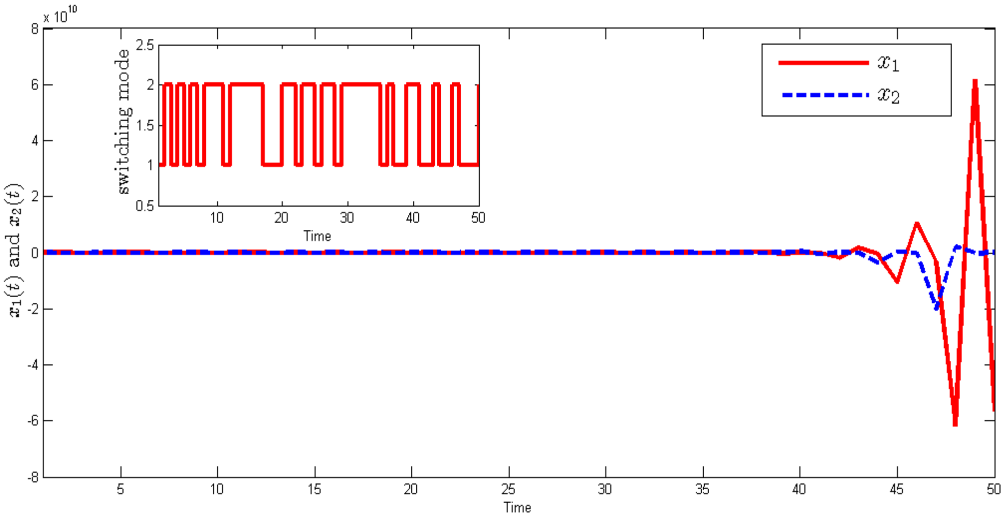



FIGURE 1.1 – State responses for Example .

**Example 1.3.12.** *The following counterexample, taken from [2] , proves that even for switched linear systems GUES does not imply the existence of a quadratic common Lyapunov function. Take $\sigma = 1,2$, and let the two matrices be*

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix}$$

*These matrices are both Hurwitz.*

*The systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ do not share a quadratic common Lyapunov function of the form $V(x) = x^T Px$. Without loss of generality, we can look for a positive definite symmetric matrix $P$ in the form*

$$P = \begin{bmatrix} 1 & q \\ q & r \end{bmatrix}$$

*which satisfies the inequality $A_i^T P + PA_i < 0 \ \forall i \in \{1,2\}$. For more detail see [2].*

### 1.3.2 Multiple Lyapunov Functions

Several stability criteria based on common quadratic Lyapunov functions exist in the literature. However, the existence of a common quadratic Lyapunov function is only a sufficient stability condition, not a necessary one. In [47], it is analytically shown that there exist switched linear systems that are asymptotically stable for which no common quadratic Lyapunov function exists. This indicates that looking for such a function may be too conservative. This motivates the search of other types of Lyapunov functions. In the literature, we can find several types of Lyapunov functions that may be classified, in a generic framework, under the name of multiple Lyapunov functions. The multiple Lyapunov functions describe a family of function of the form

$$V(x) = x^T P(\sigma,x)x$$

where the Lyapunov matrix may depend on the state vector and/or on the switching law. The concatenation of these functions composes one common, non quadratic, Lyapunov function. The idea is to reassemble matrices $P_i$ in the idea of the mode switching, we consider Lyapunov function form

$$V(t,x(t)) = x^T(t)P_{\sigma(t)}x(t). \tag{1.25}$$

**Theorem 1.3.13** ( [7]). *The following statements are equivalent :*

1. *There exists Lyapunov function under form* (1.25) *strictly decreasing along system* (1.7) *trajectories* $\forall\sigma \in \Lambda$.

2. *There exist N symmetric matrices* $P_i$*, for all* $i \in \Lambda$*, satisfying the LMIs :*

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > 0, \ \forall(i,j) \in \Lambda \times \Lambda.$$

3. *There exist N symmetric matrices* $S_i$ *and N matrices* $G_i \in \mathbb{M}_n(\mathbb{R})$ *satisfying the LMIs :*

$$\begin{bmatrix} -S_i & A_i G_i \\ G_i^T A_i^T & S_j - G_i - G_i^T \end{bmatrix} < 0, \quad \forall(i,j) \in \Lambda \times \Lambda.$$

The poly-quadratic function approach represents a generalization of the common quadratic approach. With the constraint $P_i = P$, $\forall i \in \Lambda$, we obtain the LMIs for the quadratic case. This generalization allows to relax the constraints imposed by the quadratic method and to obtain less conservative stability conditions.
This result can be seen as generally as CQLFs, in this case of $P_i = P$ for all $i \in \Lambda$ in (1.25), we obtain CQLF.

#### Euler Approximation

We have the following useful proposition that establishes the connection between a stable continuous-time system and its discrete-time Euler approximating system.

**Proposition 1.3.14.** *[48] Suppose that the continuous-time switched linear system* (1.8) *is stable. Then, the Euler approximating system defined as*

$$x(t+1) = (I + TA_\sigma)x(t) \tag{1.26}$$

*is also stable for sufficiently small* $T$ .

## 1.4 Switched Systems with Uncertain Parameters

The stability problem is very complicated when parameter uncertainty is considered. In this case, the dynamic of each mode is influenced by uncertainty. Until now, few results are concerned with robust stability in the context of switched systems with arbitrary switching law. In the case of discrete time switched linear systems, the uncertain model is described by :

$$x_{t+1} = \tilde{A}_{\sigma_t} x_t \tag{1.27}$$

where $\tilde{A}_i$ represents an uncertain matrix that belongs to the domain $\mathscr{A}_i \subset \mathbb{R}^{n\times n}, \forall i \in \Lambda$. Usually we consider compact $\mathscr{A}_i$ domains, such as

1. norm bounded uncertainties, with

$$\tilde{A}_i \in \mathscr{A}_i = \{A_{i,0} + M_i D_i(t) E_i,\ ||D_i(t)|| \leq 1\}, \tag{1.28}$$

   where the $A_{i,0}$ matrices describe a nominal behavior, $D_i(t)$ is an uncertain time varying matrix and $M_i, E_i,\ \forall i \in \Lambda$ are known matrices ;
2. polytopic uncertainties, with

$$\tilde{A}_i \in \mathscr{A}_i = co\{A_{i,1}, A_{i,2}, \ldots, A_{i,n_i}\}, \forall i \in \Lambda \tag{1.29}$$

   where the $A_{i,1}, A_{i,2}, \ldots, A_{i,n_i}$ matrices are called vertex.

An important problem is to extend the methods given for the classical switched systems case to the case when the switched models are uncertain.

## 1.5 Controllability, Observability, Stabilizability and Detectability

**Controllability** :
In general, a switched linear system is composed of a family of subsystems and a rule that governs the switching among them and is mathematically described by

$$\dot{x} = A_\sigma x + B_\sigma u \tag{1.30}$$

It is obvious that if one subsystem say $(A_1, B_1)$ is controllable, then system (1.30) is both controllable. Hence, we shall investigate the nontrivial situation where each subsystem $(A_i, B_i),\ i \in \Lambda$ is not controllable.
In the sequel, the following definition of controllability for the system (1.30) will be adopted [49] :

**Definition 1.5.1.** *The switched linear system* (1.30) *is said to be (completely) controllable if for any initial state $x_0$ and final state $x_f$ , there exist a time instance $t_f > 0$, a switching signal $\sigma : [0, t_f) \longrightarrow \Lambda$ and an input $u : [0, t_f) \longrightarrow \mathbb{R}^m$ such that $x(0) = x_0$ and $x(t_f) = x_f$.*

For the controllability of switched linear systems, a matrix rank condition was given in [49].

**Proposition 1.5.2.** *[49] if the matrix :* $[B_1, B_2, \ldots, B_m, A_1B_1, \ldots, A_mB_1, \ldots, A_1B_m, \ldots, A_mB_m, A_1^2B_1, A_2A_1, \ldots, A_mA_1B_1, A_1A_2B_1, A_2^2B_1, \ldots, A_mA_2B_1, \ldots, A_1A_mB_m, A_2A_mB_m, A_m^2B_m, \ldots, A_1^{n-1}B_1, A_2A_1^{n-2}B_1, \ldots, A_nA_1^{n-1}B_1, A_1A_2A_1^{n-3}B_1, A_2^2A_1^{n-3}B_1, \ldots, A_1^{n-3}B_1, \ldots, A_1A_m^{n-2}B_m, A_2A_m^{n-2}B_m, \ldots, A_m^{n-1}B_m]$ *has full row rank n, then the switched linear system* (1.30) *is controllable, and vice versa.*

**Remark 1.5.3.** *When $\sigma = 1$, we obtain the following criterion of controllability of linear continuous systems*

$$\dot{x} = Ax + Bu \tag{1.31}$$

*and discrete time*

$$x_{t+1} = Ax_t + Bu_t \tag{1.32}$$

— *The pair $(A,B)$ is controllable if and only if the following $n \times nm$ matrix (called a controllability matrix) has rank $n$ :*

$$[B\, AB\, \ldots\, A^{n-1}B].$$

**Observability** :
Consider a switched linear control system with outputs given by

$$\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma u \\ y_t &= C_\sigma x_t \end{aligned} \tag{1.33}$$

**Definition 1.5.4.** *The switched linear system* (1.33) *is (completely) observable, if there exists a time $t_1 > 0$ and a switching path $\sigma : [0,t_1) \longrightarrow \Lambda$, such that state $x_0$ can be determined from knowledge of the output $y_t$, $t \in [0,t_1]$ and the input $u_t$, $t \in [0,t_1]$.*

**Definition 1.5.5.** *The switched linear system* (1.33) *is (completely) determinable, if there exist a time $t_1 < 0$ and a switching path $\sigma : [t_1,0] \longrightarrow \Lambda$, such that state $x_0$ can be determined from knowledge of the output $y_t$, $t \in [t_1,0]$ and the input $u_t$, $t \in [t_1,0]$.*

**Theorem 1.5.6.** *[49] For switched linear system* (1.33)*, the following statements are equivalent :*

1. *The system is completely observable ;*
2. *The system is completely determinable ; and*
3. $\boldsymbol{O} = \mathbb{R}^n$,
   *where subspace $\boldsymbol{O} = \sum_{j=1}^{\infty} \boldsymbol{O}_j$ is defined by*

$$\boldsymbol{O}_1 = ImC_1^T + \cdots + ImC_m^T,$$

$$\boldsymbol{O}_{j+1} = \Gamma_{A_1^T}\boldsymbol{O}_j + \cdots + \Gamma_{A_m^T}\boldsymbol{O}_j, \qquad j = 1,2,\ldots$$

*where $\Gamma_A \boldsymbol{O} = \boldsymbol{O} + A^T\boldsymbol{O} + \cdots + A^{n-1}\boldsymbol{O}$*

■

**Remark 1.5.7.** *When $\sigma = 1$, we obtain the following criterion of controllability of linear continuous systems*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y_t &= Cx_t \end{aligned} \tag{1.34}$$

*and discrete time*

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t \end{aligned} \tag{1.35}$$

— *The pair $(C,A)$ is observable if and only if the following $pn \times n$ matrix (called an observability matrix) has rank $n$ :*

$$\begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}.$$

We also define the concept of detectability, which is dual to the definition of stabilizability.

**Definition 1.5.8.** *[50](**Detectability**). Switched system* (1.34) *is said to be detectable, if $y_t = 0$ for all $t \in \mathbb{N}$, it holds that $x(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. The system is called undetectable otherwise.*

**Remark 1.5.9.** *The pair $(C,A)$ is said to be detectable if there exists $L \in \mathbb{R}^{p\times n}$ such that $r_\sigma(A+LC) < 1$.*

The idea behind the concept of controllability is rather simple. It deals with answering the following question : for a certain pair $(A,B)$, is it possible to apply a sequence of $u_t$ in order to drive the system from any $x_0$ to a specified final state $x_f$ in a finite time.

**Definition 1.5.10.** *[51][Switching sequence] A switching sequence is a finite set of pairs $\pi := \{(i_1,h_1),\ldots,(i_M,h_M)\}$ where the positive integer $M \leq \infty$ is the length of $\pi$, $i_m \in \{1,\ldots,N\}$ the index of the mth realization, $h_m \in (0,\infty)$ is the time interval of the mth realization, for $m = 1,\ldots,M$. The switching sequence is called infinite if $M < \infty$, otherwise, is called infinite.*

**Definition 1.5.11.** *[51][Stabilizability] Given an infinite switching sequence, system $(A_1,B_1),\ldots,(A_N,B_N)$ is said to be stabilizable under $\pi$ if for any nonzero state $x_0$, there exists $u(t), t \in [0,+\infty)$ such that $\lim_{t\to+\infty} x(t) = 0$.*

**Definition 1.5.12.** *The control $u$ in closed-loop, also called a feedback, or a bouclage is an application $x \mapsto u = \gamma(x)$ defined on the space $\mathscr{M} = \mathbb{R}^m$ with values in the set of admissible controls $\mathscr{U}$.*

In this chapter, we will consider the following three types of output feedback controller :

— Static feedback :

$$u_t = K_\sigma x_t$$

where $K_\sigma$ is the controller gain.

— Output feedback :

$$u_t = K_\sigma y_t.$$

— Observer-based controller :

$$u_t = K_\sigma \hat{x}_t$$

where $\hat{x}_t$ is the estimate $x_t$.

Suppose that for some $K_\sigma$, we apply $u = K_\sigma x$ in System (1.30), yielding

$$\dot{x} = (A_\sigma + B_\sigma K_\sigma)x.$$

According to the theorem above, an adequate choice of $K_\sigma$ (for $A_\sigma, B_\sigma$ and $K_\sigma$ real) would allow us to perform pole placement for the closed loop system $(A_\sigma + B_\sigma K_\sigma)$. For instance we could use state feedback to stabilize an unstable system.

The case in which the state feedback can only change the unstable eigenvalues of the system is of great interest and leads us to the introduction of the concept of stabilizability.

**Definition 1.5.13.** *[52](**Stabilizability**). The pair $(A_\sigma, B_\sigma)$ is said to be stabilizable if there exists $K_\sigma$ such that $r_\sigma(A_\sigma + B_\sigma K_\sigma) < 1$.*

The observer structure considered in the remainder of this thesis corresponds to that of a Luenberger type observer.

**Definition 1.5.14 (Luenberger observer).** *A Luenberger observer $\hat{x}$ of $x$ is a solution of a system of type*

$$\begin{aligned}\hat{x}_{t+1} &= \underbrace{f(\hat{x}, u)}_{(I)} + \underbrace{L(y_t - \hat{y}_t)}_{(II)},\\ \hat{y}_t &= h(\hat{x}_t, u)\end{aligned} \tag{1.36}$$

*The part $(I)$ is that corresponding to the dynamics of the system and part $(II)$is the corrective. By definition of an observer, the matrix $L \in \mathscr{M}_{n,p}$ is such that*

$$\forall x(0), \hat{x}(0) \in \mathbb{R}^n,\ x(t) - \hat{x}(t) \underset{t \to +\infty}{\longrightarrow} 0. \tag{1.37}$$

Before starting an observer design procedure for a dynamic system, it is important and necessary to ensure that the condition of the latter can be estimated from information on input and output. The observability of a system is the property ie the state can be estimated only from the knowledge of the input and output signals. exit. In the case of non-linear systems, the notion of observability is linked to inputs and to the initial conditions.

## 1.6 Linear Matrix Inequalities Based Approach

A large number of automatic problems concerning performance and robustness can be translated in the form of a convex optimization with inequalities constraints. More recently, progress in the field of convex optimization and the introduction of LMIs by Yakubovich in the control of systems a new stage that combines the mathematical possibilities with computer computing power. The reference still unavoidable in LMI is the book [33] where is done a history and a directory of the many problems using LMIs.

**Definition 1.6.1** (Linear Matrix Inequality). *An LMI constraint LMI* [1] *1 is a constraint on a real vector $x \in \mathbb{R}^m$ of the form*

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq 0, \tag{1.38}$$

*where the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n\times n}$ are given, and the symbol of inequality in the broad sense (rep.strict) means that $F$ is semi-defined (defined) positive, ie $u^T F u \geq 0$ for all $u \in \mathbb{R}^n$.*

The LMI constraint is equivalent to a set of $n$ polynomial inequalities constraints (corresponding to the principal minor of $F(x)$).

**Definition 1.6.2** (Bilinear Matrix Inequality). *a constraint BMI is a constraint on $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^r$ which is of the form*

$$F(x,y) := F_{0,0} + \sum_{i=1}^{m} \sum_{j=1}^{r} x_i y_j F_{i,j} \geq 0, \tag{1.39}$$

*with $F_{0,0} \in \mathbb{R}^{n\times n}$ and $F_{i,j} \in \mathbb{R}^{n\times n}$.*

This is the generalization of the notion of LMI. A wide variety of automatic problems can be formulated as BMIs. However, BMIs are not convex, which generates difficulties in their resolution.

## 1.7 Useful Lemmas

In the following, we collect some mathematical inequalities and useful lemmas that have been widely used in the manuscript. Let us first recall the "Finsler's Lemma" and "Schur's complement" which makes it possible to transform certain nonlinear matrix inequalities into LMI.

**Lemma 1.7.1** (Finsler's Lemma). *[53] Let $x \in \mathbb{R}^n, P \in \mathbb{S}^{n\times n}$, and $U \in \mathbb{R}^{m\times n}$ such that $rank(U) = r < n$. The following statements are equivalent :*

1. $x^T P x < 0,\ \forall U x = 0, x \neq 0,$
2. $U^{\perp T} P U^{\perp} < 0.$
3. $\exists \mu \in \mathbb{R} : P - \mu U^T U < 0.$
4. $\exists X \in \mathbb{R}^{n\times m}$ *such that* $P + XU + U^T X^T < 0.$

1. The term "Linear Matrix Inequality" is not consistent with the expression $F(x) \geq 0$ since $F(x)$ ) is not necessarily linear in the variable $x$ but affine. The term AMI for Maternal Affine Inequalities would seem more appropriate.

**Proof 1.** $1 \Rightarrow 2$ : *For all $x$ such that $Ux = 0$ can be written as $x = U^{\perp}y$. Consequently*

$$x^T P x = y^T {U^{\perp}}^T P U^{\perp} y < 0$$

*for all $y \neq 0$ which implies ${U^{\perp}}^T P U^{\perp} < 0$.*
$2 \Rightarrow 1$ : *By multiplied ${U^{\perp}}^T P U^{\perp} < 0$ pre and post by $y^T$, we obtain $y^T {U^{\perp}}^T P U^{\perp} y < 0$ and using the change of variable $x = U^{\perp}y$ then we get $x^T P x < 0$.*
$3 \Rightarrow 2$ : *We multiply 2 pre-and post by ${U^{\perp}}^T$ we obtain ${U^{\perp}}^T P U^{\perp} < 0$.*

$4 \Rightarrow 2$ : *We multiply 4 pre-and post by ${U^{\perp}}^T$ we obtain ${U^{\perp}}^T P U^{\perp} < 0$.*
$3 \Rightarrow 4$ : *We choose $X = -\frac{\mu}{2} U^T$.*
$2 \Rightarrow 3$ : *We have $U$ in the full rank factors $U = U_l U_r$, on define $H = U_r^T (U_r U_r^T)^{-1} (U_l^T U_l)^{-\frac{1}{2}}$ and using the congruence principle $\begin{bmatrix} H & U^{\perp} \end{bmatrix}$*

$$\begin{bmatrix} H^T \\ {U^{\perp}}^T \end{bmatrix} (P - \mu U^T U) \begin{bmatrix} H & U^{\perp} \end{bmatrix} = \begin{bmatrix} H^T P H - \mu I & H^T P U^{\perp} \\ (\star) & {U^{\perp}}^T P U^{\perp} \end{bmatrix} < 0$$

*Since the ${U^{\perp}}^T P U^{\perp}$ is negative definite by assumption, a sufficiently large $\mu$ exists so that the whole matrix is negative definite.*

■

**Lemma 1.7.2** (Schur's Complement ). *Let $Q_1$, $Q_2$ and $Q_3$ be three matrices of appropriate dimensions such as $Q_1 = Q_1^T$ and $Q_3 = Q_3^T$. Then,*

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} < 0 \tag{1.40}$$

*if and only if*

$$Q_3 < 0, \text{ et } Q_1 - Q_2 Q_3^{-1} Q_2^T < 0$$

*or equivalent*

$$Q_1 < 0, \text{ et } Q_3 - Q_2^T Q_1^{-1} Q_2 < 0$$

One of the main ideas we have adopted in the linearization of bilinear terms consists in applying the Young inequalities described in the following lemma.

**Lemma 1.7.3** (Young inequality). *Let $X$ and $Y$ be two matrices of appropriate dimensions. have for every invertible matrix $S$ and scalar $\varepsilon > 0$,*

$$X^T Y + Y^T X \leq \varepsilon X^T S X + \frac{1}{\varepsilon} Y^T S^{-1} Y. \tag{1.41}$$

## 1.8 Stabilization of Switched Linear Systems with Parameter Uncertainties

### 1.8.1 Introduction

The stabilisation of switching system discrete time is a subject that has aroused much and motivated the interest of the automation community [54], [55], [38], [34], [8], [37], [27], [1], [56]. A variety of stabilization conditions have been established for different classes of systems.

### 1.8.2 Problem Formulation

Consider a class of switched discrete-time linear systems described by the following equations :

$$\begin{cases} x_{t+1} = (A_\sigma + \Delta A_\sigma)x_t + (B_\sigma + \Delta B_\sigma)u_t \\ y_t = (C_\sigma + \Delta C_\sigma)\, x_t \end{cases} \tag{1.42}$$

where $x_t \in \Re^n$ is the state vector, $y_t \in \Re^m$ is the output measurement vector, $u_t \in \Re^p$ is the control input vector, and $\sigma : \mathbb{N} \to \Lambda \triangleq \{1, 2, \ldots, N\}$, $t \mapsto \sigma_t$, is a switching rule. If there is no ambiguity, we simply write $\sigma$ instead of $\sigma_t$. $A_\sigma$, $B_\sigma$, and $C_\sigma$, $\sigma \in \Lambda$, are $n \times n$, $n \times m$ and $p \times n$ real matrices, respectively. Assume that

$$[\Delta A_\sigma, \Delta B_\sigma, \Delta C_\sigma] \triangleq M_\sigma D_\sigma [E_{\sigma 1}, E_{\sigma 2}, E_{\sigma 3}], \tag{1.43}$$

where, for each $\sigma \in \Lambda$, the uncertainty $D_\sigma$ satisfies

$$D_\sigma^T D_\sigma \leq I. \tag{1.44}$$

$M_\sigma, E_{\sigma 1}, E_{\sigma 2}, E_{\sigma 3}$ are constant matrices characterizing the structure of the uncertainties. Note that such a model can be used to describe a large class of practical systems, such as cognitive radio networks [57], stepper motors [58], and control of an F-16 aircraft [4]. The references [2], [8], [20] provide a general and accurate modeling framework for many relevant real-world models and processes. In particular, a discrete-time version of the Lipschitz nonlinear switched system modeling the longitudinal dynamics of an F-18 aircraft [11] can be viewed as a switched linear discrete-time system with parameter uncertainties. Indeed, any Lipschitz system can be transformed to a linear system with structured and norm-bounded parameter uncertainties [59].
Throughout this thesis, the following assumption is needed [1] :

**Assumptions 1.8.1.** *The switching function $\sigma$ satisfies the two following items :*

- *i) $\sigma$ is unknown a priori, but it is available in real-time ;*
- *ii) the switching of the observer should coincide exactly with the switching of the system.*

As speculated in [25], assuming an unknown switching rule $\sigma$ can be very useful in many practical applications such as the case when $\sigma$ is computed via complex algorithms by a higher level supervisor or when it is generated by a human operator (for instance the switch of gears in a car).

The observer-based controller we consider in this thesis is under the form :

$$\begin{cases} \hat{x}_{t+1} = A_\sigma \hat{x}_t + L_\sigma (y_t - C_\sigma \hat{x}_t) + B_\sigma u_t \\ \hat{y}_t = C_\sigma \hat{x}_t \\ u_t = K_\sigma \hat{x}_t \end{cases} \tag{1.45}$$

where $\hat{x}_t \in \mathbb{R}^n$ is the estimate of $x_t$, $L_\sigma \in \mathbb{R}^{n\times p}$, $K_\sigma \in \mathbb{R}^{m\times n}$, $\sigma \in \Lambda$, are the observer-based controller gains. Consider the generalized state vector

$$\bar{x}_t = [\hat{x}_t^T \quad e_t^T]^T,$$

where $e_t = \hat{x}_t - x_t$ is the estimation error. Then the closed-loop system resulted from (1.42) and (5.4) can be written as :

$$\bar{x}_{t+1} = \overline{A}_\sigma \bar{x}_t \tag{1.46}$$

where

$$\overline{A}_\sigma = \begin{bmatrix} A_\sigma + B_\sigma K_\sigma + L_\sigma \Delta C_\sigma & -L_\sigma (C_\sigma + \Delta C_\sigma) \\ -(\Delta A_\sigma + \Delta B_\sigma K_\sigma - L_\sigma \Delta C_\sigma) & A_\sigma + \Delta A_\sigma - L_\sigma (C_\sigma + \Delta C_\sigma) \end{bmatrix}. \tag{1.47}$$

The objective is to design output feedback matrices $K_\sigma$ and $L_\sigma$, $\sigma \in \Lambda$, so that the closed-loop system (1.46) is asymptotically stable. Let us define the indicator function

$$\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_N(t)]^T$$

as follows :

$$\xi_i(t) = \begin{cases} 1, & \sigma_t = i\,; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, system (1.46) can be rewritten in the unified form :

$$\bar{x}_{t+1} = \sum_{i=1}^{N} \xi_i(t) \overline{A}_i \bar{x}_t, \tag{1.48}$$

where $\overline{A}_i$ is defined in (1.47), when $\sigma_t = i$.
To analyze stability of the closed-loop system (1.48), we use the switched Lyapunov function defined as :

$$\begin{aligned} V(\bar{x}_t, \xi(t)) &= \bar{x}_t^T \hat{P}(\xi(t)) \bar{x}_t \\ &= \sum_{i=1}^{N} \xi_i(t) \bar{x}_t^T \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \bar{x}_t. \end{aligned} \tag{1.49}$$

Notice that the Lyapunov function (1.49) is well known in the literature, (see for instance [54] and [60]). For shortness we use $\sigma_t = i$ and $\sigma_{t+1} = j$. This means that $\xi_i(t) = 1$ and $\xi_j(t+1) = 1$. Then we get

$$\Delta V \triangleq V(\bar{x}_{t+1}, \xi(t+1)) - V(\bar{x}_t, \xi(t))$$

$$
\begin{aligned}
&= \bar{x}_{t+1}^T \left( \sum_{j=1}^{N} \xi_j(t+1)\hat{P}_j \right) \bar{x}_{t+1} - \bar{x}_t^T \left( \sum_{i=1}^{N} \xi_i(t)\hat{P}_i \right) \bar{x}_t \\
&= \sum_{j=1}^{N} \xi_j(t+1) \left( \sum_{i=1}^{N} \xi_i(t) \begin{bmatrix} \bar{x}_t \\ \bar{x}_{t+1} \end{bmatrix}^T \begin{bmatrix} -\hat{P}_i & 0 \\ 0 & \hat{P}_j \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ \bar{x}_{t+1} \end{bmatrix} \right).
\end{aligned} \tag{1.50}
$$

We have to show that, under suitable conditions, $\Delta V < 0$, which means that the closed-loop system (1.48) is asymptotically stable. Thanks to Finsler's lemma and the principle convexity, inequality $\Delta V < 0$ holds if, and only if, for all $i, j \in \Lambda$, there exists $X_{ij}$ satisfying

$$
P_{ij} + X_{ij}H_i + H_i^T X_{ij}^T < 0 \tag{1.51}
$$

for all $i, j \in \Lambda$, where

$$
P_{ij} = \begin{bmatrix} -\hat{P}_i & 0 \\ 0 & \hat{P}_j \end{bmatrix}, H_i = \begin{bmatrix} \overline{A}_i & -I \end{bmatrix}. \tag{1.52}
$$

**The main question that we address in this thesis is to determine a judicious choice of the matrix $X_{ij}$ so as to eliminate the bilinear terms of the Finsler inequality** (1.51)**. Which also amounts to linearize inequality** (1.51)**.**

### 1.8.3 State of the Art on LMI techniques

This section is devoted to the some LMI techniques for stabilisation problem for both switched linear discrete-time systems and LPV systems, with which we will compare the proposed main contribution of this thesis.
We begin our presentation by recalling the approaches using Finsler's lemma.
Finsler's lemma has been used previously in the control literature mainly in order to eliminate some unlike matrix terms, see e.g. [33]. Switched quadratic Lyapunov functions combined with Finsler's lemma have been used in [34] to get necessary and sufficient LMI conditions for the asymptotic stability issue. However, based on the pioneering work in [34], some attempts using Finsler's lemma-based approach have been presented in [1] and [27]. Unfortunately, the obtained LMI conditions still remain very conservative. Indeed, they are either subjected to strong equality constraints [27], or require particular choices of the decision variables [1].

#### Approach by Li and Liu [1]

In [1], the Finsler lemma has been used with the following parameters

$$
x = \begin{bmatrix} \bar{x}_t \\ \bar{x}_{t+1} \end{bmatrix}, P_{ij} = \begin{bmatrix} -\hat{P}_i & 0 \\ 0 & \hat{P}_j \end{bmatrix}, H_i = \begin{bmatrix} \overline{A}_i & -I \end{bmatrix}, X_i = \begin{bmatrix} \hat{F}_i \\ \hat{G}_i^T \end{bmatrix}, \tag{1.53}
$$

which lead to $\Delta V < 0$ if the following second Finsler inequality :

$$
P_{ij} + X_i H_i + H_i^T X_i^T < 0 \tag{1.54}
$$

holds for all $i, j \in \Lambda$. By substituting (2.1) in (2.2) we get the detailed inequality :

$$\begin{bmatrix} \hat{F}_i\overline{A}_i + \overline{A}_i^T \hat{F}_i^T - \hat{P}_i & -\hat{F}_i + \overline{A}_i^T \hat{G}_i \\ -\hat{F}_i^T + \hat{G}_i^T \overline{A}_i & \hat{P}_j - \hat{G}_i - \hat{G}_i^T \end{bmatrix} < 0, \tag{1.55}$$

instead of [1, Inequality (9)] which is erroneous.
We enounce the following theorem in [1].

**Theorem 1.8.2.** *[1] For the closed-loop switched system* (1.46)*, if there exist $P_i, Q_i \in \mathbb{S}_+^{n\times n}$ and matrices $F_i, G_i, R_i, L_i \in \mathbb{R}^{n\times n}$ $(i = 1,2,\ldots,N)$, satisfying for $\forall i, j \in M = \{1,2,\ldots,N\}$,*

$$\begin{bmatrix} \Pi_{i1} & -L_iC_i & \Pi_{i2} & -L_iC_i \\ (\star) & \Pi_{i3} & 0 & \Pi_{i4} \\ (\star) & (\star) & \Pi_{i5} & 0 \\ (\star) & (\star) & (\star) & \Pi_{j6} \end{bmatrix} < 0, \tag{1.56}$$

*where*

$$\begin{aligned}
\Pi_{i1} &= A_iF_i^T + F_iA_i^T + B_iR_i + R_i^T B_i^T - P_i, \\
\Pi_{i2} &= A_iG_i + B_iR_i - F_i, \\
\Pi_{i3} &= (A_i - L_iC_i) + (A_i - L_iC_i)^T - Q_i, \\
\Pi_{i4} &= A_i - L_iC_i - I, \\
\Pi_{i5} &= P_j - G_i - G_i^T, \\
\Pi_{i6} &= Q_j - 2I,
\end{aligned}$$

*and the static output feedback matrices are taken as*

$$K_i = R_iG_i^{-1}, \ \forall i \in M = \{1,2,\ldots,N\}. \tag{1.57}$$

*Then the closed-loop switched system* (1.46) *is asymptotically stable under an arbitrary switching rule.*

**Approach by Mahmoud and Xia [27]**

In what follows, we introduce the approach presented in the paper [27].
The authors considered a Lyapunov function with a triangular Lyapunov matrix in the stability analysis ie $\hat{P}_j = \begin{bmatrix} \mathscr{P}_{1j} & 0 \\ \mathscr{P}_{2j} & \mathscr{P}_{3j} \end{bmatrix}$. The technique uses several principle congruence and the choice of slack variable $\hat{G}_j = \begin{bmatrix} \Upsilon_s & 0 \\ \Upsilon_o & \Upsilon_c \end{bmatrix}$, and uses the constraint equality $C_j\Upsilon_c^{-1} = E_1C_j$
In [27], the Finsler lemma has been used with the following parameters

$$x = \begin{bmatrix} \overline{x}_t \\ \overline{x}_{t+1} \end{bmatrix}, P_{js} = \begin{bmatrix} -\hat{P}_j & 0 \\ 0 & \hat{P}_s \end{bmatrix}, H_j = \begin{bmatrix} \overline{A}_j & -I \end{bmatrix}, X_j = \begin{bmatrix} 0 \\ \hat{G}_j^T \end{bmatrix}, \tag{1.58}$$

These stabilization conditions are expressed in LMIs form and formulated in the theorem below.

**Theorem 1.8.3.** *[27] The switched system* (1.46) *with* $\Delta A_j = \Delta B_j = \Delta C_j = 0$ *is asymptotically stable. if there exist matrices* $0 < \mathscr{X}_j^T = \mathscr{X}_j, 0 < \mathscr{X}_s^T = \mathscr{X}_s, Y_j, \hat{\mathscr{S}}^T = \hat{\mathscr{S}} > 0, Y_j$ *satisfying the following LMIs, for* $(j,s) \in \Lambda$

$$\begin{bmatrix} \Pi_{11} & -\Lambda_o^T & \Pi_{13} & \Pi_{14} \\ -\Lambda_o - \mathscr{X}_{2s} & \Pi_{22} & 0 & \Pi_{24} \\ (\star) & (\star) & -\mathscr{P}_{1s} & 0 \\ (\star) & (\star) & -\mathscr{P}_{2s} & -\mathscr{P}_{3s} \end{bmatrix} < 0, \tag{1.59}$$

*where*

$$\begin{aligned} \Pi_{11} &= -\Lambda_s - \Lambda_s^T - \mathscr{X}_{1s}, \\ \Pi_{13} &= A_j \Lambda_s^T + B_j Y_j, \\ \Pi_{14} &= A_j \Lambda_o^T + B_j K_j \Lambda_o^T + G_j C_j \lambda_c^T, \\ \Pi_{22} &= -\Lambda_c - \Lambda_c^T - \mathscr{X}_{3s}, \\ \Pi_{24} &= A_j \Lambda_c^T - L_{1j} C_j. \end{aligned}$$

*Moreover the gain matrices are given by*

$$K_j = Y_j \Lambda_s^{-T}, \qquad G_j = L_{1j} E_s^{-1}. \tag{1.60}$$

This next paragraph is devoted to some LMI techniques for stabilisation problem for a class of LPV systems, with which we will compare the proposed main contribution of this thesis.

### Approach by Jetto and Orsini [38] : ITV-based approach for LPV systems

There are other approaches dealing with stabilization of uncertain linear systems in time discrepancy with other types of uncertainties such as LPV type uncertainties, (see for instance [37,38]).

**Remark 1.8.4.** *It should be mentionned that observer-based control design problem for switched linear systems can be seen as an observer-based control design problem for polytopic uncertain linear time-variant systems. Indeed, as stated by Lin and Antsaklis [8], asymptotic stability problem for switched linear systems with arbitrary switching is equivalent to the asymptotic stability problem for polytopic uncertain linear time-variant systems.*

Consider the system governed by the following equation

$$x_{k+1} = A(\rho)x_k + B(\rho)u_k \tag{1.61a}$$

$$y_k = C(\rho)x_k \tag{1.61b}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^m$ is the input control. $A(\rho), B(\rho), C(\rho)$ are assumed to belong to convex polytopes with a finite number of vertices. Indeed, the results related to this type of uncertainty are generally obtained under certain constraints [61], [62], [63], [54]. L. Jetto and V. Orsini have developed in

[38] a new approach to avoid these additional constraints and reduce the number of of the LMIs to be resolved so that they do not depend on the number of variant parameters. To this end, they proposed an approach called the ITV-based approach (matrices defined in time varying intervals). Under the following assumptions :

— the vector $\rho(.) = \left[\rho_1(.),\ldots,\rho_p(.)\right]^T$ takes its values in a compact set;

— the extreme vectors $\rho^- \triangleq \left[\rho_1^-,\ldots,\rho_p^-\right]^T$ and $\rho^+ \triangleq \left[\rho_1^+,\ldots,\rho_p^+\right]$ are known;

— the triplet $(C,\bar{A},B)$ is detectable and observable;

Jetto and Orsini proposed sufficient system stabilization conditions (1.61) with $B(\rho) = B$, $C(\rho) = C$, via the following observer-based control :

$$z(k+1) = (A(\rho(k)) + LC)z(k) + Bu(k) - Ly(k) \tag{1.62}$$

$$u(k) = Kz(k). \tag{1.63}$$

These stabilization conditions are expressed in LMIs form and formulated in the theorem below.

