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On Observer-Based Differentiator Design

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In loving memory of my late father

Abstract

Observer-based differentiators are model-based algorithms that estimate the derivatives of a measured time-varying signal, which are not available in some cases, for control use purposes.

The following is a review of recent design techniques of observer-based differentiators. First of all, the design procedure of the high gain differentiator, which relies on a high-gain asymptotic observer is introduced. Then the finite time sliding mode differentiators are studied more particularly the Levant first and higher-order sliding modes differentiators. Finally the design process of prescribed-time sliding mode differentiators based on modulating functions is considered. Numerous simulation examples of the previously mentioned differentiators were conducted using MATLAB/Simulink to better illustrate the related theoretical results.

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I am filled with deep gratitude for my mother, whose love and support have been my constant source of strength and advice. I carry in my heart the memory of my father, who, though no longer with us, continues to inspire me. It is my heartfelt aspiration to make them proud through my endeavors and achievements.

Contents

List of Figures

List of Tables

Introduction

The development in hard sensors has lead to access a huge amount of measured data. The data analysis and extracting informations from these data allows to understand, control, diagnose and monitor systems. However, physical sensors present several limitations; some variables that are needed are not accessible directly which is the case of the neural activity for example, in addition these sensors are generally expensive which is the case in electric vehicles. Furthermore, physical sensors can be slow and sensitive to hostile environment which make them inaccurate. This is to say, physical sensors need to be replaced by some alternatives that we call virtual sensors which are the bridge between measured data and the desired information, they combine available data with some estimation algorithms in order to provide robust, fast, accurate and low cost sensing strategies. For virtual sensors, two main approaches exist, the model based approach requires mathematical model that needs to be validated which allow us to develop estimation methods to estimate the variables of interest. In contrast, model free approach uses directly the data by mean of data driven methods or signal processing methods to extract informations about the variables of interest [65].

One of the estimation topics is real-time differentiation which consists of estimating the successive derivatives of a time-varying signal by means of differentiators. It takes application in control theory especially in nonlinear systems case, system identification, signal and image processing [2], [13], [14], [56], [57], [58], [59], [60], [61], [62]. In the literature, differentiation methods, as methods estimating the derivatives, are grouped in two majoring categories: non asymptotic algebraic methods and observer based methods. The first category is a model free approach that do not rely on a dynamical model but rather uses directly data and signal information, it includes methods like digital differentiation and filtering, and numerical differentiation based on algebraic parameters estimation using taylor expansion. These methods result is very sensitive to noise and generally biased. Observer based methods on the other side, introduce a dynamical model whose output is the signal to be differentiated and then design an observer to estimate signal derivatives.

High-gain observers are used as real time differentiators, with gains taken sufficiently large, it provides for fast and accurate estimation. However this leads also to transient peaking phenomenon and sensitivity to measurement noise [1], [6], [8], [9], [10], [11], [12], [15], [16], [41], [48], [51], [53] .

The sliding mode observers are also used in differentiator design, they are valued for their finite-time exact estimation for noise free signal and provide an optimal estimation error for noisy signals, robustness to model uncertainties and to small Lebesgue measurable noise [3], [18], [23], [24], [25], [26], [27], [31], [43], [46], [54],[55], [63], [64]. The concept of fixed-time stability allow for fixed-time convergence independently of initial estiation errors. This is still insufficient for practical real-time differentiation particularly in applications where convergence time needs to meet an exact desired value which is the case for autonomous vehicles and for vehicle driving assisstance. In that regard, predefined-time and prescribed-time are introduced and allow for convergence before a given upper bound in the first case and within an exact user-selected time in the second [20], [32], [34], [36], [37], [38], [39].

Observer based methods require then some compromise when designing differentiation algorithms between estimation, precision and noise sensitivity.

Thesis outline

The hoped-for goal of this monograph is to introduce observer-based differentiator design techniques and it is subdivided into three chapters

Chapter I presents observability of nonlinear systems and related criteria followed by high-gain observer design. Performance of such observer is discussed through simulation examples at the end of the chapter.

Chapter II provides an overview of sliding mode control concept followed by a presentation of sliding mode control algorithms. The sliding mode differentiators are then discussed starting from the first order sliding mode differentiator based on the supertwisting algorithm and then higher order sliding mode differentiator. Simulation examples are provided to show performance of presented differentiators.

Chapter III discuss the prescribed-time sliding mode differentiator based on modulating functions with simulation examples showing its performance and behaviour in both noise free and noisy signal case.

Potential future work is announced by an outlook describing open problems and possible future research directions at the end of the manuscript.

Chapter 1

Nonlinear High-Gain Observers

High-Gain observers (HGO) are initially introduced in linear feedback control [50] for robust observer design and in [51] on H_∞ control. Late 1980's, High-gain observers have evolved to handle nonlinear dynamics and used in nonlinear feedback control [48], [49]. The work by Gauthier [8] in 1992 initiated a line of research directions covering a large class of nonlinear systems. On the other hand, Khalil in [53] gave attention to the peaking phenomenon and how it can destabilize the closed loop system as the observer gain is driven high. The first separation principle and tools for semiglobal stabilization are introduced by Teel and Prally [47], [52]. HGOs are widely used after in a wide range of nonlinear control problems including stabilization, regulation, tracking and adaptive control, fault detection and diagnosis. It finds applications across various fields such as robotics, aerospace, electric drives and biomedical engineering.

In this chapter, we describe observability property of nonlinear systems followed by high-gain observer and HGO-based differentiator design.

1.1 Nonlinear Observability

In linear systems case, observer design is guaranteed whenever the system is observable (the Kalman rank condition is verified). However, in the nonlinear systems case, the observability property can be expected in general to depend on the input and on specific structure of the considered system.

1.1.1 The Observation Problem Statement

The general form of a state space representation for a nonlinear system is given by:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned} \tag{1.1}$$

where:

$x \in R^n$ is the state vector,
$u \in R^m$ is the vector of inputs,
$y \in R^p$ is the vector of measured outputs.

Let $\chi_u(t, x_0)$ denote its solution at time t, with initial condition x_0 at time $t = 0$, $u(.)$ are assumed to be measurable and bounded functions.

In order to control system 1.1 we need to know $x(t)$ while generally in practice we have only access to u and y. The observation problem as formulated in [11] is as follows:

Observation problem *Given a system described by a representation of (1.1), find an estimate* $\hat{x}(t)$ *for* $x(t)$ *from the knowledge of* $u(\tau)$ *and* $y(\tau)$ *for* $0 \leq t \leq \tau$.

1.1.2 Nonlinear Observability Criteria

Observability is a necessary condition for possible design of nonlinear observers.

In general, a nonlinear system is said to be observable if and only if the unmeasured state $x(t)$ at a specific time t can be uniquely determined [10] from the knowledge of inputs u and outputs y. In other words, one must be able to distinguish different initial states by the value of the output measurement.

Definition 1.1 Indistinguishability *(Hermann, [41])*
A pair $(x_0, \tilde{x_0})$ *will be said to be indistiguishable by* u *if* $\forall t \geq 0$, $h(\chi_u(t, x_0)) = h(\chi_u(t, \tilde{x_0}))$. *The pair is just said to be indistinguishable, if it is so for any* u.

Definition 1.2 Observability *[2]*
A nonlinear system 1.1 is observable if it does not have any indistinguishable pair of states.

It is clear from the above definition that there may exist inputs for which some states are indistinguishable and thus observability can depend on inputs. To fix this problem of inputs, one may consider the case of inputs for which no indistinguishable pair of states can be found.

Definition 1.3 Universal inputs *[12]*
An input u *is universal on* $[0, t]$ *if for evry pair of distinct states* $x_0 \neq \bar{x}_0$, *there exists* $\tau \in [0, t]$ *such that* $h(\chi_u(\tau, x_0)) \neq h(\chi_u(\tau, \bar{x}_0))$. *If* u *is universal on* $\mathbf{R}^+$, *it is just said to be universal.*

Definition 1.4 Singular inputs *[12]*
A non universal input is called singular.

Definition 1.5 Uniformly observable systems *(Besançon, [11])*
A system is uniformly observable if every input is universal.

Definition 1.6 Weak observability *(Besançon, [11])*
A system (1.1) is weakly observable (resp. at x_0*) if there exists a neighborhood* $\mathbf{U}$ *of any* x *(resp. of* x_0*) such that there is no indistinguishable state from* x *(resp.* x_0*) in* $\mathbf{U}$.

Definition 1.7 Local observability *(Besançon, [11])*
A system (1.1) is locally observable if there exists a neighborhood $\mathbf{U}$ *of any* x *such that for any neighborhood* $\mathbf{V}$ *of* x *contained in* $\mathbf{U}$, *there is no indistinguishable state from* x *in* $\mathbf{V}$ *when considering time intervals for which trajectories remain in* $\mathbf{V}$.

This means that one can distinguish every state from neighbors on its surroundings. Local observability in other words, stands for a property of a system having all its inputs as universal within a limited time of interval $[t_0,\ t_0 + T]$.

This notions allows us to introduce an 'observation space' for which we can attribute some 'rank condition' to verify observability.

Definition 1.8 Observation space *(Besançon, [11])*
The observation space for a system (1.1) is defined as the smallest real vector space (denoted by $\mathcal{O}(h)$) of C^∞ functions containing the components of h and closed under Lie derivation along $f_u := f(.,u)$ for any constant $u \in \mathbf{R^m}$ (namely such that for any $\varphi \in \mathcal{O}(h)$, $L_{f_u}\varphi = \frac{\partial \varphi}{\partial t} f(x,u)$).

Definition 1.9 Observability rank condition *(Besançon, [11])*
A system (1.1) is said to satisfy the observability rank condition (resp. at x_0) if:

$$\forall x,\, dim\, d\mathcal{O}(h)|_x = n \quad [\,resp.\;\, dim\; d\mathcal{O}(h)|_{x_0} = n\,]$$

where:
$d\mathcal{O}(h)|_x$ is the set of $d\varphi(x) \in \mathcal{O}(h)$
such that:

$$\mathcal{O}(h) = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} h \\ L_f h \\ \vdots \\ {L_f}^{(n-1)} h \end{pmatrix}$$

and for the gradient $d(.)$

$$d\mathcal{O}(h) = \frac{\partial \mathcal{O}(h)}{\partial x} = \begin{pmatrix} dy \\ d\dot{y} \\ \vdots \\ dy^{(n-1)} \end{pmatrix} = \begin{pmatrix} dh \\ dL_f h \\ \vdots \\ d{L_f}^{(n-1)} \end{pmatrix} \quad \text{is the jacobian of } \mathcal{O}h$$

A system (1.1) satisfying the observability rank condition [resp. at x_0] is locally weakly [11] observable [resp. at x_0].
System (1.1) is said to be globally observable if it is observable for all $x \in R^n$ and $det(d\mathcal{O}h)$ do not depend on x.

1.1.3 Considered Nonlinear Normal Forms

The major issue with nonlinear observer design is the lack of genenrality; there do not exist a unified and systematic method for the design of such observers. Available results in the litterature are restricted to specefic classes, each making specefic assumptions on the considered structures. This is to say that nonlinear observer design is strightforward depending on the coordinates we choose to express the system's dynamics.

Normal forms are specefic structures of nonlinear systems for which observer design is direct and easier. Our focus herein will be in state affine normal form and triangular form which are the most popular as presented in [10],[11]:

- **State-Affine Normal Form**

$$\begin{aligned} \dot{x}(t) &= A(u)\,x(t) + B(u) \\ y(t) &= C(u)\,x(t) \end{aligned} \tag{1.2}$$

- **Triangular Form**

$$\begin{cases} \dot{x}(t) = x_2(t) + \Phi_1(u, x_1) \\ \quad \vdots \\ \dot{x}_i(t) = x_{i+1}(t) + \Phi_i(u, x_1, \dots, x_i) \\ \quad \vdots \\ \dot{x}_n(t) = \Phi_n(x, u) \end{cases} \quad ; \quad y(t) = x_1(t) \tag{1.3}$$

which is equivalent to the following

$$\begin{cases} \dot{x}(t) = Ax(t) + \Phi(u, x) \\ y(t) = Cx(t) \end{cases} \tag{1.4}$$

such that

$$A = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ \vdots & & 1 & & \\ 0 & & & \ddots & 1 \\ 0 & \dots & & 0 & 0 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

and the triangular nonlinearities matrix given by $\Phi(x, u) = \begin{bmatrix} \Phi_1(u, x_1) \\ \vdots \\ \Phi_{n-1}(u, x_1, \dots, x_{n-1}) \\ \Phi_n(x, u) \end{bmatrix}$

(A,C) is observable under Brunowski's form.

A particular case of the triangular form when only Φ_n is nonzero is known as the phase-variable form and takes the following structure:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \quad \vdots \\ \dot{x}_n(t) = \Phi_n(x, u) \end{cases} \quad ; \quad y = x_1(t) \tag{1.5}$$

1.2 High-Gain Observer Design

The high gain observer design require observability of the nonlinear system. We present next, two nonlinear structures and their corresponding HGO design techniques.

Performance of such observers is discussed through example simulations showing main features of this observer. Limitations of such observers, as the measurement noise sensitivity in particular, are discussed.

1.2.1 High-Gain Observer Design for Lipschitz Triangular Form

The system (1.4) is uniformly observable [11]. This allows for observer design which is independent of the inputs. In order to design a high gain observer, the following assumption will be required:

The nonlinear function Φ_n is a global Lipschitz function:

$$\exists c > 0,\ \forall x, \acute{x} \in R^n, \text{we have} \parallel \Phi_n(x,u) - \Phi_n(\acute{x},u) \parallel \leq c \parallel x - \acute{x} \parallel \tag{1.6}$$

where $\parallel . \parallel$ denotes the euclidean norm of R^n.