**Theorem 1.8.5** ( [38]). *for $\theta_1$, $\theta_2 \in (0,1]$, there exist matrices $U_1$, $U_2$, and two matrices diagonals $S_1 > 0_n$ and $S_2 > 0_n$ such that the following LMI conditions are feasible*

$$\begin{bmatrix} S_1 & \frac{1}{\theta_1}(\bar{A}S_1 + BU_1)^T \\ (\star) & S_1 \end{bmatrix} > 0, \tag{1.64}$$

$$\begin{bmatrix} S_2 & \frac{1}{\theta_2}(\bar{A}^T S_2 + C^T U_2)^T \\ (\star) & S_2 \end{bmatrix} > 0, \tag{1.65}$$

$$\bar{A}S_1 + BU_1 \succeq 0_n, \bar{A}S_2 + C^T U_2 \succeq 0_n,\ -BU_1 \succeq 0_n, \tag{1.66}$$

$$-\bar{A}S_1 - 2BU_1 - A^- S_1 \preceq 0_n, \tag{1.67}$$

$$-\bar{A}^T S_2 - 2C^T U_2 - A^{-T} S_2 \preceq 0_n, \tag{1.68}$$

*with*

$$\bar{a}_{ij} = \max\{|a_{ij}^-|, |a_{ij}^+|\}$$

*then the system* (1.61) *is asymptotically stabilizable via the observer-based control* (1.62)-(1.63).

### Approach by Heemels et al. [37]

In what follows, we present the approach developed in the paper [37].
The observer based is governed by

$$\hat{x}_{t+1} = A(\hat{\rho}_t)\hat{x}_t + Bu_t + L(\hat{\rho}_t)(y - C\hat{x}_t) \tag{1.69}$$

The estimation error $e_t = x_t - \hat{x}_t$ is governed by

$$e_{t+1} = \mathscr{A}_e(\hat{\rho}_t)e_t + \upsilon_t \tag{1.70}$$

with $\mathscr{A}_e(\hat{\rho}_t) := \sum_{i=1}^N \xi^i(\hat{\rho}_t)\tilde{A}_i$, where $\tilde{A}_i = A_i - L_iC_i$ and

$$\upsilon_t = \underbrace{(A(\rho_t) - A(\hat{\rho}_t))}_{:=\Delta A(\rho_t,\hat{\rho}_t)}x_t \tag{1.71}$$

**Theorem 1.8.6** ( [37]). *Assume that there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $G_i \in \mathbb{R}^{n\times n}$, $F_i \in \mathbb{R}^{n\times m}$, $i = 1,\dots,N$ and a scalar $\sigma_{ev} \geq 1$ satisfying for all $i, j = 1,\dots,N$ the LMIs*

$$\begin{bmatrix} G_i^T + G_i - P_j & 0 & G_iA_i - F_iC_i & G_i \\ (\star) & I & I & 0 \\ (\star) & (\star) & P_i & 0 \\ (\star) & (\star) & (\star) & \sigma_{ev}I \end{bmatrix} > 0 \tag{1.72}$$

*then the error dynamics* (1.70) *with uncertainty set $\Theta$ for $\hat{\rho}$ and $L_i = G_i^{-1}F_i$ is ISS with respect to $\upsilon$ with ISS gain $\gamma(s) = \sigma_{ev}s, s \in \mathbb{R}^+$.*

We now focus on the design of a state feedback. This results in the closed loop

$$x_{t+1} = \mathscr{A}_x(\hat{\rho}_t)x_t + \upsilon_t - BK(\hat{\rho}_t)e_t \tag{1.73}$$

with, as before, $\upsilon_t$ given by (1.71) and

$$\mathscr{A}_x(\hat{\rho}_t) = \sum_{i=1}^{N} \xi^i(\hat{\rho}_t)\underbrace{(A_i + BK_i)}_{A_{BK_i}}.$$

**Theorem 1.8.7** ( [37]). *Assume that there exist symmetric matrices $Y_i \in \mathbb{R}^{n\times n}$, matrices $Z_i \in \mathbb{R}^{n\times n}$, $i = 1,\dots,N$ and a scalar $\sigma_{xv}, \sigma_{xe}, \mu$ with $\mu > 0$ and $\sigma_{xv} \geq 1$ satisfying for all $i, j = 1,\dots,N$ the LMI conditions*

$$\begin{bmatrix} Y_i & 0 & 0 & Y_iA_i^T + Z_i^TB^T & Y_i \\ (\star) & \sigma_{xv}I & 0 & I & 0 \\ (\star) & (\star) & \sigma_{xe}I & -Z_i^TB^T & 0 \\ (\star) & (\star) & (\star) & Y_j & 0 \\ (\star) & (\star) & (\star) & (\star) & I \end{bmatrix} > 0 \tag{1.74}$$

*and for $i = 1,\dots,N$*

$$Y_i \geq \mu I \tag{1.75}$$

*then the closed-loop system* (1.73) *with uncertainty set $\Theta$ for $\hat{\rho}$ and $K_i = Z_iY_i^{-1}$, $i = 1,\dots,N$, is ISS with respect to $e$ and $\upsilon$.*

Next, we will show that the separate design of the observer and a state feedback leads to a stabilizing output-based controller for some nontrivial level of uncertainty $\delta := \sup||\Delta A(\rho,\hat{\rho})||, ||\rho - \hat{\rho}||_\infty \leq \Delta$. The closed-loop system is given by

$$\begin{pmatrix} x_{t+1} \\ e_{t+1} \end{pmatrix} = \begin{bmatrix} A(\rho_t) + BK(\hat{\rho}_t) & -BK(\hat{\rho}_t) \\ A(\rho_t) - A(\hat{\rho}_t) & A(\hat{\rho}_t) - L(\hat{\rho}_t)C \end{bmatrix} \begin{pmatrix} x_t \\ e_t \end{pmatrix}. \tag{1.76}$$

**Theorem 1.8.8.** *[37] Let an observer* (1.36) *that satisfies the hypotheses of Theorem* (1.8.6) *and a state feedback law that satisfies the hypotheses of Theorem* (1.8.7) *be given. Then for any $\max 1 - 1/\sigma_{ev}, 1 - 1/\sigma_{xv} \leq \varepsilon < 1$ and any $0 < \beta \leq (1 - (1-\varepsilon)\sigma_{ev})/\mu^{-2}\sigma_{xe}$ the closed-loop system* (1.76) *is GES with decay factor equal to $\sqrt{\varepsilon}$ for all uncertainties satisfying*

$$||\Delta A(\rho,\hat{\rho})|| \leq \delta := \sqrt{\frac{\beta(1-(1-\varepsilon)\sigma_{x\upsilon})}{\sigma_{e\upsilon} + \beta\sigma_{x\upsilon}}}.$$

**Approach by Zemouche et al. [39]**

This approach is based on the use of a diagonal Lyapunov function and a particular choice of the slack variable in the stability analysis. The authors use several transformations Schur's lemma and an application of the linearization Ibrir lemma [64], to arrive at a inequality which has a structure a diagonal structure equivalent to the two conditions LMIs formulated in the following theorem

**Theorem 1.8.1** ( [39]). *The observer-based controller* (1.69) *stabilizes asymptotically the system* (1.61a) *if the there exist symmetric positive definite matrices $P_j$, matrices $\tilde{K}_j, \tilde{L}_j$ of appropriate dimensions, and positive scalars $\alpha_j, \varepsilon_j,\ j = 1, \dots, N$ such that LMIs* (1.77) *are feasible :*

$$\begin{bmatrix} -P_l & \begin{bmatrix} \alpha_j A_j - B\tilde{K}_j & B\tilde{K}_j \\ 0 & \alpha_j A_j - \tilde{L}_j C \end{bmatrix} & 0 & \alpha_j \begin{bmatrix} I \\ I \end{bmatrix} \\ (\star) & -2\alpha_j I_{2n} + \begin{bmatrix} \varepsilon_j \Delta^2 \Gamma^T \Gamma & 0 \\ 0 & 0 \end{bmatrix} & P_j & 0 \\ (\star) & (\star) & -P_j & 0 \\ (\star) & (\star) & (\star) & -\varepsilon_j I_n \end{bmatrix} < 0. \tag{1.77}$$

## 1.8.4 Conclusion

This chapter gave an overview of some important stability/stabilizability criteria encountered in the context of switched linear systems. In the following chapters we intend to provide robust control methods for stability analysis and control design for switched linear systems.

Chapitre 2

# Observer-based Stabilization of Switching Discrete-time Linear Systems with Parameter Uncertainties

## Sommaire



## 2.1 Introduction

The main objective of this chapter consists in developing new and less conservative LMI synthesis conditions for the observer-based stabilization problem for switched discrete-time linear systems with parameter uncertainties. As mentioned previously, the addressed problem has been investigated in [1] and [27] in the LMI context by using switched Lyapunov functions combined with Finsler's lemma. However, the obtained LMI conditions are conservative because of the particular use of Finsler's lemma in its basic formulation. In addition, the switched Lyapunov matrices used in [1] are assumed to be diagonal, and the LMI synthesis conditions proposed in [27] require additional equality constraints, which turns out to be very conservative. To show the importance of the work we proposed in this paper, we summarize the contributions in the following items :

— A new linearization scheme is proposed, thanks to a convenient use of Finsler's lemma.
— Analytical developments to show the superiority of the proposed design method compared to the existing techniques in the literature are proposed.
— As compared to the Young inequality based approach introduced recently in [35] and [36], the proposed technique in this paper allows to eliminate some bilinear terms arising from the use of the Young relation-based approach, and then leads to less conservative LMI conditions.
— The proposed LMI method is more general than those established in the literature for the same stabilization problem. We essentially demonstrated that the way to select the matrices inferred from Finsler's lemma plays an important role in the feasibility of the obtained LMIs.

## 2.2 Preliminary results

This section is devoted to two LMI techniques reported in the literature, with which we will compare the proposed main contribution of this chapter. On the other hand, it is worth mentioning that these two techniques can be considered as preliminary results because they are not available for the same class of systems. First, we will recall the standard Finsler lemma based approach in [1] that we will correct because of some erroneous mathematical decompositions in [1]. Second, we will generalize the Young inequality based approach introduced in [35, 36] to switched linear systems in the presence of parameter uncertainties.

### 2.2.1 Comments on standard Finsler's lemma based approach [1]

The aim of this subsection is to provide a discussion about the result in [1]. Indeed, the result in [1] is erroneous because of a significant mistake in [1, Inequality (9)], which removes many bilinear terms. Throughout this section, we will give a correct version of the result in [1] and we will provide some comments on the way Finsler's lemma has been used.

In [1], the Finsler lemma has been used with the following parameters

$$x = \begin{bmatrix} \bar{x}_t \\ \bar{x}_{t+1} \end{bmatrix}, P_{ij} = \begin{bmatrix} -\hat{P}_i & 0 \\ 0 & \hat{P}_j \end{bmatrix}, H_i = \begin{bmatrix} \overline{A}_i & -I \end{bmatrix}, X_i = \begin{bmatrix} \hat{F}_i \\ \hat{G}_i^T \end{bmatrix}, \tag{2.1}$$

which leads to $\Delta V < 0$ if the following second Finsler inequality :

$$P_{ij} + X_i H_i + H_i^T X_i^T < 0 \tag{2.2}$$

holds for all $i, j \in \Lambda$. By substituting (2.1) in (2.2) we get the detailed inequality :

$$\begin{bmatrix} \hat{F}_i \overline{A}_i + \overline{A}_i^T \hat{F}_i^T - \hat{P}_i & -\hat{F}_i + \overline{A}_i^T \hat{G}_i \\ -\hat{F}_i^T + \hat{G}_i^T \overline{A}_i & \hat{P}_j - \hat{G}_i - \hat{G}_i^T \end{bmatrix} < 0, \tag{2.3}$$

instead of [1, Inequality (9)] which is erroneous.
Now, using the same matrices as in [1], defined as follows :

$$\hat{F}_i = \mathrm{diag}(F_i, I), \hat{G}_i = \mathrm{diag}(G_i, I), \hat{P}_i = \mathrm{diag}(\hat{P}_i^{11}, \hat{P}_i^{22}),$$

with $F_i = G_i^T$.
To simplify the presentation and to understand more the corrected version of the result, we consider, as in [1], systems without uncertainties, i.e :

$$\Delta A_i = 0, \Delta B_i = 0, \Delta C_i = 0, \ \forall i \in \Lambda.$$

By substituting $\hat{F}_i$, $\hat{G}_i$, and $\hat{P}_i$ in (2.3), and after developing, we get the following detailed inequalities :

$$\begin{bmatrix} \Omega_{11}^i & -F_i L_i C_i & \Omega_{13}^i & 0 \\ (\star) & \Omega_{22}^i & -C_i^T L_i^T G_i & \Omega_{24}^i \\ (\star) & (\star) & \hat{P}_j^{11} - G_i^T - G_i & 0 \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2I \end{bmatrix} < 0, \ i, j \in \Lambda \tag{2.4}$$

where

$$\begin{aligned} \Omega_{11}^i &= -\hat{P}_i^{11} + \mathrm{He}\Big(F_i A_i + F_i B_i K_i\Big), \\ \Omega_{13}^i &= -F_i + A_i^T G_i + K_i^T B_i^T G_i, \\ \Omega_{22}^i &= -\hat{P}_i^{22} + \mathrm{He}\Big(A_i - L_i C_i\Big), \\ \Omega_{24}^i &= -I + A_i^T - C_i^T L_i^T. \end{aligned}$$

Note that, for each $i, \ j \in \Lambda$, inequality (2.4) is a BMI, which cannot be linearized by choosing $F_i = G_i^T$ as in [1]. This difficulty is due to the presence of the coupling $F_i B_i K_i$ and $G_i^T B_i K_i$, which are vanished from [1, Inequality (9)] because of the mistake.
To linearize such a BMI, we use several steps. First, we pre- and post-multiply (2.4) by $\mathrm{diag}\Big(F_i^{-1}, I, G_i^{-T}, I\Big)$, we obtain :

$$\begin{bmatrix} \hat{\Omega}_{11}^i & -L_i C_i & \hat{\Omega}_{13}^i & 0 \\ (\star) & \Omega_{22}^i & -C_i^T L_i^T & \Omega_{24}^i \\ (\star) & (\star) & G_i^{-T} \hat{P}_j^{11} G_i^{-1} - G_i^{-T} - G_i^{-1} & 0 \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2I \end{bmatrix} < 0, \tag{2.5}$$

where

$$\hat{\Omega}^i_{11} = -F_i^{-1}\hat{P}_i^{11}F_i^{-T} + \mathrm{He}\Big(A_iF_i^{-T} + B_iK_iF_i^{-T}\Big),$$
$$\hat{\Omega}^i_{13} = -G_i^{-1} + F_i^{-1}A_i^{\,T} + F_i^{-1}K_i^TB_i^T.$$

There are still two bilinear terms in (2.5), namely $-F_i^{-1}\hat{P}_i^{11}F_i^{-T}$ and $G_i^{-T}\hat{P}_j^{11}G_i^{-1}$. To avoid these terms, we first introduce the following change of variables

$$F_i^{-1} = \tilde{F}_i,\; (\hat{P}_i^{11})^{-1} = \tilde{P}_i^{11},\; G_i^{-1} = \tilde{G}_i,\; R_i = K_iF_i^{-T}.$$

Then, using the inequality

$$-F_i^{-1}\hat{P}_i^{11}F_i^{-T} \leq \tilde{P}_i^{11} - \tilde{F}_i - \tilde{F}_i^T,$$

it follows that (2.5) is fulfilled if the following inequality holds :

$$\begin{bmatrix} \Upsilon^i_{11} & -L_iC_i & \Upsilon^i_{13} & 0 \\ (\star) & \Omega^i_{22} & -C_i^TL_i^T & \Omega^i_{24} \\ (\star) & (\star) & \tilde{G}_i^T\hat{P}_j^{11}\tilde{G}_i - \tilde{G}_i^T - \tilde{G}_i & 0 \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2I \end{bmatrix} < 0, \tag{2.6}$$

where

$$\Upsilon^i_{11} = \tilde{P}_i^{11} + \mathrm{He}\Big(-\tilde{F}_i + A_i\tilde{F}_i^T + B_iR_i\Big),$$
$$\Upsilon^i_{13} = -\tilde{G}_i + \tilde{F}_iA_i^{\,T} + R_i^TB_i^T.$$

Finally, a simple application of Schur lemma [33] on (2.6) leads to the following theorem, which is a corrected version of [1, Theorem 1].

**Theorem 2.2.1.** *Assume that there exist matrices $\tilde{P}_i^{11}, \hat{P}_i^{22} \in \mathbb{S}_+^{n\times n}$, invertible matrices $\tilde{F}_i \in \mathbb{R}^{n\times n}$, $\tilde{G}_i \in \mathbb{R}^{n\times n}$, and arbitrary matrices $R_i \in \mathbb{R}^{m\times n}$, $L_i \in \mathbb{R}^{n\times p}$, $i \in \Lambda$, such that the following LMIs are fulfilled :*

$$\begin{bmatrix} \Upsilon^i_{11} & -L_iC_i & \Upsilon^i_{13} & 0 & 0 \\ (\star) & \Upsilon^i_{22} & -C_i^TL_i^T & -I + A_i^T - C_i^TL_i^T & 0 \\ (\star) & (\star) & -\tilde{G}_i - \tilde{G}_i^T & 0 & \tilde{G}_i^T \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2I & 0 \\ (\star) & (\star) & (\star) & (\star) & -\tilde{P}_j^{11} \end{bmatrix} < 0,\; i,j \in \Lambda \tag{2.7}$$

*with*

$$\Upsilon^i_{11} = \tilde{P}_i^{11} + \mathrm{He}\Big(-\tilde{F}_i + A_i\tilde{F}_i^T + B_iR_i\Big),$$
$$\Upsilon^i_{13} = -\tilde{G}_i + \tilde{F}_iA_i^{\,T} + R_i^TB_i^T,$$
$$\Upsilon^i_{22} = -\hat{P}_i^{22} + \mathrm{He}\Big(A_i - L_iC_i\Big).$$

*Then, the closed-loop system* (1.46) *without uncertainties is asymptotically stable for the observer-based controller gains*

$$K_i = R_i\tilde{F}_i^{-T},\; i \in \Lambda \tag{2.8}$$

*and $L_i$, $i \in \Lambda$, are free solutions of* (2.7).

**Proof 2.** *The rest of the proof is omitted. It is based on the use of the Schur's lemma on* (2.6) *to linearize the remaining bilinear term $G_i^{-T}\hat{P}_j^{11}G_i^{-1}$.*

### 2.2.2 Young's inequality based approach

This section is dedicated to the application of Young's relation based approach introduced in [35, 36] to solve the problem of observer-based stabilization problem of linear uncertain systems. The Young inequality based approach in [35] corresponds to $\hat{F}_i = 0$ and $\hat{G}_i = \mathrm{diag}(G^{11}, G_i^{22})$, with $G_i^{22} = (G_i^{22})^T$. It follows that inequality (2.3) has the same structure than [35, Inequality (11)].
The crucial linearization problem lies in the presence of the isolated term $(C_i + \Delta C_i)^T L_i^T$, while the matrix $L_i$ is elsewhere coupled with the matrix $G_i^{22}$. Then, to retrieve the term $L_i^T G_i^{22}$ and eliminate the isolated term related to $L_i$, in order the make a change of variables, a solution has been proposed in [35, 36], which provides straightforwardly the next theorem valid for linear switched systems (1.42).

**Theorem 2.2.2.** *Assume that for some fixed positive scalars $\varepsilon_i$, $\gamma_i$ and $\mu_i$, $i \in \Lambda$, there exist positive definite matrices* $\mathscr{D}_i \triangleq \begin{bmatrix} \tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ *and* $G_i^{22}$, $\tilde{G}^{11} \in \mathbb{R}^{n\times n}$, $\tilde{K}_i \in \mathbb{R}^{m\times n}$, $\hat{L}_i \in \mathbb{R}^{n\times p}$*, for $i \in \Lambda$, such that the LMI* (2.9) *holds for all $i, j \in \Lambda$*

$$\begin{bmatrix}
-\tilde{P}_i^{11} & -\tilde{P}_i^{12} & \mathbf{(1.3)} & 0 & 0 & 0 & \mathbf{(1.7)} & 0 & \mathbf{(1.9)} & 0 \\
(\star) & -\hat{P}_i^{22} & 0 & \mathbf{(2.4)} & -C_i^T \hat{L}_i^T & 0 & E_{i1}^T & 0 & -E_{i3}^T & 0 \\
(\star) & (\star) & \mathbf{(3.3)} & \tilde{P}_j^{12} & 0 & I & 0 & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & \mathbf{(4.4)} & 0 & 0 & 0 & G_i^{22} M_i & 0 & \hat{L}_i M_i \\
(\star) & (\star) & (\star) & (\star) & -\varepsilon_i G_i^{22} & 0 & 0 & 0 & 0 & \hat{L}_i M_i \\
(\star) & (\star) & (\star) & (\star) & (\star) & -\varepsilon_i^{-1} G_i^{22} & 0 & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\gamma_i I & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\gamma_i^{-1} I & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\mu_i I & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\mu_i^{-1} I
\end{bmatrix} < 0 \tag{2.9}$$

$$\begin{array}{ll}
\mathbf{(1.3)} = (\tilde{G}^{11})^T A_i^T + \tilde{K}_i^T B_i^T, & \mathbf{(1.7)} = -(\tilde{G}^{11})^T E_{i1}^T - \tilde{K}_i^T E_{i1}^T \\
\mathbf{(1.9)} = (\tilde{G}^{11})^T E_{i3}^T, & \mathbf{(2.4)} = A_i^T G_i^{22} - C_i^T \hat{L}_i^T \\
\mathbf{(3.3)} = \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T, & \mathbf{(4.4)} = \hat{P}_j^{22} - 2G_i^{22}
\end{array} \tag{2.10}$$

*Then the closed-loop system* (1.46) *is asymptotically stable with the observer-based controller gains :*

$$K_i = \tilde{K}_i \tilde{G}^{11}, \; L_i = (G_i^{22})^{-1} \hat{L}_i. \tag{2.11}$$

**Proof 3.** *The proof is omitted. It is straightforward and follows exactly the same steps than [35]. The matrix $\mathscr{D}_i$ comes from the change of variable :*

$$\mathscr{D}_i \triangleq \begin{bmatrix} (\tilde{G}^{11})^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix} \begin{bmatrix} \tilde{G}^{11} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix}.$$

Although Theorem 2.2.1 and Theorem 2.2.2 provide solutions to the observer-based stabilization problem for switched linear systems, the obtained LMIs still remain conservative, and then there are some possibilities for improvements from LMI feasibility point of view. This is the objective of the next section, where new and enhanced LMI conditions will be proposed by exploiting the Finsler's lemma in a non-standard way.

## 2.3 Main Results : Enhanced LMI Conditions

In this section, we introduce the main result of this paper, which consists in new LMI conditions to solve the problem of robust observer-based stabilization for switched systems. We will show that thanks to the use of convenient matrices in the Finsler's lemma, we get more general and less conservative LMIs compared to the those presented in the previous section.

### 2.3.1 Introductory developments

We will analyze all the bilinear terms in (2.3) by considering the detailed structures of $\hat{F}_i, \hat{G}_i$, and $\hat{P}_i$ as follows :

$$\hat{F}_i \triangleq \begin{bmatrix} F_i^{11} & F_i^{12} \\ F_i^{21} & F_i^{22} \end{bmatrix}, \ \hat{G}_i \triangleq \begin{bmatrix} G_i^{11} & G_i^{12} \\ G_i^{21} & G_i^{22} \end{bmatrix}, \ \hat{P}_i \triangleq \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix}. \tag{2.12}$$

By substituting (4.15) in (2.3) and after developing, we get the new inequality :

$$\begin{bmatrix} \Omega_{11}^i & \Omega_{12}^i & \Omega_{13}^i & \Omega_{14}^i \\ (\star) & \Omega_{22}^i & \Omega_{23}^i & \Omega_{24}^i \\ (\star) & (\star) & \hat{P}_j^{11} - (G_i^{11})^T - G_i^{11} & \hat{P}_j^{12} - (G_i^{21})^T - G_i^{12} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \tag{2.13}$$

where

$$\begin{aligned}
\Omega_{11}^i &= -\hat{P}_i^{11} + \mathrm{He}\Big(F_i^{11}A_i + F_i^{11}B_iK_i + (F_i^{11} + F_i^{12})L_i\Delta C_i - F_i^{12}\Delta A_i - F_i^{12}\Delta B_iK_i\Big), \\
\Omega_{12}^i &= -\hat{P}_i^{12} + F_i^{12}(A_i + \Delta A_i) - (F_i^{11} + F_i^{12})L_i(C_i + \Delta C_i) + A_i^T(F_i^{21})^T, \\
&\quad + K_i^T B_i^T (F_i^{21})^T + \Delta C_i^T L_i^T (F_i^{21} + F_i^{22})^T - \Delta A_i^T (F_i^{22})^T - K_i^T \Delta B_i^T (F_i^{22})^T, \\
\Omega_{13}^i &= -F_i^{11} + A_i{}^T G_i^{11} + K_i^T B_i^T G_i^{11} - \Delta A_i^T G_i^{21} - K_i^T \Delta B_i^T G_i^{21} + \Delta C_i^T L_i^T (G_i^{11} + G_i^{21}), \\
\Omega_{14}^i &= -F_i^{12} + A_i^T G_i^{12} + K_i^T B_i^T G_i^{12} - \Delta A_i{}^T G_i^{22} - K_i^T \Delta B_i^T G_i^{22} + \Delta C_i^T L_i^T (G_i^{12} + G_i^{22}), \\
\Omega_{22}^i &= -\hat{P}_i^{22} + \mathrm{He}\Big(F_i^{22}(A_i + \Delta A_i) - (F_i^{21} + F_i^{22})L_i(C_i + \Delta C_i)\Big), \\
\Omega_{23}^i &= -F_i^{21} - (C_i + \Delta C_i)^T L_i^T (G_i^{11} + G_i^{21}) + A_i^T G_i^{21} + \Delta A_i^T G_i^{21}, \\
\Omega_{24}^i &= -F_i^{22} + A_i^T G_i^{22} + \Delta A_i^T G_i^{22} - (C_i + \Delta C_i)^T L_i^T (G_i^{12} + G_i^{22}).
\end{aligned}$$

As we can see, the linearization problem is a hard challenge due to the presence of twelve bilinear terms without counting the bilinearities related to the uncertainties. We cannot use change of variables because the matrices $L_i$, $i \in \Lambda$, are coupled with eight different matrices, namely $G_i^{11}$, $G_i^{12}$, $G_i^{22}$, $G_i^{21}$, $F_i^{11}$, $F_i^{12}$, $F_i^{22}$, and $F_i^{21}$. The strategy consists in exploiting the invertibility of the matrices $G_i^{11}$, and $G_i^{22}$, which is a consequence of (2.13). Then we use the congruence principle with convenient matrices. To do this, we first start by linearizing the bilinear terms related to the gains $K_i$. The linearization procedure is presented in the next section.

### 2.3.2 A new linearization procedure

To enhance the clarity of the contributions and to simplify the understanding of the main ideas, the proposed linearization strategy is shared into three steps.

**First step : Linearization with respect to $K_i$**

Since the matrices $G_i^{11}$ and $G_i^{22}$ are necessarily invertible, then using a congruence transformation on (2.13) by pre- and post-multiplying by $\mathrm{diag}\Big((G_i^{11})^{-T}, I, (G_i^{11})^{-T}, I\Big)$, we get

$$\begin{bmatrix} \hat{\Omega}_{11}^i & \hat{\Omega}_{12}^i & \hat{\Omega}_{13}^i & \hat{\Omega}_{14}^i \\ (\star) & -\hat{\Omega}_{22}^i & \hat{\Omega}_{23}^i & \hat{\Omega}_{24}^i \\ (\star) & (\star) & \hat{\Omega}_{33}^{ij} & \hat{\Omega}_{34}^{ij} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \tag{2.14}$$

where

$$\begin{aligned}
\hat{\Omega}_{11}^i =& -(G_i^{11})^{-T}\hat{P}_i^{11}(G_i^{11})^{-1} + \mathrm{He}\Big((G_i^{11})^{-T}F_i^{11}A_i(G_i^{11})^{-1} + (G_i^{11})^{-T}F_i^{11}B_iK_i(G_i^{11})^{-1} \\
& + ((G_i^{11})^{-T}F_i^{11} + (G_i^{11})^{-T}F_i^{12})L_i\Delta C_i(G_i^{11})^{-1} - (G_i^{11})^{-T}F_i^{12}\Delta A_i(G_i^{11})^{-1} \\
& - (G_i^{11})^{-T}F_i^{12}\Delta B_iK_i(G_i^{11})^{-1}\Big), \\
\hat{\Omega}_{12}^i =& -(G_i^{11})^{-T}\hat{P}_i^{12} + (G_i^{11})^{-T}F_i^{12}(A_i + \Delta A_i) - ((G_i^{11})^{-T}F_i^{11} + (G_i^{11})^{-T}F_i^{12})L_i(C_i + \Delta C_i) \\
& + (G_i^{11})^{-T}A_i^T(F_i^{21})^T + (G_i^{11})^{-T}K_i^TB_i^T(F_i^{21})^T + (G_i^{11})^{-T}\Delta C_i^TL_i^T(F_i^{21} + F_i^{22})^T \\
& - (G_i^{11})^{-T}\Delta A_i^T(F_i^{22})^T - (G_i^{11})^{-T}K_i^T\Delta B_i^T(F_i^{22})^T, \\
\hat{\Omega}_{13}^i =& -(G_i^{11})^{-T}F_i^{11}(G_i^{11})^{-1} + (G_i^{11})^{-T}A_i{}^T + (G_i^{11})^{-T}K_i^TB_i^T - (G_i^{11})^{-T}\Delta A_i^TG_i^{21}(G_i^{11})^{-1} \\
& - (G_i^{11})^{-T}K_i^T\Delta B_i^TG_i^{21}(G_i^{11})^{-1} + (G_i^{11})^{-T}\Delta C_i^TL_i^T(I + G_i^{21}(G_i^{11})^{-1}), \\
\hat{\Omega}_{14}^i =& -(G_i^{11})^{-T}F_i^{12} + (G_i^{11})^{-T}A_i^TG_i^{12} + (G_i^{11})^{-T}K_i^TB_i^TG_i^{12} - (G_i^{11})^{-T}\Delta A_i{}^TG_i^{22} \\
& - (G_i^{11})^{-T}K_i^T\Delta B_i^TG_i^{22} + (G_i^{11})^{-T}\Delta C_i^TL_i^T(G_i^{12} + G_i^{22}), \\
\hat{\Omega}_{22}^i =& \Omega_{22}^i, \\
\hat{\Omega}_{23}^i =& -F_i^{21}(G_i^{11})^{-1} - (C_i + \Delta C_i)^TL_i^T(I + G_i^{21}(G_i^{11})^{-1}) + A_i^TG_i^{21}(G_i^{11})^{-1} + \Delta A_i^TG_i^{21}(G_i^{11})^{-1}, \\
\hat{\Omega}_{24}^i =& \Omega_{24}^i, \\
\hat{\Omega}_{33}^{ij} =& (G_i^{11})^{-T}\hat{P}_j^{11}(G_i^{11})^{-1} - (G_i^{11})^{-1} - (G_i^{11})^{-T} \\
\hat{\Omega}_{34}^{ij} =& (G_i^{11})^{-T}\hat{P}_j^{12} - (G_i^{11})^{-T}G_i^{12} - (G_i^{11})^{-T}(G_i^{21})^T.
\end{aligned}$$

Then, we can be see, there are two "similar" bilinear terms in inequality (2.14), namely, $(G_i^{11})^{-T}\hat{P}_i^{11}(G_i^{11})^{-1}$ and $(G_i^{11})^{-T}\hat{P}_j^{11}(G_i^{11})^{-T}$, in the expressions of $\hat{\Omega}_{11}^i$, and $\hat{\Omega}_{33}^{ij}$, respectively. By choosing $G_i^{11} = G^{11}$ for all $i$, then we can introduce a suitable change of variables. On the other hand, in order to avoid some bilinear terms containing $K_i$, we focus on the case where $F_i^{11} = 0$. To sum up, we introduce the convenient change of variables :

$$(G^{11})^{-1} \triangleq \tilde{G}^{11}, \quad \tilde{K}_i \triangleq K_i\tilde{G}^{11}, \; (\tilde{G}^{11})^T\hat{P}_i^{11}\tilde{G}^{11} \triangleq \tilde{P}_i^{11}, \; (\tilde{G}^{11})^T\hat{P}_i^{12} \triangleq \tilde{P}_i^{12}.$$

Therefore, inequality (2.14) becomes :

$$\begin{bmatrix} \tilde{\Omega}^i_{11} & \tilde{\Omega}^i_{12} & \tilde{\Omega}^i_{13} & \tilde{\Omega}^i_{14} \\ (\star) & \tilde{\Omega}^i_{22} & \tilde{\Omega}^i_{23} & \tilde{\Omega}^i_{24} \\ (\star) & (\star) & \tilde{P}^{11}_j - \tilde{G}^{11} - (\tilde{G}^{11})^T & \tilde{P}^{12}_j - \tilde{G}^{11} G^{12}_i - \tilde{G}^{11}(G^{21}_i)^T \\ (\star) & (\star) & (\star) & \hat{P}^{22}_j - G^{22}_i - (G^{22}_i)^T \end{bmatrix} < 0, \tag{2.15}$$

where

$$\begin{aligned}
\tilde{\Omega}^i_{11} &= -\tilde{P}^{11}_i + \mathrm{He}\Big((\tilde{G}^{11})^T F^{12}_i L_i \Delta C_i \tilde{G}^{11} - (\tilde{G}^{11})^T F^{12}_i \Delta A_i (\tilde{G}^{11}) - (\tilde{G}^{11})^T F^{12}_i \Delta B_i \tilde{K}_i\Big),\\
\tilde{\Omega}^i_{12} &= -\tilde{P}^{12}_i + (\tilde{G}^{11})^T F^{12}_i (A_i + \Delta A_i) - (\tilde{G}^{11})^T F^{12}_i L_i (C_i + \Delta C_i) + (\tilde{G}^{11})^T A^T_i (F^{21}_i)^T + \tilde{K}^T_i B^T_i (F^{21}_i)^T\\
&\quad + (\tilde{G}^{11})^T \Delta C^T_i L^T_i (F^{21}_i + F^{22}_i)^T - (\tilde{G}^{11})^T \Delta A^T_i (F^{22}_i)^T - \tilde{K}^T_i \Delta B^T_i (F^{22}_i)^T,\\
\tilde{\Omega}^i_{13} &= (\tilde{G}^{11})^T A^T_i + \tilde{K}^T_i B^T_i - (\tilde{G}^{11})^T \Delta A^T_i G^{21}_i \tilde{G}^{11} - \tilde{K}^T_i \Delta B^T_i G^{21}_i \tilde{G}^{11} + (\tilde{G}^{11})^T \Delta C^T_i L^T_i (I + G^{21}_i \tilde{G}^{11}),\\
\tilde{\Omega}^i_{14} &= -(\tilde{G}^{11})^T F^{12}_i + (\tilde{G}^{11})^T A^T_i G^{12}_i + \tilde{K}^T_i B^T_i G^{12}_i - (\tilde{G}^{11})^T \Delta A_i{}^T G^{22}_i - \tilde{K}^T_i \Delta B^T_i G^{22}_i\\
&\quad + (\tilde{G}^{11})^T \Delta C^T_i L^T_i (G^{12}_i + G^{22}_i),\\
\tilde{\Omega}^i_{22} &= -\hat{P}^{22}_i + \mathrm{He}\Big(F^{22}_i (A_i + \Delta A_i) - (F^{21}_i + F^{22}_i) L_i (C_i + \Delta C_i)\Big),\\
\tilde{\Omega}^i_{23} &= -F^{21}_i \tilde{G}^{11} - (C_i + \Delta C_i)^T L^T_i (I + G^{21}_i \tilde{G}^{11}) + A^T_i G^{21}_i \tilde{G}^{11} + \Delta A^T_i G^{21}_i \tilde{G}^{11},\\
\tilde{\Omega}^i_{24} &= -F^{22}_i + A^T_i G^{22}_i + \Delta A^T_i G^{22}_i - (C_i + \Delta C_i)^T L^T_i (G^{12}_i + G^{22}_i).
\end{aligned}$$

Inequality (2.15) is still a BMI with respect to $\tilde{K}_i$, even in the uncertainty free case. This is due to their coupling with the matrices $F^{21}_i$ and $G^{12}_i$. Then, these bilinear terms vanish if $G^{12}_i = F^{21}_i = 0$. Then, in such a case, inequality (2.15) is equivalent to the following one :

$$\begin{bmatrix} \Theta^i_{11} & \Theta^i_{12} & \Theta^i_{13} & \Theta^i_{14} \\ (\star) & \Theta^i_{22} & \Theta^i_{23} & \Theta^i_{24} \\ (\star) & (\star) & \tilde{P}^{11}_j - \tilde{G}^{11} - (\tilde{G}^{11})^T & \tilde{P}^{12}_j - \tilde{G}^{11}(G^{21}_i)^T \\ (\star) & (\star) & (\star) & \hat{P}^{22}_j - G^{22}_i - (G^{22}_i)^T \end{bmatrix} < 0, \tag{2.16}$$

where

$$\begin{aligned}
\Theta^i_{11} &= -\tilde{P}^{11}_i + \mathrm{He}\Big((\tilde{G}^{11})^T F^{12}_i L_i \Delta C_i \tilde{G}^{11} - (\tilde{G}^{11})^T F^{12}_i \Delta A_i (\tilde{G}^{11}) - (\tilde{G}^{11})^T F^{12}_i \Delta B_i \tilde{K}_i\Big),\\
\Theta^i_{12} &= -\tilde{P}^{12}_i + (\tilde{G}^{11})^T F^{12}_i (A_i + \Delta A_i) - (\tilde{G}^{11})^T F^{12}_i L_i (C_i + \Delta C_i) + (\tilde{G}^{11})^T \Delta C^T_i L^T_i (F^{22}_i)^T\\
&\quad - (\tilde{G}^{11})^T \Delta A^T_i (F^{22}_i)^T - \tilde{K}^T_i \Delta B^T_i (F^{22}_i)^T,\\
\Theta^i_{13} &= (\tilde{G}^{11})^T A^T_i + \tilde{K}^T_i B^T_i - (\tilde{G}^{11})^T \Delta A^T_i G^{21}_i \tilde{G}^{11} - \tilde{K}^T_i \Delta B^T_i G^{21}_i \tilde{G}^{11} + (\tilde{G}^{11})^T \Delta C^T_i L^T_i (I + G^{21}_i \tilde{G}^{11}),\\
\Theta^i_{14} &= -(\tilde{G}^{11})^T F^{12}_i - (\tilde{G}^{11})^T \Delta A_i{}^T G^{22}_i - \tilde{K}^T_i \Delta B^T_i G^{22}_i + (\tilde{G}^{11})^T \Delta C^T_i L^T_i G^{22}_i,\\
\Theta^i_{22} &= -\hat{P}^{22}_i + \mathrm{He}\Big(F^{22}_i (A_i + \Delta A_i) - F^{22}_i L_i (C_i + \Delta C_i)\Big),\\
\Theta^i_{23} &= -(C_i + \Delta C_i)^T L^T_i (I + G^{21}_i \tilde{G}^{11}) + A^T_i G^{21}_i \tilde{G}^{11} + \Delta A^T_i G^{21}_i \tilde{G}^{11},\\
\Theta^i_{24} &= -F^{22}_i + A^T_i G^{22}_i + \Delta A^T_i G^{22}_i - (C_i + \Delta C_i)^T L^T_i G^{22}_i.
\end{aligned}$$

Now that the BMI (2.13) is linearized with respect to the controller matrices $\tilde{K}_i$, we will proceed to the linearization with respect to the observer gains $L_i$. This is the aim of the next linearization step.

**Second step : Linearization of (2.16) with respect to $L_i$**

Throughout this step, we aim to linearize all the bilinear terms related to the observer gains $L_i$, namely the terms $(\tilde{G}^{11})^T F_i^{12} L_i C_i$, $(I+G_i^{21}\tilde{G}^{11})^T L_i C_i$, $F_i^{22} L_i C_i$, and $(G_i^{22})^T L_i C_i$. The other terms containing the uncertainties will be handled in the third linearization step. To avoid all the previous bilinear terms, the strategy consists in taking

$$(\tilde{G}^{11})^T F_i^{12} = I + G_i^{21}\tilde{G}^{11} = (G_i^{22})^T \text{ and } F_i^{22} = 0,$$

these identities lead to

$$F_i^{12} = (G^{11})^T (G_i^{22})^T,\, G_i^{21} = G_i^{22} G^{11} - G^{11} \text{ and } F_i^{22} = 0,$$

which means that the matrices $\hat{F}_i$ and $\hat{G}_i$ have the following structures :

$$\hat{F}_i = \begin{bmatrix} 0 & (G^{11})^T (G_i^{22})^T \\ 0 & 0 \end{bmatrix},\ \hat{G}_i = \begin{bmatrix} G^{11} & 0 \\ G_i^{22} G^{11} - G^{11} & G_i^{22} \end{bmatrix}. \tag{2.17}$$

It follows that the following change of variable

$$\hat{L}_i = (G_i^{22})^T L_i$$

is possible.
By substituting (2.17) in (2.16) we get the new inequality :

$$\begin{bmatrix} \hat{\Theta}_{11}^i & \hat{\Theta}_{12}^i & \hat{\Theta}_{13}^i & \hat{\Theta}_{14}^i \\ (\star) & -\hat{P}_i^{22} & \hat{\Theta}_{23}^i & \hat{\Theta}_{24}^i \\ (\star) & (\star) & \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T & \tilde{P}_j^{12} + I - (G_i^{22})^T \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \tag{2.18}$$

where

$$\begin{aligned}
\hat{\Theta}_{11}^i =& -\tilde{P}_i^{11} + \mathrm{He}\left(\hat{L}_i \Delta C_i \tilde{G}^{11} - (G_i^{22})^T \Delta A_i \tilde{G}_i^{11} - (G_i^{22})^T \Delta B_i \tilde{K}_i\right), \\
\hat{\Theta}_{12}^i =& -\tilde{P}_i^{12} + (G_i^{22})^T (A_i + \Delta A_i) - \hat{L}_i (C_i + \Delta C_i), \\
\hat{\Theta}_{13}^i =& (\tilde{G}^{11})^T A_i{}^T + \tilde{K}_i^T B_i^T - (\tilde{G}^{11})^T \Delta A_i{}^T (G_i^{22} - I) - \tilde{K}_i^T \Delta B_i^T (G_i^{22} - I) + (\tilde{G}^{11})^T \Delta C_i^T \hat{L}_i^T, \\
\hat{\Theta}_{14}^i =& -(G_i^{22})^T - (\tilde{G}^{11})^T \Delta A_i{}^T G_i^{22} - \tilde{K}_i^T \Delta B_i^T G_i^{22} + (\tilde{G}^{11})^T \Delta C_i^T \hat{L}_i^T, \\
\hat{\Theta}_{23}^i =& -(C_i + \Delta C_i)^T \hat{L}_i^T + (A_i^T + \Delta A_i^T)(G_i^{22} - I), \\
\hat{\Theta}_{24}^i =& A_i^T G_i^{22} + \Delta A_i^T G_i^{22} - (C_i + \Delta C_i)^T \hat{L}_i^T.
\end{aligned} \tag{2.19}$$

**Third step : Full linearization**

This step is classic and well-known in the literature, see in particular [35]. By developing $\Delta A_i$, $\Delta B_i$, and $\Delta C_i$, we can rewrite (2.18) in the following convenient from :

$$\Xi^{ij} + \mathrm{He}\left(Z_{i1}^T D_i^T Z_{i2} + Z_{i3}^T D_i^T Z_{i4}\right) < 0, \tag{2.20}$$

where

$$Z_{i1} = \begin{bmatrix}(\Theta_{15}^i)^T & E_{i1} & 0 & 0\end{bmatrix}^T, Z_{i3} = \begin{bmatrix}E_{i3}\tilde{G}^{11} & -E_{i3} & 0 & 0\end{bmatrix}^T,$$
$$Z_{i2} = \begin{bmatrix}M_i^T G_i^{22} & 0 & M_i^T(G_i^{22}-I) & M_i^T G_i^{22}\end{bmatrix},$$
$$Z_{i4} = \begin{bmatrix}M_i^T \hat{L}_i^T & 0 & M_i^T \hat{L}_i^T & M_i^T \hat{L}_i^T\end{bmatrix},$$
$$\Xi^{ij} = \begin{bmatrix}-\tilde{P}_i^{11} & \tilde{\Theta}_{12}^i & \tilde{\Theta}_{13}^i & -(G_i^{22})^T \\ (\star) & -\hat{P}_i^{22} & \tilde{\Theta}_{23}^i & \tilde{\Theta}_{24}^i \\ (\star) & (\star) & \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T & \tilde{P}_j^{12} + I - (G_i^{22})^T \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T\end{bmatrix},$$

$$\begin{aligned}
\tilde{\Theta}_{12}^i &= -\tilde{P}_i^{12} + (G_i^{22})^T A_i - \hat{L}_i C_i,\\
\tilde{\Theta}_{13}^i &= (\tilde{G}^{11})^T A_i^T + \tilde{K}_i^T B_i^T,\\
\tilde{\Theta}_{15}^i &= -(\tilde{G}^{11})^T E_{i1}^T - \tilde{K}_i^T E_{i2}^T,\\
\tilde{\Theta}_{23}^i &= -C_i^T \hat{L}_i^T + A_i^T(G_i^{22} - I),\\
\tilde{\Theta}_{24}^i &= A_i^T G_i^{22} - C_i^T \hat{L}_i^T,\\
\Theta_{15}^i &= -(\tilde{G}^{11})^T E_{i1}^T - \tilde{K}_i^T E_{i2}^T.
\end{aligned}$$

Using the Young inequality [33] and the fact that $D_\sigma^T D_\sigma \leq I$, we deduce that inequality (4.32) is fulfilled if the following one holds :

$$\Xi^{ij} + \alpha_i^{-1} Z_{i1}^T Z_{i1} + \alpha_i Z_{i2}^T Z_{i2} + \lambda_i^{-1} Z_{i3}^T Z_{i3} + \lambda_i Z_{i4}^T Z_{i4} < 0, \tag{2.21}$$

where $\alpha_i$ and $\lambda_i$ are some positive scalars. Now, it remains to use Schur lemma on the right hand side of (2.21) to get an LMI. This LMI is stated in the next theorem.

**Theorem 2.3.1.** *Assume that there exist positive definite matrices*

$$\mathscr{D}_i \triangleq \begin{bmatrix}\tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22}\end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

*invertible matrices $G_i^{22}$ and $\tilde{G}^{11} \in \mathbb{R}^{n\times n}$, and matrices $\tilde{K}_i \in \mathbb{R}^{m\times n}$,$\hat{L}_i \in \mathbb{R}^{n\times p}$, for $i \in \Lambda$, such that the LMI* (2.22) *holds for some positive constants $\alpha_i$ and $\lambda_i$, for all $i, j \in \Lambda$.*
*Then the closed-loop system* (1.46) *is asymptotically stable with the observer-based controller gains :*

$$K_i = \tilde{K}_i \tilde{G}^{11},\ L_i = (G_i^{22})^{-T} \hat{L}_i,\ i \in \Lambda. \tag{2.23}$$

**Proof 4.** *The proof is done in the three previous linearization steps. It remains to apply the Schur lemma on the right hand side of* (2.21) *to get the LMI* (2.22)*. The matrices $\mathscr{D}_i$, for $i \in \Lambda$, come from the change of variable :*

$$\mathscr{D}_i \triangleq \begin{bmatrix}\tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22}\end{bmatrix} = \begin{bmatrix}(\tilde{G}^{11})^T & 0 \\ 0 & I\end{bmatrix} \begin{bmatrix}\hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22}\end{bmatrix} \begin{bmatrix}\tilde{G}^{11} & 0 \\ 0 & I\end{bmatrix}, \quad \forall i \in \Lambda.$$

$$\begin{bmatrix} -\tilde{P}_i^{11} & \mathbf{(1.2)} & \mathbf{(1.3)} & -(G_i^{22})^T & \mathbf{(1.5)} & (G_i^{22})^T M_i & (\tilde{G}^{11})^T E_{i3}^T & \hat{L}_i M_i \\ (\star) & -\hat{P}_i^{22} & \mathbf{(2.3)} & \mathbf{(2.4)} & E_{i1}^T & 0 & -E_{i3}^T & 0 \\ (\star) & (\star) & \mathbf{(3.3)} & \mathbf{(3.4)} & 0 & \mathbf{(3.6)} & 0 & \hat{L}_i M_i \\ (\star) & (\star) & (\star) & \mathbf{(4.4)} & 0 & (G_i^{22})^T M_i & 0 & \hat{L}_i M_i \\ (\star) & (\star) & (\star) & (\star) & -\alpha_i I & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\alpha_i^{-1} I & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\lambda_i I & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\lambda_i^{-1} I \end{bmatrix} < 0 \qquad (2.22)$$

$$\begin{aligned} &\mathbf{(1.2)} = -\tilde{P}_i^{12} + (G_i^{22})^T A_i - \hat{L}_i C_i, && \mathbf{(1.3)} = (\tilde{G}^{11})^T A_i^T + \tilde{K}_i^T B_i^T \\ &\mathbf{(1.5)} = -(\tilde{G}^{11})^T E_{i1}^T - \tilde{K}_i^T E_{i2}^T, && \mathbf{(2.3)} = -C_i^T \hat{L}_i^T + A_i^T (G_i^{22} - I) \\ &\mathbf{(3.3)} = \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T, && \mathbf{(2.4)} = A_i^T G_i^{22} - C_i^T \hat{L}_i^T \\ &\mathbf{(3.4)} = \tilde{P}_j^{12} + I - (G_i^{22})^T, && \mathbf{(4.4)} = \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \\ &\mathbf{(3.6)} = (G_i^{22} - I)^T M_i, && \end{aligned}$$

## 2.4 Numerical Design Aspects and Some Comments

### 2.4.1 On the Optimization of the uncertainty bounds

Notice that the scalars $\alpha_i$ and $\lambda_i$ are to be fixed *a priori* to render linear the condition (2.22). Moreover, in order to overcome this drawback and to maximize the uncertainty bounds tolerated by (2.22), the uncertainties are replaced by the more general form :

$$\Delta A_i = M_A^i D_i(t) E_{i1}, \Delta B_i = M_B^i F_i(t) E_{i2}, \Delta C_i = M_C^i H_i(t) E_{i1}. \qquad (2.24)$$

The uncertain matrices, containing the uncertainty bounds, are replaced by :

$$D_i^T(t) D_i(t) \leq \delta_i^2 I, F_i^T(t) F_i(t) \leq \beta_i^2 I, H_i^T(t) H_i(t) \leq \gamma_i^2 I \qquad (2.25)$$

instead of (1.44). This formulation is often used in decentralized stabilization problem of interconnected systems. The objective consists in maximizing the bounds $\delta_i, \beta_i$, and $\gamma_i$. Such a strategy leads to an LMI without *a priori* choice of the scalars $\alpha_i$ and $\lambda_i$.
Under these new considerations, inequality (4.32) becomes

$$\Xi^{ij} + \mathrm{He}\Big(Z_{i1}^T D_i^T Z_{i2} + Z_{i3}^T F_i^T Z_{i4} + Z_{i5}^T H_i^T Z_{i6}\Big) < 0, \qquad (2.26)$$

where

$$\begin{aligned} Z_{i1} &= \begin{bmatrix} -E_{i1}\tilde{G}^{11} & E_{i1} & 0 & 0 \end{bmatrix}^T, Z_{i2} = {M_A^i}^T \begin{bmatrix} G_i^{22} & 0 & (G_i^{22} - I) & G_i^{22} \end{bmatrix} \\ Z_{i3} &= \begin{bmatrix} -E_{i2}\tilde{K}_i & 0 & 0 & 0 \end{bmatrix}^T, Z_{i4} = {M_B^i}^T \begin{bmatrix} G_i^{22} & 0 & (G_i^{22} - I) & G_i^{22} \end{bmatrix} \\ Z_{i5} &= \begin{bmatrix} E_{i3}\tilde{G}^{11} & -E_{i3} & 0 & 0 \end{bmatrix}^T, Z_{i6} = \begin{bmatrix} {M_C^i}^T \hat{L}_i^T & 0 & {M_C^i}^T \hat{L}_i^T & {M_C^i}^T \hat{L}_i^T \end{bmatrix}. \end{aligned}$$

Using the classical Young's relation and taking into account (2.25), we deduce that (2.26) holds if the following one is fulfilled :

$$\Xi^{ij} + \left(a_i\delta_i^2 Z_{i1}^T Z_{i1} + a_i^{-1} Z_{i2}^T Z_{i2} + b_i\beta_i^2 Z_{i3}^T Z_{i3} + b_i^{-1} Z_{i4}^T Z_{i4} + c_i\gamma_i^2 Z_{i5}^T Z_{i5} + c_i^{-1} Z_{i6}^T Z_{i6}\right) < 0.$$

Finally, with the change of variables

$$\xi_i = \frac{1}{a_i\delta_i^2},\ \nu_i = \frac{1}{b_i\beta_i^2},\ \kappa_i = \frac{1}{c_i\gamma_i^2}$$

and by using the Schur lemma, we get the following enhanced version of Theorem 2.3.1.

**Theorem 2.4.1.** *Assume that there exist positive definite matrices* $\mathscr{D}_i \in \mathbb{R}^{2n\times 2n}$ *and* $G_i^{22},\ \tilde{G}^{11} \in \mathbb{R}^{n\times n}, \tilde{K}_i \in \mathbb{R}^{m\times n}, \hat{L}_i \in \mathbb{R}^{n\times p}$*, for* $i \in \Lambda$*, such that the following convex optimization problem holds :*

$$\min \mathrm{Trace}(\Gamma_i) \quad \text{subject to}$$

$$\begin{bmatrix} \Xi^{ij} & \begin{bmatrix} Z_{i1}^T & Z_{i2} & Z_{i3} & Z_{i4} & Z_{i5} & Z_{i6} \end{bmatrix} \\ (\star) & -\Gamma_i \end{bmatrix} < 0,\ i, j \in \Lambda \qquad (2.27)$$

$$\Gamma_i = \mathrm{diag}\left\{\xi_i I,\ a_i I,\ \nu_i I,\ b_i I,\ \kappa_i I,\ c_i I\right\}.$$

*Then the closed-loop system* (1.46) *is asymptotically stable with the observer-based controller gains :*

$$K_i = \tilde{K}_i \tilde{G}^{11},\ L_i = (G_i^{22})^{-T}\hat{L}_i, \qquad (2.28)$$

*for all* $\delta_i$, $\beta_i$, $\gamma_i$, $i \in \Lambda$*, satisfying*

$$\delta_i \le \frac{1}{\sqrt{a_i\xi_i}},\ \beta_i \le \frac{1}{\sqrt{b_i\nu_i}},\ \gamma_i \le \frac{1}{\sqrt{c_i\kappa_i}}.$$

## 2.4.2 Some comments

This section is dedicated to some constructive remarks, which may be helpful and useful for any application of the proposed enhanced LMI methodology.

### On the a priori choice of some scalar variables

Conditions (2.22) and (2.9) are LMIs if the positive scalars $\alpha_i, \lambda_i, \varepsilon_i, \gamma_i$ and $\mu_i$ are fixed a priori. Then to get LMIs we need to use some techniques providing these a priori choices of the scalar variables. One of the famous techniques can be found in [35, Remark 3], namely the gridding method. It is worth noticing that this alternative solution will be used in case where the bounds of the uncertainties are fixed and not to be maximized. Indeed, this latter may be handled by using Theorem 2.4.1.

**Comparison with the Young relation based approach**

As compared to the Young inequality based approach, the judicious choice of the slack variable in (4.15) coming from Finsler's lemma (especially the triangular structure of $\hat{G}_i$), has eliminated the isolated term $(C_i + \Delta C_i)^T L_i^T$ arising from the diagonal structure of $\hat{G}_i$ used in the Young inequality based approach. Hence, the proposed enhanced LMI design methodology based on a convenient use of Finsler's inequality allows to avoid all these bilinear terms without using Young's inequality several times, which leads to conservative LMI conditions like in the Young relation based approach [35].

**Handling the uncertainties to get full linearization**

It should be mentioned that Young's relation is almost unavoidable when dealing with uncertainties satisfying equations (1.43)-(1.44). This is due mainly to their structure and the condition (1.44), namely $D_\sigma^T(t)D_\sigma(t) \leq I$. This technique is standard and well known in the literature. We can proceed otherwise if we are dealing with other uncertainties, such as LPV uncertainties. These latter can be handled more easily, thanks to the use of the convexity principle. This LPV reformulation of the uncertainties is not suitable in the context of the paper dealing with switched systems. Indeed, in case of switched systems with large number of subsystems we have a large number of LMIs to solve. Then the LPV reformulation of the uncertainties leads to a higher number of LMIs, which may causes numerical problems from computational point of view.

On the other hand, the Young inequality based approach may be used even in the uncertainty free case to handle the BMI coming from some coupling between decision variables. Young's inequality based approach is more conservative than the new proposed enhanced LMI methodology based on the novel and non standard use of the Finsler lemma. As we have mentioned above, the source of the conservatism is the diagonal structure of the variables $G_i$. The conservatism comes also crucially from the manner to handle the term $(C_i + \Delta C_i)^T L_i^T$.

### 2.4.3 On the numerical complexity of the proposed LMI techniques

The numerical complexity associated with the proposed LMI conditions can be computed in terms of the number of scalar variables and number of LMI to be solved. As for the relaxed algorithm proposed in Theorem 2.4.1, the computational complexity can easily be evaluated. Indeed, we must solve $N^2$ LMIs conditions to get $12N+1$ decision variables, or $N(3n^2 + n(m+p+1)+6)+n^2$ scalar variables. As compared with the other conditions presented in this paper, following the comment in subsection 2.4.2, if we use the gridding method, we must solve conditions (2.9) by scaling the parameters $\varepsilon_i, \gamma_i, \mu_i$ via the change of variables $s_i := \varepsilon_i/(1+\varepsilon_i)$, $t_i := \gamma_i/(1+\gamma_i)$, $\kappa_i := \mu_i/(1+\mu_i)$, with $s_i, t_i, \kappa_i \in (0,1)$. Thus, for each mode $i$, we have to make a (uniform) mesh of the interval $(0,1)$ with length equal to $\Delta_i^{s_i}$, $\Delta_i^{t_i}$ and $\Delta_i^{\kappa_i}$, respectively. If conditions (2.9) are found feasible for $(s_i^*, t_i^*, \kappa_i^*)$, then this means that we solved, for each $i, j \in \Lambda$, the following number of LMI conditions :

$$\left[\frac{s_i^*}{\Delta_i^{s_i}}\right]\left[\frac{t_i^*}{\Delta_i^{t_i}}\right]\left[\frac{\kappa_i^*}{\Delta_i^{\kappa_i}}\right]$$

where $[x]$ denotes the integer part of a real number $x$. This amounts to solving a number of LMI equal to

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\left[\frac{s_i^*}{\Delta_i^{s_i}}\right]\left[\frac{t_i^*}{\Delta_i^{t_i}}\right]\left[\frac{\kappa_i^*}{\Delta_i^{\kappa_i}}\right].$$

It should be noted that this number is greater than or equal to $N^2$. We have $N^2$ LMIs to solve only if the LMI (2.9) is found feasible at the first step when the gridding method is applied, i.e : $\left[\frac{s_i^*}{\Delta_i^{s_i}}\right]=\left[\frac{t_i^*}{\Delta_i^{t_i}}\right]=\left[\frac{\kappa_i^*}{\Delta_i^{\kappa_i}}\right]=1$. The computational complexity of the algorithm given by (2.22) can be evaluated similarly. Table 2.1 shows the number, $\sharp$SV, of the scalar variables, the number, $\sharp$DV, of decision variables, and the number, $\sharp$LMI, of LMI conditions to be solved for the three tests presented here. From a numerical complexity point of view, the superiority of (2.27) is quite clear.

| **Algorithm** | $\sharp$SV | $\sharp$DV | $\sharp$LMIs |
|---|---|---|---|
| **LMI** (2.27) | $N\Big(3n^2+n(m+p+1)+6\Big)+n^2$ | $12N+1$ | $N^2$ |
| **LMI** (2.22) | $N\Big(3n^2+n(m+p+1)+2\Big)+n^2$ | $8N+1$ | $N\sum_{i=1}^{N}\left[\frac{\tau_i^*}{\Delta_i^{\tau_i}}\right]\left[\frac{\upsilon_i^*}{\Delta_i^{\upsilon_i}}\right]$ |
| **LMI** (2.9) | $N\Big(3n^2+n(m+p+1)+3\Big)+n^2$ | $9N+1$ | $N\sum_{i=1}^{N}\left[\frac{s_i^*}{\Delta_i^{s_i}}\right]\left[\frac{t_i^*}{\Delta_i^{t_i}}\right]\left[\frac{\kappa_i^*}{\Delta_i^{\kappa_i}}\right]$ |

TABLE 2.1 – Numerical complexity associated with the proposed algorithms

## 2.5 Numerical examples and comparisons

In this section, we present numerical examples to show the validity and effectiveness of the proposed design methodology. For a comparison reason, we reconsider the examples given in [35] and [1]. We will also provide a Monte Carlo simulation to evaluate the superiority of the enhanced LMI conditions (2.22) in the uncertainty free case.

### 2.5.1 Example 1

Here we consider the example proposed in [1]. First, we take exactly the same example (given without uncertainties). That is $\Delta A_i=\Delta B_i=\Delta C_i=0$ and the other parameters are described as follows :

$$A_1=\begin{bmatrix}1.5 & 1\\ 0 & 2\end{bmatrix}, B_1=\begin{bmatrix}1 & 2\\ 4 & 0\end{bmatrix}, C_1=\begin{bmatrix}3 & 2\\ 0 & -1\end{bmatrix}, \tag{2.29a}$$

$$A_2 = \begin{bmatrix} 1.7 & 1 \\ 0.5 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{2.29b}$$

It is clear that all the matrices $A_1$ and $A_2$ are unstable. It should be noticed that the LMI conditions proposed in [1] are not feasible for this example, contrarily to what has been speculated in [1]. Indeed, first, the LMI conditions given in [1] are false because the authors made a mistake in [1, Inequality (9)]. This mistake removes many bilinear terms and conducted the authors to very simple LMIs. On the other hand, despite this error, the LMIs in [1] are not feasible for this example because of the particular choice $\hat{G}_i = \hat{F}_i^T$ and a conservative way of using Finsler's inequality. The same goes to the approach presented in [27], which is found infeasible due to a strong equality constraint. However, using Matlab LMI toolbox, we get that both LMI (2.22) and LMI (2.9) are feasible. Note that the solvability of (2.9) is performed via the gridding technique with respect to $epsilon_i$. Indeed, by scaling $\varepsilon_i$, $i \in \{1,2\}$, by defining $s_i = \varepsilon_i/(1+\varepsilon_i)$, with $s_i \in [0.1, 0.9]$, then with a uniform subdivision of the interval $[0.1, 0.9]$ of length equal to $\Delta_i^{s_i} = 0.1$, $\forall i \in \{1,2\}$, we get LMI (2.9) feasible for $s_1 = 0.7, s_2 = 0.8$, i.e., $\varepsilon_1 = 2.3333$, $\varepsilon_2 = 4$. The observer-based controller gains are given in Table 2.2. Notice that the symbol (!) means that the corresponding LMI condition is found infeasible.

| | **LMI** (2.7) | **LMI [27]** | **LMI** (2.22) | **LMI** (2.9) |
|---|---|---|---|---|
| | | | | $\varepsilon_1 = 2.3333$, $\varepsilon_2 = 4$ |
| $K_1$ | ( ! ) | ( ! ) | $\begin{bmatrix} -0.1737 & -0.6394 \\ -0.2222 & -0.0530 \end{bmatrix}$ | $\begin{bmatrix} -0.1373 & -0.5938 \\ -0.2100 & -0.0146 \end{bmatrix}$ |
| $K_2$ | ( ! ) | ( ! ) | $\begin{bmatrix} -2.3124 & -4.2526 \\ -0.1131 & 1.4825 \end{bmatrix}$ | $\begin{bmatrix} -2.0309 & -3.6428 \\ -0.1534 & 1.3106 \end{bmatrix}$ |
| $L_1$ | ( ! ) | ( ! ) | $\begin{bmatrix} +0.3877 & -0.3601 \\ 0.0103 & -1.9046 \end{bmatrix}$ | $\begin{bmatrix} +0.3403 & -0.5823 \\ 0.0435 & -1.7723 \end{bmatrix}$ |
| $L_2$ | ( ! ) | ( ! ) | $\begin{bmatrix} 0.5731 & -1.0100 \\ -1.0873 & -1.7078 \end{bmatrix}$ | $\begin{bmatrix} -0.7444 & -1.7542 \\ -0.9839 & -1.6725 \end{bmatrix}$ |

TABLE 2.2 – Observer-based controllers for the proposed LMI design applied to system (2.29)

The simulation results corresponding to these observer-based controller gains obtained by solving LMIs (2.22) are given in Figure 2.1. These simulations are done for an horizon $T = 40s$, with $x_0 = \begin{bmatrix} -3 & -5 \end{bmatrix}^T$ and $\hat{x}_0 = \begin{bmatrix} 7 & -15 \end{bmatrix}^T$. The switching rule is taken in this form :

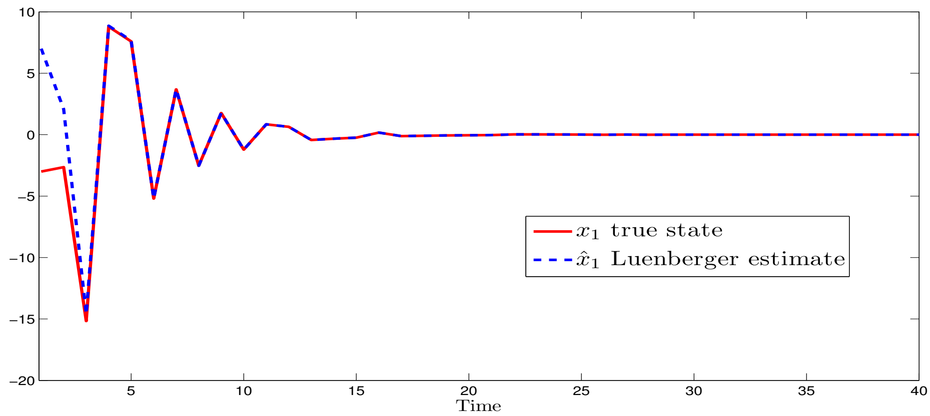
$$\sigma_t = 1 + \text{round}\,(\omega_t) \tag{2.30}$$

for $t = 1$ to $T$,

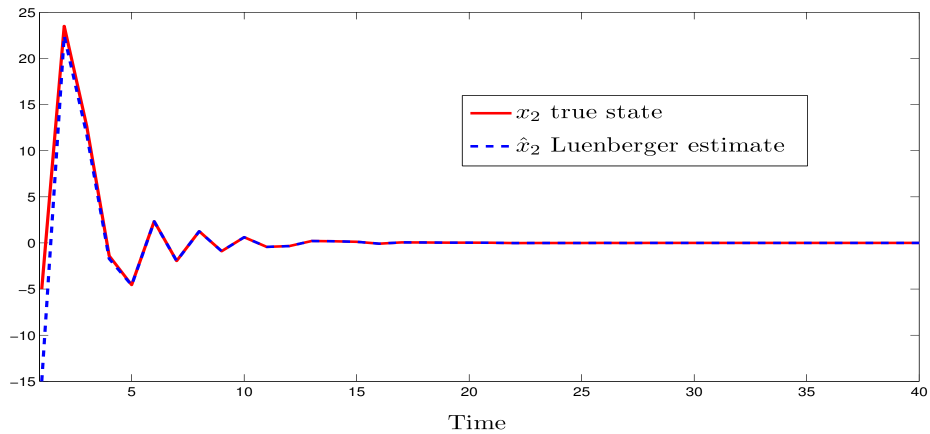
— Algorithm 1 (switching rule generation) : for $t = 1$ to $T$,

$$\sigma_t = 1 + \text{round}\,((N-1) * \text{rand}(t))$$

where $\omega_t$ is an uniformly distributed random variable on the interval $[0,1]$, and $\text{round}(x)$ is the nearest integer function of real number $x$. Then, the switching signal can be




(a) Example 1 : Time-behaviors of $x_1$ and $\hat{x}_1$ in the uncertainty free case.




(b) Example 1 : Time-behaviors of $x_2$ and $\hat{x}_2$ in the uncertainty free case.

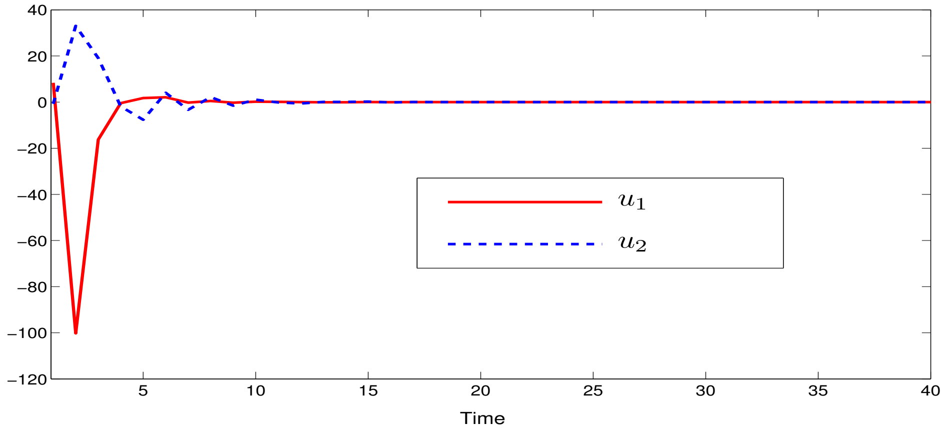
realized by Matlab and a possible case is shown in Figure 2.1(d). Note that the switching instants in Figure 2.1(d) are arbitrary.
In order to boost comparisons between the proposed LMI conditions (2.22), (2.9) and (2.27), we add to the previous example parameter uncertainties as follows :

$$M_1 = \begin{bmatrix} 0.35 & 0.2 \\ 0.3 & 0.15 \end{bmatrix}, M_2 = \begin{bmatrix} 0.3 & -0.1 \\ 0.3 & 0.2 \end{bmatrix}, \tag{2.31a}$$

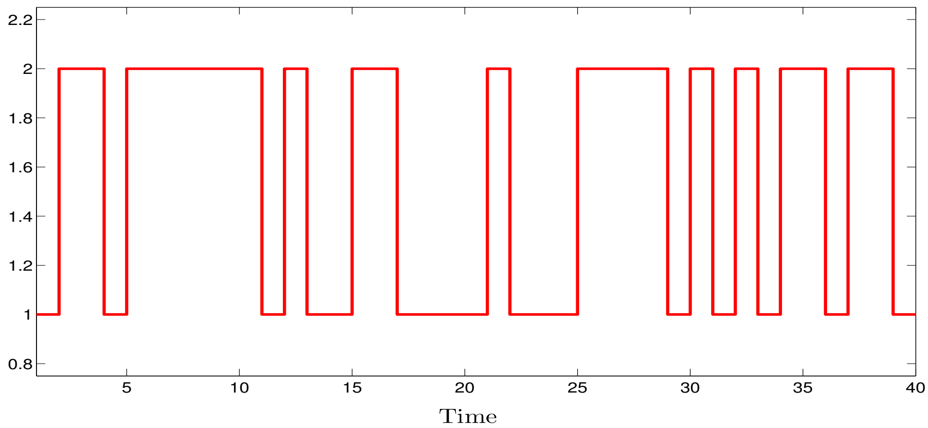
$$E_{11} = \begin{bmatrix} 0.22 & 0.22 \\ 0.2 & 0.25 \end{bmatrix}, E_{12} = \begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.5 \end{bmatrix}, E_{13} = \begin{bmatrix} 0.2 & 0.21 \\ 0.15 & 0.25 \end{bmatrix}, \tag{2.31b}$$

$$E_{21} = \begin{bmatrix} 0.25 & 0.15 \\ 0.31 & 0.25 \end{bmatrix}, E_{22} = \begin{bmatrix} 0.15 & 0.25 \\ 0.15 & 0.2 \end{bmatrix}, E_{23} = \begin{bmatrix} 0.15 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}. \tag{2.31c}$$

Then, the proposed LMIs (2.22) and (2.27) work successfully. LMI (2.22) is found feasible for $\tau_1^* = \tau_2^* = 0.5$, $\upsilon_1^* = 0.7$, $\upsilon_2^* = 0.6$, and $\Delta_i^{\tau_i} = \Delta_i^{\upsilon_i} = 0.1$, for all $i$. We obtain then, $\alpha_1 = \alpha_2 = \frac{\tau_1^*}{1-\tau_1^*} = 1$ and $\lambda_1 = \frac{\upsilon_1^*}{1-\upsilon_1^*} = 2.3333$, $\lambda_2 = \frac{\upsilon_2^*}{1-\upsilon_2^*} = 1.5$. However, the Young inequality-based approach is found infeasible for the same values of $\varepsilon_i$ given in Table 2.2, and for the same step of discretization. The results are summarized in Table 2.3.