Theorem 1.1 *(Hammouri, [11])*

Under Lipschitzness of triangular nonlinearities, the system:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \Phi(\hat{x},u) + \Delta_\theta K(y - \hat{y}) \\ \quad \hat{y} = C\hat{x} \end{cases} \tag{1.7}$$

forms an exponential observer for system (1.4).

Where y is the measured output associated to the unknown states of system (1.3), $\theta > 1$

$$\Delta_\theta = \begin{pmatrix} \theta & 0 & \dots & 0 \\ 0 & \theta^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \theta^n \end{pmatrix}; \quad K = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$$

Proof of theorem (1.1)

Set $e_x(t) = x - \hat{x}$ then

$$\dot{e}_x = Ae_x - \Delta_\theta KCe_x + \Phi(x,u) - \Phi(x,u)$$

let $e_x = \Delta_\theta \tilde{e}_x$ thus $\tilde{e}_x = \Delta_\theta^{-1} e_x$
from which

$$\begin{aligned} \dot{\tilde{e}}_x &= \Delta_\theta^{-1}\dot{e}_x \\ &= \theta[A - KC]\tilde{e}_x + \Delta_\theta^{-1}[\Phi(x,u) - \Phi(\hat{x},u)] \end{aligned} \tag{1.8}$$

K is determined such that A-KC is a Hurwitz matrix. In (1.4) we assumed Φ to be a global Lipschitz function which yields

$$\parallel \Delta_\theta^{-1}(\Phi(x,u) - \Phi(\hat{x},u)) \parallel \leq c \parallel x - \hat{x} \parallel$$

Now we introduce the Lyapunov function $V(\tilde{e}_x) = \tilde{e}_x^T.P.\tilde{e}_x$
where P is a symetric positive definite matrix and let $\delta = \Phi(x,u) - \Phi(\hat{x},u)$
Then

$$\dot{V} = \theta\tilde{e}_x^T[(A - KC)^T P + P(A - KC)]\tilde{e}_x + 2\delta^T P\tilde{e}_x \tag{1.9}$$

Since A-KC is Hurwitz there exist a symetric positive definite matrix Q such that

$$(A - KC)^T + P(A - KC) = -Q \tag{1.10}$$

replacing (1.10) in (1.9):

$$\dot{V} = -\theta\, \tilde{e}_x^T\, Q\, \tilde{e}_x + 2\, \delta(t)^T\, P\, \tilde{e}_x \tag{1.11}$$

With $\parallel \delta(t) \parallel \leq c \parallel \tilde{e}_x \parallel$ and applying Raileigh inequality on (1.11) yields

$$\begin{gathered}
\dot{V} \leq -\theta\lambda_{min}(Q) \parallel \tilde{e}_x \parallel^2 + 2c \parallel \tilde{e}_x \parallel \parallel P \parallel \parallel \tilde{e}_x \parallel \\
\dot{V} \leq -\theta\lambda_{min}(Q) \parallel \tilde{e}_x \parallel^2 + 2\lambda_{max}(P)c \parallel \tilde{e}_x \parallel^2 \\
\dot{V} \leq -[\theta\lambda_{min}(Q) - 2\lambda_{max}(P)c] \parallel \tilde{e}_x \parallel^2
\end{gathered}$$

this is true if $(\theta\lambda_{min}(Q) - 2\lambda_{max}(P)c) > 0$ which means

$$\theta > \frac{2\lambda_{max}(P)c}{\lambda_{min}(Q)} = \theta_0 \tag{1.12}$$

then for every $\theta > \theta_0$, it follows that $V(\tilde{e}_x) \leq e^{-(\theta-\theta_0)t}\, V(\tilde{e}(0))$ which exponentially converges to zero.

Example 1.1

Consider the nonlinear system state space representation:

$$\begin{cases} \dot{x}_1 = x_2 - x_1^3 u \\ \dot{x}_2 = x_3 + x_2 x_1 u \\ \dot{x}_3 = -3x_3 - 3x_2 - x_1^3 - u \end{cases} \quad ; \quad y = x_1$$

with input u taken as $u = 0.2\, cos(t)$.

The system is under the triangular form and can be written as follow:

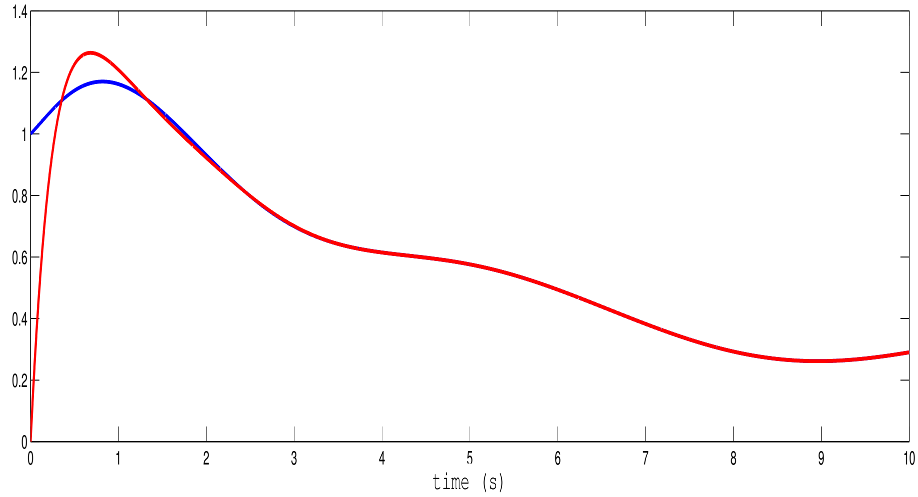
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}^{\text{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} -x_1^3 u \\ x_2 x_1 u \\ -3x_3 - 3x_2 - x_1^3 - u \end{bmatrix}}^{\Phi} ; \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It is clear according to definintion 1.5 that the above system is uniformly observable as it satisfies the observability rank condition and u is a universal input. we also have system initial conditions $x(0) = [1\, 0.5\, 0.5]^T$, and the poles to calculates K are $P = [-0.1\ -0.2\ -0.3]^T$

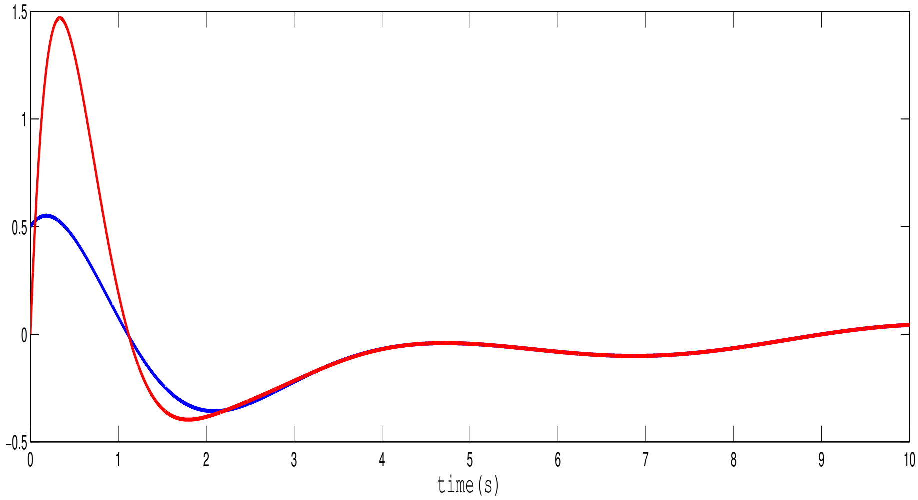
A high-gain observer is taken as in (1.7) with $\theta = 10$ and null initial conditions $\hat{x}(0) = 0$.

Simulations under MATLAB/Simulink of the system and its observer are shown in figure 1.1.

(a)



(b)



(c)

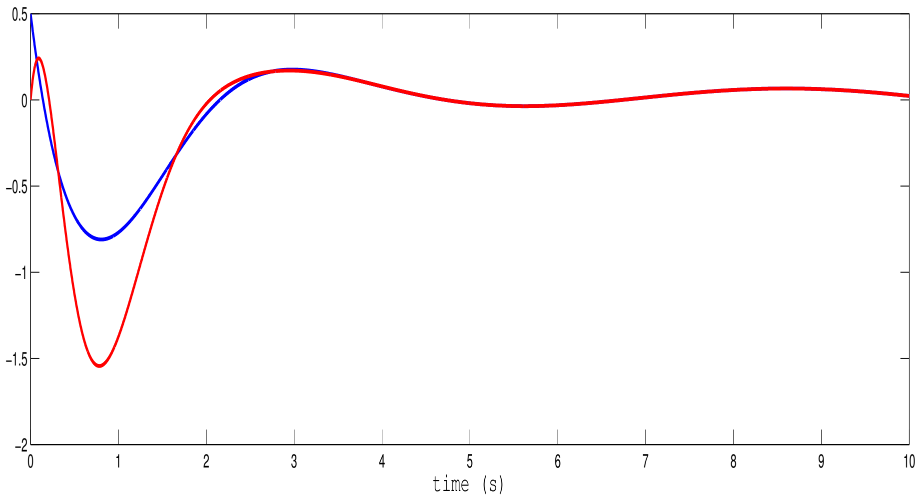


Figure 1.1: Simulation results of example 1.1: (a) — x_1 — $\hat{x}_1$ (b)— x_2 — $\hat{x}_2$ (c)— x_3 — $\hat{x}_3$

Figure 1.1 shows performance of the high-gain observer (1.6). One can notice that the estimates $\hat{x}_1, \hat{x}_2, \hat{x}_3$ converge exactly asymptotically to the real system states.

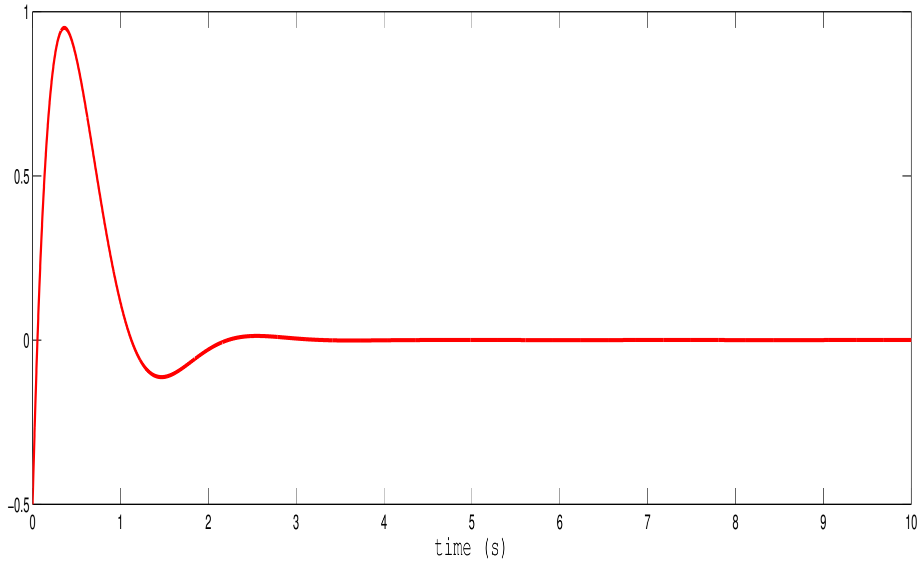


Figure 1.2: Simulation results of example 1.1: estimation error e_{x2}

Figure 1.2 shows the estimation error $e_{x2} = x_2 - \hat{x}_2$ that converges to zero. The estimation error exibits some impulsive behaviour during transient response due to the initial error before it decays rapidly toward zero, this feature of the high-gain observer is known as the 'Peaking Phenomenon'.

In order to evaluate the performance of the HGO when the measured output is subject to noise, simulations are carried out by considering an additive white gaussian noise. Figures 1.3, 1.4, 1.5 illustrate the high-gain observer sensitivity to measurement noise (SNR=40 dB).