(c) Example 1 : Time-behaviors of $u_1$ and $u_2$ in the uncertainty free case.




(d) Example 1 : Switching mode

FIGURE 2.1 – Example 1 : Controlled states and their estimates in the uncertainty free case.

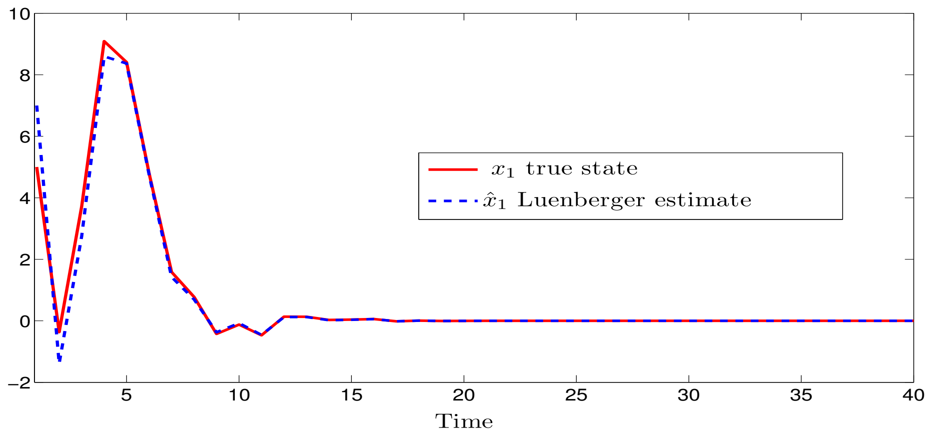
The simulation results corresponding to these observer-based controller gains returned by LMIs (2.22) are shown in Figure 2.2. These simulations shown in Figure 2.2 are done over an horizon of length $T = 40s$ with $x_0 = \begin{bmatrix} 5 & 6.5 \end{bmatrix}^T$, $\hat{x}_0 = \begin{bmatrix} 7 & 4.5 \end{bmatrix}^T$. The switching rule is generated randomly as in (2.30).

### 2.5.2 Example 2 (Evaluation of maximum admissible uncertainty)

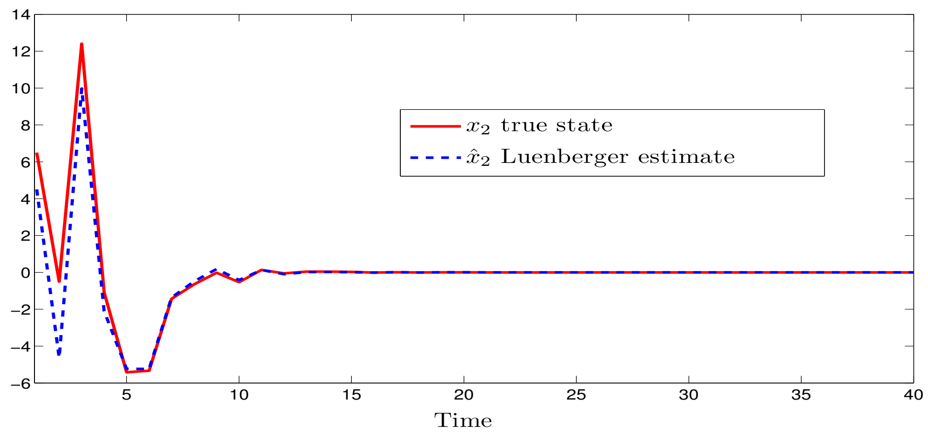
Through this example, we will show that the proposed LMI conditions are less conservative than those provided in [35]. We reconsider the same system as in [35, Example 1]. Obviously, this example can be viewed as a switching system under the form (1.42) with only one mode (there is no switching). The system is described by the following

| | **LMI** (2.22) | **LMI** (2.27) | **LMI** (2.9) |
|---|---|---|---|
| | $\Delta_i^{\tau_i} = \Delta_i^{\upsilon_i} = 0.1, \alpha_1 = \alpha_2 = 1$ $\lambda_1 = 2.3333, \lambda_2 = 1.5$ | | $\Delta_i^{t_i} = \Delta_i^{\kappa_i} = 0.1$ $\varepsilon_1 = 2.3333,\ \varepsilon_2 = 4$ |
| $K_1$ | $\begin{bmatrix} -0.2532 & -0.6666 \\ -0.1582 & 0.1107 \end{bmatrix}$ | $\begin{bmatrix} -0.4175 & -0.9023 \\ 0.3077 & 0.7274 \end{bmatrix}$ | ( ! ) |
| $K_2$ | $\begin{bmatrix} -2.9813 & -5.7095 \\ 0.5477 & 2.6315 \end{bmatrix}$ | $\begin{bmatrix} -3.6037 & -6.2027 \\ 1.2277 & 3.0381 \end{bmatrix}$ | ( ! ) |
| $L_1$ | $\begin{bmatrix} +0.4065 & -0.2117 \\ 0.2954 & -0.8230 \end{bmatrix}$ | $\begin{bmatrix} +0.3752 & -0.2373 \\ 0.1966 & -0.9884 \end{bmatrix}$ | ( ! ) |
| $L_2$ | $\begin{bmatrix} +1.1317 & -0.7526 \\ 0.1649 & -1.3045 \end{bmatrix}$ | $\begin{bmatrix} 0.6219 & -0.8185 \\ -0.5084 & -1.3906 \end{bmatrix}$ | ( ! ) |

TABLE 2.3 – Observer-based controllers for the proposed LMI design applied to system (2.29)-(2.31)



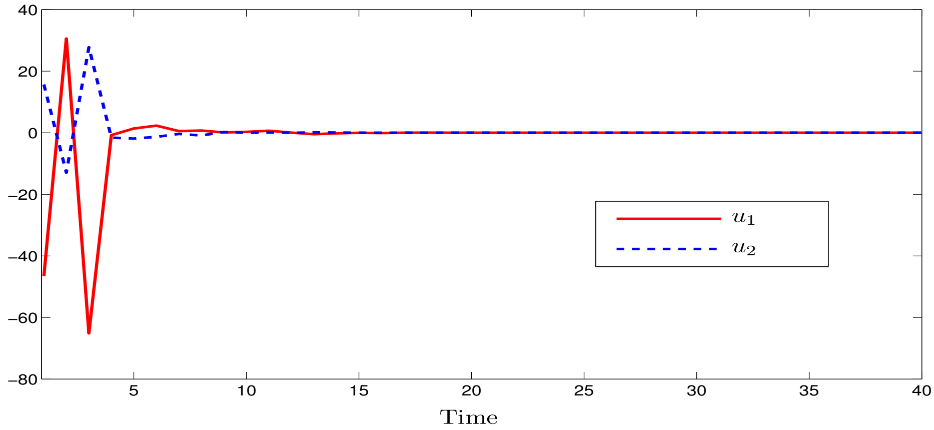

(a) Example 1 : Time-behaviors of $x_1$ and $\hat{x}_1$ in the presence of uncertainties



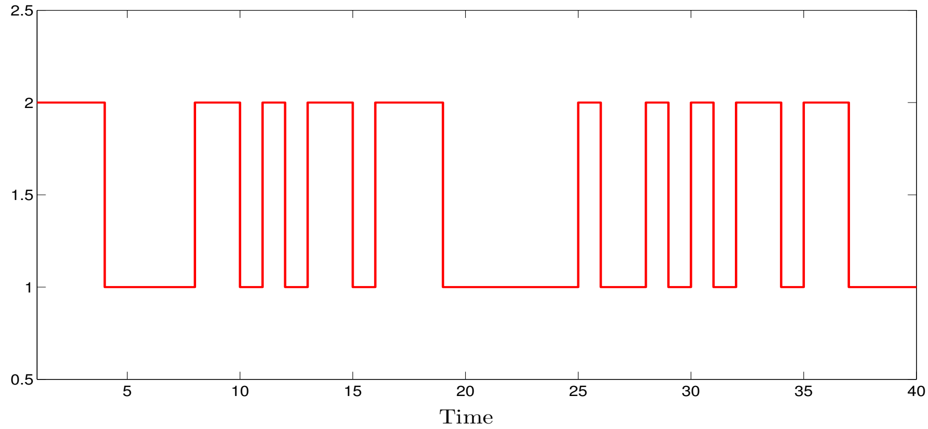

(b) Example 1 : Time-behaviors of $x_2$ and $\hat{x}_2$ in the presence of uncertainties

matrices :

$$A = \begin{bmatrix} 1 & 0.1 & 0.4 \\ 1 & 1 & 0.5 \\ -0.3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0.3 \\ -0.4 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$




(c) Example 1 : Time-behaviors of $u_1$ and $u_2$ in the presence of uncertainties




(d) Example 1 : Switching mode

FIGURE 2.2 – Example 1 : Simulation results in the presence of uncertainties.

$$M_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0 \end{bmatrix}, N_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0 \end{bmatrix},$$

$$M_C = \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0 & 0.8 \end{bmatrix}, N_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}.$$

The proposed design methodology works successfully. Solving the LMI (2.22) of Theorem 2.3.1 with $\alpha_1 = 1$ and $\lambda_1 = 0.5$, we get the following gains :

$$K = \begin{bmatrix} 1.2322 & 0.8710 & -0.8064 \\ -1.9374 & -1.1776 & -2.0770 \end{bmatrix}, L = \begin{bmatrix} 0.6909 & -0.2591 \\ 0.9596 & -0.3599 \\ 0.6405 & -0.2402 \end{bmatrix}.$$

To show the superiority of the proposed design methodology as compared with [35], we considered uncertain matrices, scaled by the parameters $\gamma_1$ and $\gamma_2$, as follows :

$$M_A = \gamma_1 \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0 \end{bmatrix}, \quad M_C = \gamma_2 \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0 & 0.8 \end{bmatrix}.$$

We look for the maximum values of $\gamma_1$ and $\gamma_2$ that satisfy LMI (2.7) and LMI (2.22). The results summarized in Table 4.2, reflect the superiority of the proposed methodology as compared to the Young inequality based approach [35] and the approach in [1].

| | **LMI** (2.7) [1] | **LMI (9) in [35]** | **LMI** (2.22) |
|---|---|---|---|
| | | $\varepsilon_1 = 2.33$<br>$\varepsilon_3 = 1.42$<br>$\varepsilon_4 = 0.08$ | $\alpha = 6$<br>$\lambda = 51.594$ |
| $\max \gamma_1$ | ( !) | 4.64 | 5.4 |
| $\max \gamma_2$ | ( !) | $10^{13}$ | $10^{15}$ |

TABLE 2.4 – Comparison between different LMI design methods

### 2.5.3 Numerical evaluation by Monte Carlo in the uncertainty free case

Here we investigate the uncertainty-free case. The aim consists in evaluating numerically the necessary conditions required by each method. For this, we generate randomly 1000 stabilizable and detectable systems of dimension $n = 3; p = 2$ and ranging from 1 to $n$ (with switching rule $\sigma_t \in \{1,2\}$). The results are summarized in Table 2.5, which gives the percentage of systems for which the different methods addressed in this note succeeded for each value of $m$.

| Method | **LMI** (2.7) | **LMI** (2.22) | **LMI** (2.9) | **LMI (60) in [27]** |
|---|---|---|---|---|
| | | | with $\varepsilon_i = 10$ | |
| $m = 1$ | 0 % | 31.5 % | 28.6 % | 0.5% |
| $m = 2$ | 0 % | 100 % | 84.1% | 1.5% |
| $m = 3$ | 0 % | 90.8 % | 78.7 % | 2 % |

TABLE 2.5 – Superiority of the proposed LMI methodology

## 2.6 Conclusions

This chapter developed new LMI conditions for the problem of stabilization of discrete-time uncertain switched linear systems. First, we revisited and corrected the approach proposed in [1] that combines Finsler's lemma and the switched Lyapunov function approach. A general theoretical method was proposed, which leads to less conservative LMI conditions. This is due to the use of Finsler's inequality in a new and convenient way. Illustrative examples are presented to demonstrate the effectiveness and superiority of the proposed design methodology.

# CHAPITRE 3 General Scenarios of Using Finsler's inequality for the Stabilization of Switched Linear Systems

## Sommaire



## 3.1 Introduction

The main objective of this chapter is to revisit the stabilization problem for a class of switching discrete-time linear systems with parameters uncertainties. Indeed, this question is treated in [1] in the LMI framework context by leaning on the use of switching Lyapunov functions and a false application of Finsler's lemma in the stability analysis. However, even with a correct application of Finsler's lemma, the obtained results remain conservative. Thus, our efforts focused on the reduction of the conservatism by adding slack variables allowing more degree of freedom in the obtained LMI condition.

This study allowed us to propose several ways to linearize Finsler's inequality, using a more general structure of the switching Lyapunov matrix and the slack variables from Finsler's Lemma. The validity and the effectiveness of the proposed design methodology are shown through an illustrative example.

In what follows, we will discuss several manners of choosing the matrix $F_i, G_i$ and $P_i$ in order to linearize Finsler's inequalities (2.3) (or equivalently (2.13)). This problem presents several difficulties. First, we must linearize the terms $G_i^{11}B_iK_i$, $G_i^{21}B_iK_i$, $F_i^{11}B_iK_i$ and $F_i^{21}B_iK_i$. Second, we can not use a change of variables because the gain matrices $L_i$, $i \in \Lambda$, are attached to eight different matrices $G_i^{11}$, $G_i^{11}$, $G_i^{22}$, $G_i^{21}$, $F_i^{11}$, $F_i^{12}$, $F_i^{22}$, and $F_i^{21}$. In order to linearize the bilinear terms attached to the gains matrices $K_i$, we use the congruence principle. We study several possibilities of choosing the matrices which appear in the expression of $X_{ij}$, namely, $F_i = \begin{bmatrix} F_i^{11} & F_i^{12} \\ F_i^{21} & F_i^{22} \end{bmatrix}$ and $G_i = \begin{bmatrix} G_i^{11} & G_i^{12} \\ G_i^{21} & G_i^{22} \end{bmatrix}$. Also, we will take into account the structure of the Lyapunov matrix, $\hat{P}_i$. Our methodology is summarized in the diagram below (see next page).

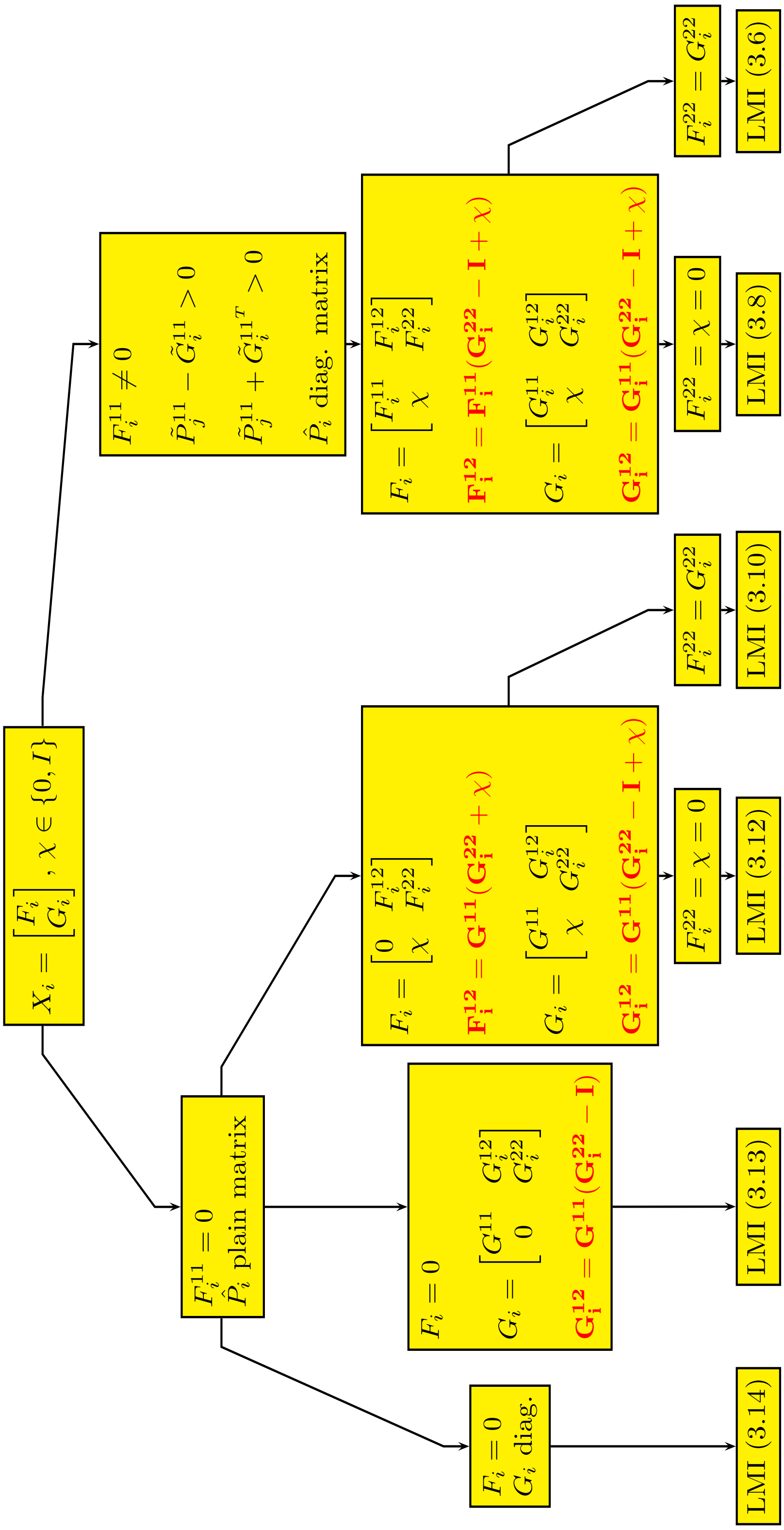



FIGURE 3.1 – Diagram of different scenarios of application of Finsler's Lemma

The next sections are devoted to new LMI design algorithms. Following the diagram in page 4, two case studies will be presented, the first one takes into account the fact that $F_i^{11} \neq 0$ and switching Lyapunov matrix $\hat{P}_i$ has a diagonal structure, and the second one considers $F_i^{11} = 0$ with general Lyapunov matrix $\hat{P}_i$. For both cases, the general structure of both $F_i$ and $G_i$ is taking into account.

## 3.2 LMI Designs : Case when $F_i^{11} \neq 0$ and $\hat{P}_i$ diagonal

Note that in view of (2.13), the matrices $G_i^{11}$, and $G_i^{22}$ $i = 1 \dots n$ are necessarily invertible. We multiply pre and post inequality (2.13) by $diag\Big((F_i^{11})^{-1}, I, (G_i^{11})^{-1}, I\Big)$, and we use the following changes of variables :

$$(G_i^{11})^{-1} = \tilde{G}_i^{11}, \quad (F_i^{11})^{-1} = \tilde{F}_i^{11},\ \tilde{K}_i = K_i(\tilde{F}_i^{11})^T,$$

$$(\hat{P}_i^{11})^{-1} = \tilde{P}_i^{11}.$$

We focus on the particular case $F_i^{21} = G_i^{21} = \chi \in \{0, I\}$. Hence, applying a result of de Oliveira et al. [65] or [7], we obtain the equivalent form of (2.13) :

$$\begin{bmatrix} \tilde{\Omega}_{11}^i & \tilde{\Omega}_{12}^i & \tilde{\Omega}_{13}^i & \tilde{\Omega}_{14}^i \\ (\star) & \tilde{\Omega}_{22}^i & \tilde{\Omega}_{23}^i & \tilde{\Omega}_{24}^i \\ (\star) & (\star) & \tilde{\Omega}_{33}^{ij} & \tilde{G}_i^{11}\hat{P}_j^{12} - \tilde{G}_i^{11}G_i^{12} - \tilde{G}_i^{11}\chi \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \tag{3.1}$$

where

$$\begin{aligned}
\tilde{\Omega}_{11}^i =& \tilde{P}_i^{11} + \mathrm{He}\Big( - \tilde{F}_i^{11} + A_i(\tilde{F}_i^{11})^T + B_i\tilde{K}_i + (I + \tilde{F}_i^{11}F_i^{12})L_i\Delta C_i(\tilde{F}_i^{11})^T - \tilde{F}_i^{11}F_i^{12}\Delta A_i(\tilde{F}_i^{11})^T \\
& - \tilde{F}_i^{11}F_i^{12}\Delta B_i\tilde{K}_i\Big), \\
\tilde{\Omega}_{12}^i =& \tilde{F}_i^{11}F_i^{12}(A_i + \Delta A_i) - (I + \tilde{F}_i^{11}F_i^{12})L_i(C_i + \Delta C_i) + \tilde{K}_i^T B_i^T \chi + \tilde{F}_i^{11}A_i^T \chi \\
& + \tilde{F}_i^{11}\Delta C_i^T L_i^T (\chi + F_i^{22})^T - \tilde{F}_i^{11}\Delta A_i^T (F_i^{22})^T - \tilde{K}_i^T \Delta B_i^T (F_i^{22})^T - \tilde{F}_i^{11}\hat{P}_i^{12}, \\
\tilde{\Omega}_{13}^i =& -(\tilde{G}_i^{11})^T + \tilde{F}_i^{11}A_i^T + \tilde{K}_i^T B_i^T - \tilde{F}_i^{11}\Delta A_i^T (\tilde{G}_i^{11}G_i^{12})^T - \tilde{K}_i^T \Delta B_i^T (\tilde{G}_i^{11}G_i^{12})^T \\
& + \tilde{F}_i^{11}\Delta C_i^T L_i^T (I + \tilde{G}_i^{11}G_i^{12})^T, \\
\tilde{\Omega}_{14}^i =& -\tilde{F}_i^{11}F_i^{12} + \tilde{F}_i^{11}A_i^T \chi + \tilde{K}_i^T B_i^T \chi - \tilde{F}_i^{11}\Delta A_i^T (G_i^{22})^T - \tilde{K}_i^T \Delta B_i^T (G_i^{22})^T \\
& + \tilde{F}_i^{11}\Delta C_i^T L_i^T (\chi + G_i^{22})^T, \\
\tilde{\Omega}_{22}^i =& -\hat{P}_i^{22} + He\Big(F_i^{22}A_i + F_i^{22}\Delta A_i - (F_i^{22} + \chi)L_i(C_i + \Delta C_i)\Big), \\
\tilde{\Omega}_{23}^i =& -\chi(\tilde{G}_i^{11})^T - (C_i + \Delta C_i^T)^T L_i^T (I + \tilde{G}_i^{11}G_i^{12})^T + (A_i^T + \Delta A_i^T)(\tilde{G}_i^{11}G_i^{12})^T, \\
\tilde{\Omega}_{24}^i =& -F_i^{22} + A_i^T (G_i^{22})^T + \Delta A_i^T (G_i^{22})^T - (C_i + \Delta C_i)^T L_i^T (G_i^{22} + \chi)^T, \\
\tilde{\Omega}_{33}^{ij} =& \tilde{G}_i^{11}\hat{P}_j^{11}(\tilde{G}_i^{11})^T - \tilde{G}_i^{11} - (\tilde{G}_i^{11})^T.
\end{aligned}$$

Therefore, we will consider two cases, each one in a separated subsection.

### 3.2.1 First case

In this case, we choose

$$F_i^{12} = F_i^{11} G_i^{22} - F_i^{11}(I - \chi), G_i^{12} = G_i^{11} G_i^{22} - G_i^{11}(I - \chi)$$

where $\chi \in \{0, I\}$. Note that (4.28) is still a BMI because of presence of the coupling term $\tilde{G}_i^{11} \hat{P}_j^{11} (\tilde{G}_i^{11})^T$. On the other hand, the gain matrix $L_i$ is attached to different variables, so we must identify the four terms $(I + \tilde{F}_i^{11} F_i^{12}) L_i C_i$, $(I + \tilde{G}_i^{11} G_i^{12}) L_i C_i$, $(F_i^{22} + \chi) L_i C_i$ and $(G_i^{22} + \chi) L_i C_i$, which amounts to put :

$$G_i^{22} = I + \tilde{F}_i^{11} F_i^{12} - \chi = I + \tilde{G}_i^{11} G_i^{12} - \chi = F_i^{22},$$

thus, we can introduce the change of variable

$$\hat{L}_i = (G_i^{22} + \chi) L_i.$$

Note that such a change of variable requiers a stronger condition on $G_i^{22}$ to be well defined when $\chi = I$. In fact, the invertibility of $(G_i^{22} + \chi)$ is note guaranteed even if $G_i^{22}$ is. But under the additional "stronger" condition $-I < G_i^{22} \leq I$, the matrix $G_i^{22} + \chi$, for each $i \in \Lambda$, is well and truly invertible.
Now, in order to deal with the bilinear term $\tilde{G}_i^{11} \hat{P}_j^{11} (\tilde{G}_i^{11})^T$, we state the following lemma :

**Lemma 3.2.1.** *If $(\hat{P}_j^{11})^{-1} - \tilde{G}_i^{11}$ and $(\hat{P}_j^{11})^{-1} + (\tilde{G}_i^{11})^T$ are positive definite matrices (or negative definite matrices), then*

$$\tilde{G}_i^{11} \hat{P}_j^{11} (\tilde{G}_i^{11})^T \leq (\hat{P}_j^{11})^{-1} + (\tilde{G}_i^{11})^T - \tilde{G}_i^{11}. \tag{3.2}$$

**Proof 5.** *Since $(\hat{P}_j^{11})^{-1} - \tilde{G}_i^{11}$ and $(\hat{P}_j^{11})^{-1} + (\tilde{G}_i^{11})^T$ and $\hat{P}_j^{11}$ are positive define matrices, we get*

$$\left((\hat{P}_j^{11})^{-1} - \tilde{G}_i^{11}\right) \hat{P}_j^{11} \left((\hat{P}_j^{11})^{-1} + (\tilde{G}_i^{11})^T\right) \geq 0$$

*by developing, we get*

$$(\hat{P}_j^{11})^{-1} + (\tilde{G}_i^{11})^T - \tilde{G}_i^{11} - \tilde{G}_i^{11} \hat{P}_j^{11} (\tilde{G}_i^{11})^T \geq 0$$

*Which is equivalent to*

$$\tilde{G}_i^{11} \hat{P}_j^{11} (\tilde{G}_i^{11})^T \leq (\hat{P}_j^{11})^{-1} + (\tilde{G}_i^{11})^T - \tilde{G}_i^{11}.$$

*In our study, we didn't take the case $(\hat{P}_j^{11})^{-1} - \tilde{G}_i^{11}$ and $(\hat{P}_j^{11})^{-1} + (\tilde{G}_i^{11})^T$ are negative define matrices i.e $(\hat{P}_j^{11})^{-1} < \tilde{G}_i^{11}$ and $(\hat{P}_j^{11})^{-1} < -(\tilde{G}_i^{11})^T$. Indeed,*

- *If $(\hat{P}_j^{11})^{-1} < \tilde{G}_i^{11}$ then $\tilde{G}_i^{11} > 0$, and we have $(\hat{P}_j^{11})^{-1} < -(\tilde{G}_i^{11})^T$, i.e $(\hat{P}_j^{11})^{-1} < 0$. Thus, we get a contradiction.*
- *If $(\hat{P}_j^{11})^{-1} < -(\tilde{G}_i^{11})^T$ then $(\tilde{G}_i^{11})^T < 0$, we also have $(\hat{P}_j^{11})^{-1} < \tilde{G}_i^{11}$ i.e $(\hat{P}_j^{11})^{-1} < 0$. A contradiction.*

Now come back to the linearization problem with respect the bilinear term $\tilde{G}_i^{11}\hat{P}_j^{11}(\tilde{G}_i^{11})^T$. We use (3.2) in the right hand side of (4.28), we deduce that (4.28) satisfied if the following inequality :

$$\begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i & \Theta_{13}^i & \Theta_{14}^i \\ (\star) & \Theta_{22}^i & \Theta_{23}^i & \Theta_{24}^i \\ (\star) & (\star) & \tilde{P}_j^{11} - 2\tilde{G}_i^{11} & \Theta_{34}^{ij} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix} < 0, \tag{3.3}$$

holds, where

$$\begin{aligned}
\Theta_{11}^i =& \tilde{P}_i^{11} + \mathrm{He}\Big( -\tilde{F}_i^{11} + A_i(\tilde{F}_i^{11})^T + B_i\tilde{K}_i + \hat{L}_i\Delta C_i(\tilde{F}_i^{11})^T - (G_i^{22}+\chi-I)\Delta A_i(\tilde{F}_i^{11})^T \\
& - (G_i^{22}+\chi-I)\Delta B_i\tilde{K}_i\Big),\\
\Theta_{12}^i =& -\tilde{F}_i^{11}\hat{P}_i^{12} + (G_i^{22}+\chi-I)A_i + (G_i^{22}+\chi-I)\Delta A_i - \hat{L}_i(C_i+\Delta C_i) + \tilde{K}_i^T B_i^T\chi + \tilde{F}_i^{11}A_i^T\chi \\
& + \tilde{F}_i^{11}\Delta C_i^T\hat{L}_i^T - \tilde{F}_i^{11}\Delta A_i^T(G_i^{22})^T - \tilde{K}_i^T\Delta B_i^T(G_i^{22})^T,\\
\Theta_{13}^i =& -(\tilde{G}_i^{11})^T + \tilde{F}_i^{11}A_i^T + \tilde{K}_i^T B_i^T - \tilde{F}_i^{11}\Delta A_i^T(G_i^{22}+\chi-I)^T - \tilde{K}_i^T\Delta B_i^T(G_i^{22}+\chi-I)^T \\
& + \tilde{F}_i^{11}\Delta C_i^T\hat{L}_i^T,\\
\Theta_{14}^i =& -(G_i^{22}+\chi-I) + \tilde{F}_i^{11}A_i^T\chi + \tilde{K}_i^T B_i^T\chi - \tilde{F}_i^{11}\Delta A_i^T(G_i^{22})^T - \tilde{K}_i^T\Delta B_i^T(G_i^{22})^T + \tilde{F}_i^{11}\Delta C_i^T\hat{L}_i^T,\\
\Theta_{22}^i =& -\hat{P}_i^{22} + \mathrm{He}\Big(G_i^{22}A_i + G_i^{22}\Delta A_i - \hat{L}_i(C_i+\Delta C_i)\Big),\\
\Theta_{23}^i =& -\chi(\tilde{G}_i^{11})^T - (C_i+\Delta C_i^T)^T\hat{L}_i^T + A_i^T(G_i^{22}+\chi-I)^T + \Delta A_i^T(G_i^{22}+\chi-I)^T,\\
\Theta_{24}^i =& -G_i^{22} + A_i^T(G_i^{22})^T + \Delta A_i^T(G_i^{22})^T - (C_i+\Delta C_i)^T\hat{L}_i^T,\\
\Theta_{34}^{ij} =& \tilde{G}_i^{11}\hat{P}_j^{12} - (G_i^{22}+\chi-I) - \tilde{G}_i^{11}\chi.
\end{aligned}$$

By developing $\Delta A_i, \Delta B_i$ and $\Delta C_i$, we can rewrite the previous inequality in the following more suitable form :

$$\Omega_{ij} = \Xi_{ij} + \mathrm{He}\left(Z_{i1}^T D_i^T Z_{i2} + Z_{i3}^T D_i^T Z_{i4}\right), \tag{3.4}$$

where :

$$\Xi_{ij} = \begin{bmatrix} \Upsilon_{11}^i & \Upsilon_{12}^i & \Upsilon_{13}^i & \Upsilon_{14}^i \\ (\star) & \Upsilon_{22}^i & \Upsilon_{23}^i & \Upsilon_{24}^i \\ (\star) & (\star) & \tilde{P}_j^{11} - 2\tilde{G}_i^{11} & \Upsilon_{34}^{ij} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix},$$

$$\begin{aligned}
\Upsilon_{11}^i =& \tilde{P}_i^{11} - \tilde{F}_i^{11} - (\tilde{F}_i^{11})^T + \mathrm{He}\Big(A_i(\tilde{F}_i^{11})^T + B_i\tilde{K}_i\Big),\\
\Upsilon_{12}^i =& -\tilde{F}_i^{11}\hat{P}_i^{12} + (G_i^{22}+\chi-I)A_i - \hat{L}_iC_i + \tilde{K}_i^T B_i^T\chi + \tilde{F}_i^{11}A_i^T\chi,\\
\Upsilon_{13}^i =& \tilde{F}_i^{11}A_i^T + \tilde{K}_i^T B_i^T - (\tilde{G}_i^{11})^T,\\
\Upsilon_{14}^i =& I - G_i^{22} - \chi + \tilde{F}_i^{11}A_i^T\chi + \tilde{K}_i^T B_i^T\chi,\\
\Upsilon_{22}^i =& -\hat{P}_i^{22} + \mathrm{He}\Big(G_i^{22}A_i - \hat{L}_iC_i\Big),\\
\Upsilon_{23}^i =& -\chi(\tilde{G}_i^{11})^T + A_i^T(G_i^{22}+\chi-I)^T - C_i^T\hat{L}_i^T,
\end{aligned}$$

$$\Upsilon_{24}^i = A_i^T (G_i^{22})^T - C_i^T \hat{L}_i^T - G_i^{22},$$
$$\Upsilon_{34}^{ij} = \tilde{G}_i^{11} \hat{P}_j^{12} + I - G_i^{22} - \chi - \tilde{G}_i^{11} \chi,$$
$$\Upsilon_{19}^i = -\tilde{F}_i^{11} E_{i1}^T - \tilde{K}_i^T E_{i2}^T,$$

and

$$\begin{aligned}
Z_{i1} &= \begin{bmatrix} (\Upsilon_{19}^i)^T & E_{i1} & 0 & 0 \end{bmatrix}^T \\
Z_{i3} &= \begin{bmatrix} E_{i3}(\tilde{F}_i^{11})^T & -E_{i3} & 0 & 0 \end{bmatrix} \\
Z_{i2} &= M_i^T \mathscr{Z}_{i2} \\
\mathscr{Z}_{i2} &= \begin{bmatrix} (G_i^{22} + \chi - I)^T & (G_i^{22})^T & (G_i^{22} + \chi - I)^T & (G_i^{22})^T \end{bmatrix} \\
Z_{i4} &= \begin{bmatrix} M_i^T \hat{L}_i^T & M_i^T \hat{L}_i^T & M_i^T \hat{L}_i^T & M_i^T \hat{L}_i^T \end{bmatrix}
\end{aligned}$$

Consequently, applying the classical Young inequality (see e.g. [33]), we get the following inequality :

$$\Omega_{ij} \leq \Xi_{ij} + \gamma_i Z_{i1}^T Z_{i1} + \gamma_i^{-1} Z_{i2}^T Z_{i2} + \lambda_i Z_{i3}^T Z_{i3} + \lambda_i^{-1} Z_{i4}^T Z_{i4}, \tag{3.5}$$

where $\gamma_i$ and $\lambda_i$ are some positive scalars. Using Schur's Lemma (see, [33]), we deduce that the inequalities $\Omega_{ij} < 0$, $i, j \in \Lambda$ are satisfied (after assuming that $\hat{P}_i^{12} = 0$ (i.e, $\hat{P}_i$ are diagonal matrices)), if the LMI (4.21) is feasible. So we just proved the following theorem :

**Theorem 3.2.2.** *For the closed-loop switched system* (1.46)*, if there exist positive definite matrices* $\tilde{P}_i^{11}, \hat{P}_i^{22} \in \mathbb{R}^{n \times n}$*, invertible matrices* $G_i^{22}, \tilde{G}_i^{11}, \tilde{F}_i^{11} \in \mathbb{R}^{n \times n}$ *so that* $\tilde{P}_j^{11} + (\tilde{G}_i^{11})^T > 0$ *and* $\tilde{P}_j^{11} - \tilde{G}_i^{11} > 0$ *for all* $i, j \in \Lambda$*, matrices* $\tilde{K}_i \in \mathbb{R}^{p \times n}, \hat{L}_i \in \mathbb{R}^{n \times m}$*, and positives scalars* $\lambda_i, \gamma_i$ *such that* (3.6) *holds for all* $i, j \in \Lambda$*,*

$$\begin{bmatrix} \Xi_{ij} & \mathbb{S}_i \\ (\star) & \mathbb{D}_i \end{bmatrix} < 0, \tag{3.6}$$

*where :*

$$\Xi_{ij} = \begin{bmatrix} \Upsilon_{11}^i & \Upsilon_{12}^i & \Upsilon_{13}^i & \Upsilon_{14}^i \\ (\star) & \Upsilon_{22}^i & \Upsilon_{23}^i & \Upsilon_{24}^i \\ (\star) & (\star) & \tilde{P}_j^{11} - 2\tilde{G}_i^{11} & \Upsilon_{34}^{ij} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix},$$

$$\Upsilon_{11}^i = \tilde{P}_i^{11} + \mathrm{He}\Big( -\tilde{F}_i^{11} + A_i (\tilde{F}_i^{11})^T + B_i \tilde{K}_i \Big),$$
$$\Upsilon_{12}^i = (G_i^{22} + \chi - I) A_i - \hat{L}_i C_i + \tilde{K}_i^T B_i^T \chi + \tilde{F}_i^{11} A_i^T \chi,$$
$$\Upsilon_{13}^i = \tilde{F}_i^{11} A_i^T + \tilde{K}_i^T B_i^T - (\tilde{G}_i^{11})^T,$$
$$\Upsilon_{14}^i = I - G_i^{22} - \chi + \tilde{F}_i^{11} A_i^T \chi + \tilde{K}_i^T B_i^T \chi,$$
$$\Upsilon_{22}^i = -\hat{P}_i^{22} + \mathrm{He}\Big( G_i^{22} A_i - \hat{L}_i C_i \Big),$$
$$\Upsilon_{23}^i = -\chi (\tilde{G}_i^{11})^T + A_i^T (G_i^{22} + \chi - I)^T - C_i^T \hat{L}_i^T,$$

$$\Upsilon_{24}^{i} = A_i^T (G_i^{22})^T - C_i^T \hat{L}_i^T - G_i^{22},$$
$$\Upsilon_{34}^{ij} = I - G_i^{22} - \chi - \tilde{G}_i^{11}\chi, \Upsilon_{19}^{i} = -\tilde{F}_i^{11} E_{i1}^T - \tilde{K}_i^T E_{i2}^T,$$

*and*

$$\mathbb{S}_i = \begin{bmatrix} \Upsilon_{19}^{i} & (G_i^{22}+\chi-I)M_i & \tilde{F}_i^{11}E_{i3}^T & \hat{L}_iM_i \\ E_{i1}^T & G_i^{22}M_i & -E_{i3}^T & \hat{L}_iM_i \\ 0 & (G_i^{22}+\chi-I)M_i & 0 & \hat{L}_iM_i \\ 0 & G_i^{22}M_i & 0 & \hat{L}_iM_i \end{bmatrix},$$
$$\mathbb{D}_i = diag(-\gamma_i^{-1}I, -\gamma_i I, -\lambda_i^{-1}I, -\lambda_i I)$$

*Then the closed-loop switched system* (1.46) *is asymptotically stable under an arbitrary switching with the gains*

$$K_i = \tilde{K}_i(\tilde{F}_i^{11})^{-T}, and L_i = (G_i^{22}+\chi)^{-1}\hat{L}_i, i \in \Lambda. \tag{3.7}$$

### 3.2.2 Second case

In what follows, we assume that $F_i^{22} = 0$ and $F_i^{21} = G_i^{21} = 0$. Let us choose

$$F_i^{12} = F_i^{11}G_i^{22} - F_i^{11}, \quad G_i^{12} = G_i^{11}G_i^{22} - G_i^{11}.$$

We replace the corresponding $F_i, G_i$ in (4.28) and by using the same steps as in the previous case, we conclude the following theorem :

**Theorem 3.2.3.** *For the closed-loop switched system* (1.46)*, if there exist positive definite matrices $\tilde{P}_i^{11}, \hat{P}_i^{22} \in \mathbb{R}^{n\times n}$, invertible matrices $G_i^{22}$, $\tilde{G}_i^{11}$, $\tilde{F}_i^{11} \in \mathbb{R}^{n\times n}$ so that $\tilde{P}_j^{11} + (\tilde{G}_i^{11})^T > 0$ and $\tilde{P}_j^{11} - \tilde{G}_i^{11} > 0$ for all $i, j \in \Lambda$, matrices $\tilde{K}_i \in \mathbb{R}^{p\times n}, \hat{L}_i \in \mathbb{R}^{n\times m}$, and positives scalars $\lambda_i, \gamma_i$ such that* (3.8) *holds for all $i, j \in \Lambda$,*

$$\begin{bmatrix} \Xi_1^{ij} & \mathbb{S}_{12}^{i} \\ (\star) & \mathbb{D}_i \end{bmatrix} < 0, \tag{3.8}$$

*where :*

$$\Xi_1^{ij} = \begin{bmatrix} \Upsilon_{11}^{i} & \Upsilon_{12}^{i} & \Upsilon_{13}^{i} & I - G_i^{22} \\ (\star) & -\hat{P}_i^{22} & \Upsilon_{23}^{i} & A_i^T(G_i^{22})^T - C_i^T\hat{L}_i^T \\ (\star) & (\star) & \tilde{P}_j^{11} - 2\tilde{G}_i^{11} & I - G_i^{22} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T \end{bmatrix},$$

$$\Upsilon_{11}^{i} = \tilde{P}_i^{11} - \tilde{F}_i^{11} - (\tilde{F}_i^{11})^T + \mathrm{He}\left(A_i(\tilde{F}_i^{11})^T + B_i\tilde{K}_i\right),$$
$$\Upsilon_{12}^{i} = (G_i^{22} - I)A_i - \hat{L}_iC_i,$$
$$\Upsilon_{13}^{i} = \tilde{F}_i^{11}A_i^T + \tilde{K}_i^TB_i^T - (\tilde{G}_i^{11})^T,$$
$$\mathbb{S}_{12}^{i} = \begin{bmatrix} -\tilde{F}_i^{11}E_{i1}^T - \tilde{K}_iE_{i2}^T & (G_i^{22}-I)M_i & \tilde{F}_i^{11}E_{i3}^T & \hat{L}_iM_i \\ E_{i1}^T & 0 & -E_{i3}^T & 0 \\ 0 & (G_i^{22}-I)M_i & 0 & \hat{L}_iM_i \\ 0 & G_i^{22}M_i & 0 & \hat{L}_iM_i \end{bmatrix},$$

$$\mathbb{D}_i = diag(-\gamma_i^{-1}I, -\gamma_i I, -\lambda_i I, -\lambda_i^{-1}I)$$

*Then the closed-loop switched system* (1.46) *is asymptotically stable under an arbitrary switching with the gains*

$$K_i = \tilde{K}_i(\tilde{F}_i^{11})^{-T} \text{ and } L_i = (G_i^{22})^{-1}\hat{L}_i, i \in \Lambda. \tag{3.9}$$

**Remark 3.2.4.** *1. Notice that the study corresponding to $F_i = G_i = I$ is not interesting. Indeed, from a numerical point of view, we did not get any result.*

*2. In the previous cases, we chose $\hat{P}_i$ as a diagonal matrix because $F_i^{11} \neq G_i^{11}$, and in case where $F_i^{11} = G_i^{11}$, we can choose $\hat{P}_i$ as a plain matrix.*

## 3.3 LMI Designs : Case when $F_i^{11} = 0$ and $\hat{P}_i$ non diagonal

In this case we cannot use the technique in of slack variable in [65] or [7] and property (3.2). As an alternative, we choose $G_i^{11} = G^{11}$ , that is independent of $i$. Recall that $G^{11}$ and $G_i^{22}$, for $i = 1 \dots N$ are necessarily invertible. Here, we consider four cases.

### 3.3.1 First case : $F_i \neq 0$

We use the same steps as in the previous subsection. First, we pre-and post multiply (2.13) by $diag\Big((G^{11})^{-1}, I, (G^{11})^{-1}, I\Big)$ and we are going to substitute some terms as follows :

$$(G^{11})^{-1} = \tilde{G}^{11}, \quad \tilde{K}_i = K_i(\tilde{G}^{11})^T, \tilde{G}^{11}\hat{P}_i^{11}(\tilde{G}^{11})^T = \tilde{P}_i^{11},$$

$$F_i^{21} = G_i^{21} = \chi(\chi = 0 \, or \, I), \;\; \tilde{G}^{11}\hat{P}_i^{12} = \tilde{P}_i^{12}.$$

We will observe that the gain matrix $L_i$ is attached to $\tilde{G}^{11}F_i^{12}$, $\chi + F_i^{22}$, $I + \tilde{G}^{11}G_i^{12}$ and $G_i^{22} + \chi$, so we identify

$$\tilde{G}^{11}F_i^{12} = \chi + F_i^{22} = I + \tilde{G}^{11}G_i^{12} = G_i^{22} + \chi.$$

This leads to the following choices of $F_i^{22} = G_i^{22}$,

$$F_i^{12} = G^{11}(G_i^{22} + \chi) \text{ and } G_i^{12} = G^{11}(G_i^{22} + \chi - I).$$

Thus after linearizing with respect to the uncertainties, we get the following theorem.

**Theorem 3.3.1.** *For the closed-loop switched system* (1.46)*, if there exist positive definite matrices $D_i \in \mathbb{R}^{2n\times 2n}$, invertible matrices $G_i^{22}, \tilde{G}^{11}, \tilde{F}_i^{11} \in \mathbb{R}^{n\times n}$, matrices $\tilde{K}_i \in \mathbb{R}^{p\times n}, \hat{L}_i \in$*

$\mathbb{R}^{n\times m}$*, and positives scalars* $\lambda_i, \gamma_i$ *so that* (3.10) *holds for all* $i, j \in \Lambda$*,*

$$\begin{bmatrix} -\tilde{P}_i^{11} & \Theta_{12}^i & \Theta_{13}^i & \Theta_{14}^i & \Theta_{15}^i & (G_i^{22}+\chi)M_i & \tilde{G}^{11}E_{i3}^T & \hat{L}_iM_i \\ (\star) & \Theta_{22}^i & \Theta_{23}^i & \Theta_{24}^i & E_{i1}^T & G_i^{22}M_i & -E_{i3}^T & \hat{L}_iM_i \\ (\star) & (\star) & \Theta_{33}^j & \Theta_{34}^{ij} & 0 & (G_i^{22}+\chi-I)M_i & 0 & \hat{L}_iM_i \\ (\star) & (\star) & (\star) & \hat{P}_j^{22}-G_i^{22}-(G_i^{22})^T & 0 & G_i^{22}M_i & 0 & \hat{L}_iM_i \\ (\star) & (\star) & (\star) & (\star) & -\alpha_iI & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\alpha_i^{-1}I & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\lambda_iI & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\lambda_i^{-1}I \end{bmatrix} < 0. \tag{3.10}$$

*where*

$$\Theta_{12}^i = -\tilde{P}_i^{12} + (G_i^{22}+\chi)A_i - \hat{L}_iC_i + (\tilde{K}_i^TB_i^T + \tilde{G}^{11}A_i^T)\chi,\ \Theta_{13}^i = \tilde{G}^{11}A_i^T + \tilde{K}_i^TB_i^T,$$
$$\Theta_{14}^i = -(G_i^{22}+\chi) + \tilde{G}^{11}A_i^T\chi + \tilde{K}_i^TB_i^T\chi,\ \Theta_{22}^i = -\hat{P}_i^{22} + He\Big(G_i^{22}A_i - \hat{L}_iC_i\Big),$$
$$\Theta_{23}^i = -(\tilde{G}^{11})^T\chi - C_i^T\hat{L}_i^T + A_i^T(G_i^{22}+\chi-I)^T,\ \Theta_{24}^i = A_i^T(G_i^{22})^T - C_i^T\hat{L}_i^T - G_i^{22},$$
$$\Theta_{33}^j = \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T,\ \Theta_{34}^{ij} = \tilde{P}_j^{12} + I - G_i^{22} - \chi - \tilde{G}^{11}\chi,$$
$$D_i = \begin{bmatrix} \tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix},\ \Theta_{15}^i = -\tilde{G}^{11}E_{i1}^T - \tilde{K}_i^TE_{i2}^T, \forall i \in \Lambda,$$

*Then the closed-loop switched system* (1.46) *is asymptotically stable under an arbitrary switching rule with the gains*

$$K_i = \tilde{K}_i(\tilde{G}^{11})^{-T} \text{ and } L_i = (G_i^{22}+\chi)^{-1}\hat{L}_i,\ i \in \Lambda. \tag{3.11}$$

### 3.3.2 Second case

Let us put : $F_i = \begin{bmatrix} 0 & G^{11}G_i^{22} \\ 0 & F_i^{22} \end{bmatrix}$ with $F_i^{22} = \chi = 0$ and choose $G_i = \begin{bmatrix} G^{11} & G^{11}G_i^{22} - G^{11} \\ 0 & G_i^{22} \end{bmatrix}$. We replace $F_i, G_i$ in inequality (2.13), and using the same step as previously, we get the following theorem

**Theorem 3.3.2.** *For the closed-loop switched system* (1.46)*, if there exist positive definite matrices* $D_i \in \mathbb{R}^{2n\times 2n}$ *and* $G_i^{22}$, $\tilde{G}^{11} \in \mathbb{R}^{n\times n}$, $\tilde{K}_i \in \mathbb{R}^{p\times n}$, $\hat{L}_i \in \mathbb{R}^{n\times m}$ *and a positives scalars* $\lambda_i, \alpha_i$ *satisfying, for* $\forall i, j \in \Lambda = 1, 2, \ldots, N$*, the LMI*(3.12)*,*

$$\begin{bmatrix} -\tilde{P}_i^{11} & \Theta_{12}^i & \Theta_{13}^i & -G_i^{22} & \Theta_{15}^i & G_i^{22}M_i & \tilde{G}^{11}E_{i3}^T & \hat{L}_iM_i \\ (\star) & -\hat{P}_i^{22} & \Theta_{23}^i & A_i^T(G_i^{22})^T - C_i^T\hat{L}_i^T & E_{i1}^T & 0 & -E_{i3}^T & 0 \\ (\star) & (\star) & \Theta_{33}^i & \tilde{P}_j^{12}+I-G_i^{22} & 0 & (G_i^{22}-I)M_i & 0 & \hat{L}_iM_i \\ (\star) & (\star) & (\star) & \hat{P}_j^{22}-G_i^{22}-(G_i^{22})^T & 0 & G_i^{22}M_i & 0 & \hat{L}_iM_i \\ (\star) & (\star) & (\star) & (\star) & -\alpha_iI & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\alpha_i^{-1}I & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\lambda_iI & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\lambda_i^{-1}I \end{bmatrix} < 0. \tag{3.12}$$

*where*

$$\Theta_{12}^i = -\tilde{P}_i^{12} + G_i^{22}A_i - \hat{L}_iC_i,\ \Theta_{15}^i = -\tilde{G}^{11}E_{i1}^T - \tilde{K}_i^T E_{i2}^T,\ \Theta_{13}^i = \tilde{G}^{11}A_i^T + \tilde{K}_i^T B_i^T,$$
$$\Theta_{23}^i = -C_i^T\hat{L}_i^T + A_i^T(G_i^{22}-I)^T,\ \Theta_{33}^i = \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T,$$
$$D_i = \begin{bmatrix} \tilde{P}_i^{11} & \tilde{P}_i^{12} \\ (\star) & \tilde{P}_i^{22} \end{bmatrix} \qquad \forall i \in \Lambda,$$

*then the closed-loop switched system* (1.46) *is asymptotically stable under an arbitrary switching rule with the gains $L_i = (G_i^{22})^{-1}\hat{L}_i$ and $K_i = \tilde{K}_i(\tilde{G}^{11})^{-T}$, for $i \in \Lambda$.*

**Remark 3.3.3.** *When $F_i = 0$, the approach of the Finsler's lemma coincides with the standard Lyapunov approach for stability analysis.*

### 3.3.3 Third case

Let us choose $F_i = 0$ and $G_i = \begin{bmatrix} G^{11} & G_i^{12} \\ 0 & G_i^{22} \end{bmatrix}$, we replace $F_i$ and $G_i$ in (2.13), we pre- and post multiply the obtained BMI by $diag\Big((G^{11})^{-1}, I, (G^{11})^{-1}, I\Big)$ and we introduce the following notations and changes of variables :

$$(G^{11})^{-1} = \tilde{G}^{11},\ (G^{11})^{-1}P_i^{11}(G^{11})^{-T} = \tilde{P}_i^{11}, \tilde{K}_i = K_i(G^{11})^{-T},$$

$$\hat{L}_i = G_i^{22}L_i, \quad G_i^{22} = I + \tilde{G}^{11}G_i^{12}, \tilde{P}_i^{12} = (G^{11})^{-1}P_i^{12}.$$

By using the standard Young inequality and Schur lemma (see, [33]), we get LMI (3.13).

$$\begin{bmatrix}
-\tilde{P}_i^{11} & -\tilde{P}_i^{12} & \Theta_{13}^i & 0 & \Theta_{15}^i & 0 & \tilde{G}^{11}E_{i3}^T & 0 \\
(\star) & -\hat{P}_i^{22} & \Theta_{23}^i & A_i^T(G_i^{22})^T - C_i^T\hat{L}_i^T & E_{i1}^T & 0 & -E_{i3}^T & 0 \\
(\star) & (\star) & \Theta_{33}^j & \hat{P}_j^{12} - G_i^{22} + I & 0 & (G_i^{22}-I)M_i & 0 & \hat{L}_iM_i \\
(\star) & (\star) & (\star) & P_j^{22} - \mathrm{He}\Big(G_i^{22}\Big) & 0 & G_i^{22}M_i & 0 & \hat{L}_iM_i \\
(\star) & (\star) & (\star) & (\star) & -\varepsilon_i^{-1}I & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & -\varepsilon_iI & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\alpha_iI & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\alpha_i^{-1}I
\end{bmatrix} < 0. \tag{3.13}$$

where :

$$\Theta_{13}^i = \tilde{G}^{11}A_i^T + \tilde{K}_i^T B_i^T,\ \Theta_{15}^i = -\tilde{G}^{11}E_{i1}^T - \tilde{K}_i^T E_{i2}^T,$$
$$\Theta_{23}^i = -C_i^T\hat{L}_i^T + A_i^T(G_i^{22}-I)^T,$$
$$\Theta_{33}^j = \tilde{P}_j^{11} - \mathrm{He}\Big(\tilde{G}^{11}\Big).$$

### 3.3.4 Fourth case

We choose $F_i = 0$, and $G_i = \text{diag}(G^{11}, G_i^{22})$ and $G^{11}$ independent of $i$ (*i.e.* $G_i^{11} = G^{11}$). In this case, we apply the Young inequality approach as in [66]. We will get LMI (3.14).

$$\begin{bmatrix} -\tilde{P}_i^{11} & -\tilde{P}_i^{12} & \Theta_{13}^i & 0 & 0 & 0 & \Theta_{17}^i & 0 & \tilde{G}^{11}E_{i3}^T & 0 \\ (\star) & -\hat{P}_i^{22} & 0 & \Theta_{24}^i & -C_i^T\hat{L}_i^T & 0 & E_{i1}^T & 0 & -E_{i3}^T & 0 \\ (\star) & (\star) & \Theta_{33}^j & \tilde{P}_j^{12} & 0 & I & 0 & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - 2G_i^{22} & 0 & 0 & 0 & G_i^{22}M_i & 0 & \hat{L}_iM_i \\ (\star) & (\star) & (\star) & (\star) & -\mu_iG_i^{22} & 0 & 0 & 0 & 0 & \hat{L}_iM_i \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\mu_i^{-1}G_i^{22} & 0 & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\gamma_iI & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\gamma_i^{-1}I & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\varepsilon_iI & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\varepsilon_i^{-1}I \end{bmatrix} < 0. \tag{3.14}$$

where :

$$\begin{aligned} \Theta_{13}^i &= \tilde{G}^{11}A_i^T + \tilde{K}_i^TB_i^T, \\ \Theta_{17}^i &= -\tilde{G}^{11}E_{i1}^T - \tilde{K}_i^TE_{i2}^T, \\ \Theta_{33}^j &= \tilde{P}_j^{11} - \tilde{G}^{11} - (\tilde{G}^{11})^T, \\ \Theta_{24}^i &= A_i^TG_i^{22} - C_i^T\hat{L}_i^T. \end{aligned}$$

**Remark 3.3.4.** *Conditions* (3.6)*,* (3.8)*,* (3.10)*,* (3.12)*,* (3.13) *and* (3.14) *are LMIs if the scalars* $\alpha_i, \lambda_i, \varepsilon_i, \gamma_i, \mu_i$*, for all* $i \in \Lambda$*, are fixed a priori. The computational issue involved by each LMI can be tackled by linearization technique proposed in [35, Remark 3].*

## 3.4 Numerical examples and comparisons

In this section, we present numerical examples to show the validity and effectiveness of the proposed design methodology. The first one is especially related with a comparative study of the different obtained LMI conditions. The second one serves to demonstrate the validity of our approach.

### Example 1 : Numerical evaluation by Monte Carlo in uncertainties free case

We randomly generate 1000 stabilization and detectable systems via a Monte Carlo simulation ( with switching rule $\sigma \in \{1,2\}$), and compute the percentage of feasible LMIs for each method with $\mu_i = 10$ in LMI (3.14), $\Delta A_i = \Delta B_i = \Delta C_i = 0$, and $\chi = 0$. The results are summarized in Table (3.15).

| Method | Finsler's Approach ($F_i \neq 0$) | | | Lyapunov Approach ($F_i = 0$) | |
|---|---|---|---|---|---|
| | **LMI** (3.6) | **LMI** (3.8) | **LMI** (3.12) | **LMI** (3.13) | **LMI** (3.14) + **Young inequality** |
| Structure of $F_i$ | $\begin{bmatrix} F_i^{11} & F_i^{11}G_i^{22} - F_i^{11} \\ 0 & G_i^{22} \end{bmatrix}$ | $\begin{bmatrix} F_i^{11} & F_i^{11}G_i^{22} - F_i^{11} \\ 0 & 0 \end{bmatrix}$ | $F_i = \begin{bmatrix} 0 & G^{11}G_i^{22} \\ 0 & 0 \end{bmatrix}$ | $F_i = 0$ | $F_i = 0$ |
| Structure of $G_i$ | $\begin{bmatrix} G_i^{11} & G_i^{11}G_i^{22} - G_i^{11} \\ 0 & G_i^{22} \end{bmatrix}$ | $\begin{bmatrix} G_i^{11} & G_i^{11}G_i^{22} - G_i^{11} \\ 0 & G_i^{22} \end{bmatrix}$ | $\begin{bmatrix} G^{11} & G^{11}G_i^{22} - G^{11} \\ 0 & G_i^{22} \end{bmatrix}$ | $\begin{bmatrix} G^{11} & G^{11}G_i^{22} - G^{11} \\ 0 & G_i^{22} \end{bmatrix}$ | $\begin{bmatrix} G^{11} & 0 \\ 0 & G_i^{22} \end{bmatrix}$ |
| Structure of $\hat{P}_i$ | $\hat{P}_i$ diag | $\hat{P}_i$ diag | $\hat{P}_i$ plain | $\hat{P}_i$ plain | $\hat{P}_i$ plain |
| $n = m = p = 2$ | 0 % | 92% | 94.4% | 94.6% | 93.3% |
| $n = m = p = 3$ | 1% | 98.2% | 91.7% | 91.9% | 89.3% |
| $n = m = 2, p = 1$ | 0% | 79.4 % | 80.7 % | 81.4% | 71.6% |
| $n = 3, m = p = 2$ | 0.9% | 100 % | 100 % | 100% | 89% |
| $n = 3, m = 2, p = 1$ | 0% | 34.6 % | 36.1 % | 37.4% | 15.2% |

(3.15)

The percentage of feasibility is weak in LMI (3.6) and LMI (3.10). Similarly, the simulation results obtained (not reported here) in the case where $\chi = I$ are scarcely interesting, because of the constraint of invertibility of $G_i^{22} + I$. In all cases, our study shows that the structure of the matrices $\hat{P}_i$, $F_i$ and $G_i$ plays a crucial role for the feasibility of the obtained LMI conditions.

## Example 2

For an $F-18$ aircraft, it is necessary to stabilize the longitudinal dynamics independent of heights since the fact that the $F-18$ aircraft needs to operate at all possible heights, and the stability has to be persevered for all the possible operation points in the presence of faulty actuators. Now, we apply the proposed methods to the following longitudinal dynamics of the F-18 aircraft operating on two different heights [11],

$$A_1 = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.2296 & 0.9931 \\ 0 & 0.2436 & -2.046 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 \\ -0.0434 & 0.0115 \\ -1.73 & -0.517 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -0.2423 & 0.9964 \\ 0 & -2.342 & -0.1737 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ -0.0416 & 0.0114 \\ -2.595 & -0.8161 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Delta A_i = \begin{bmatrix} 0.35\sin(x_1 - x_2) & -0.35\sin(x_1 - x_2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Delta C_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and thus with

$$M_1 = M_2 = \begin{bmatrix} \sqrt{0.35} & \sqrt{0.35} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_{11} = E_{21} = \begin{bmatrix} \sqrt{0.35} & 0 & 0 \\ 0 & \sqrt{0.35} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Euler discretisation is applied to such a system to obtain a discrete-time model with sample period inferior to $\delta = 1.4$ we get LMI feasible, we chose $\delta = 1.4$, we need to solve the (2.22) with system matrices $\tilde{A}_i = I + \delta A_i$, $\tilde{B}_i = \delta B_i$, $\tilde{M}_i = \sqrt{\delta} M_i$, $\tilde{E}_{i1} = \sqrt{\delta} E_{i1}$. instead of $A_i$, $B_i$, $M_i$, $E_{i1}$.
Applying our design methodology, the solution of the LMI (2.22) with $\alpha_1 = 1.5, \alpha_2 = 1.5$, provides the following gains :

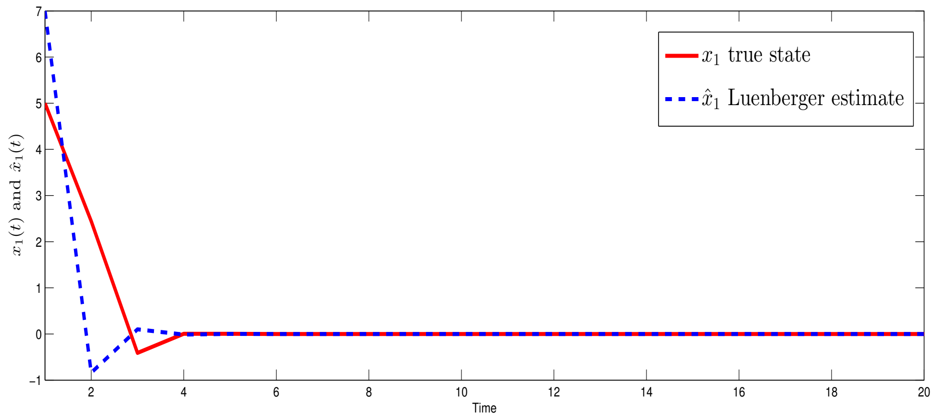
$$K_1 = \begin{bmatrix} 0 & 6.8676 & 11.8050 \\ 0 & -22.4156 & -42.0954 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 0 \\ 0.5638 & 0.9982 \\ 0.3410 & -1.8526 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0 & 6.4509 & 12.9395 \\ 0 & -23.9305 & -40.4780 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 \\ 0.5638 & 0.9982 \\ 0.3410 & -1.8526 \end{bmatrix}.$$

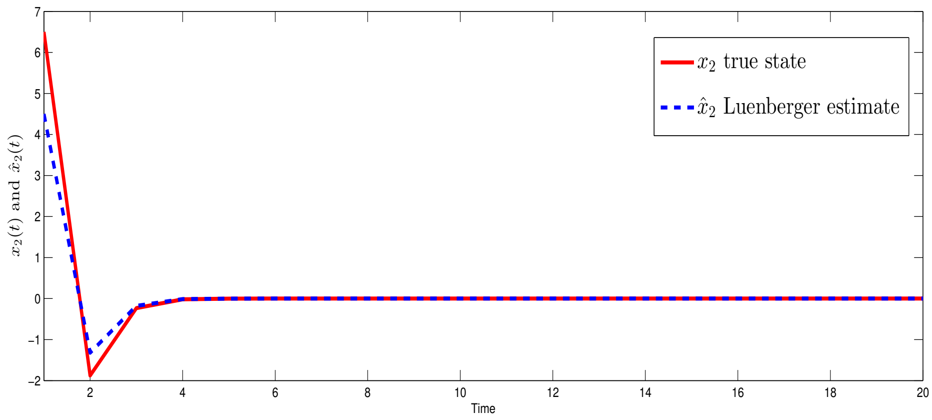
The simulation results are illustrated in Figure 3.2 -3.3.

## 3.5 Conclusion

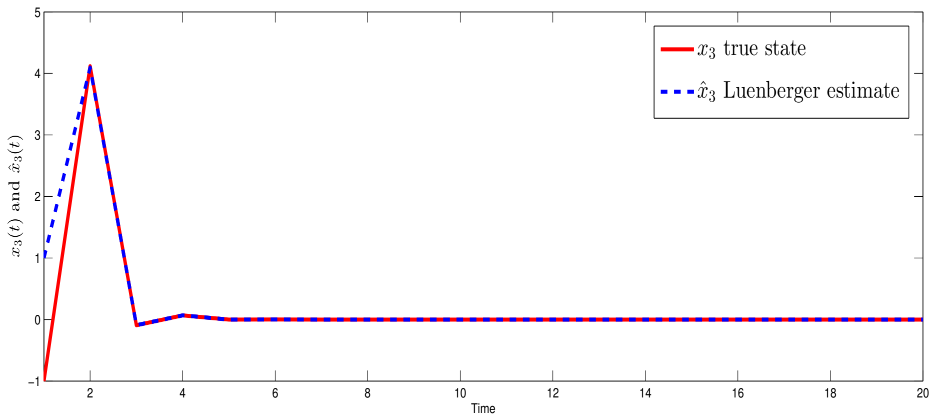
This chapter has developed new sufficient linear matrix inequality conditions for the problem of stabilization of discrete-time uncertain switched linear systems under arbitrary switching rule. We revisited and corrected the approach proposed in [1] that combines the Finsler's Lemma and the Switching Lyapunov function approach. A global theoretical study was performed, in particular, we obtained Lemma 2 that could lead to quite interesting applications in the field of LMI. This study has allowed us to offer several ways to linearize Finsler's inequality, by choosing judiciously the slack variables coming from Finsler's Lemma. An example has been investigated, and a simulation results are presented to demonstrate the effectiveness of the proposed methods.




(a) Example 2 : Time-behaviors of $x_1$ and $\hat{x}_1$.

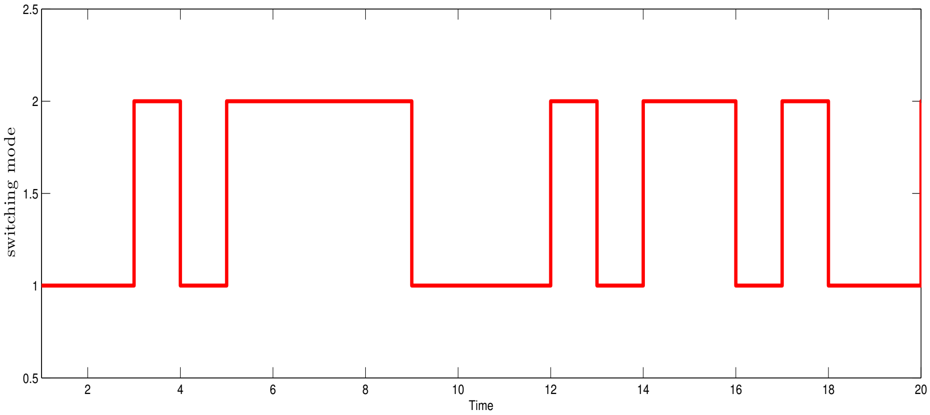



(b) Example 2 : Time-behaviors of $x_2$ and $\hat{x}_2$.




(c) Example 2 : Time-behaviors of $x_3$ and $\hat{x}_3$.

FIGURE 3.2 – Observer-based stabilized states in example $F-18$ aircraft




(a)

FIGURE 3.3 – Switching mode

Chapitre 4

# Further Contributions to the $\mathscr{H}_\infty$ Output Feedback Stabilization for a Class of Switched Discrete-time Linear Systems with Uncertain Parameters

## Sommaire



## 4.1 Introduction

This chapter presents a robust observer-based $\mathscr{H}_\infty$ controller design method via LMIs for a class of switched discrete-time linear systems with $l_2$-bounded disturbances and parameter uncertainties. The main contribution of this chapter may be viewed as :

1. an extension of the technique presented in previous chapters to systems with with $l_2$-bounded disturbances in the dynamical equations and the output measurements ;
2. an improvement of the LMI techniques in [67] by introducing a more general structure of the slack variables coming from Finsler's lemma.

More precisely, we propose a new and judicious use of the slack variables coming from Finsler's lemma. We show analytically how the proposed slack variables allow to eliminate some bilinear matrix coupling. The validity and effectiveness of the proposed design methodology are shown through two numerical examples.

## 4.2 Formulation of the Problem

### 4.2.1 System description

Let us consider the class of switching discrete-time linear systems described by :

$$x_{t+1} = (A_{\sigma_t} + \Delta A_{\sigma_t})x_t + B_{\sigma_t}u_t + E_{\sigma_t}\omega_t \tag{4.1a}$$

$$y_t = (C_{\sigma_t} + \Delta C_{\sigma_t})x_t + S_{\sigma_t}\omega_t \tag{4.1b}$$

$$z_t = H_{\sigma_t}x_t + D_{\sigma_t}u_t + J_{\sigma_t}\omega_t \tag{4.1c}$$

where $t \in \mathbb{N}$, $x_t \in \mathbb{R}^n$ is the state vector, $y_t \in \mathbb{R}^p$ is the output measurement, $u_t \in \mathbb{R}^m$ is the control input, $w_t \in \mathbb{R}^v$ is an unknown exogenous disturbance, $z_t \in \mathbb{R}^q$ is the controlled output, and $\sigma : \mathbb{N} \to \Lambda = \{1, 2, \ldots, N\}$, $t \mapsto \sigma_t$, is a switching rule.
Without ambiguity and for shortness, we write $\sigma$ instead of $\sigma_t$. The matrices $A_\sigma \in \mathbb{R}^{n\times n}$, $B_\sigma \in \mathbb{R}^{n\times m}$, $E_\sigma \in \mathbb{R}^{n\times v}$, $C_\sigma \in \mathbb{R}^{p\times n}$, $S_\sigma \in \mathbb{R}^{p\times v}$, $H_\sigma \in \mathbb{R}^{q\times n}$, $D_\sigma \in \mathbb{R}^{q\times m}$, and $J_\sigma \in \mathbb{R}^{q\times v}$, $\sigma \in \Lambda$, are constant with real coefficients.
The uncertainties $\Delta A_\sigma$ and $\Delta C_\sigma$ are structured and norm-bounded in the sense of conditions

$$[\Delta A_\sigma,\ \Delta C_\sigma] = [M_\sigma,\ N_\sigma]\,\Gamma_\sigma\,[\mathscr{E}_\sigma,\ \mathscr{S}_\sigma], \tag{4.2}$$

$$\Gamma_\sigma^T \Gamma_\sigma \leq I, \tag{4.3}$$

where the matrices $M_\sigma, N_\sigma, \mathscr{E}_\sigma, \mathscr{S}_\sigma$ are known constant which characterize the structure of the uncertainty, and the normalized matrix $\Gamma_\sigma$ contains the uncertain parameters.
The pairs $(A_\sigma, B_\sigma)$ and $(A_\sigma, C_\sigma)$ are assumed to be stabilizable and detectable, respectively. Throughout the paper, the coming assumptions are to build (see e.g. [1], [68]).
Assume without loss of generality that the switching rule $\sigma_t$ satisfies the following two items :

— The switching rule $\sigma$ is not known a priori, but its instantaneous value is available in real time.