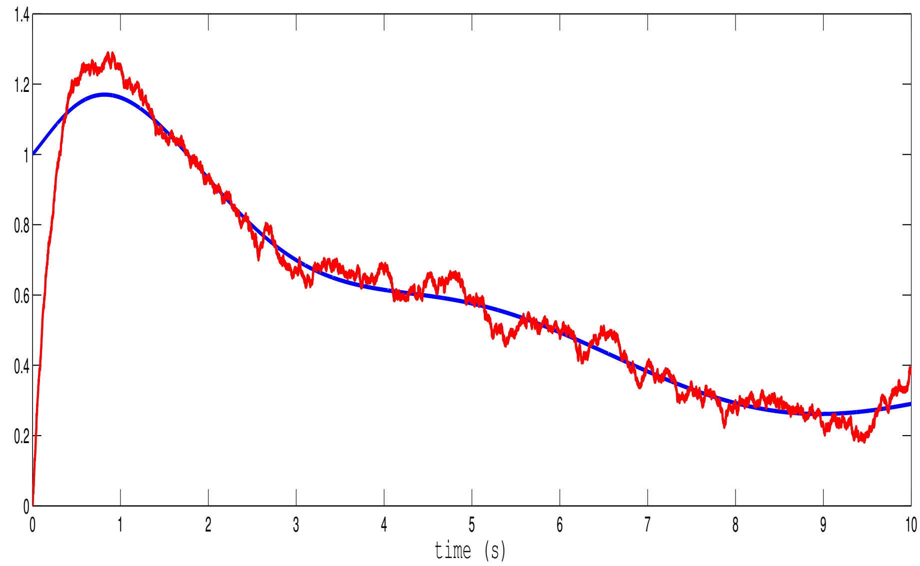


Figure 1.3: Behaviour x_1 and $\hat{x}_1$ in presence of measurement noise : — $\hat{x}_1$ — x_1

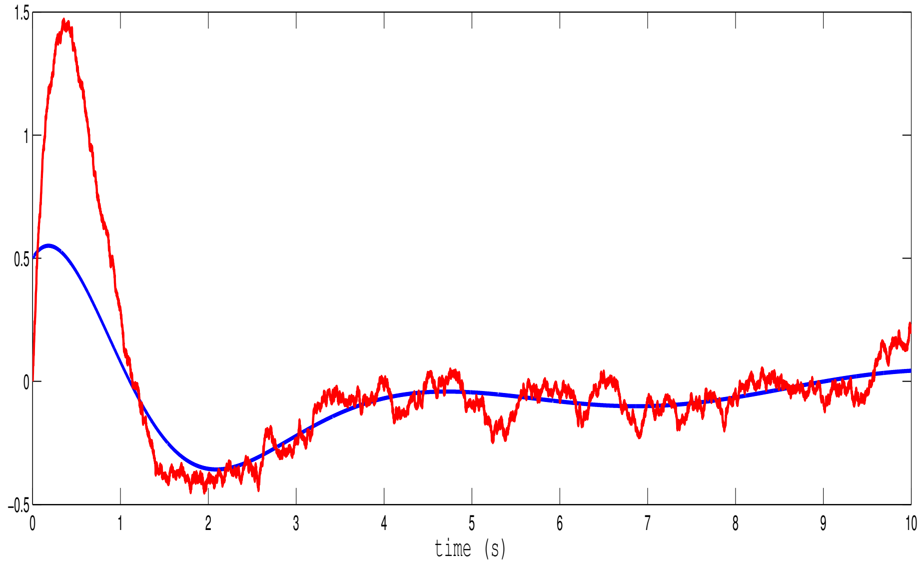


Figure 1.4: Behaviour of x_2 and $\hat{x}_2$ in presence of measurement noise : — $\hat{x}_2$ — x_2

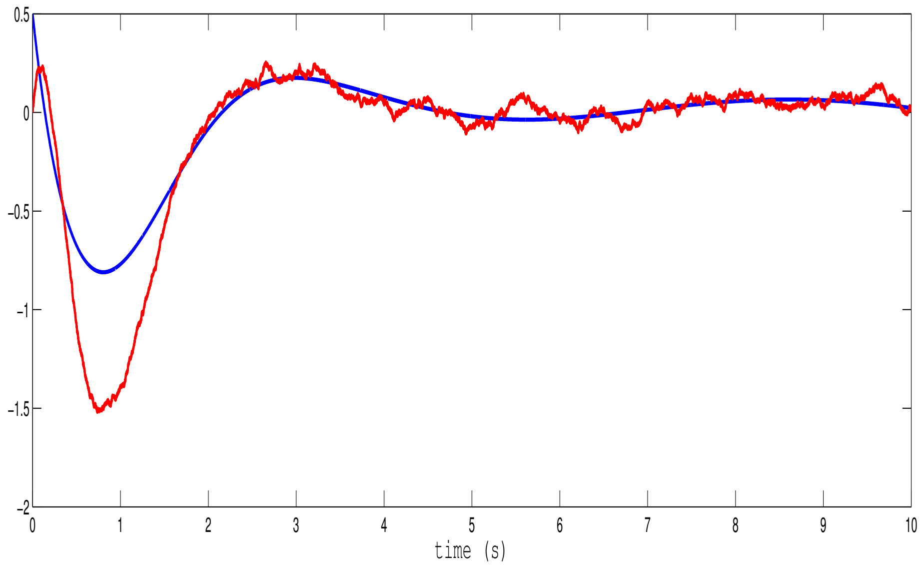


Figure 1.5: Behaviour of x_3 and $\hat{x}_3$ in presence of measurement noise : — $\hat{x}_3$ — x_3

In the presence of measurement noise, the high gain observer (1.7) loose its robustness performance and the estimates are severly affected.

1.2.2 High-Gain Observer Design for Nonlinear Variable-Phase Normal Form

Consider again a nonlinear system under the variable-phase form (1.5). We assume the unknown function $\Phi_n(x, u)$ is bounded [10] and his nominal expression $\Phi_0(x, u)$.
System (1.5) satisfy the observability rank condition stated in definition 1.9 and thus it is globally observable.

A high-gain observer for system (1.5) is as presented in [1]

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1} + \dfrac{\alpha_i}{\epsilon^i}(y - \hat{x}_1) \;\; for \; 1 \leq i < n \\ \dot{\hat{x}}_n = \Phi_0(\hat{x}, u) + \dfrac{\alpha_n}{\epsilon^n}(y - \hat{x}_1) \end{cases} \quad ; \quad \hat{y} = \hat{x}_1 \tag{1.13}$$

where ϵ is a sufficiently small positive constant and α_1 *to* α_n are chosen such that the polynomial

$$S^n + \alpha_1 S^{n-1} + \cdots + \alpha_{n-1} S + \alpha_n \tag{1.14}$$

is Hurwitz.
The estimation error is given by:

$$\begin{bmatrix} \dot{e}_{x1} \\ \vdots \\ \dot{e}_{xn} \end{bmatrix} = \begin{bmatrix} \dfrac{-\alpha_1}{\epsilon} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & 0 & \dots & \dots & 1 \\ \dfrac{-\alpha_n}{\epsilon^n} & 0 & \dots & \dots & 0 \end{bmatrix} \begin{bmatrix} e_{x1} \\ \vdots \\ e_{xn} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \delta(t) \tag{1.15}$$

such that

$$\delta(t) = \Phi_n(x, u) - \Phi_0(\hat{x}, u) \tag{1.16}$$

is the modeling error assumed to be bounded.

Example 1.2

The two dimensional system under variable-phase form (1.4) is described by the state space representation:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_2^3 + u \end{cases} \quad ; \quad y = x_1$$

with input $u = -x_2^3 - x_1 - x_2$ and initial states $x(0) = [1,\ 1]^T$

The proposed high-gain observer to estimte the above system states takes the form (1.13) and is written under the following expression:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \dfrac{\alpha_1}{\epsilon}(y - \hat{y}) \\ \dot{\hat{x}}_2 = -\hat{x}_1 - \hat{x}_2 + \dfrac{\alpha_2}{\epsilon^2}(y - \hat{y}) \end{cases} \quad ; \quad \hat{y} = \hat{x}_1$$

Initial conditions of the observer are $\hat{x}(0) = [0.8 \;\; 1.5]^T$ and the gains $\alpha_1 = 2$ and $\alpha_2 = 1$ and $\epsilon = 0.1$

The results of simulation under MATLAB/Simulink of the system and observer are presented in figures 1.6, 1.7.

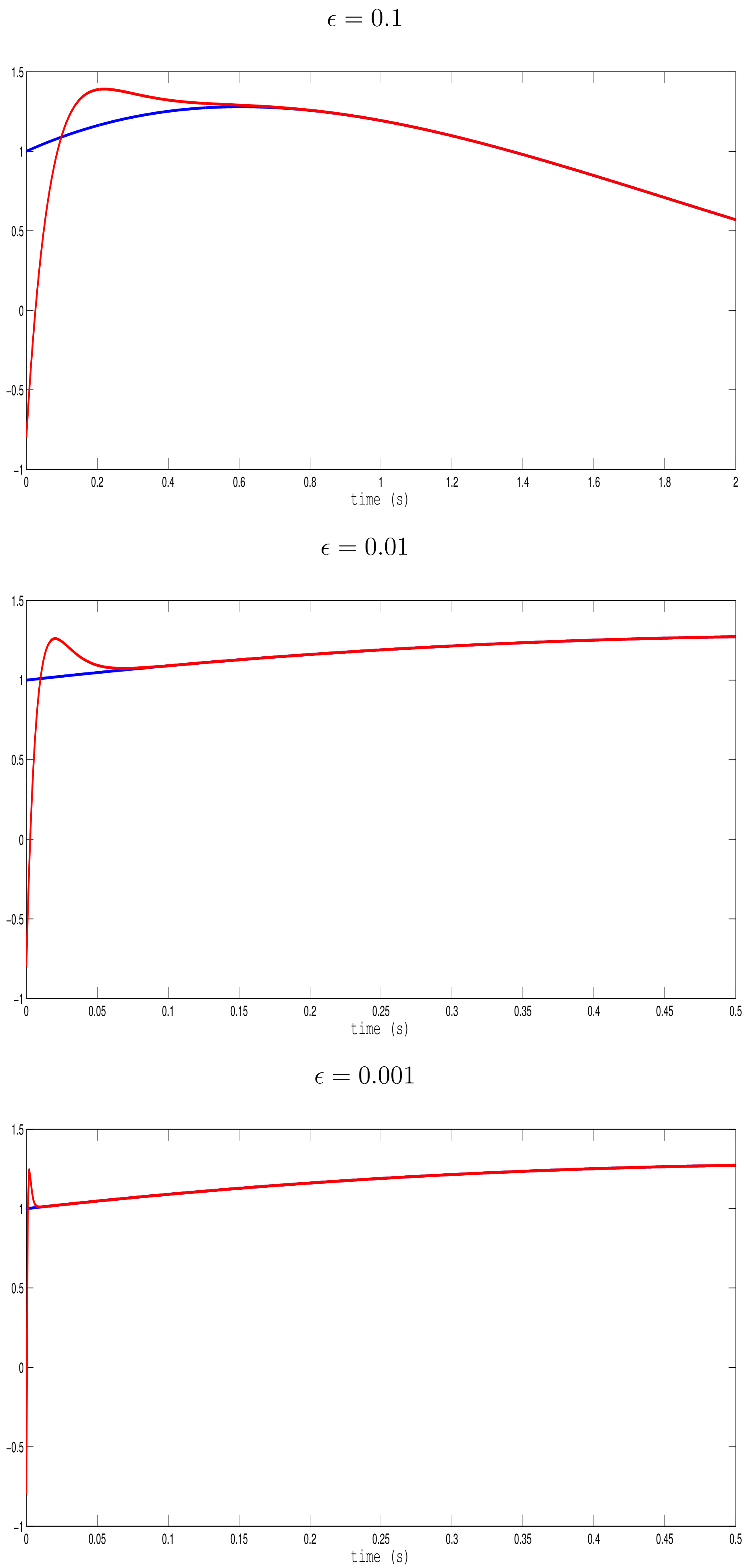


Figure 1.6: Simulation results of exampe 1.2: —— $\hat{x}_1$ —— x_1 for different values of ϵ

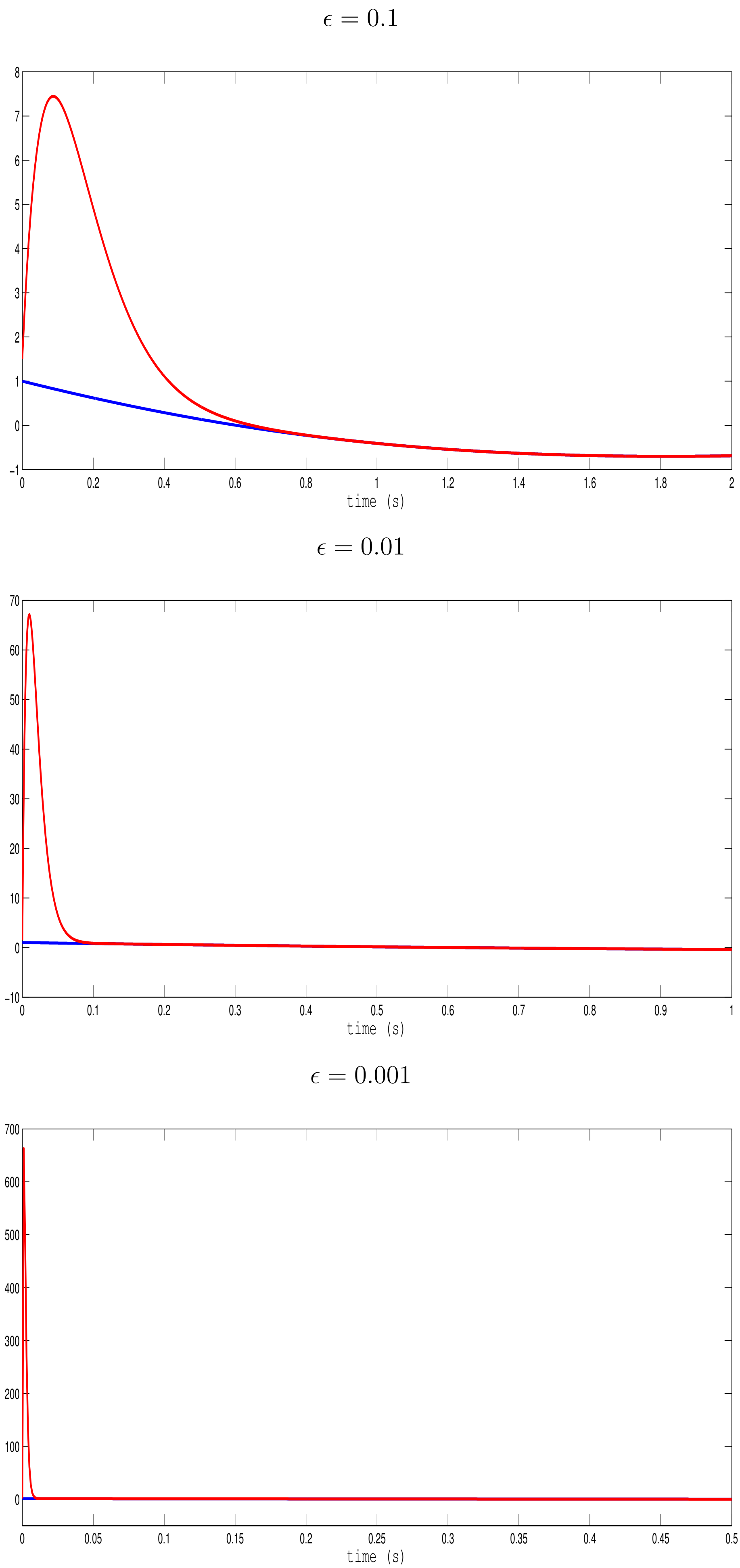


Figure 1.7: Simulation results of exampe 1.2: — $\hat{x}_2$ — x_2 for different values of ϵ

The estimated states by the high-gain observer (1.13) converges exactly to the system's real states after some transient dynamic, we have then asymptotic convergence and the more the observer's gain is getting higher (small values of ϵ) the more the convergence is getting faster.