— The switching of the observer for systems should coincide exactly with the switching of the system.

**Remark 4.2.1.** *It is worth to notice that when $\Lambda = \{1\}$, the class of systems* (4.1) *is reduced to the class of linear discrete-time systems with parameter uncertainties widely investigated* *in recent literature [69], [70]. For instance, in the uncertainty free case, we get the class of systems investigated in [71].*

**Remark 4.2.2.** *Without loss of generality, we can consider an arbitrary switching rule $\sigma_t$. That is the instantaneous values of the switching rule are not available in real time, and then the switching rule in the observer, $\hat{\sigma}_t$, does not coincide with $\sigma_t$. Indeed, for instance, the simple class of systems*

$$x_{t+1} = A_{\sigma_t} x_t + B u_t \tag{4.4a}$$
$$y_t = C_{\sigma_t} x_t \tag{4.4b}$$

*can be rewritten under the form*

$$x_{t+1} = \Big(A_{\hat{\sigma}_t} + \overbrace{(A_{\sigma_t} - A_{\hat{\sigma}_t})}^{\Delta A}\Big) x_t + B u_t \tag{4.5a}$$
$$y_t = \Big(C_{\hat{\sigma}_t} + \overbrace{(C_{\sigma_t} - C_{\hat{\sigma}_t})}^{\Delta C}\Big) x_t \tag{4.5b}$$

*where $\hat{\sigma}_t$ is the estimated switching mode $\sigma_t$ that will be used in the observer-based controller. The class of systems* (4.4) *is often investigated in the literature, some times in switching context [28], [27], [32], and some times in the LPV context with inexact parameters [37], [38]. Notice that switching systems with arbitrarily switching rule may be very useful in many practical applications. For instance, this is the case when $\sigma_t$ is computed via complex algorithms by a higher level supervisor; or when it is generated by a human operator [25] (for example the switch of gears in a car). On the other hand, notice that the class of systems* (1.42) *can be generalized to systems with $B_{\sigma_t} + \Delta B_{\sigma_t}, E_{\sigma_t} + \Delta E_{\sigma_t}, S_{\sigma_t} + \Delta S_{\sigma_t}, H_{\sigma_t} + \Delta H_{\sigma_t}$, and $J_{\sigma_t} + \Delta J_{\sigma_t}$ instead of $B_{\sigma_t}, E_{\sigma_t}, S_{\sigma_t}, H_{\sigma_t}$, and $J_{\sigma_t}$, respectively. For simplicity of the presentation and more clarity of the new proposed tools, we consider only the class of systems* (1.42) *in this chapter.*

### 4.2.2 $\mathscr{H}_\infty$ Observer-based stabilization problem

The state observer-based controller we consider in this chapter has the following standard structure :

$$\hat{x}_{t+1} = A_\sigma \hat{x}_t + B_\sigma u_t + L_\sigma \Big(y_t - C_\sigma \hat{x}_t\Big) \tag{4.6a}$$
$$u_t = K_\sigma \hat{x}_t \tag{4.6b}$$

where $\hat{x}_t \in \mathbb{R}^n$ is the estimate of $x_t$, and for each $\sigma \in \Lambda$, $L_\sigma \in \mathbb{R}^{n\times p}, K_\sigma \in \mathbb{R}^{m\times n}$ is the observer-based controller gains to be determined such that the estimation error $e_t = x_t - \hat{x}_t$ and the state $x_t$ satisfy a prescribed performance criterion, namely the $\mathscr{H}_\infty$ criterion considered in this chapter.

From (4.1) and (4.6), the dynamics of the augmented vector $\bar{x}_t = [\hat{x}_t^T \quad e_t^T]^T$ is given by :

$$\begin{aligned} \bar{x}_{t+1} &= \underbrace{\begin{bmatrix} \Omega_{11}(\sigma) & \Omega_{12}(\sigma) \\ \Omega_{21}(\sigma) & \Omega_{22}(\sigma) \end{bmatrix}}_{\Omega_\sigma} \bar{x}_t + \underbrace{\begin{bmatrix} L_\sigma S_\sigma \\ L_\sigma S_\sigma - E_\sigma \end{bmatrix}}_{\Pi_\sigma} w_t \\ &\triangleq \Omega_\sigma \bar{x}_t + \Pi_\sigma w_t \end{aligned} \tag{4.7}$$

where

$$\Omega_{11}(\sigma) = A_\sigma + B_\sigma K_\sigma + L_\sigma \Delta C_\sigma, \tag{4.8a}$$
$$\Omega_{12}(\sigma) = -L_\sigma (C_\sigma + \Delta C_\sigma), \tag{4.8b}$$
$$\Omega_{21}(\sigma) = -(\Delta A_\sigma - L_\sigma \Delta C_\sigma), \tag{4.8c}$$
$$\Omega_{22}(\sigma) = A_\sigma + \Delta A_\sigma - L_\sigma (C_\sigma + \Delta C_\sigma). \tag{4.8d}$$

Let us define the indicator function

$$\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_N(t)]^T$$

as follows :

$$\xi_i(t) = \begin{cases} 1, & \sigma_t = i\,; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, system (4.7) and $z_t$ in (4.1c) can be rewritten in the unified form :

$$\begin{bmatrix} \bar{x}_{t+1} \\ z_t \end{bmatrix} = \sum_{i=1}^{N} \xi_i(t) \begin{bmatrix} \Omega_i & \Pi_i \\ H_i + D_i K_i & -H_i \end{bmatrix} \begin{bmatrix} \bar{x}_t \\ w_t \end{bmatrix}, \tag{4.9}$$

where $\Omega_i$ are defined in (4.7)-(4.8), when $\sigma_t = i$.
In the aim to analyze stability of the closed-loop system (4.9), we use the switched Lyapunov function (1.49). Notice that the Lyapunov function (1.49) is well known in the literature, (see for instance [54] and [60]). For shortness we use $\sigma_t = i$ and $\sigma_{t+1} = j$. This means that $\xi_i(t) = 1$ and $\xi_j(t+1) = 1$. Then we have

$$\begin{aligned} \Delta V_{ij}(t) &\triangleq V(\bar{x}_{t+1}, \xi(t+1)) - V(\bar{x}_t, \xi(t)) \\ &= \bar{x}_{t+1}^T \Big(\sum_{i=1}^{N} \xi_i(t+1) \hat{P}_i\Big) \bar{x}_{t+1} - \bar{x}_t^T \Big(\sum_{i=1}^{N} \xi_i(t) \hat{P}_i\Big) \bar{x}_t \\ &= [\bar{x}_t^T \quad \bar{x}_{t+1}^T] \begin{bmatrix} -\hat{P}_i & 0 \\ 0 & \hat{P}_j \end{bmatrix} [\bar{x}_t^T \quad \bar{x}_{t+1}^T]^T \end{aligned} \tag{4.10}$$

Hence the $\mathscr{H}_\infty$ performance criterion is fulfilled if the following inequality holds :

$$\vartheta_{ij}(t) := \Delta V_{ij}(t) + z_t^T z_t - \mu w_t^T w_t < 0. \tag{4.11}$$

It is worth to notice that the inequalities (4.11) are sufficient to ensure the $\mathscr{H}_\infty$ criterion

$$\|z\|_{\ell_2^n} \leq \sqrt{\mu \|w\|_{\ell_2^q}^2 + \nu \|z_0\|^2} \tag{4.12}$$

where $\sqrt{\mu}$ is the disturbance attenuation level, representing the disturbance gain from $w$ to $z$, and $\nu > 0$ is to be determined. To show how (4.11) implies the classical $\mathscr{H}_\infty$ criterion (4.12), we refer the reader to [56] and [70] for instance. This criterion is well known in the literature and the use of Lyapunov analysis like in (4.11) is standard.

### 4.2.3 Application of Finsler's Lemma

Now we will exploit the Finsler's lemma to get sufficient conditions ensuing $\vartheta_{ij}(t) < 0$ for all $t \geq 0$.
Let us introduce the following notations :

$$\zeta_t = \begin{bmatrix} \bar{x}_t \\ \bar{x}_{t+1} \\ w_t \end{bmatrix}, P_{ij} = \begin{bmatrix} -\hat{P}_i & 0 & 0 & \Upsilon_i \\ (\star) & \hat{P}_j & 0 & 0 \\ (\star) & (\star) & -\mu I & J_i^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix},$$

$$U_i = \begin{bmatrix} \Omega_i & -I & \Pi_i \end{bmatrix},$$

$$\Upsilon_i = \begin{bmatrix} K_i^T D_i^T + H_i^T \\ -H_i^T \end{bmatrix}, \ \forall i, j \in \Lambda$$

We have $U_i \zeta_t = 0$ and $\vartheta(t) = \zeta_t^\top P_{ij} \zeta_t$. Then, from Lemma **??** (Finsler's Lemma), we deduce that

$$\vartheta(t) < 0, \ \forall U_i \zeta_t = 0, \zeta_t \neq 0$$

if there exists

$$X_{i,j} = \begin{bmatrix} F_{i,j} \\ G_{i,j} \\ T_{i,j} \end{bmatrix}$$

such that

$$P_{ij} + X_{i,j} U_i + U_i^T X_{i,j}^T < 0. \tag{4.13}$$

After developing the calculations, we get the following equivalent detailed form of (4.13) :

$$\begin{bmatrix} \Im_{ij} & -F_{ij} + \Omega_i^T G_{ij}^T & F_{ij}\Pi_i + \Omega_i^T T_{ij}^T & \Upsilon_i \\ (\star) & \hat{P}_j - \mathrm{He}(G_{ij}) & G_{ij}\Pi_i - T_{ij}^T & 0 \\ (\star) & (\star) & -\mu I + \mathrm{He}(T_{ij}\Pi_i) & J_i^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} < 0, \tag{4.14}$$

for all $i, j \in \Lambda$, where $\Im_{ij} = \mathrm{He}(F_{ij}\Omega_i) - \hat{P}_i$, and $\mathrm{He}(Y) = Y + Y^T$, for any matrix $Y$.
Rewriting the matrices $F_{ij}, G_{ij}, T_{ij}$ and $\hat{P}_i$ under the detailed forms :

$$F_{ij} = \begin{bmatrix} F_{ij}^{11} & F_{ij}^{12} \\ F_{ij}^{21} & F_{ij}^{22} \end{bmatrix}, \tag{4.15}$$

$$G_{ij} = \begin{bmatrix} G_{ij}^{11} & G_{ij}^{12} \\ G_{ij}^{21} & G_{ij}^{22} \end{bmatrix}, \tag{4.16}$$

$$\hat{P}_i = \begin{bmatrix} \hat{P}_i^{11} & \hat{P}_i^{12} \\ (\star) & \hat{P}_i^{22} \end{bmatrix}, \tag{4.17}$$

$$T_{ij} = \begin{bmatrix} T_{ij}^1 & T_{ij}^2 \end{bmatrix}, \tag{4.18}$$

we get the equivalent detailed form of (4.14) :

$$\begin{bmatrix} \Psi_{ij} & \begin{bmatrix} \Upsilon_i^T & 0 & 0 & J_i \end{bmatrix}^T \\ (\star) & -I \end{bmatrix} < 0, \tag{4.19}$$

for all $i, j \in \Lambda$, where

$$\Psi_{ij} = \begin{bmatrix} \Omega_{11}^{ij} & \Omega_{12}^{ij} & \Omega_{13}^{ij} & \Omega_{14}^{ij} & \Omega_{15}^{ij} \\ (\star) & \Omega_{22}^{ij} & \Omega_{23}^{ij} & \Omega_{24}^{ij} & \Omega_{25}^{ij} \\ (\star) & (\star) & \Omega_{33}^{ij} & \hat{P}_j^{12} - G_{ij}^{12} - (G_{ij}^{21})^T & \Omega_{35}^{ij} \\ (\star) & (\star) & (\star) & \hat{P}_j^{22} - G_{ij}^{22} - (G_{ij}^{22})^T & \Omega_{45}^{ij} \\ (\star) & (\star) & (\star) & (\star) & \Omega_{55}^{ij} \end{bmatrix},$$

$$\begin{aligned}
\Omega_{11}^{ij} =& -\hat{P}_i^{11} + \mathrm{He}\Big(F_{ij}^{11}A_i + F_{ij}^{11}B_iK_i + (F_{ij}^{11} + F_{ij}^{12})L_i\Delta C_i - F_{ij}^{12}\Delta A_i\Big), \\
\Omega_{12}^{ij} =& -\hat{P}_i^{12} + F_{ij}^{12}(A_i + \Delta A_i) - (F_{ij}^{11} + F_{ij}^{12})L_i(C_i + \Delta C_i) + K_i^T B_i^T (F_{ij}^{21})^T + A_i^T (F_{ij}^{21})^T \\
& + \Delta C_i^T L_i^T (F_{ij}^{21} + F_{ij}^{22})^T - \Delta A_i^T (F_{ij}^{22})^T, \\
\Omega_{13}^{ij} =& -F_{ij}^{11} + A_i^T (G_{ij}^{11})^T + K_i^T B_i^T (G_{ij}^{11})^T - \Delta A_i^T (G_{ij}^{12})^T + \Delta C_i^T L_i^T (G_{ij}^{11} + G_{ij}^{12})^T, \\
\Omega_{14}^{ij} =& -F_{ij}^{12} + A_i^T (G_{ij}^{21})^T + K_i^T B_i^T (G_{ij}^{21})^T - \Delta A_i^T (G_{ij}^{22})^T + \Delta C_i^T L_i^T (G_{ij}^{21} + G_{ij}^{22})^T, \\
\Omega_{15}^{ij} =& A_i^T (T_{ij}^1)^T + K_i^T B_i^T (T_{ij}^1)^T + \Delta C_i^T L_i^T (T_{ij}^1 + T_{ij}^2)^T - \Delta A_i^T (T_{ij}^2)^T + (F_{ij}^{11} + F_{ij}^{12})L_iS_i - F_{ij}^{12}E_i, \\
\Omega_{22}^{ij} =& -\hat{P}_i^{22} + \mathrm{He}\Big(F_{ij}^{22}A_i + F_{ij}^{22}\Delta A_i - (F_{ij}^{22} + F_{ij}^{21})L_i(C_i + \Delta C_i)\Big), \\
\Omega_{23}^{ij} =& -F_{ij}^{21} - (C_i + \Delta C_i)^T L_i^T (G_{ij}^{11} + G_{ij}^{12})^T + A_i^T (G_{ij}^{12})^T + \Delta A_i^T (G_{ij}^{12})^T, \\
\Omega_{24}^{ij} =& -F_{ij}^{22} + A_i^T (G_{ij}^{22})^T + \Delta A_i^T (G_{ij}^{22})^T - (C_i + \Delta C_i)^T L_i^T (G_{ij}^{22} + G_{ij}^{21})^T, \\
\Omega_{25}^{ij} =& -(C_i + \Delta C_i)^T L_i^T (T_{ij}^1 + T_{ij}^2)^T + A_i^T (T_{ij}^2)^T + \Delta A_i^T (T_{ij}^2)^T + (F_{ij}^{21} + F_{ij}^{22})L_iS_i - F_{ij}^{22}E_i, \\
\Omega_{33}^{ij} =& \hat{P}_j^{11} - (G_{ij}^{11})^T - G_{ij}^{11}, \\
\Omega_{35}^{ij} =& (G_{ij}^{11} + G_{ij}^{12})L_iS_i - G_{ij}^{12}E_i - (T_{ij}^1)^T, \\
\Omega_{45}^{ij} =& (G_{ij}^{21} + G_{ij}^{22})L_iS_i - G_{ij}^{22}E_i - (T_{ij}^2)^T, \\
\Omega_{55}^{ij} =& -\mu I + \mathrm{He}\Big((T_{ij}^1 + T_{ij}^2)L_iS_i - T_{ij}^2E_i\Big).
\end{aligned}$$

In the next section we will provide some techniques allowing to handle the BMI problem (4.19). By exploiting the Finsler's lemma, we will show that convenient choices of some matrices in $X_{ij}$ lead to less conservative LMI conditions compared to the existing results in the literature.

## 4.3 Main contribution : New LMI design

This section is devoted tot the main contribution of this chapter. We will propose new and less conservative LMI conditions to handle the observer-based stabilization problem for a class of linear switched systems in the presence of $\mathscr{L}_2$ bounded disturbances and norm-bounded parameter uncertainties.

### 4.3.1 Main Theorem

This subsection is devoted to the main result of the chapter. We will provide less conservative LMI synthesis conditions ensuring the $\mathscr{H}_\infty$ criterion (4.12).

**Theorem 4.3.1.** *If for $i, j \in \Lambda$ there exist positive definite matrices $\tilde{P}_i^{11}, \hat{P}_i^{22} \in \mathbb{R}^{n\times n}$, invertible matrices $G_i^{22}, \tilde{G}_{ij}^{11}, \tilde{F}_i^{11} \in \mathbb{R}^{n\times n}$, matrices $\tilde{K}_i \in \mathbb{R}^{m\times n}, \hat{L}_i \in \mathbb{R}^{n\times p}$, such that the following convex optimization problem holds for some positive constants $\varepsilon_i$ and $\lambda_i$ :*

$$\min(\mu) \ \textit{subject to} \tag{4.20}$$

$$\begin{bmatrix} \Xi_{ij} & \mathbb{S}_i \\ (\star) & \mathbb{D}_i \end{bmatrix} < 0, \forall i, j \in \Lambda, \tag{4.21}$$

*where*

$$\Xi_{ij} = \begin{bmatrix} \Upsilon_{11}^i & -A_i & \Upsilon_{13}^{ij} & I & E_i & \Upsilon_{16}^i & 0 \\ (\star) & -\hat{P}_i^{22} & -A_i^T & \Upsilon_{24}^i & 0 & -H_i^T & 0 \\ (\star) & (\star) & \Upsilon_{33}^{ij} & I & E_i & 0 & \tilde{G}_{ij}^{11} \\ (\star) & (\star) & (\star) & \Upsilon_{44}^{ij} & \Upsilon_{45}^i & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -\mu I & J_i^T & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -I & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\tilde{P}_j^{11} \end{bmatrix}, \tag{4.22}$$

$$\begin{aligned} \Upsilon_{11}^i &= \tilde{P}_i^{11} + \mathrm{He}\Big( - \tilde{F}_i^{11} + A_i (\tilde{F}_i^{11})^T + B_i \tilde{K}_i \Big), \\ \Upsilon_{13}^{ij} &= \tilde{F}_i^{11} A_i^T + \tilde{K}_i^T B_i^T - (\tilde{G}_{ij}^{11})^T, \\ \Upsilon_{16}^i &= \tilde{K}_i^T D_i^T + \tilde{F}_i^{11} H_i^T, \\ \Upsilon_{24}^i &= A_i^T (G_i^{22})^T - C_i^T \hat{L}_i^T, \\ \Upsilon_{33}^{ij} &= -\mathrm{He}(\tilde{G}_{ij}^{11}), \\ \Upsilon_{45}^i &= \hat{L}_i S_i - G_i^{22} E_i, \\ \Upsilon_{44}^{ij} &= \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T, \end{aligned} \tag{4.23}$$

$$\mathbb{S}_i = \begin{bmatrix} -\tilde{F}_i^{11} \mathscr{E}_i^T & -M_i & \tilde{F}_i^{11} \mathscr{S}_i^T & 0 \\ \mathscr{E}_i^T & 0 & -\mathscr{S}_i^T & 0 \\ 0 & -M_i & 0 & 0 \\ 0 & G_i^{22} M_i & 0 & \hat{L}_i N_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.24}$$

$$\mathbb{D}_i = \mathrm{diag}(-\varepsilon_i^{-1} I, -\varepsilon_i I, -\lambda_i^{-1} I, -\lambda_i I), \tag{4.25}$$

*then the $\mathscr{H}_\infty$ criterion* (4.12) *is satisfied with the obtained minimum attenuation level $\mu$ and the observer-based controller gains :*

$$K_i = \tilde{K}_i (\tilde{F}_i^{11})^{-T}, \ L_i = (G_i^{22})^{-1} \hat{L}_i. \tag{4.26}$$

### 4.3.2 Proof of Theorem 4.3.1

The proof is too long and uses different tools. To enhance the clarity of the contributions, we shared it into three steps. We will present the linearization of the BMI (4.19) to get the LMI (4.21) step by step until a full linearization.

**First step : Linearization with respect to $K_i$**

From inequality (4.19), we can deduce that the matrices $G_{ij}^{11}$, and $G_{ij}^{22}$ are invertible. Let us focus on the case where $F_{ij}^{11}$ is invertible and independent of $j$. This is mainly due to the following principle of congruence. Indeed, by pre- and post-multiply the left hand side of (4.19) by

$$\operatorname{diag}\Big((F_i^{11})^{-1}, I, (G_{ij}^{11})^{-1}, I\Big)$$

and by using the change of variables

$$\tilde{G}_{ij}^{11} = (G_{ij}^{11})^{-1}, \tilde{F}_i^{11} = (F_i^{11})^{-1}, \tilde{K}_i = K_i(\tilde{F}_i^{11})^T$$

we get the equivalent inequality

$$\begin{bmatrix} \tilde{\Omega}_{11}^{ij} & \tilde{\Omega}_{12}^{ij} & \tilde{\Omega}_{13}^{ij} & \tilde{\Omega}_{14}^{ij} & \tilde{\Omega}_{15}^{ij} & \tilde{K}_i^T D_i^T + \tilde{F}_i^{11} H_i^T \\ (\star) & \Omega_{22}^{ij} & \tilde{\Omega}_{23}^{ij} & \Omega_{24}^{ij} & \Omega_{25}^{ij} & -H_i^T \\ (\star) & (\star) & \tilde{\Omega}_{33}^{ij} & \tilde{\Omega}_{34}^{ij} & \tilde{\Omega}_{35}^{ij} & 0 \\ (\star) & (\star) & (\star) & \Omega_{44}^{ij} & \Omega_{45}^{ij} & 0 \\ (\star) & (\star) & (\star) & (\star) & \Omega_{55}^{ij} & J_i^T \\ (\star) & (\star) & (\star) & (\star) & (\star) & -I \end{bmatrix} < 0, \tag{4.27}$$

where

$$\begin{aligned}
\tilde{\Omega}_{11}^{ij} =& -\tilde{F}_i^{11}\hat{P}_i^{11}(\tilde{F}_i^{11})^T + \mathrm{He}\Big(A_i(\tilde{F}_i^{11})^T + B_i\tilde{K}_i + (I+\tilde{F}_i^{11}F_{ij}^{12})L_i\Delta C_i(\tilde{F}_i^{11})^T \\
& -\tilde{F}_i^{11}F_{ij}^{12}\Delta A_i(\tilde{F}_i^{11})^T\Big), \\
\tilde{\Omega}_{12}^{ij} =& \tilde{F}_i^{11}F_{ij}^{12}(A_i+\Delta A_i) - (I+\tilde{F}_i^{11}F_{ij}^{12})L_i(C_i+\Delta C_i) + \tilde{K}_i^T B_i^T (F_{ij}^{21})^T + \tilde{F}_i^{11}A_i^T(F_{ij}^{21})^T - \tilde{F}_i^{11}\hat{P}_i^{12} \\
& + \tilde{F}_i^{11}\Delta C_i^T L_i^T (F_{ij}^{21}+F_{ij}^{22})^T - \tilde{F}_i^{11}\Delta A_i^T(F_{ij}^{22})^T, \\
\tilde{\Omega}_{13}^{ij} =& -(\tilde{G}_{ij}^{11})^T + \tilde{F}_i^{11}A_i^T + \tilde{K}_i^T B_i^T - \tilde{F}_i^{11}\Delta A_i^T(\tilde{G}_{ij}^{11}G_{ij}^{12})^T + \tilde{F}_i^{11}\Delta C_i^T L_i^T (I+\tilde{G}_{ij}^{11}G_{ij}^{12})^T, \\
\tilde{\Omega}_{14}^{ij} =& -\tilde{F}_i^{11}F_{ij}^{12} + \tilde{F}_i^{11}A_i^T(G_{ij}^{21})^T + \tilde{K}_i^T B_i^T (G_{ij}^{21})^T - \tilde{F}_i^{11}\Delta A_i^T(G_{ij}^{22})^T + \tilde{F}_i^{11}\Delta C_i^T L_i^T (G_{ij}^{21}+G_{ij}^{22})^T, \\
\tilde{\Omega}_{15}^{ij} =& \tilde{F}_i^{11}A_i^T(T_{ij}^1)^T + \tilde{K}_i^T B_i^T (T_{ij}^1)^T + \tilde{F}_i^{11}\Delta C_i^T L_i^T (T_{ij}^1+T_{ij}^2)^T - \tilde{F}_i^{11}\Delta A_i^T(T_{ij}^2)^T \\
& + (I+\tilde{F}_i^{11}F_{ij}^{12})L_iS_i - \tilde{F}_i^{11}F_{ij}^{12}E_i, \\
\tilde{\Omega}_{23}^{ij} =& -F_{ij}^{21}(\tilde{G}_{ij}^{11})^T - (C_i+\Delta C_i)^T L_i^T (I+\tilde{G}_{ij}^{11}G_{ij}^{12})^T + (A_i^T+\Delta A_i^T)(\tilde{G}_{ij}^{11}G_{ij}^{12})^T, \\
\tilde{\Omega}_{33}^{ij} =& \tilde{G}_{ij}^{11}\hat{P}_j^{11}(\tilde{G}_{ij}^{11})^T - \tilde{G}_{ij}^{11} - (\tilde{G}_{ij}^{11})^T,
\end{aligned}$$

$$\tilde{\Omega}_{34}^{ij} = \tilde{G}_{ij}^{11}\hat{P}_j^{12} - \tilde{G}_{ij}^{11}G_{ij}^{12} - \tilde{G}_{ij}^{11}(G_{ij}^{21})^T,$$
$$\tilde{\Omega}_{35}^{ij} = (I+\tilde{G}_{ij}^{11}G_{ij}^{12})L_iS_i - \tilde{G}_{ij}^{11}G_{ij}^{12}E_i - \tilde{G}_{ij}^{11}(T_{ij}^{1})^T,$$
$$\Omega_{44}^{ij} = \hat{P}_j^{22} - G_{ij}^{22} - (G_{ij}^{22})^T.$$

Note that $F_{ij}^{11}$ must depend only on $i$, otherwise the previous congruence principle reveals the term $K_i(\tilde{F}_{ij}^{11})^T$, which prevents a change of variables with respect to the gain $K_i$.
There are still some bilinear terms related to $\tilde{K}_i$, which need to be avoided, namely the coupling of $\tilde{K}_i$ with $F_{ij}^{21}, G_{ij}^{21}, T_{ij}^{1}$ and $T_{ij}^{2}$. To do this, the best way we propose is the following convenient and judicious choice :

$$G_{ij}^{21} = F_{ij}^{21} = 0, T_{ij}^{1} = T_{ij}^{2} = 0.$$

This is mainly do to the presence of bilinear terms $\tilde{F}_i^{11}A_i^T(T_{ij}^1)^T, \tilde{F}_i^{11}\Delta A_i^T(T_{ij}^2)^T$, which may lead to very conservative conditions if they are not null. Consequently, by using the change of variable $\tilde{P}_i^{11} = (\hat{P}_i^{11})^{-1}$ and the following Young's inequality $-\tilde{F}_i^{11}\hat{P}_i^{11}(\tilde{F}_i^{11})^T \leq \tilde{P}_i^{11} - \tilde{F}_i^{11} - (\tilde{F}_i^{11})^T$, we deduce that (4.27) is fulfilled if the following inequality holds :

$$\begin{bmatrix} \hat{\Omega}_{11}^{ij} & \hat{\Omega}_{12}^{ij} & \hat{\Omega}_{13}^{ij} & \hat{\Omega}_{14}^{ij} & \hat{\Omega}_{15}^{ij} & \tilde{K}_i^TD_i^T + \tilde{F}_i^{11}H_i^T \\ (\star) & \hat{\Omega}_{22}^{ij} & \hat{\Omega}_{23}^{ij} & \hat{\Omega}_{24}^{ij} & \hat{\Omega}_{25}^{ij} & -H_i^T \\ (\star) & (\star) & \tilde{\Omega}_{33}^{ij} & \hat{\Omega}_{34}^{ij} & \hat{\Omega}_{35}^{ij} & 0 \\ (\star) & (\star) & (\star) & \Omega_{44}^{ij} & \hat{\Omega}_{45}^{ij} & 0 \\ (\star) & (\star) & (\star) & (\star) & -\mu I & J_i^T \\ (\star) & (\star) & (\star) & (\star) & (\star) & -I \end{bmatrix} < 0, \tag{4.28}$$

where

$$\hat{\Omega}_{11}^{ij} = \tilde{P}_i^{11} + \mathrm{He}\Big(-\tilde{F}_i^{11} + A_i(\tilde{F}_i^{11})^T + B_i\tilde{K}_i + (I+\tilde{F}_i^{11}F_{ij}^{12})L_i\Delta C_i(\tilde{F}_i^{11})^T - \tilde{F}_i^{11}F_{ij}^{12}\Delta A_i(\tilde{F}_i^{11})^T\Big),$$
$$\hat{\Omega}_{12}^{ij} = \tilde{F}_i^{11}F_{ij}^{12}(A_i+\Delta A_i) - (I+\tilde{F}_i^{11}F_{ij}^{12})L_i(C_i+\Delta C_i) + \tilde{F}_i^{11}\Delta C_i^TL_i^T(F_{ij}^{22})^T - \tilde{F}_i^{11}\Delta A_i^T(F_{ij}^{22})^T - \tilde{F}_i^{11}\hat{P}_i^{12},$$
$$\hat{\Omega}_{13}^{ij} = -(\tilde{G}_{ij}^{11})^T + \tilde{F}_i^{11}A_i^T + \tilde{K}_i^TB_i^T - \tilde{F}_i^{11}\Delta A_i^T(\tilde{G}_{ij}^{11}G_{ij}^{12})^T + \tilde{F}_i^{11}\Delta C_i^TL_i^T(I+\tilde{G}_{ij}^{11}G_{ij}^{12})^T,$$
$$\hat{\Omega}_{14}^{ij} = -\tilde{F}_i^{11}F_{ij}^{12} - \tilde{F}_i^{11}\Delta A_i^T(G_{ij}^{22})^T + \tilde{F}_i^{11}\Delta C_i^TL_i^T(G_{ij}^{22})^T,$$
$$\hat{\Omega}_{15}^{ij} = (I+\tilde{F}_i^{11}F_{ij}^{12})L_iS_i - \tilde{F}_i^{11}F_{ij}^{12}E_i,$$
$$\hat{\Omega}_{22}^{ij} = -\hat{P}_i^{22} + \mathrm{He}\Big(F_{ij}^{22}A_i + F_{ij}^{22}\Delta A_i - F_{ij}^{22}L_i(C_i+\Delta C_i)\Big),$$
$$\hat{\Omega}_{23}^{ij} = -(C_i+\Delta C_i)^TL_i^T(I+\tilde{G}_{ij}^{11}G_{ij}^{12})^T + (A_i^T+\Delta A_i^T)(\tilde{G}_{ij}^{11}G_{ij}^{12})^T,$$
$$\hat{\Omega}_{24}^{ij} = -F_{ij}^{22} + A_i^T(G_{ij}^{22})^T + \Delta A_i^T(G_{ij}^{22})^T - (C_i+\Delta C_i)^TL_i^T(G_{ij}^{22})^T,$$
$$\hat{\Omega}_{25}^{ij} = F_{ij}^{22}L_iS_i - F_{ij}^{22}E_i,$$
$$\tilde{\Omega}_{33}^{ij} = \tilde{G}_{ij}^{11}\hat{P}_j^{11}(\tilde{G}_{ij}^{11})^T - \tilde{G}_{ij}^{11} - (\tilde{G}_{ij}^{11})^T,$$
$$\hat{\Omega}_{34}^{ij} = \tilde{G}_{ij}^{11}\hat{P}_j^{12} - \tilde{G}_{ij}^{11}G_{ij}^{12},$$
$$\hat{\Omega}_{35}^{ij} = (I+\tilde{G}_{ij}^{11}G_{ij}^{12})L_iS_i - \tilde{G}_{ij}^{11}G_{ij}^{12}E_i,$$

$$\hat{\Omega}_{45}^{ij} = G_{ij}^{22} L_i S_i - G_{ij}^{22} E_i.$$

Now that the BMI (4.19) is linearized with respect to the matrices $\tilde{K}_i$, we will proceed to the linearization with respect to the observer gains $L_i$. The next second step is dedicated to this issue.

## Second step : Semi-linearization with respect to $L_i$

The gains $L_i$ are coupled with the matrices $(I + \tilde{F}_i^{11} F_{ij}^{12}) L_i C_i$, $(I + \tilde{G}_{ij}^{11} G_{ij}^{12}) L_i C_i$, $F_{ij}^{22} L_i C_i$, and $G_{ij}^{22} L_i C_i$. Since $G_{ij}^{22}$ is necessarily invertible and $L_i$ depends only on $i$, then to avoid complicated bilinearities, we take $G_{ij}^{22} = G_i^{22}$ independent of $j$. With the following convenient matrices :

$$F_i = \begin{bmatrix} F_i^{11} & -F_i^{11} \\ 0 & 0 \end{bmatrix}, \tag{4.29a}$$

$$G_{ij} = \begin{bmatrix} G_{ij}^{11} & -G_{ij}^{11} \\ 0 & G_i^{22} \end{bmatrix}, \tag{4.29b}$$

$$\hat{P}_i^{12} = 0, \tag{4.29c}$$

and the change of variable $\hat{L}_i = G_i^{22} L_i$, we get the following semi-linearized version of (4.28) with respect to $L_i$ :

$$\begin{bmatrix} \Theta_{11}^i & \Theta_{12}^i & \Theta_{13}^{ij} & \Theta_{14}^i & E_i & \Theta_{16}^i & 0 \\ (\star) & -\hat{P}_i^{22} & \Theta_{23}^i & \Theta_{24}^i & 0 & -H_i^T & 0 \\ (\star) & (\star) & \Theta_{33}^{ij} & I & E_i & 0 & \tilde{G}_{ij}^{11} \\ (\star) & (\star) & (\star) & \Theta_{44}^{ij} & \Theta_{45}^i & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -\mu I & J_i^T & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -I & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\tilde{P}_j^{11} \end{bmatrix} < 0, \tag{4.30}$$

where

$$\begin{aligned}
\Theta_{11}^i &= \tilde{P}_i^{11} + \mathrm{He}\Big( -\tilde{F}_i^{11} + A_i (\tilde{F}_i^{11})^T + B_i \tilde{K}_i + \Delta A_i (\tilde{F}_i^{11})^T \Big), \\
\Theta_{12}^i &= -(A_i + \Delta A_i), \\
\Theta_{13}^{ij} &= -(\tilde{G}_{ij}^{11})^T + \tilde{F}_i^{11} A_i^T + \tilde{K}_i^T B_i^T + \tilde{F}_i^{11} \Delta A_i^T, \\
\Theta_{14}^i &= I - \tilde{F}_i^{11} \Delta A_i^T (G_i^{22})^T + \tilde{F}_i^{11} \Delta C_i^T \hat{L}_i^T, \\
\Theta_{16}^i &= \tilde{K}_i^T D_i^T + \tilde{F}_i^{11} H_i^T, \\
\Theta_{23}^i &= -(A_i^T + \Delta A_i^T), \qquad \Theta_{33}^{ij} = -\tilde{G}_{ij}^{11} - (\tilde{G}_{ij}^{11})^T, \\
\Theta_{24}^i &= A_i^T (G_i^{22})^T + \Delta A_i^T (G_i^{22})^T - (C_i + \Delta C_i)^T \hat{L}_i^T, \\
\Theta_{44}^{ij} &= \hat{P}_j^{22} - G_i^{22} - (G_i^{22})^T, \qquad \Theta_{45}^i = \hat{L}_i S_i - G_i^{22} E_i.
\end{aligned}$$

In fact, equations (4.29a)-(4.29b) imply

$$I+\tilde{F}_i^{11}F_{ij}^{12}=0, \tag{4.31a}$$

$$I+\tilde{G}_{ij}^{11}G_{ij}^{12}=0, \tag{4.31b}$$

which mean that the bilinear terms $(I+\tilde{F}_i^{11}F_{ij}^{12})L_iC_i$, $(I+\tilde{G}_{ij}^{11}G_{ij}^{12})L_iC_i$ related to $L_i$ vanish. Finally, with (4.29c) and from Schur Lemma applied on the term $\tilde{G}_{ij}^{11}\hat{P}_j^{11}(\tilde{G}_{ij}^{11})^T$, we get easily (4.30).
Hence all the bilinear terms except those related to $\Delta A_i$ and $\Delta C_i$, are avoided. The linearization of the terms related to the uncertainties is the aim of the next and last step of the proof.