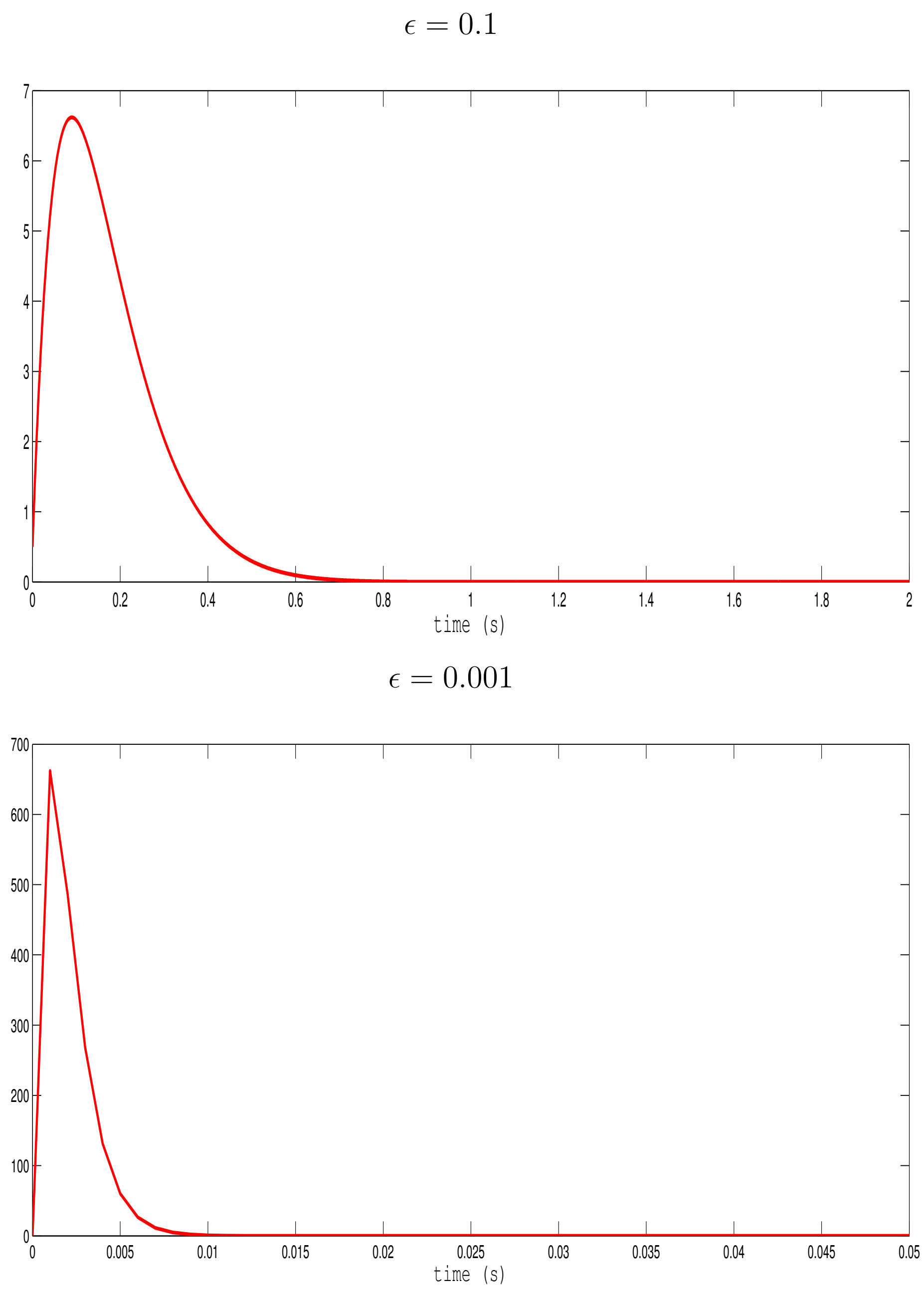
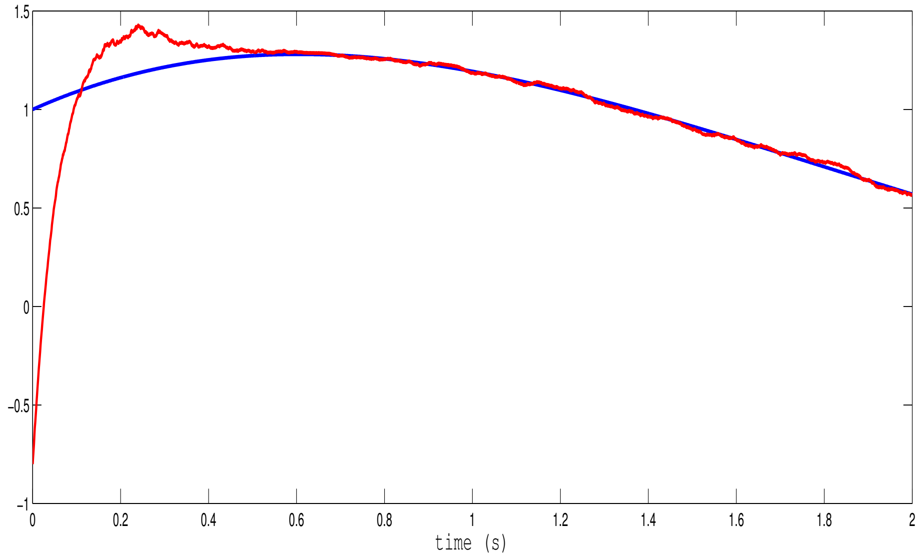


Figure 1.8: Simulation results of example 1.2: estimation error e_{x2} for different values of ϵ

Figure 1.8 shows the estimation error $e_{x2} = x_2 - \hat{x}_2$ for different value of ϵ. We can notice for small value of ϵ the convergence time decrease and the estimation error vanishes in a faster way. On the other side, the higher the observer gain is, the more the peaking is getting important.

In order to evaluate the performance of the high-gain observer 1.13 with respect to measurement noise, we consider the measured output is subject to an additive white noise. Simuations are carried out for the value of $\epsilon = 0.1$ and the results are illustrated in figure 1.9.

(a)



(b)

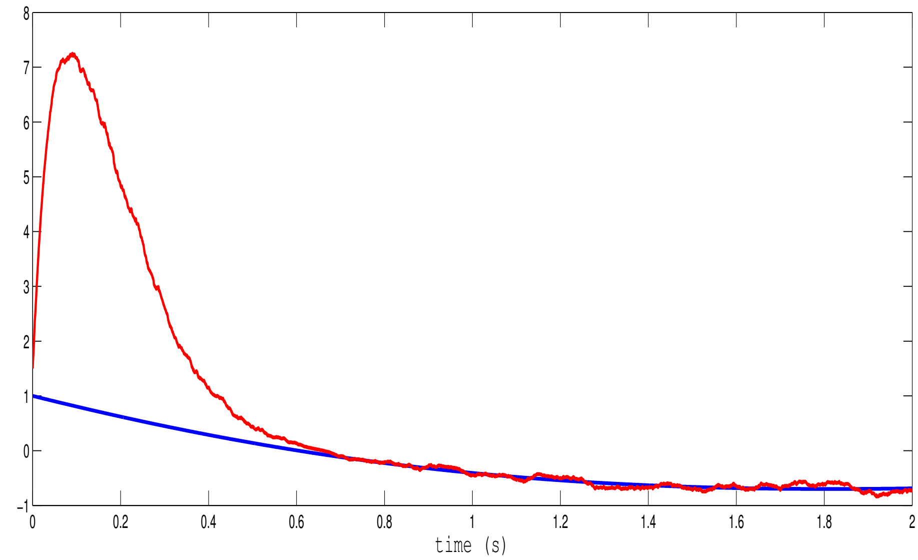


Figure 1.9: Simulation results of example 1.2: (a) — x_1 — $\hat{x}_1$ (b) — x_2 — $\hat{x}_2$

In the presence of measurement noise, the high gain observer (1.13) loose its robustness performance and the estimates are severly affected.

1.3 High-Gain Differentiator Design

The high gain observer can be used for signal differentiation purposes. It allows in fact to estimate the derivatives of a given signal. In this section we explore the advantages and drawbacks of using such observers.

1.3.1 Observer-Based Differentiation Problem Formulation

Let the signal $f(t), t \geq 0$ be composed of a base signal $f_0(t)$ and an additive Lebesgue-measurable noise $w(t)$.

$$f(t) = f_0(t) + w(t) \tag{1.17}$$

The noise signal $w(t)$ is assumed to be bounded, i.e $\|w(t)\| \leq \epsilon_w$.

The differentiation problem consists on estimating successive derivatives of $f_0(t)$ from the knowledge of $f(t)$.

A differentiator is called n^{th}-order differentiator if it provides the estimates for the n^{th} first derivatives $f_0^{(1)}(t)$, $f_0^{(2)}(t)$, ..., $f_0^{(n)}(t)$ with $f_0^{(i)}(t) = \dfrac{d^i f_0(t)}{dt^i}$, $i = 1, 2, \ldots, k$ and $f_0^{(0)}(t) = f_0(t)$.

The $(n+1)^{th}$ derivative of $f_0(t)$ is bounded, i.e for all $t \geq 0$, $\quad \|f_0^{(n+1)}(t)\| \leq L$ where L denotes a known finite positive Lipschitz constant.

We consider the following choice of state variables $x_1(t) = f_0(t)$, $x_2(t) = f_0^{(1)}(t), \ldots, x_{n+1}(t) = f_0^{(n)}(t)$.

Observer-bassed differentiation methods transform the differentiation problem into an observer design problem for the following observable state space model

$$\begin{aligned} \dot{x}_i(t) = x_{i+1}(t); \quad i = 1, n \\ \dot{x}_{n+1}(t) = u(t) \\ y = x_1(t) + w(t) = f(t) \end{aligned} \tag{1.18}$$

where $u(t) = f_0^{(n+1)}(t)$ is assumed unknown and bounded, $\|u(t)\| \leq L$, $\forall t \geq 0$ and $y(t)$ is the measured output.

1.3.2 High-Gain Differentiator

For high-gain differentiator design we consider the above observable state space model (1.20). We assume $u(t) = 0$. A high gain differentiator (observer) of the form (1.14) may be designed to estimate the derivatives of $y(t)$.

Example 1.3

To illustrate the use of high-gain differentiators, we consider the signal $y(t) = sin(3\pi t)$. The objective here is to estimate the first derivative of this signal by mean of a high-gain observer. We define the following state variables $x_1 = y(t)$, $x_2 = \dot{y}(t)$, from which we obtain the following model:

$$\begin{aligned} \dot{x}_1 &= x2 \\ \dot{x}_2 &= \ddot{y} = -9\pi^2 \sin(3) = l(t) \end{aligned}$$

such that $l(t)$ is an unknown function and initial conditions are taken equal to zero.

We can then write a high-gain observer as in (1.14) for the system above as follow:

$$\dot{\hat{x}}_1 = \hat{x}_2 + \frac{\alpha_1}{\epsilon}(y - \hat{x}_1)$$
$$\dot{\hat{x}}_2 = \frac{\alpha_2}{\epsilon^2}(y - \hat{x}_1)$$

Simulation results under MATLAB/Simulink for the following values $\alpha_1 = \alpha_2 = 5$ and null observer initial conditions are shown in figures 1.10, 1.11

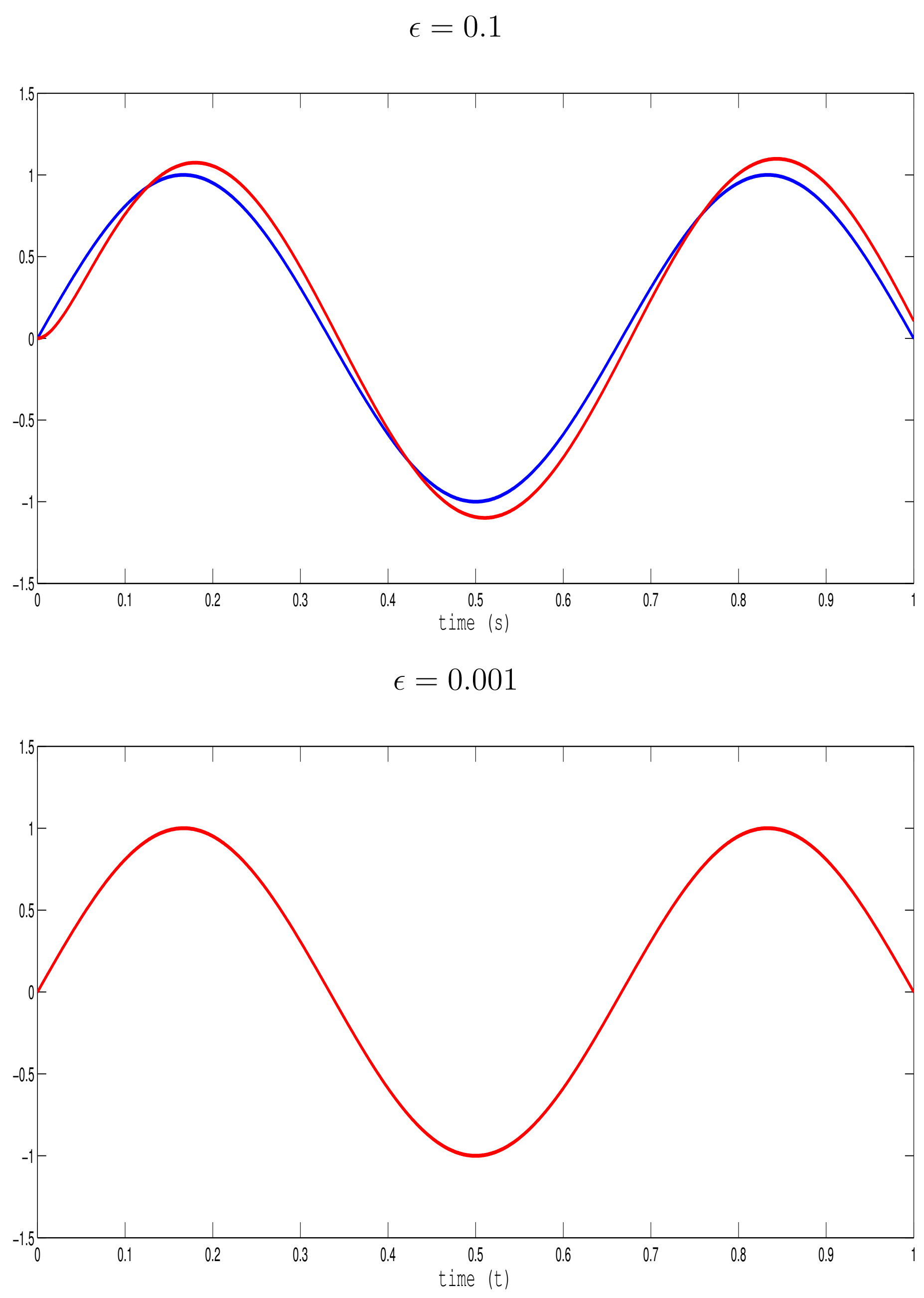


Figure 1.10: Simulation results of example 1.3: — $y(t)$ — $\hat{y}(t)$ for different values of ϵ

The signal $y(t)$ in noise-free case is estimated by $\hat{y}(t)$ and the convergence is faster. The estimated signal is more precise with lower values of ϵ which means higher gains of the differentiator. Higher gains of the observer allow to cope with Lipchitz nonlinearities.

Estimation of the the first derivative by the high-gain diffrentiator is presented in figure 1.11

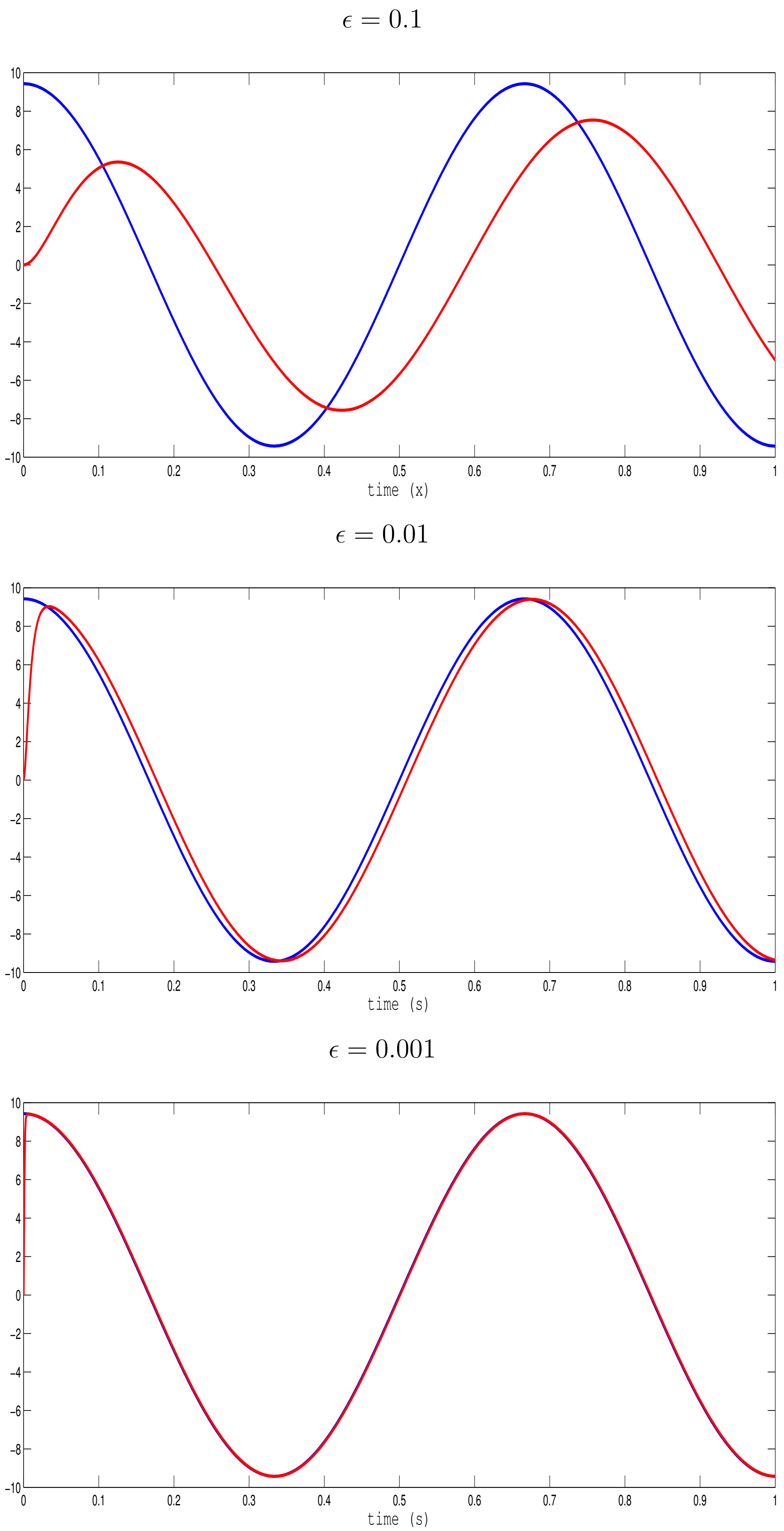


Figure 1.11: Simulation results of example 1.3: — $\dot{y}(t)$ — $\dot{\hat{y}}(t)$

Although high gains of the observer allow for better estimation of the derivatives of the signal in a much more faster way, it amplifies the peaking phenomenon on the estimation errors as we can notice in the following figure 1.12 for a small value of $\epsilon = 0.0001$

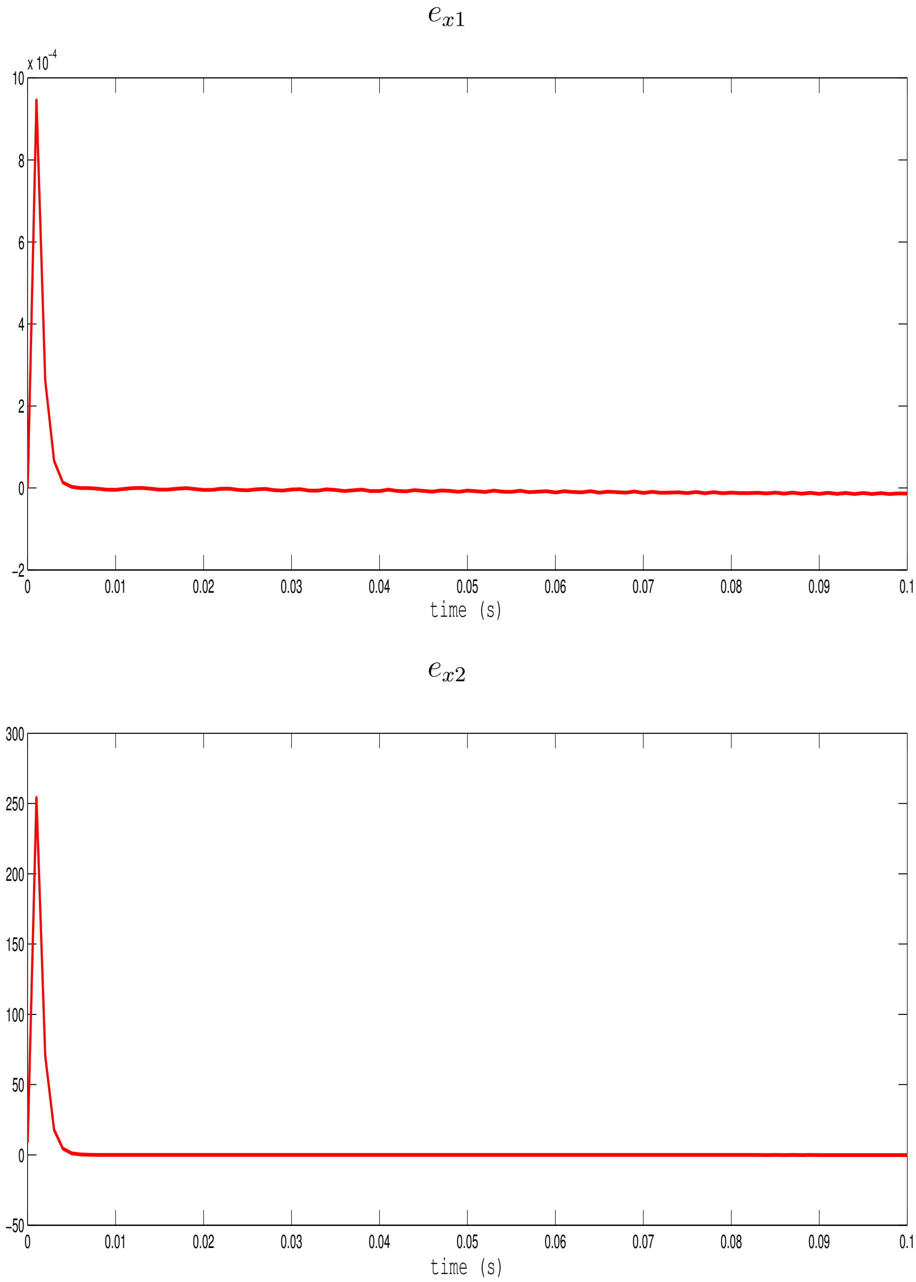
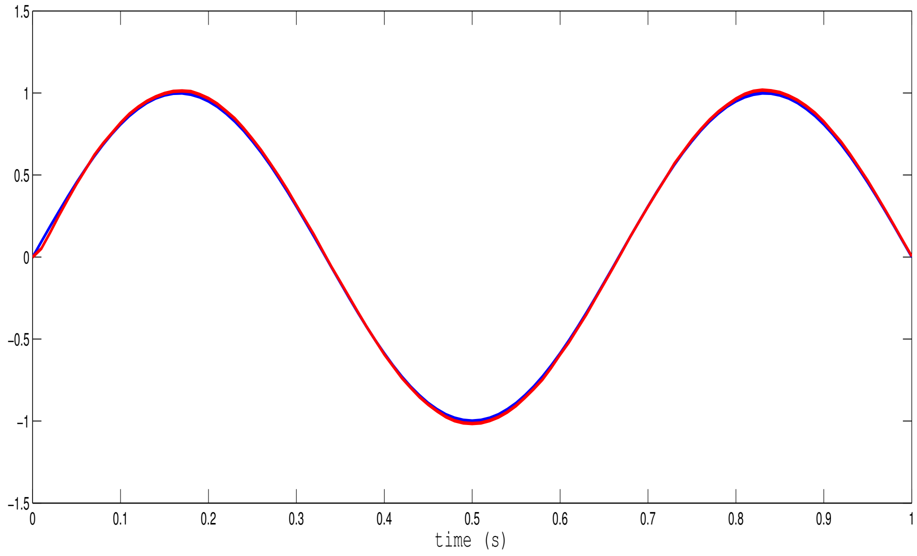


Figure 1.12: Simulation results of example 1.3: estimation errors e_{x1} and e_{x2} with $\epsilon = 0.001$

In order to evaluate the robustness of the high gain observer with respect to measurement noise, we assume the signal $y(t)$ is corrupted with an additive white noise (with Signal-Noise Ratio SNR=40 dB). The effect of measurement noise on the differentiation process is presented in figure 1.13

(a)



(b)

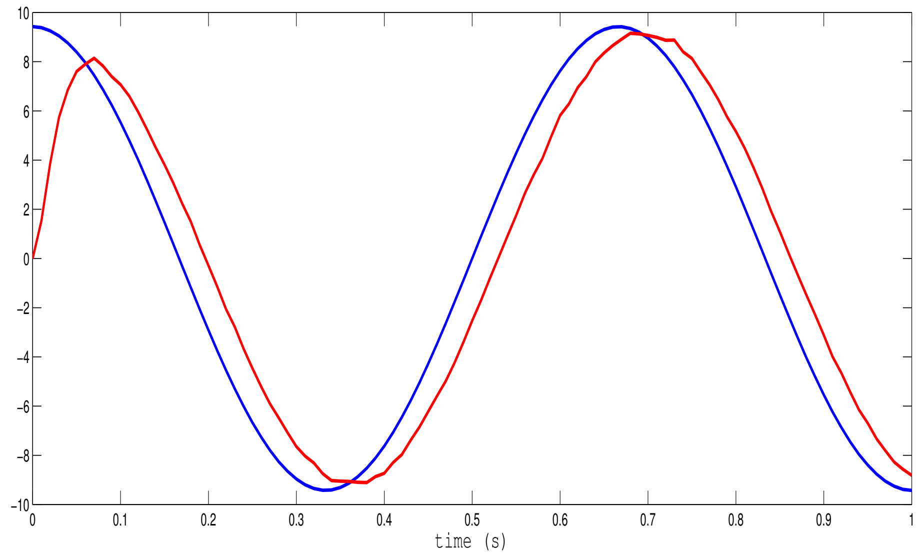


Figure 1.13: Simulation results of example 1.3: (a) — $y(t)$ — $\hat{y}(t)$ (b) — $\dot{y}(t)$ — $\dot{\hat{y}}(t)$, subject to measurement noise.