**Third step : Full linearization**

In this subsection, we use Young relation for linearize uncertainties. By developing $\Delta A_i$ and $\Delta C_i$ , we can rewrite the previous inequality in the following more suitable form : Inequality (4.30) may be rewritten under the form :

$$\Xi_{ij}+\mathrm{He}\Big(Z_{i1}^T\Gamma_i^T Z_{i2}+Z_{i3}^T\Gamma_i^T Z_{i4}\Big)<0, \tag{4.32}$$

where

$$\begin{aligned}
Z_{i1} &= \begin{bmatrix}-\mathscr{E}_i(\tilde{F}_i^{11})^T & \mathscr{E}_i & 0 & 0 & 0 & 0 & 0\end{bmatrix},\\
Z_{i3} &= \begin{bmatrix}\mathscr{S}_i(\tilde{F}_i^{11})^T & -\mathscr{S}_i & 0 & 0 & 0 & 0 & 0\end{bmatrix},\\
Z_{i2} &= \begin{bmatrix}-M_i^T & 0 & -M_i^T & M_i^T(G_i^{22})^T & 0 & 0 & 0\end{bmatrix},\\
Z_{i4} &= \begin{bmatrix}0 & 0 & 0 & N_i^T\hat{L}_i^T & 0 & 0 & 0\end{bmatrix},
\end{aligned}$$

and $\Xi_{ij}$ is defined in (4.22). Consequently, applying the well known Young's inequality, we deduce that (4.32) holds if the following one is satisfied :

$$\Xi_{ij}+\varepsilon_i Z_{i1}^T Z_{i1}+\varepsilon_i^{-1}Z_{i2}^T Z_{i2}+\lambda_i Z_{i3}^T Z_{i3}+\lambda_i^{-1}Z_{i4}^T Z_{i4}, \tag{4.33}$$

where $\varepsilon_i$, $\lambda_i$ are some positive scalars coming from Young's relation. finally, by using again Schur Lemma (1.7.2), inequality (4.33) is fulfilled if the LMI (4.21) is feasible. This ends the proof of Theorem 4.3.1.

## 4.4 Numerical Examples

In this section, we present two numerical examples to show the validity and effectiveness of the proposed design methodology. The superiority of the new LMI synthesis conditions (4.21) is quite clear from these examples.

### 4.4.1 Example 1

We reconsider the example presented in [29] and modify it by adding parameter uncertainties and disturbances. The system is described under the form (4.1) with the matrices given in Table 4.1.

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $A_i$ | $\begin{bmatrix}0.7786 & 0.9908 & 0.1270\\0.1616 & 0.8443 & 0.8144\\0.9214 & 0.9747 & 0.7825\end{bmatrix}$ | $\begin{bmatrix}0.3894 & 0.3263 & 0.7746\\0.7806 & 0.9886 & 0.1297\\0.8814 & 0.4718 & 0.3110\end{bmatrix}$ | $\begin{bmatrix}0.3049 & 0.4247 & 0.8979\\0.8448 & 0.2485 & 0.6921\\0.7558 & 0.9160 & 0.3636\end{bmatrix}$ | $\begin{bmatrix}0.1194 & 0.3964 & 0.2454\\0.1034 & 0.2515 & 0.4983\\0.6981 & 0.8655 & 0.2403\end{bmatrix}$ |
| $B_i$ | $\begin{bmatrix}0.2458 & 0.7409\\0.2501 & 0.5257\\0 & 0\end{bmatrix}$ | $\begin{bmatrix}0.2722 & 0.6055\\0.1576 & 0.1580\\0 & 0\end{bmatrix}$ | $\begin{bmatrix}0.4945 & 0.3020\\0.9237 & 0.9118\\0 & 0\end{bmatrix}$ | $\begin{bmatrix}0.9894 & 0.7205\\0.1709 & 0.1519\\0 & 0\end{bmatrix}$ |
| $C_i$ | $\begin{bmatrix}0.3815 & 0.6916 & 0.7183\end{bmatrix}$ | $\begin{bmatrix}0.0591 & 0.8258 & 0.4354\end{bmatrix}$ | $\begin{bmatrix}0.5204 & 0.8010 & 0.9708\end{bmatrix}$ | $\begin{bmatrix}0.6995 & 0.3081 & 0.8767\end{bmatrix}$ |
| $E_i$ | $\begin{bmatrix}0.1 & 0\\0.2 & 0.1\\0 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0.2\\0.1 & 0.1\\0 & 0.1\end{bmatrix}$ | $\begin{bmatrix}0 & 0\\0.1 & 0.1\\0.1 & 0\end{bmatrix}$ | $\begin{bmatrix}0.1 & 0.2\\-0.2 & 0.1\\0.1 & 0\end{bmatrix}$ |
| $S_i$ | $\begin{bmatrix}0.6 & 0\end{bmatrix}$ | $\begin{bmatrix}0.2 & 0.2\end{bmatrix}$ | $\begin{bmatrix}0.4 & 0.1\end{bmatrix}$ | $\begin{bmatrix}0.2 & 0.5\end{bmatrix}$ |
| $J_i$ | $\begin{bmatrix}0.2 & 0.1\end{bmatrix}$ | $\begin{bmatrix}0.2 & 0\end{bmatrix}$ | $\begin{bmatrix}0.3 & 0.2\end{bmatrix}$ | $\begin{bmatrix}0.5 & 0.1\end{bmatrix}$ |
| $H_i$ | $\begin{bmatrix}0 & 0.1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0.2 & 0\end{bmatrix}$ | $\begin{bmatrix}0.2 & 0 & 0.1\end{bmatrix}$ | $\begin{bmatrix}0.1 & 0.2 & 0\end{bmatrix}$ |
| $M_i$ | $\begin{bmatrix}0.1 & 0 & 0.1\\0 & 0.3 & 0.1\\0 & 0.2 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0.1 & 0\\0.1 & 0 & 0\\0 & 0.1 & 0.1\end{bmatrix}$ | $\begin{bmatrix}0.1 & 0 & 0.3\\0 & 0.2 & 0\\0 & 0.1 & 0.2\end{bmatrix}$ | $\begin{bmatrix}0.1 & 0 & 0\\0 & 0.1 & 0\\0.2 & 0 & 0.2\end{bmatrix}$ |
| $\mathscr{E}_i$ | $\begin{bmatrix}0.3 & 0 & 0.1\\0 & 0.2 & 0\\0.5 & 0 & 0.1\end{bmatrix}$ | $\begin{bmatrix}0.1 & 0 & 0.1\\0 & 0.1 & 0.2\\0.3 & 0.1 & 0\end{bmatrix}$ | $\begin{bmatrix}0.2 & 0 & 0.1\\0 & 0.1 & 0\\0.1 & 0 & 0.2\end{bmatrix}$ | $\begin{bmatrix}0.3 & 0.2 & 0\\0.2 & 0 & 0.1\\0 & 0.3 & 0\end{bmatrix}$ |
| $N_i$ | $\begin{bmatrix}0.1 & 0.1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0.1 & 0.2\end{bmatrix}$ | $\begin{bmatrix}0 & 0.1 & 0.1\end{bmatrix}$ | $\begin{bmatrix}0.1 & 0 & 0.2\end{bmatrix}$ |
| $\mathscr{S}_i$ | $\begin{bmatrix}0.1 & 0 & 0\\0 & 0.1 & 0.1\\0.2 & 0 & 0.1\end{bmatrix}$ | $\begin{bmatrix}0.1 & 0 & 0.1\\0 & 0.1 & 0\\0.1 & 0 & 0.15\end{bmatrix}$ | $\begin{bmatrix}0 & 0 & 0.1\\0.1 & 0 & 0\\0 & 0.1 & 0\end{bmatrix}$ | $\begin{bmatrix}0 & 0 & 0.1\\0.1 & 0 & 0\\0 & 0 & 0.1\end{bmatrix}$ |

TABLE 4.1 – The matrices of the system in Example 1.

All the matrices $A_1, A_2, A_3$ and $A_4$ are unstable. Assume that the system is disturbed by a noise $w_t = \chi(t)\omega_t$, where $\omega_t$ is an uniformly distributed random variable on the interval

[0 1], and $\chi(.)$ is defined by

$$\chi(t) = \begin{cases} 2 & \text{if } t \in [0, 20[; \\ -2 & \text{if } t \in [40, 70[; \\ 0 & \text{elsewhere,} \end{cases}$$

After solving the LMI (4.21) with $\varepsilon_i = 0.5, \lambda_i = \frac{5}{6}$ for $i \in \{1, \ldots, 4\}$, we get the optimal disturbance attenuation level $\mu_{\min} = 2.2998$, and the observer-based controller gains :

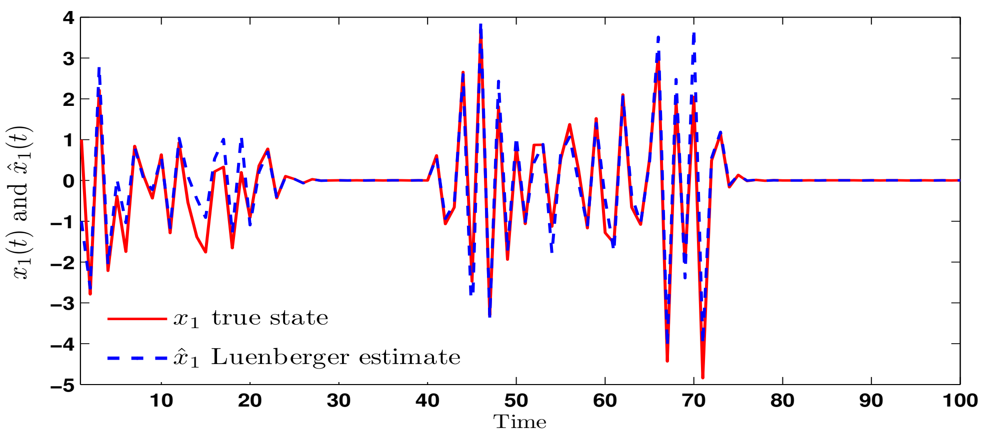
$$K_1 = \begin{bmatrix} 4.5728 & 0.5119 & -9.0280 \\ -3.3328 & -2.1843 & 2.2872 \end{bmatrix}, L_1 = \begin{bmatrix} 0.8025 \\ 0.9772 \\ 1.1982 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -10.1939 & -11.9644 & -0.4646 \\ 2.9566 & 4.2608 & -1.4921 \end{bmatrix}, L_2 = \begin{bmatrix} 0.8025 \\ 0.9772 \\ 1.1982 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -2.3231 & -3.6350 & -4.5955 \\ 1.2054 & 3.1349 & 3.7249 \end{bmatrix}, L_3 = \begin{bmatrix} 0.7091 \\ 0.7368 \\ 0.7709 \end{bmatrix},$$

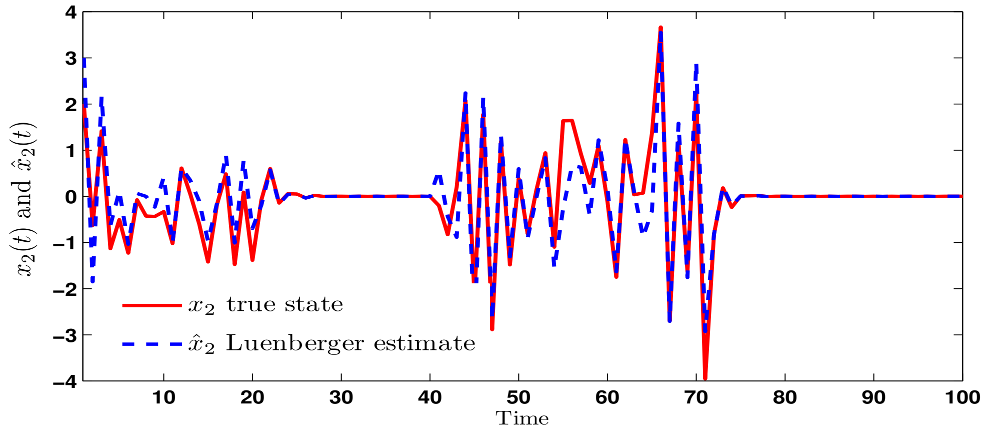
$$K_4 = \begin{bmatrix} 2.2343 & 5.5158 & 11.4936 \\ -3.7789 & -8.7247 & -16.3481 \end{bmatrix}, L_4 = \begin{bmatrix} 0.4002 \\ 0.3955 \\ 0.7685 \end{bmatrix}.$$

The simulation results corresponding to these observer-based controller gains are given in Fig. 4.1(a)-4.1(f). These simulations are done for an horizon $T = 100$, with $x = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}^T, \hat{x} = \begin{bmatrix} -1 & 3 & 6 \end{bmatrix}^T$.

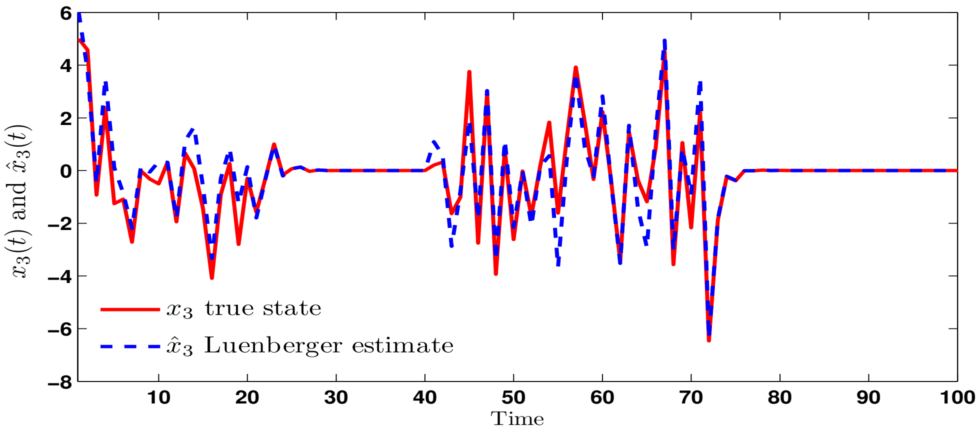



(a) Example 1 : Time-behaviors of $x_1$ and $e_1$ in the presence of uncertainties

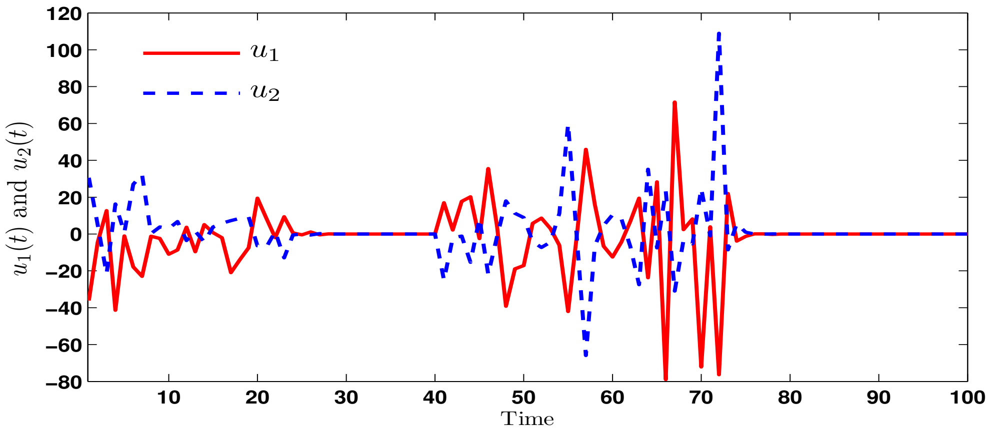
In the goal to show the superiority of the proposed design methodology as compared to [68], we replace $M_i$ by $\gamma_i M_i$ and without disturbance. We look for the maximum values of $\gamma_i$ for $i \in \{1, \ldots, 4\}$ which satisfy the LMI (4.21). The results of the comparisons are summarized in Table 4.2, which shows the superiority of the LMI (4.21).




(b) Example 1 : Time-behaviors of $x_2$ and $e_2$ in the presence of uncertainties



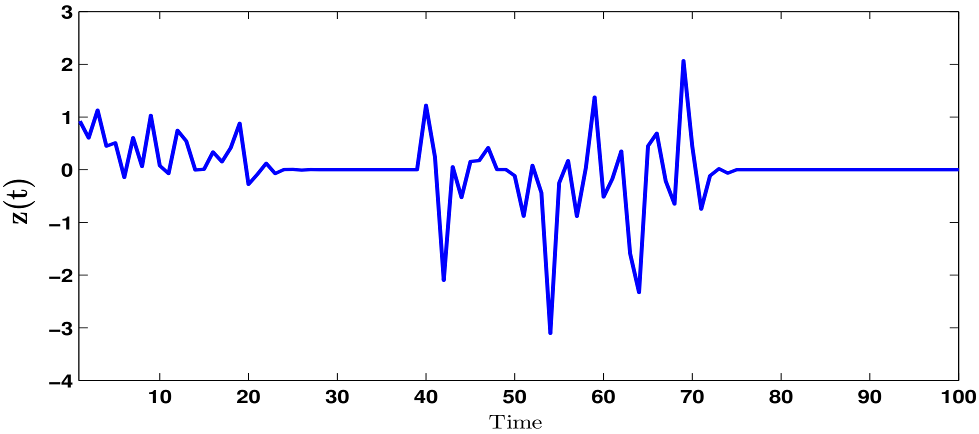

(c) Example 1 : Time-behaviors of $x_3$ and $e_3$ in the presence of uncertainties



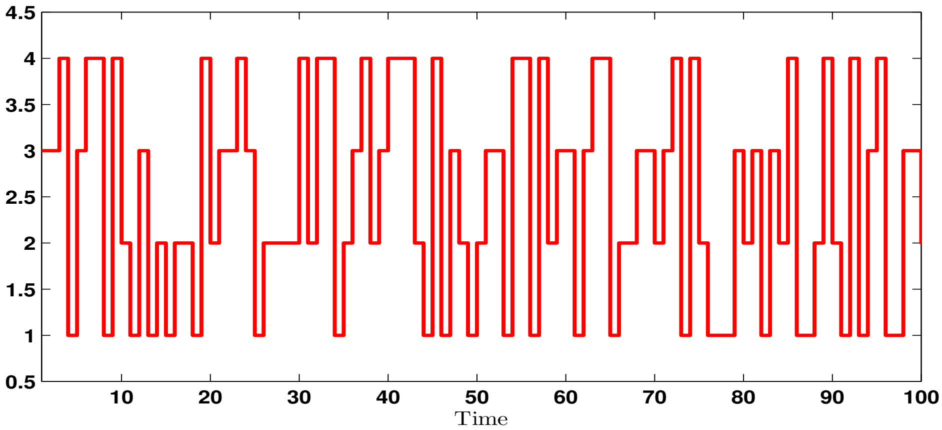

(d) Example 1 : Control input $u_1$ and $u_2$

### 4.4.2 Example 2

Through this example, we will show that the proposed LMIs (4.21) are less conservative than those provided in [71]. Then, we reconsider the same example given in [71],



(e) Control output $z_t$



(f) Switching mode $\sigma_t$

which is a linear system without parameter uncertainties. The system is described by the following matrices :

$$A = \begin{bmatrix} \delta & 0.8 & -0.4 \\ -0.5 & 0.4 & 0.5 \\ 1.2 & 1.1 & 0.8 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 1.3 \end{bmatrix},$$

$$E^T = \begin{bmatrix} 0.1 & 0.4 & 0.1 \end{bmatrix}, C = \begin{bmatrix} -1 & 1.2 & 1 \\ 0 & -3 & 1 \end{bmatrix}, D = 0,$$

$$S^T = \begin{bmatrix} 0.1 & 0.4 \end{bmatrix}, H = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}, J = 0.3$$

Obviously, this example can be viewed as a switching system under the form (4.1) with only one mode (there are no switching) and with $\Delta A_i = \Delta C_i = 0$. We test the feasibility for different values of $\delta$ and we look for the minimum value of $\mu_{\min}$ provided by each method. Table 4.3 summarizes the results.

| **LMI Methods** | **LMI (18) in [68]** | **LMI** (4.21) |
|---|---|---|
| $\gamma_{1\max}$ | 0.6 | 1.3 |
| $\gamma_{2\max}$ | 0.4 | 1.4 |
| $\gamma_{3\max}$ | 0.8 | 1 |
| $\gamma_{4\max}$ | 0.6 | 1.1 |

TABLE 4.2 – Example 1 : Comparison between two LMI design methods.

| $\delta$ | Theorem 1 in [71] | | LMI (4.21) |
|---|---|---|---|
| | $(\alpha, \beta)$ | $\mu_{\min}$ | $\mu_{\min}$ |
| 0.5 | $(-0.03, -2.85)$ | 0.5523 | 0.3703 |
| 1.2 | $(0.11,\ 3.66)$ | 0.6307 | 0.3703 |
| 1.7 | $(1.03,\ 4.08)$ | 2.9248 | 0.3703 |
| 1.9 | $(-1.20, -1.69)$ | 16.1504 | 0.3703 |

TABLE 4.3 – Example 2 : Value of $\mu_{\min}$ for two LMI design methods

## 4.5 Conclusion

In this chapter, new LMI conditions have been developed for the problem of the stabilization of a class of switching discrete-time linear systems with parameter uncertainties and $l_2$-bounded disturbances. We have shown that a judicious choice of slack variables coming from Finsler's lemma leads to less conservative LMIs. Analytical developments have been provided to clarify how the proposed choice allows to eliminate some bilinear matrix coupling without using any conservative inequality. The validity of the proposed design method is shown through two numerical examples.

CHAPITRE 5

# Contribution to Observer-based Control Design via LMI for a class of LPV Discrete-Time Linear Systems

## Sommaire



## 5.1 Introduction

Linear parameter-varying (LPV) modeling has been the subject of increasing interest in the literature with many practical applications in the areas of aerospace, automotive, . . .etc.. Indeed, LPV systems offer an alternative to handle the complexity of nonlinear systems and combine the simplicity of linear systems and the real variability of their parameters avoiding the approximation of nonlinear systems by linearization or with a transformation [40], [41]. If the problem of observer design for LPV systems with known and bounded parameters is easy to investigate, the observer-based stabilization problem is difficult from the LMI point of view. Therefore, the stabilization problem becomes more complicated when the parameters are unknown. Indeed, these unknown parameters lead to bilinear matrix coupling, which are difficult to linearize with known

mathematical tools. Some design methods have been proposed in this area in the literature, however all the proposed techniques still remain conservative ( see for instance [54], [7], [64], [72], [59]). This motivates us to develop new and less conservative LMI conditions by using new mathematical tools.
In this chapter we propose new and enhanced LMI conditions to solve the problem of observer-based stabilization for a class of discrete-time LPV systems with uncertain parameters. The proposed technique consists in designing an observer-based controller which stabilizes the LPV system, provided that the difference between the uncertain parameters and their estimates are norm-bounded with bounds not exceeding a tolerated maximum value. Hence, the observer-based gains depend on these bounds.
One of the contributions of this chapter consists in the use of a relaxed reformulation for the parameter uncertainty. This relaxation is inspired from recent design methods for observer design of nonlinear Lipschitz systems [40], [41]. Using this reformulation, two new LMI synthesis methods are developed to design observer-based controllers for a class of LPV systems with inexact but bounded parameters. The first approach is based on the use of Young's relation, while the second one uses a new congruence principle by pre- and post-multiplying the basic BMI condition by new and ingenious slack matrices. Thanks to these matrices, which can be seen as additional decision variables, some bilinear terms vanish from the BMI, which increases flexibility in the to linearization. We show analytically how particular forms of the slack variables reduce the complexity of the bilinear problem. This new use of congruence principle allows avoiding the gridding method as in [35].

## 5.2 System description and problem statement

Consider a discrete-time LPV system described by the following state-space equation :

$$\begin{cases} x_{t+1} = A(\rho(t))x_t + Bu_t \\ y_t = Cx_t \end{cases} \tag{5.1}$$

where $x_t \in \mathbb{R}^n$ is the state vector, $y_t \in \mathbb{R}^m$ is the measurement vector, $u_t \in \mathbb{R}^p$ is the control signal, for any $t \in \mathbb{Z}^+$. Further, the state matrix $A(\rho(t)) \in \mathbb{R}^{n\times n}$ depends on a bounded time-varying parameter $\rho(t) = [\rho_1(t), \ldots, \rho_N(t)]^T$, which is assumed to be not available in real time, but only an approximated $\hat{\rho}(t) \in \Theta \subset \mathbb{R}^N$, satisfying

$$\sup_{t\in\mathbb{Z}^+} ||\rho(t) - \hat{\rho}(t)|| \leq \Delta \tag{5.2}$$

is known, where $\Delta$ is some nonnegative constant indicating the uncertainty level, and $\Theta$ is some bounded subset of $\mathbb{R}^N$. $B$ and $C$ are are $n \times m$ and $p \times n$ real matrices, respectively. Throughout the paper, the following assumptions are made :

— The matrix $A(\rho(t))$ lies for each $\rho(t) \in \Theta$ in the convex hull $\mathrm{Co}(A_1, ..., A_N)$, that is there exists a finite sequence $\left(\xi^i(\rho(t))\right)_{i=1}^N$ depending on $\rho(t)$ such that $\xi^i(\rho_t) \geq 0$,

$\sum_{i=1}^{N} \xi^i(\rho(t)) = 1$, and

$$A(\rho(t)) = \sum_{i=1}^{N} \xi^i(\rho(t))A_i; \tag{5.3}$$

— The pairs $(A_i,B)$ and $(A_i,C)$, for $i = 1\ldots,N$, are respectively stabilizable and detectable.

The observer-based controller we proposed here is the same as in [37–39], and described by the following equations :

$$\begin{cases} \hat{x}_{t+1} = A(\hat{\rho}(t))\hat{x}_t + L(\hat{\rho}(t))(y_t - C\hat{x}_t) + Bu_t, \\ \hat{y}_t = C\hat{x}_t. \end{cases} \tag{5.4}$$

Based on the estimate $\hat{x}_t$, we develop a set of the controllers of the form

$$u_t = K(\hat{\rho}(t))\hat{x}_t. \tag{5.5}$$

Define $\bar{x}_t = [\hat{x}_t^T \; e_t^T]^T$, where $e_t = \hat{x}_t - x_t$ is the estimate error. Then the closed-loop system of (5.1), (5.4) and (5.5) is described by

$$\bar{x}_{t+1} = \begin{bmatrix} A(\hat{\rho}) + BK(\hat{\rho}) & -L(\hat{\rho})C \\ -\Delta A & A(\hat{\rho}) + \Delta A - L(\hat{\rho})C \end{bmatrix} \bar{x}_t, \tag{5.6}$$

where, for shortness, we set $\Delta A := A(\rho(t)) - A(\hat{\rho}(t))$, $X(\hat{\rho}) := X(\hat{\rho}(t))$ and $Y(\rho) := Y(\rho(t))$, for any parametric matrices $X$ and $Y$.

In this chapter, the aim is to design a collection of observer-based controller gains $K(\hat{\rho}) = \sum_{j=1}^{N} \xi^j(\hat{\rho})K_j$ and $L(\hat{\rho}) = \sum_{j=1}^{N} \xi^j(\hat{\rho})L_j$ such that the closed-loop system (5.6) is globally asymptotically stable.

We first formulate the non-convex optimization problem that allows us to compute the observer-based controller gains $K_j$ and $L_j$, for $j \in \Lambda$, using Lyapunov stability. For the stability analysis, we use the same quadratic and parameter dependent Lyapunov function as that in [39], namely,

$$V(\bar{x},\hat{\rho}) := \bar{x}_t^T P(\hat{\rho}(t))\bar{x}_t = \bar{x}_t^T \begin{bmatrix} P_{11}(\hat{\rho}) & P_{12}(\hat{\rho}) \\ P_{12}^T(\hat{\rho}) & P_{22}(\hat{\rho}) \end{bmatrix} \bar{x}_t,$$

where

$$P(\hat{\rho}(t)) = \sum_{j=1}^{N} \xi^j(\hat{\rho}(t))P_j.$$

One obtains after some calculations (see [39] for more details) that

$$\begin{aligned} \Delta V_t(\bar{x},\hat{\rho}) &:= V(\bar{x}_{t+1},\hat{\rho}(t+1)) - V(\bar{x}_t,\hat{\rho}(t)) \\ &= \sum_{j,l=1}^{N} \xi^j(\hat{\rho}(t))\xi^l(\hat{\rho}(t+1)\bar{x}_t^T \Omega_{jl}\bar{x}_t < 0 \end{aligned}$$

where for each $j,l \in \Lambda$,

$$\Omega_{jl} = -P_j + \Pi_j^T P_l \Pi_j$$

and

$$\Pi_j = \begin{bmatrix} A_j + BK_j & -L_jC \\ -\Delta A & A_j + \Delta A - L_jC \end{bmatrix}. \tag{5.7}$$

We have that $\Delta V_t(\bar{x}, \hat{\rho}) < 0$ for all $\bar{x}_t \neq 0$ and $\hat{\rho}(t) \in \Theta$ if

$$\Omega_{jl} = -P_j + \Pi_j^T P_l \Pi_j < 0, \forall j, l \in \Lambda, \tag{5.8}$$

or, equivalently,

$$\begin{bmatrix} -P_j & \Pi_j^T \\ \Pi_j & -P_l^{-1} \end{bmatrix} < 0, \ \forall j, l \in \Lambda. \tag{5.9}$$

## 5.3 The Proposed LMI Design Procedure

This section proposes a new way of "convexifying" the Lyapunov stability problem (5.9), which is BMI due to many coupling between the Lyapunov matrices and the observer based controller gains. Our purpose is to give two new strategies that give better solutions to (5.9) in the sense that they tolerate larger uncertainty level $\Delta$. We begin by giving an equivalent reformulation for the parameter uncertainty, and consequently a relaxation of the assumption proposed in [37] will be derived.

### 5.3.1 About the Assumption (5.2) on the uncertainty

It is easy to see that condition (5.2) is equivalent to the existence of two vectors $\Delta^{\min} \in \mathbb{R}^N$ and $\Delta^{\max} \in \mathbb{R}^N$ so that

$$\Delta_k^{\min} \leq \rho_k(t) - \hat{\rho}_k(t) \leq \Delta_k^{\max}, \quad \forall k = 1, \ldots, N. \tag{5.10}$$

Thanks to this reformulation of (5.2), it allows us to relaxe the assumptions proposed in [37], namely the matrix $\Delta A$ satisfies the following condition :

$$\Delta A^T \Delta A \leq \Delta^2 \Gamma^T \Gamma \tag{5.11}$$

where $\Gamma$ is known constant matrix of appropriate dimension. This relaxation, inspired from recent design methods for observer design of nonlinear Lipschitz systems [40], [41], is based on the use of an LPV approach to treat the uncertainties on the parameters. From now on, the condition (5.10) will be denoted by (H).
Let us start by checking that under condition (5.2) or (H), the following result, which is given in [39] without proof, holds.

**Proposition 5.3.1.** *Under condition (H), the parametric matrix $\Delta A$ belongs to a bounded convex $\mathbb{B}$, for which the set of vertices is defined by :*

$$\mathbb{V}_{\mathbb{B}} = \Big\{ \mathscr{A}(\rho) = \sum_{(i,j,k) \in \mathbb{S}} \rho_k \lambda_{ij}^k \Lambda_{ij} : \rho_k \in \{\Delta_k^{min}, \Delta_k^{max}\} \Big\}. \tag{5.12}$$

*The set $\mathbb{S}$ is defined by*

$$\mathbb{S} = \Big\{ (i,j,k) : (\rho_k - \hat{\rho}_k)\lambda_{ij}^k \neq 0 \Big\}.$$

The proof is based on the following Lemma :

**Lemma 5.3.2** ( [40]). *Consider a function $\Psi : \mathbb{R}^n \to \mathbb{R}$. Then, for all*

$$X = \left[x_1, \ldots, x_n\right]^T \in \mathbb{R}^n \, and \, Y = \left[y_1, \ldots, y_n\right]^T \in \mathbb{R}^n$$

*there exist functions $\psi_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ j = 1, \ldots, n$ so that*

$$\Psi(X) - \Psi(Y) = \sum_{j=1}^{j=n} \psi_j\left(X^{Y_{j-1}}, X^{Y_j}\right) e_n^T(j)\left(X - Y\right) \tag{5.13}$$

*where $X^{Y_j} = \left[y_1, \ldots, y_j, x_{j+1}, \ldots, x_n\right]^T$, $X^{Y_0} = X$ and $e_n(j)$ is the jth vector of the canonical basis of $\mathbb{R}^n$.*

**Proof of Proposition 1 :** We have the detailed form of $A(\rho)$ :

$$A(\rho) = \sum_{i,j} A_{ij}(\rho)\, e_n(i) e_n^T(j).$$

Thus we can write $\Delta A$ under the more suitable form :

$$\begin{aligned} \Delta A &= A(\rho) - A(\hat{\rho}) \\ &= \sum_{i,j=1}^{n} \left(A_{ij}(\rho) - A_{ij}(\hat{\rho})\right) e_n(i) e_n^T(j). \end{aligned} \tag{5.14}$$

By using Lemma 1 on $A_{ij}$, for each fixed $i, j = 1, \ldots, n$, we obtain the existence of functions $\psi_{ij}^k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ k = 1, \ldots, n$ such that

$$A_{ij}(\rho) - A_{ij}(\hat{\rho}) = \sum_{k=1}^{n} \psi_{ij}^k\left(\rho^{\hat{\rho}_{k-1}}, \rho^{\hat{\rho}_k}\right)(\rho_k - \hat{\rho}_k) \tag{5.15}$$

By replacing in (5.14) the reformulation of $A_{ij}(\rho) - A_{ij}(\hat{\rho})$ given in (5.15), we get

$$A(\rho) - A(\hat{\rho}) = \sum_{i,j,k} (\rho_k - \hat{\rho}_k) \psi_{ij}^k\left(\rho^{\hat{\rho}_{k-1}}, \rho^{\hat{\rho}_k}\right) e_n(i) e_n^T(j) \tag{5.16}$$

Thanks to the following notations

$$\lambda_{ij}^k = \psi_{ij}^k\left(\rho^{\hat{\rho}_{k-1}}, \rho^{\hat{\rho}_k}\right),\ \Lambda_{ij} = e_n(i) e_n^T(j)$$

we obtain that

$$\Delta A = \sum_{(i,j,k) \in \mathbb{S}} (\rho_k(t) - \hat{\rho}_k(t)) \lambda_{ij}^k \Lambda_{ij}, \tag{5.17}$$

from which we conclude that $\Delta A$ belongs to the bounded convex $\mathbb{B}$. Which completes the proof of Proposition 1.

■

Now, let us return to the linearization problem (5.9). To our knowledge, the best manner to linearize the Lyapunov inequality in the discrete-time systems is the introduction of slack variables (see for instance [73], [7]). Since inequality (5.9) depends on both indices $j$, $l$, let us introduce a matrix $G_{jl}$ of adequate dimension that depends on $j$ and $l$. Firstly, we use the congruence principle as follows : we pre- and post- multiply (5.9) by $\mathrm{diag}(I, G_{jl})$ and $\mathrm{diag}(I, G_{jl}^T)$ respectively. Secondly we exploit the adapted well-known inequality (see for example [37, Ineq. (12)]),

$$-G_{jl}P_l^{-1}G_{jl}^T \leq P_l^{11} - G_{jl} - G_{jl}^T, \ \forall j, l \in \Lambda.$$

One obtains that inequalities (5.9) hold, if the following ones

$$\begin{bmatrix} -P_j & \Pi_j^T G_{jl}^T \\ G_{jl}\Pi_j & P_l - G_{jl} - G_{jl}^T \end{bmatrix} < 0, \ \forall j, l \in \Lambda \tag{5.18}$$

are fulfilled. Now, we take the detailed structures of $P_l$ and $G_{jl}$ respectively :

$$P_l = \begin{bmatrix} P_l^{11} & P_l^{12} \\ (\star) & P_l^{22} \end{bmatrix}, G_{jl} = \begin{bmatrix} G_{jl}^{11} & G_{jl}^{12} \\ G_{jl}^{21} & G_{jl}^{22} \end{bmatrix}. \tag{5.19}$$

Notice that the slack variable $G_{jl}$ that we use here is more general than [28, 37–39] since it involves both indices $j$ and $l$ and not necessarily of the form $\alpha_j I$. This form comes from the dependency of inequality (5.9) on both $j$ and $l$.