In the presence of measurement noise, the observer's high gain amplifies any noise present in measurement signal, which can lead to significant estimation errors.

1.4 Conclusion

The presented observer in this chapter is called high-gain observer because its gain must be chosen sufficiently large in order to compensate for Lipschitz constant of the nonlinearities which are seen here as disturbances. This observer provides an exponential convergence whoose rate increases with the gain. The larger the gain, the larger the solution can become before converging, this is the so called peaking phenomenon. It is also shown that the larger the gain, the larger the impact of disturbances and measurement noise. A compromise has to be found before choosing these gains and solutions like saturating the control may be considered. The peaking phenomenon can destabilize the closed loop system. The high gain observer present many drawbacks and though it can be used for signal differentiation, one can only obtain asymptotic estimation which is not what really seeked for in most practical engineering situations. Finite time differentiators are then of interest and are introduced in the upcoming chapter.

Chapter 2

Sliding Mode Differentiators

In general, the dynamics of a system cannot be changed by the designer to improve system performances but rather we design a controller that generates a control law u that enables the system to achieve a desired performance. In order to design a controller, one generally proceed by modeling the system. However, there is always a mismatch between mathematical model of the system and the actual systems dynamics. This mismatch can be due to unmodeled dynamics, variation in the system parameters and to approximations of complex behaviour of the system. One of the most popular control techniques especially for variable structure systems (VSS) is the sliding mode control (SMC). Sliding mode controllers are robust and produce required performance in practice regardless mismatches apart from handling uncertainties and disturbances while maintaining high performance.

Sliding mode observers take a priveleged place in the design of differentiation algorithms [3],[18], [26]. Sliding mode differentiators proved their robustness with respect to uncerainties and small Lebesgue measurable noise and most importantly for their finite time convergence. The first order differentiator based on the super twisting algorithm is introduced to estimate the first derivative of a bounded noisy signal [24], [22]. A generalization based on higher-order sliding mode observers to estimate the n sucessive derivatives of a given noisy signal is developped [25], [26], [27] which provide finite time exact estimation in noise free case and an asymptotic optimal estimation error for noisy signals.

Sliding mode differentiators are used in a wide range of engineering applications including robotic manipulators control [57], sensorless control of electric drives [13], in electric vehicles for state of charge and state of health estimation [62], noisy signal estimation and filtering in signal processing [60], in vehicle dynamics for automotive control systems [58], in biological systems [61] and in renewable energy systems like wind turbine control [59].

2.1 Overview of Sliding Mode Control

In sliding mode control, the goal is to find a control law u that steers the nonlinear system to a given manifold (surface) and keeps it within this constraint [40]. This sliding mode control can be split into two regions (modes):

The sliding mode :

The best sliding surface we can choose is when $s(x) = 0$ because sliding on this surface allows to reach the equilibrium point which is the origin and system states are redirected to it.

The reaching mode :

starting from initial condition $s(0) = s_0$, find the reaching mode that ensure this initial condition will moves toward the sliding surface ($s = 0$).

Figure (2.1) shows The reaching phase (when the state trajectory is driven towards the sliding surface) and a sliding phase (when the state trajectory is moving towards the origin along the sliding surface).

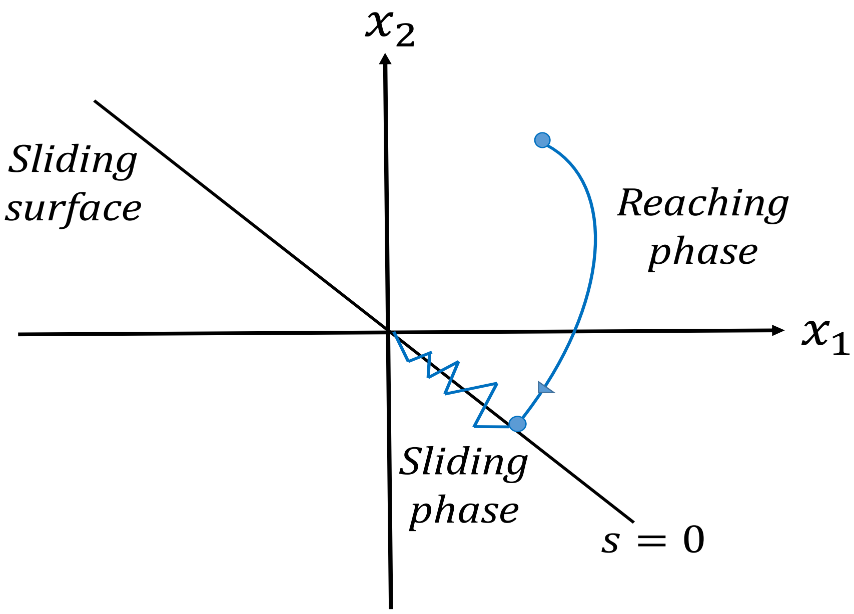


Figure 2.1: Reaching phase and sliding phase

Throughout this section, basic principles and concepts of conventional sliding mode control are presented including the definition of a sliding mode control, its motion equations and design methodes.

2.1.1 Conventional Sliding Modes

Studied since 60's, Conventional sliding modes as used by Levant [3], refers to sliding modes of the first order. The design procedure for conventional sliding modes consist of selecting a manifold (surface) and a control law enforcing system state trajectories to this manifold.

The design procedure of sliding modes control is as follow:
we consider single input single output nonlinear systems under the following control-affine form

$$\dot{x}(t) = f(x,t) + g(x,t)u + b(x,t) \tag{2.1}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ is the state verctor, $u \in R$ is the system input. Functions $f(x,t)$ and $g(x,t)$ are assumed to be sufficiently smooth. $b(x,t)$ contains all perturbations, uncertainties and unmodeled dynamics.

The system relative degree is required to be equal to one with respect to the sliding variable s.

The first step to design sliding mode is to introduce a new variable in the state space of the system which is the sliding variable s [43].

$$s = x_2 + cx_1, \quad c > 0 \tag{2.2}$$

that we have to drive to zero in finite time by means of the control u. It corresponds, for a second order system, to a straight line in the state space of the system and known as the sliding surface. This sliding surface has to be attractive and meet the reachability or existence condition that ensure the trajectories of the system are driven towards the sliding surface (2.2) and remains on it thereafter. This can be achieved using the Lyapunov method by designing a Lyapunov function that renders the sliding surface stable by satisfying the following; the equilibrium (the origin) is uniformly asymptotically stabe and system trajectories convergence toward it.

$$\begin{aligned} \dot{V} &= 0 \quad for \quad x = 0 \\ \dot{V} &= \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} < 0 \quad for \quad x \neq 0 \end{aligned} \tag{2.3}$$

We consider the Lyapunov-candidate function of the sliding surface $V(s)$ of class C^1 and defined as

$$\begin{aligned} V &= 0 \quad for \quad x = 0 \\ V &> 0 \quad for \quad x \neq 0 \end{aligned} \tag{2.4}$$

We generally choose a quadratic function $V(s) = \frac{1}{2}s^2$ such that

$$\dot{V}(s) = s\dot{s} < 0 \tag{2.5}$$

this yields the sliding mode existence condition.

The second step is then to find the control law u; a discontinuous function [3] that drives the state variables to the sliding surface $s = 0$ in finite time tr and keeps them on the surface thereafter in the presence of the bounded disturbance $b(x,t)$. u is is then generated bu a sliding mode controller and an ideal sliding mode is said to be taking place in the system (2.1) for all $t > tr$.The control law can be given as

$$u = -ksign(s) \quad k > 0 \tag{2.6}$$

The conventional sliding mode controller ensure finite-time convergence of the sliding variable to zero. Asymptotic convergence of the state variables to zero in the presence of the external bounded disturbance $b(x,t)$ is guaranteed.

2.1.2 Fillipov Approach

Consider the dynamical system described by initial value problem given by ordinary differential equations of the form

$$\dot{x}(t) = f(t, x(t)) \tag{2.7}$$

$f(t, x(t))$ is called the right hand side of (2.7), t_0 and x_0 are initial time and initial state respectively.

A continuously differentiable function $x(t)$ is called solution of the initial value problem (2.7) if and only if it satisfies (2.7) [45].

Theorem 2.1 ***Existence and uniqueness theorem***

Consider the differential equation (2.7). If $f(t, x)$ is

- *Continuous on $R^+ \times R^n$.*

- *Uniformly Lipschitz continuous with respect to its second argument x which means there exists a Lipchitz constant $L \geq 0$ such that:*

$$\forall t \in R^+, \forall x_1, x_2 \in R^n \quad \|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$$

then for each $(t_0, x_0) \in R^+ \times R^n$, the initial value problem (2.7) has unique solution $x(t)$.

For sliding modes the control law u is discontinuous and thus the differential equation describing the resulting closed loop system has discontinuous right-hand side which yields that Theorem 2.7 cannot be guaranteed anymore.

Filipov in [44] proposed a solution concept for differential equations with discontinuous right-hand side by constructing a solution as the 'average' of the solutions obtained from approaching the point of discontinuity from different directions.

We consider then the following differential equation with discontinuous right-hand side

$$\dot{x}(t) = f^c(x), \quad x \in R^n \tag{2.8}$$

f^c is discontinuous with respect to the state vector.

Define $f^c_-(x)$ and $f^c_+(x)$ as limits of $f^c(x)$ as the point of discontinuity is approached from opposite sides of s. Equation (2.8) can be written as

$$\dot{x}(t) = f^c(x) = \begin{cases} f^c_+, \ for \ s(x) > 0 \\ f^c_-, \ for \ s(x) < 0 \end{cases} \tag{2.9}$$

The Filipov solution is an average solution of the two velocity vectors at the point of discontinuity.

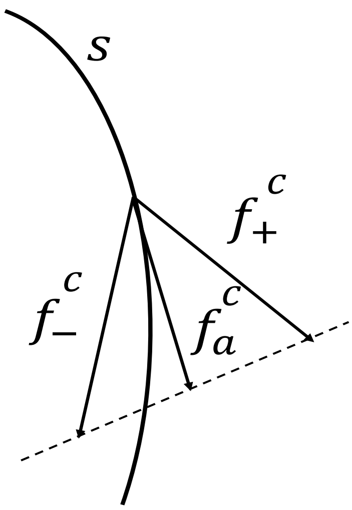


Figure 2.2: The Filipov construction scheme

This solution is given by

$$\dot{x}(t) = (1-\alpha)f_-^c(x) + \alpha f_+^c(x), \quad 0 < \alpha < 1 \tag{2.10}$$

The scalar alpha is chosen such that

$$f_a^c := (1-\alpha)f_-^c + \alpha f_+^c \tag{2.11}$$

is tangential to s.

Equation (2.8) can be written as the Filipov differential inclusion [3]

$$\dot{x}(t) \in F(x) \tag{2.12}$$

with the right hand side is a convex set defined as

$$F(x) = \begin{cases} f_-(x) & for \quad x \in s^- \\ (1-\alpha)f_-^c + \alpha f_+^c & for \quad x \in s \\ f_+(x) & for \quad x \in s^+ \end{cases} \tag{2.13}$$

Slolutions of (2.12) are defined as absolutely continuous functions of time satisfying the inclusion. These solutions always exist and have most of the well-known standard properties except the uniqueness.

2.1.3 The Chattering Phenomenon

In an ideal sliding mode, system states remains on the sliding surface once they reach it in finite time (for all $t > t_r$), this implies perfect tracking of the sliding mode surface without deviation by mean of the infinite switching frequency of the control input. These ideal sliding modes perfectly reject disturbances and compensate for modeling inaccuracies.

In practice the ideal sliding mode does not exist due to the switching imperfections such as switching time delays and small time constants in the actuators. This produces a particular behaviour in the vicinity of the sliding surface known as 'the chattering phenomenon'.

Chattering may excite high frequency oscillations and noise leading to less precise control and potentially destabilizing the system [54], in addition to increasing energy consumption and important heat loss.

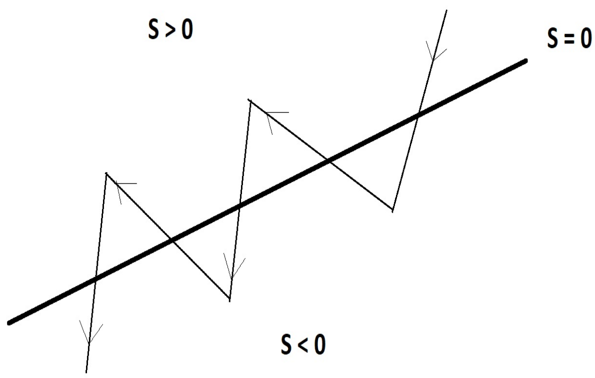


Figure 2.3: The chattering phenomenon

To mitigate chattering effects, several strategies have been proposed. Filtering, which consists on implementing filters to smooth-on the high frequency components of the control signal. Other technique consists on replacing the signum function by a continuous approximation like the sigmoid function or ssaturation function. An intersting solution to remove chattering is based on the theory of higher-order sliding modes is presented in the next section.

2.2 Higher-Order Sliding Modes (HOSM)

The core idea behind sliding mode control is to keep a properly chosen constraint (surface) at zero in sliding mode. Sliding modes are then characterized by the smoothness degree of the constraint function calculated along the system trajectories.

The standard sliding mode is of first order and may be implemented only when relative degree of the constraint s is one (which means control has to appear explicitly already in the first total time derivative of the constraint function). In addition, the high frequency control switching may cause the so-called chattering effect.

The higher order sliding modes (HOSM) approach [25],[27],[23] generalizes the basic sliding mode idea, acting on the higher order time derivatives of the system deviation from the constraint instead of influencing the first deviation derivative as it happens in standard sliding modes. It is a movement on a discontinuity set of a dynamic system understood in Filipov's sense. It allows removing both the chattering and relative degree restrictions while preserving the sliding mode features and improving its accuracy. The sliding order characterizes the dynamic smoothness degree in the vicinity of the mode and it is the number of continuous total derivatives of s (including the zero one) in the vicinity of the sliding mode and is determined by the following set

$$s = \dot{s} = \ddot{s} = \cdots = s^{(r-1)} = 0 \tag{2.14}$$

The motion on the set (2.14) is called r-sliding mode. The r-th derivative $s^{(r)}$ is mostly supposed to be discontinuous or non existed [25]. The r^{th} order sliding mode can be reffered to as r^{th} order sliding or more simply r-sliding.

HOSM ensure finite time stabilization of a differential inclusion and are robust to disturbances and uncertainties with chattering attenuation.

One of the approaches to achieve (2.14) is under the framework of homogeneity and the designed controllers are said r-sliding homogeneous [23].

2.2.1 Homogeneity Notions

Homogeneity theory provides an alternative approach to prove convergence in sliding mode control without solely relying on Lyapunov theory. The main idea behind involves levereging the properties of homogeneous functions and vector fields to establish finite time stability and convergence of the system states.

Definition 2.1 ***Homogeneity*** *[25]*
A function $f : R^n \to R$ (respectively, a vector-set field $F(x) \subset R^n,\ x \in R^n$) is called homogeneous of the degree $q \in R$ with the dilation $d_\kappa : (x_1, x_2, \ldots, x_n) \to (\kappa^{m_1}x_1, \kappa^{m_2}x_2, \ldots, \kappa^{m_n}x_n)$, where $m_1, \ldots, m_n$ are some positive numbers (weights), if for any $\kappa > 0$ the identity $f(x) = \kappa^{-q}f(d_\kappa x)$ holds (respectively, $F(x) = \kappa^{-q}d_\kappa^{-1}F(d_\kappa x)$).
The homogeneity of a vector field $f(x)$ (a vector set field $F(x)$) can equivalently be defined as the invariance of the differential equation $\dot{x} = f(x)$ (differential inclusion $\dot{x} \in F(x)$)with respect to the combined time-coordinate transformation $G_\kappa : (t, x) \to (\kappa^p t,\ d_\kappa x)$, $p = -q$.

Definition 2.2 *[25] A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called globally uniformly finite-time stable at 0, if it is Lyapunov stable and for any $R > 0$ exists $T > 0$, such that any trajectory starting within the disk $\|x\| < R$ stabilizes at zero in the time T.*

Definition 2.3 *[25] A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called globally uniformly asymptotically stable at 0, if it is Lyapunov stable and for any $R > 0, \epsilon > 0, T > 0$ exists such that any trajectory starting within the disk $\|x\| < R$ enters the disk $\|x\| < \epsilon$ in the time T to stay there forever.*
A set D is called dilation retractable if $d_\kappa D \subset D$ for any $\kappa < 1$.

Definition 2.4 *[25] A homogeneous differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called contractive if there are two compact sets D_1, D_2 and $T > 0$ such that D_2 lies in the interior of D_1 and contains the origin, D_1 is dilation retractable, and all trajectories starting at the time 0 within D_1 are localised in D_2 at the same moment T.*

Theorem 2.2 *[25] Let $\dot{x} \in F(x)$ be a homogeneous Filipov inclusion with a negative homogeneous degree $-p$. Then definitions 2.2, 2.3, 2.4 are equivalent and the maximal settling time is a continuous homogeneous function of the initial conditions of the degree p.*

Corollary 2.1 *[25] The global uniform finite-time stability of homogeneous differential equations (Filippov inclusions) with negative homogeneous degree is robust with respect to homogeneous perturbations causing locally small changes of the equation (inclusion) graph.*

2.2.2 2-Sliding Mode Controllers

Unlike conventional sliding modes, second-order sliding modes incorporates higher-order derivatives of the sliding surface; it ensures both the sliding surface and its first derivative converge to zero, providing smoother and more precise control actions.

Definition 2.5 *[3]*
Consider a discontinuous differential equation (Filippov differential inclusion (2.12)) with a smooth function s and let it be understood in the Filippov sense. Then, provided that:

- *s and the total time derivative $\dot{s}$ are continuous functions of x.*

- *The set*

$$s = \dot{s} = 0 \tag{2.15}$$

  is a nonempty integral set.

- *The Filippov set of admissible velocities at the set defined by Equation (2.15) contains more than one vector.*

  then the motion on the set (2.15) is said to exist in a 2-sliding (second-order sliding) mode and the set (2.15) is called a 2-sliding set.

Once more consider a dynamic system of the form

$$\begin{aligned} \dot{x}(t) &= f(x,t) + g(x,t)u + b(x,t) \\ y &= h(x) \end{aligned} \tag{2.16}$$

Assume the relative degree of system (2.16) with respect to the sliding variable is of maximum two and this means that for the first time u appears explicitly only in the second total derivative of s, one can then distinguish the following two cases:

- The relative degree for the system is one, the first derivative of s is given by

$$\dot{s} = a_1(x) + b_1(x)\,u \quad \textit{with } b_1(x) \neq 0 \tag{2.17}$$

  where $a_1(x) = L_f s(x)$ and $b_1(x) = L_g s(x)$.

- The relative degree of the system is equal to two, then $b_1(x) = 0$ and the second derivative of s is given by

$$\dot{s} = a_2(x) + b_2(x)u \tag{2.18}$$

For some K_M, K_m, $C > 0$ suppose that the following inequalities hold

$$0 < K_m \leq b_2(x) \leq K_M \quad and \quad |a_2(x)| \leq C$$

then (2.17), (2.18) imply the differential inclusion

$$\ddot{s} \in [-C, C] + [K_m, K_M]u. \tag{2.19}$$

The problem (2.19) is solved by contructing a control law $u = u(s, \ddot{s})$ such that all trajectories of (2.19) converge in finite time to the origin $s = \dot{s} = 0$ of the 2-sliding phase space s, $\dot{s}$.

Techniques among others, like Twisting and Super-Twisting algorithms are presented hereafter since they are commonly used to implement second order sliding modes.

Twisting Controller

The twisting controller described below is historically the first 2-sliding controller which was proposed. It is defined by the formula [3]

$$u = -(r_1 sign(s) + r_2 sign(\dot{s})) \quad with \quad r_1 > r_2 > 0 \tag{2.20}$$

Theorem 2.3 *Let r_1 and r_2 satisfy the conditions*

$$(r_1 + r_2)K_m - C > (r_1 - r_2)K_M + C \quad and \quad (r_1 - r_2)Km > C \tag{2.21}$$

then, the controller in (2.20) guarantees the appearance of a 2-sliding mode (2.15) attracting the trajectories of the sliding variable dynamics (2.19) in finite time.

Super-Twisting Controller

The super twisting controller is a 2-sliding control algorithm used to establish a stable 2-sliding mode for the sliding variable dynamics (2.15) in finite time. Applied for systems with relative degree one which means the derivative of the sliding surface is as in (2.17).

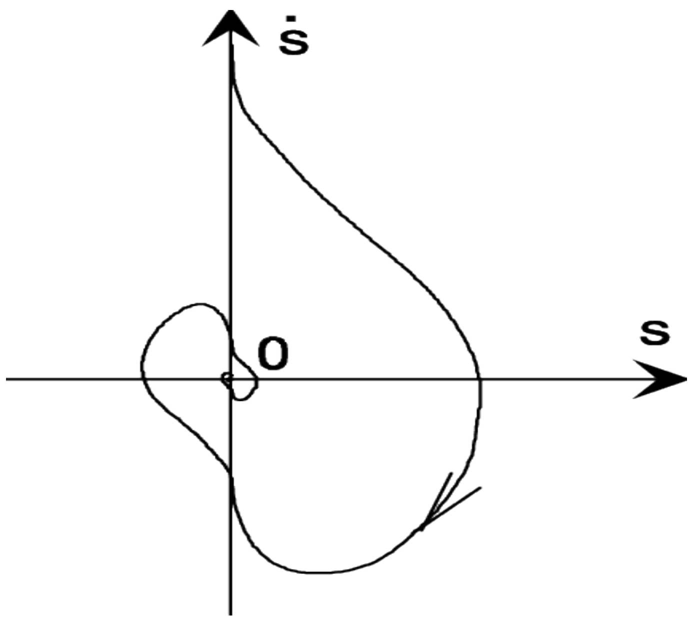


Figure 2.4: Trajectory of the super-twisting controller

Assuming that for the positive constants C, K_M, K_m, U_M, q the following inequalities hold

$$0 < K_m \le b_1(x) \le K_M \quad and \quad \left|\frac{a_1(x)}{b_1(x)}\right| < q\,U_M, \quad 0 < q < 1 \tag{2.22}$$

then the supertwisting controller is defined by the following algorithm

$$u = -\lambda\,|s|^{\frac{1}{2}}\,sign(s) + u_1, \quad \dot{u}_1 = \begin{cases} -u & for \quad u > U_M \\ -\alpha sign(s) & for \quad u \le U_M \end{cases} \tag{2.23}$$

Theorem 2.4 *With $K_m\alpha > C$ and λ sufficiently large, the controller (2.23) guarantees the appearance of a 2-sliding mode $s = \dot{s} = 0$ in system (2.17), which attracts the trajectories in finite time.*

We notice that for the supertwisting algorithm, one do not need measurements of $\dot{s}$.

Other 2-sliding controller in the litterature include suboptimal algorithm, quasi-continuous control algorithm and control algorithm with prescribed convergence law [3].

2.2.3 Higher Order Sliding Mode Controllers

The r-sliding controllers $r \ge 3$ require only the knowledge of the system relative degree. The produced control is a discontinuous function of the surface s and of its real-time calculated successive derivatives $\dot{s}$, $\ddot{s}$, ..., s^{r-1} and the rth sliding mode is determined by the equalities:

$$s = \dot{s} = \ddot{s} = \cdots = s^{r-1} = 0 \tag{2.24}$$

These controllers are robust, finite-time stabilizing and are able to remove the chattering effect.

There is a number of HOSM controllers proposed in the litterature. Among them we cite the nested r-sliding controller buit by some recursive algorithm [3]. On the other hand the r-sliding quasi-continuous controller [55] provide a control law which is continuous evrywhere except at (2.24). HOSM control via homogeneity received a lot of attention. It uses the concept of homogeneity to design sliding mode controllers of any desired order [40], [43].

2.3 Sliding Mode Differentiators

Sliding mode differentiators are robust differentiators that leverage the principles of sliding mode control. They are designed to estimate the derivatives of a signal in a way that is robust to disturbances. Convergence to the true derivatives is made in finite time.

The core idea behind sliding mode differentiator design is to apply sliding mode control that renders the estimation error dynamic, which is chosen here as a sliding surface, to vanish in finite time and select appropriate gains to ensure robust performance.

The differentiation process under sliding modes follows then the two steps:

- The differentiation error trajectory, from its inintial value, evolves toward the sliding surface under what is called attainment (reaching) mode.

- The differentiation error trajectory remains on the sliding surface with imposed dynamics to cancel any differentiation error and this step is the sliding mode.

Sliding mode algorithms, in particular higher order sliding modes, are then used in the differentiator design process as they provide more advanced differentiation capabilities with reduced chattering.

We present in the following, a first order differentiator based on the super twisting algorithm and then make some generalization on arbitrary order differentiators.

2.3.1 First-Order Differentiator

We introduce hereafter, a first order sliding mode differentiator based on the super-twisting algorithm.

We consider again the input signal $f(t)$, $t \geq 0$ be composed of a base signal $f_0(t)$ and an additive Lebesgue-measurable noise $w(t)$.

$$f(t) = f_0(t) + w(t) \tag{2.25}$$

The noise signal $w(t)$ is assumed to be bounded, i.e. $\|w(t)\| \leq \epsilon_w$.
We assume the first derivative $\dot{f}_0(t)$ is globaly Lipschitz [3] with Lipschitz constant $L > 0$.

The main objective here is to find real-time robust estimation of $f_0(t)$ and $\dot{f}_0(t)$, this estimation is exact in the absence of noise.

With a particular choise of the observable state space model as in (1.18) and a particular choice of the sliding surface s, the sliding mode super-twisting algorithm can be used to differentiate the unknown base signal $f_0(t)$ with the advantage that it depends only on s.