### 5.3.2 Young's inequality based approach

In this subsection we study the case where the slack variable $G_{jl}$ has a diagonal form. To this end, we choose

$$G_{jl} = \mathrm{diag}\{\sqrt{\varepsilon}G_l^{11}, G_j^{22}\} \tag{5.20}$$

where $\varepsilon$ is some positive scalar. Observe that if $\varepsilon = 1$ and $G_j^{11} = G_j^{22} = \alpha_j I$ we get the approach based on the linearization Lemma of Ibrir adressed in [28] and [39]. On the other hand, the introduction of the parameter $\varepsilon$ inside $G_{jl}$ will reduce considerably the conservatisme of the Young's inequality based approach introduced in [36] and [59] in the continuous case and in [35], in the discrete case. Indeed, our approach does not involve the application of the gridding method.

With the choice (5.20) combined with congruence principle : we pre- and post- multiply (5.18) by

$$\mathrm{diag}\Big((G_j^{11})^{-1}, I, (G_l^{11})^{-1}, I\Big)$$

we get by introducing the following notations and changes of variables :

$$\tilde{G}_j^{11} = (G_j^{11})^{-1}, \ \tilde{P}_j^{11} = \tilde{G}_j^{11}P_j^{11}(\tilde{G}_j^{11})^T, \ \hat{G}_j^{11} = \sqrt{\varepsilon}\tilde{G}_j^{11},$$

$$\tilde{P}_j^{12} = \tilde{G}_j^{11}P_j^{12}, \ \hat{L}_j = G_j^{22}L_j, \ \tilde{K}_j = K_j(\hat{G}_j^{11})^T,$$

that (5.18) hold if the following inequality :

$$\begin{bmatrix} -\tilde{P}_j^{11} & -\tilde{P}_j^{12} & \Omega_{13}^j & -\tilde{G}_j^{11}\Delta A^T(G_j^{22})^T \\ (\star) & -P_j^{22} & -C^T L_j^T\sqrt{\varepsilon} & \Omega_{24}^j \\ (\star) & (\star) & \Omega_{33}^l & \tilde{P}_l^{12} \\ (\star) & (\star) & (\star) & \Omega_{44}^{jl} \end{bmatrix} < 0, \tag{5.21}$$

is fulfilled for all $j, l \in \Lambda$, where

$$\begin{aligned} \Omega_{13}^j &= \hat{G}_j^{11}A_j^T + \tilde{K}_j^T B^T, \\ \Omega_{24}^j &= A_j^T(G_j^{22})^T + \Delta A^T(G_j^{22})^T - C^T\hat{L}_j^T, \\ \Omega_{33}^l &= \tilde{P}_l^{11} - \hat{G}_l^{11} - (\hat{G}_l^{11})^T, \\ \Omega_{44}^{jl} &= P_l^{22} - \mathrm{He}(G_j^{22}). \end{aligned}$$

The BMI (5.21) can be rewritten under the more suitable form :

$$\Psi_{jl} + \mathrm{He}(X_j^T Y) < 0, \forall j, l \in \Lambda \tag{5.22}$$

where

$$\Psi_{jl} = \begin{bmatrix} -\tilde{P}_j^{11} & -\tilde{P}_j^{12} & \hat{G}_j^{11}A_j^T + \tilde{K}_j^T B^T & -\tilde{G}_j^{11}\Delta A^T(G_j^{22})^T \\ (\star) & -P_j^{22} & 0 & \Omega_{24}^j \\ (\star) & (\star) & \Omega_{33}^l & \tilde{P}_l^{12} \\ (\star) & (\star) & (\star) & P_l^{22} - \mathrm{He}(G_j^{22}) \end{bmatrix},$$

$X_j = \begin{bmatrix} 0 & -\sqrt{\varepsilon}L_jC & 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix}$. Using the Young's relation as in [35] and [36], we obtain that inequalities (5.22) hold if the following

$$\Psi_{jl} - \begin{bmatrix} S_jX_j \\ Y \end{bmatrix}^T \begin{bmatrix} -\frac{1}{\varepsilon}S_j & 0 \\ 0 & -\varepsilon S_j \end{bmatrix}^{-1} \begin{bmatrix} S_jX_j \\ Y \end{bmatrix} < 0, \forall j, l \in \Lambda \tag{5.23}$$

are satisfied for $S_j = G_j^{22}$ and $\varepsilon$ is given in (5.20). Hence, using Schur complement Lemma [33], inequalities (5.23) are verified if and only if

$$\begin{bmatrix} \Psi_{jl} & \begin{bmatrix} 0 & 0 \\ -C^T\hat{L}_j^T\sqrt{\varepsilon} & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \\ (\star) & \begin{bmatrix} -\varepsilon G_j^{22} & 0 \\ (\star) & -\frac{1}{\varepsilon}G_j^{22} \end{bmatrix} \end{bmatrix} < 0, \forall j, l \in \Lambda. \tag{5.24}$$

In order to linearize the terms $\varepsilon G_j^{22}$ and $\frac{1}{\varepsilon}G_j^{22}$, we pre- and post-multiply inequality (5.24) by $\mathrm{diag}\left(I, \frac{1}{\sqrt{\varepsilon}}I, \sqrt{\varepsilon}I\right)$, we obtain the following set of BMI with respect to the uncertainty

$$\mathbb{BMI}(\Delta A) = \begin{bmatrix} \Psi_{jl} & \begin{bmatrix} 0 & 0 \\ -C^T\hat{L}_j^T & 0 \\ 0 & \sqrt{\varepsilon}I \\ 0 & 0 \end{bmatrix} \\ (\star) & \begin{bmatrix} -G_j^{22} & 0 \\ (\star) & -G_j^{22} \end{bmatrix} \end{bmatrix} < 0, \forall j, l \in \Lambda. \tag{5.25}$$

We complete our procedure by linearizing the uncertainty $\Delta A$ under assumption (H).

**Linearization of (5.25) with respect to the uncertainty.**

By virtue of the convexity principal, we get that $\mathbb{BMI}(\Delta A) < 0$ if and only if $\mathbb{BMI}(\mathscr{A}(\rho)) < 0$ holds for each $\mathscr{A}(\rho) \in \mathbb{V}_{\mathbb{B}}$. Thus inequalities (5.25) are equivalent to :

$$\tilde{\Xi}_{jl} + \mathrm{He}(Z_{1j}^T Z_{2j}) < 0, \forall l, j \in \Lambda, \forall \mathscr{A}(\rho) \in \mathbb{V}_{\mathbb{B}}. \tag{5.26}$$

where :

$$\tilde{\Xi}_{jl}(\mathscr{A}(\rho)) = \begin{bmatrix} -\tilde{P}_j^{11} & -\tilde{P}_j^{12} & \Theta_{13}^j & 0 & 0 & 0 \\ (\star) & -P_j^{22} & 0 & \Theta_{24}^j & -C^T \hat{L}_j^T & 0 \\ (\star) & (\star) & \Theta_{33}^l & \tilde{P}_l^{12} & 0 & \sqrt{\varepsilon} I \\ (\star) & (\star) & (\star) & \Theta_{44}^{jl} & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -G_j^{22} & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -G_j^{22} \end{bmatrix},$$

$$\begin{aligned} \Theta_{24}^j(\mathscr{A}(\rho)) &= (A_j + \mathscr{A}(\rho))^T G_j^{22} - C^T \hat{L}_j^T, \\ Z_{1j}(\mathscr{A}(\rho)) &= \begin{bmatrix} -\mathscr{A}(\rho)(\tilde{G}_j^{11})^T & 0 & 0 & 0 \end{bmatrix}, \\ Z_{2j} &= \begin{bmatrix} 0 & 0 & 0 & G_j^{22} \end{bmatrix}. \end{aligned}$$

By using (5.17), the Young's relation, and Schur complement Lemma, we obtain that inequalities (5.26) hold, if there exist some scalars $\gamma_j$, $j \in \Lambda$ such that the inequality

$$\begin{bmatrix} \tilde{\Xi}_{jl}(\mathscr{A}(\rho)) & \sqrt{\varepsilon} Z_{1j}^T(\mathscr{A}(\rho)) & Z_{2j}^T \\ (\star) & -\frac{\varepsilon}{\gamma_j} I & 0 \\ (\star) & (\star) & -\gamma_j I \end{bmatrix} < 0, \tag{5.27}$$

is fulfilled for all $j, l \in \Lambda$ and $\mathscr{A}(\rho) \in \mathbb{V}_{\mathbb{B}}$. To linearize the term $-\frac{\varepsilon}{\gamma_j} I$ in inequality (5.27), we use the well-known estimation $-\frac{\varepsilon}{\gamma_j} I \leq (\gamma_j - 2\sqrt{\varepsilon}) I$ which becomes optimal when taking $\gamma_j = \sqrt{\varepsilon}$. At this stage, we enounce the following theorem :

**Theorem 5.3.3.** *The observer-based controller* (5.5) *stabilizes asymptotically the system* (5.1) *if, there exist symmetric positive definite matrices* $D_l = \begin{bmatrix} \tilde{P}_l^{11} & \tilde{P}_l^{12} \\ (\star) & P_l^{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$, *invertible matrices* $\hat{G}_j^{11}$, $G_j^{22}$, *matrices* $\tilde{K}_j, \hat{L}_j$ *and positive scalars* $\gamma_j$, $\varepsilon$, *with* $l, j \in \Lambda$, *such that LMI* (5.28) *holds for all* $l, j \in \Lambda$ *and* $\mathscr{A}(\rho) \in \mathbb{V}_{\mathbb{B}}$. *Hence, the stabilizing observer-based control gains are given by*

$$L_j = (G_j^{22})^{-1} \hat{L}_j, K_j = \tilde{K}_j (\hat{G}_j^{11})^{-T}.$$

$$\begin{bmatrix} \begin{bmatrix} -\tilde{P}_j^{11} & -\tilde{P}_j^{12} & \hat{G}_j^{11}A_j^T+\tilde{K}_j^TB^T & 0 & 0 & 0 \\ (\star) & -P_j^{22} & 0 & \Theta_{24}^j & -C^T\hat{L}_j^T & 0 \\ (\star) & (\star) & \tilde{P}_l^{11}-\mathrm{He}\left(\hat{G}_l^{11}\right) & \tilde{P}_l^{12} & 0 & \sqrt{\varepsilon}I \\ (\star) & (\star) & (\star) & P_l^{22}-2G_j^{22} & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -G_j^{22} & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -G_j^{22} \end{bmatrix} & \begin{bmatrix} -\hat{G}_j^{11}\mathscr{A}^T(\rho) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & G_j^{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ (\star) & \begin{bmatrix} (\gamma_j-2\sqrt{\varepsilon})I & 0 \\ (\star) & -\gamma_jI \end{bmatrix} \end{bmatrix} < 0. \tag{5.28}$$

*where :*

$$\Theta_{24}^j = (A_j+\mathscr{A}(\rho))^TG_j^{22}-C^T\hat{L}_j^T.$$

### 5.3.3 A relaxed LMI Design Procedure

In what follows, we will discuss a new way to choose judiciously the matrix $G_{jl}$ that allows to convexity inequality (5.29). Our strategy consists in analyzing how to eliminate the bilinear terms coming from a more general structure of $G_{jl}$, without imposing a diagonal structure. By substituting (5.19) in inequality (5.18) and after some mathematical developments, one obtains the following detailed version of (5.18) :

$$\begin{bmatrix} -P_j^{11} & -P_j^{12} & \omega_{13}^{jl} & \omega_{14}^{jl} \\ (\star) & -P_j^{22} & \omega_{23}^{jl} & \omega_{24}^{jl} \\ (\star) & (\star) & \omega_{33}^{jl} & P_l^{12}-\mathrm{He}(G_{jl}^{12}) \\ (\star) & (\star) & (\star) & P_l^{22}-\mathrm{He}(G_{jl}^{22}) \end{bmatrix} < 0, \tag{5.29}$$

for all $j,l \in \Lambda$, where

$$\begin{aligned} \omega_{13}^{jl} &= A_j^T(G_{jl}^{11})^T+K_j^TB^T(G_{jl}^{11})^T-\Delta A^T(G_{jl}^{12})^T, \\ \omega_{14}^{jl} &= A_j^T(G_{jl}^{21})^T+K_j^TB^T(G_{jl}^{21})^T-\Delta A^T(G_{jl}^{22})^T, \\ \omega_{23}^{jl} &= -C^TL_j^T(G_{jl}^{11}+G_{jl}^{12})^T+(A_j+\Delta A)^T(G_{jl}^{12})^T, \\ \omega_{24}^{jl} &= (\Delta A+A_j)^T(G_{jl}^{22})^T-C^TL_j^T(G_{jl}^{21}+G_{jl}^{22})^T, \\ \omega_{33}^{j} &= P_l^{11}-\mathrm{He}(G_{jl}^{11}). \end{aligned}$$

We begin by dealing with terms coupled with $L_j$, $j \in \Lambda$, namely the bilinear terms in $\omega_{23}^{jl}$ and $\omega_{24}^{jl}$. Our first strategy consists in choosing $G_{jl}^{11} = G_l^{11}$ (independent of $j$) and $G_{jl}^{22} =$

$G_j^{22}$ (independent of $l$) and eliminate the remining bilinear terms by setting $G_{jl}^{11}+G_{jl}^{12}=0$ or $G_{jl}^{21}+G_{jl}^{22}=0$. But since $G_{jl}^{21}$ is also coupled with the matrices $K_j$, it should be taken null. So the following structure of

$$G_{jl}=\begin{bmatrix} G_l^{11} & -G_l^{11} \\ 0 & G_j^{22} \end{bmatrix} \tag{5.30}$$

is suitable. An alternative choice consists in choosing $G_{jl}^{11}=G_l^{11}$, $G_{jl}^{22}=G_j^{22}$ and $G_{jl}^{21}=0$ as symmetric matrices, but for the choice of $G_{jl}^{12}$, we use at first the congruence principal (we pre- and post- multiply (5.29) by $\mathrm{diag}\Big((G_j^{11})^{-1}, I, (G_l^{11})^{-1}, I\Big)$. One obtains by taking into account the following changes of variables and notations :

$$\tilde{G}_j^{11}=(G_j^{11})^{-1},\ \tilde{P}_j^{11}=\tilde{G}_j^{11}P_j^{11}(\tilde{G}_j^{11})^T,$$

$$\tilde{K}_j=K_j(\tilde{G}_j^{11})^T, \tilde{P}_j^{12}=\tilde{G}_j^{11}P_j^{12},$$

that (5.29) holds if (and only if)

$$\begin{bmatrix} -\tilde{P}_j^{11} & -\tilde{P}_j^{12} & \tilde{\omega}_{13}^{jl} & -\tilde{G}_j^{11}\Delta A^T(G_j^{22})^T \\ (\star) & -P_j^{22} & \tilde{\omega}_{23}^{jl} & \tilde{\omega}_{24}^{j} \\ (\star) & (\star) & \tilde{\omega}_{33}^{l} & \tilde{P}_l^{12}-\tilde{G}_l^{11}G_{jl}^{12} \\ (\star) & (\star) & (\star) & P_l^{22}-(G_j^{22})^T-G_j^{22} \end{bmatrix}<0, \tag{5.31}$$

where :

$$\begin{aligned}
\tilde{\omega}_{13}^{jl}&=\tilde{G}_j^{11}A_j^T+\tilde{K}_j^TB^T-\tilde{G}_j^{11}\Delta A^T\left(\tilde{G}_l^{11}G_{jl}^{12}\right)^T,\\
\tilde{\omega}_{23}^{jl}&=-C^TL_j^T(I+\tilde{G}_l^{11}G_{jl}^{12})^T+(A_j+\Delta A)^T\left(\tilde{G}_l^{11}G_{jl}^{12}\right)^T,\\
\tilde{\omega}_{24}^{j}&=(\Delta A+A_j)^T(G_j^{22})^T-C^TL_j^T(G_j^{22})^T,\\
\tilde{\omega}_{33}^{l}&=\tilde{P}_l^{11}-\tilde{G}_l^{11}-(\tilde{G}_l^{11})^T.
\end{aligned}$$

So in order to eliminate the remaining terms in (5.31), and since the term $L_j$ attached $I+\tilde{G}_l^{11}G_{jl}^{12}$ and $G_j^{22}$, we set

$$\hat{L}_j=G_j^{22}L_j \text{ and } G_{jl}^{12}=G_l^{11}G_j^{22}-G_l^{11}.$$

This leads to the following structure of $G_{jl}$ :

$$G_{jl}=\begin{bmatrix} G_l^{11} & G_l^{11}G_j^{22}-G_l^{11} \\ 0 & G_j^{22} \end{bmatrix}. \tag{5.32}$$

Both choices (5.30) and (5.32) allow to simplify the complexity of the initial bilinear problem. So our idea consists in combining the two choices (5.30) and (5.32) by introducing another scalar variable $\alpha_{jl}$ satisfying :

$$G_{jl}^{12}=\alpha_{jl}G_l^{11}G_j^{22}-G_l^{11} \tag{5.33}$$

which means that

$$G_{jl} = \begin{bmatrix} G_l^{11} & \alpha_{jl} G_l^{11} G_j^{22} - G_l^{11} \\ 0 & G_j^{22} \end{bmatrix}. \tag{5.34}$$

Observe that if $\alpha_{jl} = 0$, for all $j, l \in \Lambda$, we get the structure (5.30), and if $\alpha_{jl} = 1$, for all $j, l \in \Lambda$, we get the structure (5.32). On the other hand, the combination (5.34) is more than a convex combination since the scalar variable that we introduced, $\alpha_{jl}$, is a free parameter (and is not necessarily in $[0,1]$).
At this stage, we complete our design methodology by linearizing the uncertain terms $\Delta A$. By virtue of the convexity principale, we get that (5.31) with (5.34) holds for each $l, j \in \Lambda$, if and only if it holds for each $l, j \in \Lambda$, and each $\mathscr{A}(\rho) \in \mathbb{V}_{\mathbb{B}}$.

**Linearization of (5.31) with respect to the uncertainty.**

Taking into account (5.34), inequality (5.31) can be rewritten under the following more suitable from :

$$\Xi_{jl}(\mathscr{A}(\rho)) + \mathrm{He}(Z_{1j}^T\, Z_{2jl}) < 0, \forall l, j \in \Lambda, \forall \mathscr{A}(\rho) \in \mathbb{V}_{\mathbb{B}}. \tag{5.35}$$

where :

$$\Xi_{jl}(\mathscr{A}(\rho)) = \begin{bmatrix} -\tilde{P}_j^{11} & -\tilde{P}_j^{12} & \Omega_{13}^j & 0 \\ (\star) & -P_j^{22} & \Omega_{23}^{jl} & \Omega_{24}^j \\ (\star) & (\star) & \Omega_{33}^l & \tilde{P}_l^{12} + I - \alpha_{jl} G_j^{22} \\ (\star) & (\star) & (\star) & P_l^{22} - G_j^{22} - (G_j^{22})^T \end{bmatrix},$$

$$\begin{aligned}
\Omega_{13}^j &= \tilde{G}_j^{11} A_j^T + \tilde{K}_j^T B^T, \\
\tilde{\Omega}_{23}^{jl} &= -\alpha_{jl} C^T \hat{L}_j^T + (A_j + \mathscr{A}(\rho))^T (\alpha_{jl} G_j^{22} - I)^T, \\
\tilde{\Omega}_{24}^j &= (A_j + \mathscr{A}(\rho))^T (G_j^{22})^T - C^T \hat{L}_j^T, \\
\tilde{\Omega}_{33}^l &= \tilde{P}_l^{11} - \tilde{G}_l^{11} - (\tilde{G}_l^{11})^T, \\
Z_{1j} &= \begin{bmatrix} -\mathscr{A}(\rho)(\tilde{G}_j^{11})^T & 0 & 0 & 0 \end{bmatrix}, \\
Z_{2jl} &= \begin{bmatrix} 0 & 0 & (\alpha_{jl} G_j^{22} - I)^T & (G_j^{22})^T \end{bmatrix}.
\end{aligned}$$

By using Young relation, and Schur complement Lemma, we obtain that inequalities (5.35) hold, if the following inequalities

$$\begin{bmatrix} \Xi_{jl} & Z_{1j}^T \mathscr{A}^T(\rho) & Z_{2j}^T \\ (\star) & -\frac{1}{\varepsilon_j} I & 0 \\ (\star) & (\star) & -\varepsilon_j I \end{bmatrix} < 0, \forall l, j \in \Lambda, \forall \mathscr{A}(\rho) \in \mathbb{V}_{\mathbb{B}}, \tag{5.36}$$

are fulfilled. We just prove the following theorem :

**Theorem 5.3.4.** *The observer-based controller* (5.5) *stabilizes asymptotically the system* (5.1) *if, for some scalars fixed a priori* $\alpha_{j,l}$, $\varepsilon_j > 0$, *there exist symmetric positive definite* *matrices* $D_l = \begin{bmatrix} \tilde{P}_l^{11} & \tilde{P}_l^{12} \\ (\star) & P_l^{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$*, invertible matrices* $\tilde{G}_j^{11}$, $G_j^{22}$*, and matrices* $\tilde{K}_j$, $\hat{L}_j$*, with* $l, j \in \Lambda$*, such that LMI* (5.37) *holds for all* $l, j \in \Lambda$*. Hence, the stabilizing observer-based control gains are given by* $L_j = (G_j^{22})^{-1}\hat{L}_j$ *and* $K_j = \tilde{K}_j(\tilde{G}_j^{11})^{-T}$*.*

$$\begin{bmatrix} -\tilde{P}_j^{11} & -\tilde{P}_j^{12} & \tilde{G}_j^{11}A_j^T + \tilde{K}_j^T B^T & 0 & -\tilde{G}_j^{11}\mathscr{A}^T(\rho) & 0 \\ (\star) & -P_j^{22} & \Theta_{23}^{jl} & (A_j + \mathscr{A}(\rho))^T (G_j^{22})^T - C^T \hat{L}_j^T & 0 & 0 \\ (\star) & (\star) & \tilde{P}_l^{11} - \tilde{G}_l^{11} - (\tilde{G}_l^{11})^T & \tilde{P}_l^{12} + I - \alpha_{jl} G_j^{22} & 0 & \alpha_{jl}G_j^{22} - I \\ (\star) & (\star) & (\star) & P_l^{22} - G_j^{22} - (G_j^{22})^T & 0 & G_j^{22} \\ (\star) & (\star) & (\star) & (\star) & -\frac{1}{\varepsilon_j} I & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\varepsilon_j I \end{bmatrix} < 0. \tag{5.37}$$

$$\Theta_{23}^{jl} = -\alpha_{jl}C^T\hat{L}_j^T + (A_j + \mathscr{A}(\rho))^T(\alpha_{jl}G_j^{22} - I)^T.$$

**Remark 5.3.5.** *Notice* (5.37) *is an LMI if we fix a priori* $\alpha_l$, $\varepsilon_j$*. Thus it's resolution involves the gridding method with respect to* $\alpha_{j,l}$, $\varepsilon_j$*, for each* $j$ *and* $l$*. From a numerical point of view, this may generate some complexity, but an interesting results are obtained when we take* $\alpha_{j,l}$ *and* $\varepsilon_j$ *independent of* $j$ *(see Table 2 in the following section). We can also linearize* (5.37) *with respect to* $\varepsilon_j$ *by using the estimation* $-\frac{1}{\varepsilon_j}I \leq (\varepsilon_j - 2)I$*.*

## 5.4 Numerical examples

In this section, we provide numerical examples and simulations. The first example is taken from [37]. The goal of this example is to show that the proposed design methodology tolerates larger uncertainty level $\Delta$. The second example is taken from [38]. We propose to compare the performance of our methods and those proposed in [37–39].

**Example 5.4.1.** *Consider the following discrete-time LPV system [37]*

$$x_{t+1} = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.6+\rho_t \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_t \tag{5.38a}$$

$$y_t = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} x_t \tag{5.38b}$$

*with* $\rho \in [0, 0.5]$*,* $k \in \mathbb{N}$*. In this case, we can take the functions* $\xi_1(\rho) = (0.5-\rho)/0.5$ *and* $\xi_2(\rho) = \rho/0.5$ *and* $A_1 = A(0), A_2 = A(0.5)$*. The LMIs* (5.37) *return simultaneously for* $\varepsilon_j = 0.03$*,* $j = 1, 2$ *and* $(\alpha_j)_{j=1}^2 = (100, 130)$*, the controller gains*

$$K_1 = 10^{-2} * \begin{bmatrix} 0.0019 & 0.0005 & -58.8554 \end{bmatrix},$$

TABLE 5.1 – The DC Motor Model Parameters

| | |
|---|---|
| Moment of inertia of the rotor | $J = 0.01\ kg.m^2$ |
| Motor viscous friction constant | $b = 0.1\ N.m.s$ |
| Electromotive force constant | $K_e = K_m = 0.01\ V/rad/s$ |
| Motor torque constant | $K_t = K_m = 0.01\ N.m/Amp$ |
| Electric resistance | $R_m = 1\,\Omega$ |
| Electric inductance | $L_m = 0.5\ H$ |

$$K_2 = 10^{-2} * \begin{bmatrix} -0.0001 & 0.0006 & -133.5613 \end{bmatrix}$$

*and the observer gains given by*

$$L_1 = \begin{bmatrix} -0.0054 \\ 0.0055 \\ 0.2273 \end{bmatrix}, L_2 = \begin{bmatrix} -0.0284 \\ 0.0112 \\ 0.4598 \end{bmatrix}.$$

*These observer-based controller gains are obtained for the largest value of uncertainty level* $\Delta_{\max} = 0.4441$*, while the other methods in [37], [38] and [39] are found infeasible for this uncertainty level. The simulation results corresponding to these observer-based controller gains are given in Figure 5.1 with* $x_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ *and* $\hat{x}_0 = \begin{bmatrix} 5 & 5 & 5 \end{bmatrix}^T$.

**Example 5.4.2.** *Now, we consider the DC motor model given in [38] and defined by the following continuous-time equations :*

$$\dot{x}(t) = A_C x(t) + B_C u(t)$$

$$y(t) = C_C x(t)$$

*with*

$$A_C = \begin{bmatrix} -\frac{b}{J} & \frac{K_m}{J} \\ -\frac{K_m}{L_m} & -\frac{R_m}{L_m} \end{bmatrix}, B_C = \begin{bmatrix} 0 \\ \frac{1}{L_m} \end{bmatrix}, C_C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

*The parameters of the DC motor are presented in Table 5.1.*
*The resistance* $R_m$ *is assumed to be a time-varying, which leads to an LPV system. That is,* $R_m(.) = \rho(.)$*. As in [38], we assume that* $\rho(.) \in \begin{bmatrix} 1 & 5 \end{bmatrix}$*. Approximating the derivative with Euler method and choosing a sampling period* $T_s = 0.1$*, we obtain the discredited version of the system, described by the following parameters :*

$$C \triangleq C_C = \begin{bmatrix} 1 & 0 \end{bmatrix}, B \triangleq B_C = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix},$$

$$A(\rho) \triangleq I + T_s A_C(\rho) = \begin{bmatrix} 0 & 0.1 \\ -0.002 & 1 - \frac{T_s}{L_m}\rho \end{bmatrix}.$$

*Hence, we can compute* $A_1$ *and* $A_2$ *as*

$$A_1 = A(5) = \begin{bmatrix} 0 & 0.1 \\ -0.002 & 0 \end{bmatrix},$$

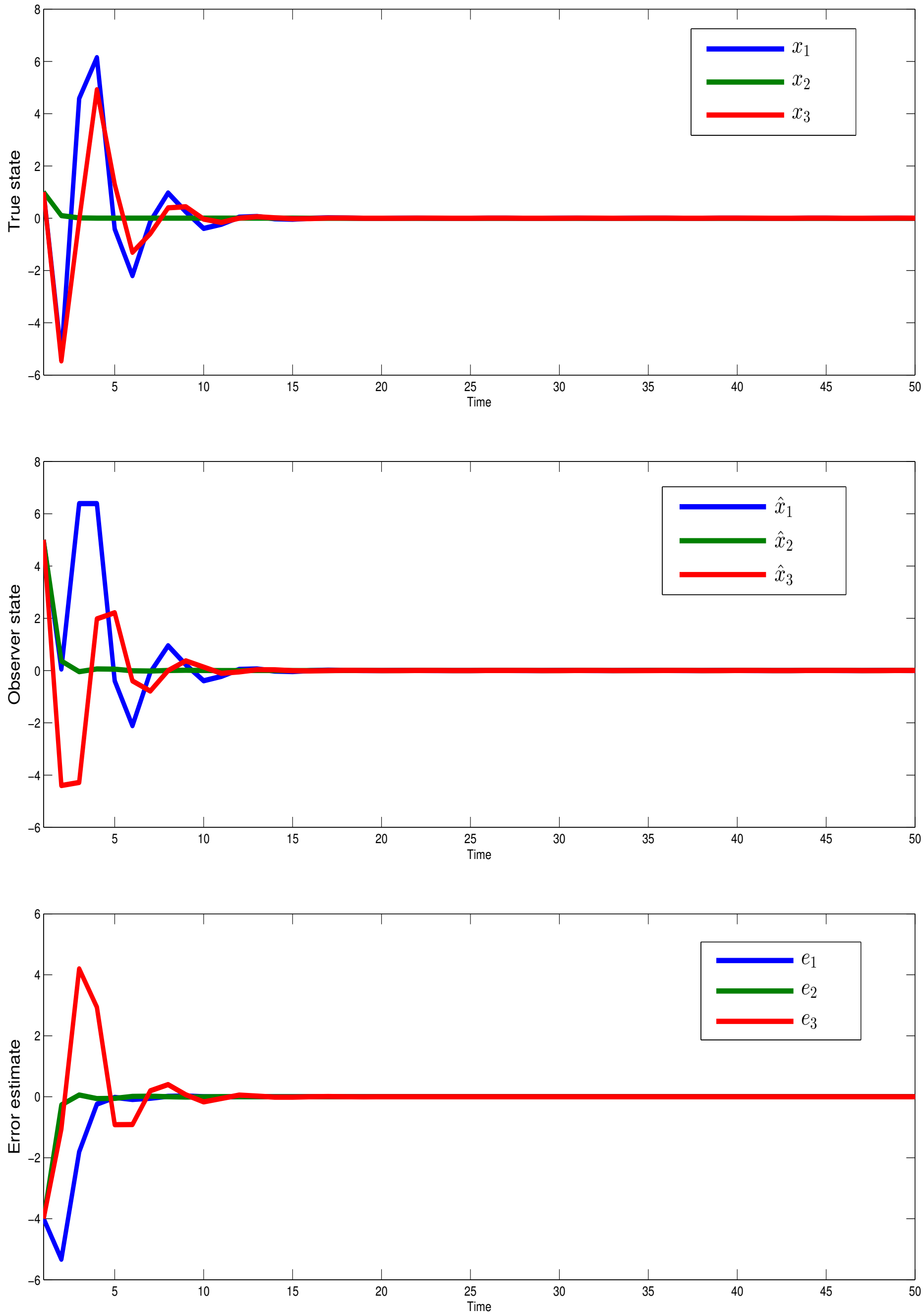



FIGURE 5.1 – The controlled states $x$, its estimate $\hat{x}$ and the estimation error $e$

$$A_2 = A(1) = \begin{bmatrix} 0 & 0.1 \\ -0.002 & 0.8 \end{bmatrix}.$$

*We have compared the feasibility of the proposed design methods and those established in [37–39], by increasing the uncertainty level* $\Delta$ *until obtaining infeasibility. The superiority of the proposed LMIs* (5.37) *is quite clear from the results presented in Table* (5.2).

TABLE 5.2 – $\Delta_{\max}$ tolerated for different methods in Example 1 and 2

| Method | $\Delta_{\max}$ in Example 1 | $\Delta_{\max}$ in Example 2 |
|---|---|---|
| LMI in [37] | 0.1786 | 1.0855 |
| LMI in [38] | (!) | 0.0800 |
| LMI 18 in [39] | 0.2123 | 0.9999 |
| LMI 24 in [39] | 0.3107 | 0.9999 |
| LMI (5.28) | 0.3415 | 1.22 |
| | $(\alpha_j)_{j=1,2} = (100, 130)$ $(\varepsilon_j)_{j=1,2} = 0.03$ | $(\alpha_j)_{j=1,2} = 1$ $(\varepsilon_j)_{j=1,2} = 24$ |
| LMI (5.37) | 0.4441 | 1.4 |

*With $\Delta_{\max} = 1.4$, LMIs (5.37) return simultaneously the observer-based controller gains*

$$K_1 = \begin{bmatrix} 0.0100 & -4.0310 \end{bmatrix}, K_2 = \begin{bmatrix} 0.0100 & -0.0067 \end{bmatrix},$$

*and*

$$L_1 = \begin{bmatrix} 0.3031 \\ 1.9758 \end{bmatrix}, L_2 = \begin{bmatrix} 0.2415 \\ -0.0407 \end{bmatrix}.$$

*We see through these comparisons that the proposed method solves the stabilization problem with a maximum uncertainty level. The gains of the observer-based controller are computed by solving only a single set of LMIs running with only one-step algorithm. This demonstrates the simplicity and the efficiency of the proposed methodology.*

## 5.5 Conclusion

In this chapter, we have presented two new LMI synthesis methods to design observer-based controllers for a class of LPV systems with inexact but bounded parameters. The first approach is based on the use of Young's relation, while the second one uses a new congruence principle by pre- and post-multiplying the basic BMI by new and ingenious matrices. Thanks to these matrices, some bilinear terms vanish from the BMI, which becomes more flexible for the linearization. To show the validity and superiority of the proposed design methods, two numerical examples from the literature have been reconsidered in this paper. The comparisons show that the proposed methodologies provide less conservative LMI conditions compared to LMI techniques reported previously in the literature.

# Conclusion and Future Works

Through this thesis, we have been interested in the problem of robust stability analysis and control design for both switched linear systems and LPV systems, with uncertain parameters. We addressed this problem by using the approach of Finsler's Lemma. The aim was to reduce the conservatism of some existing methods in the literature. We have proposed several scenarios of linearization of the Finsler inequality. We have shown analytically that our judicious choices of the slack variables coming from Finsler's lemma eliminate some bilinear terms. A numerical evaluation of the complexity of the proposed LMIs methods has been carried out. We also tested the validity and superiority of the proposed methods on several practical examples.

**Perspectives**

Many questions remain in this area. As future research perspectives on the results presented in this manuscript, we intend to develop and improve the following points :

- Extend the applicability of our approach to other types of systems, such as continuous time switching systems, Lipschitz nonlinear switching systems, switching systems without a priori knowledge of mode and switching law , and interconnected systems.
- Search for new Lyapunov functions resulting in less conservative synthesis conditions.
- Issues related to the observability of nonlinear systems.
- Study of stochastic systems.
- Study of T-S fuzzy switched system.

Finally, we would like our contribution is of some interest and be the subject of further studies in this area, where, as we said, many fundamental questions remain open.

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## Bibliographie


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## Résumé

Le problème de stabilisation par retour d'état dynamique pour les systèmes linéaires à paramètres incertains se traduit par la résolution d'une inégalité bilinéaire matricielle (BMI), qui, d'un point de vue numérique, est un problème NP-difficile. Cette thèse aborde ce problème pour des systèmes à commutations à temps discrets et à paramètres incertains, en se basant sur le Lemme de Finsler. Un scénario de méthodes d'optimisation exprimées par des inégalités linéaires matricielles (LMIs) sont développées. La supériorité des méthodes proposées par rapport aux résultats existants dans la littérature est démontrée analytiquement. Une application aux systèmes Linéaires à Paramètres Variants (LPV) est aussi abordée.



**Mots-clés:** Systèmes à commutations à temps discret ; Fonction de Lyapunov commutée ; Lemme de Finsler ; Observateur de Luenberger ; Stabilisation ; Inégalités Matricielles Linéaires (LMI) ; Incertitudes



## Abstract

The problem of stabilization by dynamic state feedback for linear systems with uncertain parameters is expressed by a bilinear matrix inequality (BMI), which, from a numerical point of view, is an NP-hard problem. This thesis addresses this issue for switched discrete-time linear systems with with $l_2$-bounded disturbances and parameter uncertainties, using the Finsler Lemma. A scenario of optimization methods expressed by linear matrix inequalities (LMIs) is developed. The superiority of the proposed methods with respect to previous results in the literature is analytically demonstrated. An application to Linear Parameters Variant Systems (LPV) is also discussed.