Set $x_1 = f_0(t)$ and $x_2 = \dot{f}_0(t)$. Now consider the auxilary system

$$\dot{z}_0 = u \tag{2.26}$$

where u is the control input.

We choose the sliding surface to be the error between z_0 and the measured signal $f(t)$ defined as

$$\begin{aligned} s &= z_0 - f(t) \\ \dot{s} &= \dot{z}_0 - \dot{f}(t) \\ &= -\dot{f}(t) + u \end{aligned} \tag{2.27}$$

The objective here is to find some control law u that keeps $s = 0$ in a 2-sliding mode which means $s = \dot{s} = 0$, so when the control $u \to \dot{f}_0$ the variable $z_0 \to f_0(t)$.
By applying the modified super-twisting algorithm we obtain

$$\begin{aligned} u &= u_1 - \lambda_1 \, |s|^{\frac{1}{2}} \, sign(s) \\ \dot{u}_1 &= -\lambda_0 \, sign(s) \end{aligned} \tag{2.28}$$

by taking z_0 and z_1 as the estimates for $f_0(t)$ and $\dot{f}_0(t)$ respectively, the resulting form of the differentiator is

$$\begin{aligned} \dot{z}_0 &= z_1 - \lambda_0 \, |s|^{\frac{1}{2}} \, sign(s) \\ \dot{z}_1 &= -\lambda_1 \, sign(s) \end{aligned} \tag{2.29}$$

where both z_0 and z_1 are the outputs of the differentiator.

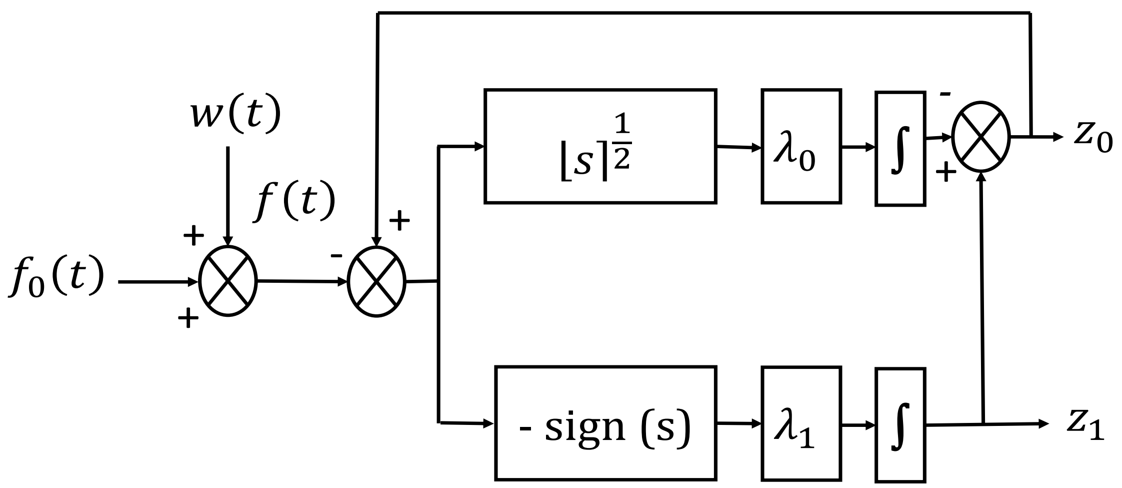


Figure 2.5: Levant first order differentiator scheme based on the super-twisting algorithm

Theorem 2.5 *[3]*
In the absence of noise, for any $\lambda_1 > L$, and for every sufficiently larger λ_0, both u and z_1 converge in finite time to $\dot{f}_0(t)$, while z_0 converges to $f_0(t)$.

This finite time convergence is guaranteed by the following sufficient conditions:

$$\lambda_1 > L \quad and \quad \lambda_0^2 > 4\,L\,\frac{\lambda_1 + L}{\lambda_1 - L}. \tag{2.30}$$

The above algorithm is proposed by Levant in [24]. The chattering phenomenon comes from the term $\lambda_0|s|^{\frac{1}{2}} sign(s)$ which, in theory, should vanish in sliding mode. However, in practice this is different than zero and the sign function makes it oscillate at high frequancy. Chattering phenomenon deteriorates the differentiator accuracy by amplifying the noise of the input signal. A good compromise should be respected when choosing λ_0 values, as high values of this gain speed-up transient response but at the same time degrades the exactness of the estimation.

Proof of Theorem 2.5

Finite time convergence of the proposed differentiator can be demonstrated by invistigating the Lyapunov approach to obtain sufficient explicit conditions on the two parameters λ_0 and λ_1 for finite time convergence.

Considering the noise free case, Assume u is bounded $|u| < \delta$ and let the estimation errors $e_1 = z_0 - f(t)$ and $e_2 = z_1 - \dot{f}(t)$ from which we obtain the following error system

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) - \lambda_0|e_1|^{\frac{1}{2}} sign(e_1) \\ \dot{e}_2(t) &= -u - \lambda_1 sign(e_1) \end{aligned} \tag{2.31}$$

As a candidate Lyapunov function we choose

$$V(e_1,\, e_2) = 2\,\lambda_1\,|e_1| + \frac{1}{2}\,e_2^2 + \frac{1}{2}(\lambda_0\,|e_1|^{\frac{1}{2}}\,sign(e_1) - e_1)^2 \tag{2.32}$$

which is under the quadratic form

$$V(e_1,\, e_2) = \Phi^T\,P\,\Phi \tag{2.33}$$

where $\Phi^T = \begin{bmatrix} |e_1|^{\frac{1}{2}}\,sign(e_1) & e_2 \end{bmatrix}$

and the symetric positive definite matrix $P = \dfrac{1}{2}\begin{bmatrix} 4\lambda_1 + \lambda_0^2 & -\lambda_0 \\ -\lambda_0 & 2 \end{bmatrix}$

then

$$\dot{V} = -\frac{1}{|e_1|^{\frac{1}{2}}}\,\Phi^T\,Q\,\Phi + u\,q^T\,\Phi \tag{2.34}$$

with $q^T = \begin{bmatrix} -\lambda_0 & 2 \end{bmatrix}$ and $Q = \dfrac{\lambda_0}{2}\begin{bmatrix} 2\lambda_1 + \lambda_0^2 & -\lambda_0 \\ -\lambda_0 & 1 \end{bmatrix}$

from which yields

$$\dot{V} < -\frac{1}{|e_1|^{\frac{1}{2}}}\,\Phi^T\,\tilde{Q}\,\Phi \tag{2.35}$$

with $\tilde{Q} = \dfrac{\lambda_0}{2}\begin{bmatrix} 2\lambda_1 + \lambda_0^2 & -2\delta \\ -(\lambda_0 + \dfrac{2\delta}{\lambda_0}) & 1 \end{bmatrix}$

then $\dot{V} < 0$ which means that the estimation errors vanish in finite time.

Theorem 2.6 *([22])*
Assuming u is bounded $|u| < \delta$. If constants λ_0 and λ_1 satisfy the conditions

$$\lambda_1 > 3\,\delta + \frac{2\delta^2}{\lambda_0}, \quad \lambda_0 > 0 \tag{2.36}$$

then the estimation errors e_1, e_2 vanish in finite time $T = \dfrac{\sqrt{V(e_1(0), e_2(0))}}{\lambda}$ with λ a function of λ_0, λ_1 and δ.

Theorem 2.7 *([3])*
Let the input noise $w(t)$ in (2.25) satisfy the inequality $|f(t) - f_0(t)| \leq \varepsilon$. Then the following inequalities are established in finite time for some positive constants μ_1, μ_2, μ_3, depending exclusively on the parameters of the differentiator and L:

$$|z_0 - f_0(t)| \leq \mu_1 \varepsilon, \quad |z_1 - \dot{f}_0(t)| \leq \mu_2 \varepsilon^{\frac{1}{2}}, \quad |u - \dot{f}_0(t)| \leq \mu_3 \varepsilon^{\frac{1}{2}} \tag{2.37}$$

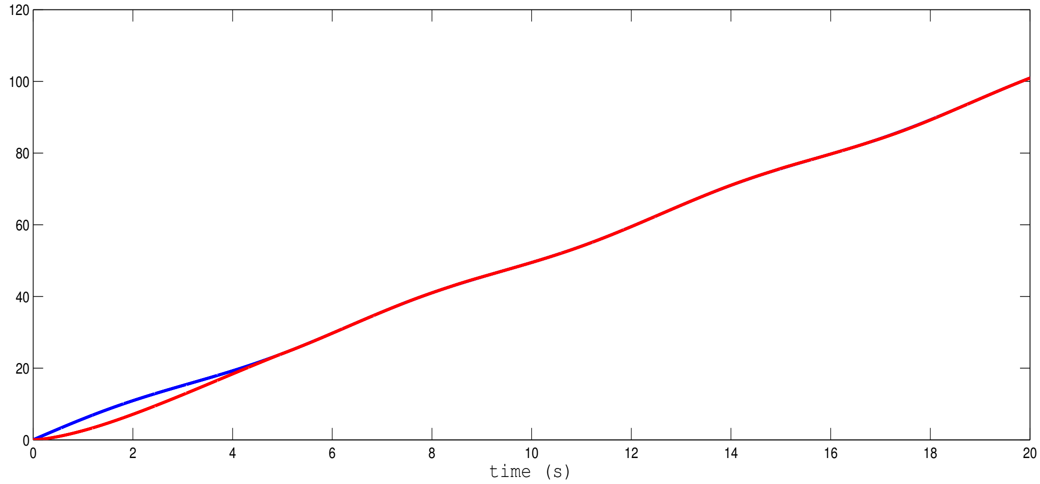
Moreover, these asymptotics cannot be improved.

Example 2.1

For simulations we consider the base signal $f_0(t) = 5t + \sin(t)$. We propose a first order sliding mode differentiator based on the super-twisting algorithm as in (2.29) to estimate the first derivative $\dot{f}_0(t)$.

In noise free case, the differentiator is simulated for different values of λ_0, λ_1 and results are shown in the following figures.

(a) $\lambda_0 = 1.1 \quad \lambda_1 = 1.5$



(b) $\lambda_0 = 12 \quad \lambda_1 = 8$

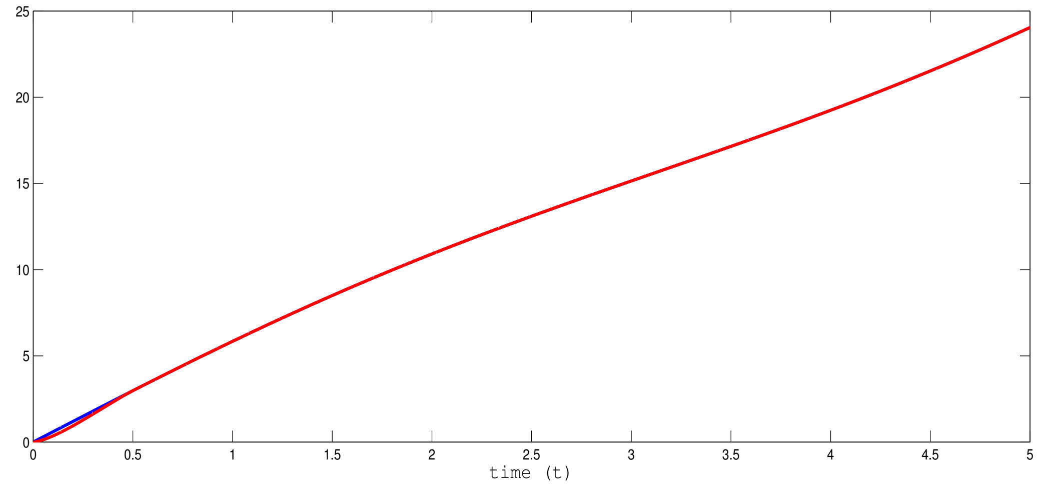
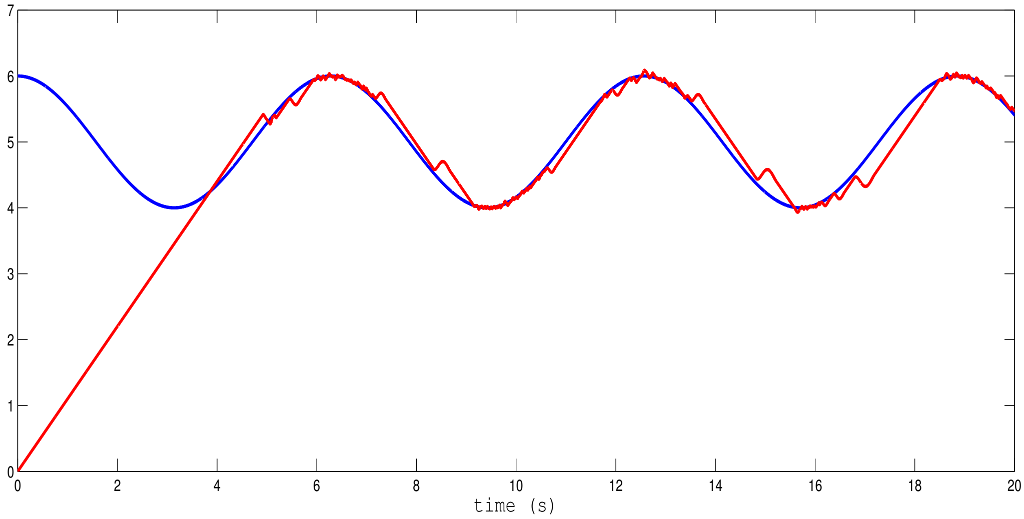
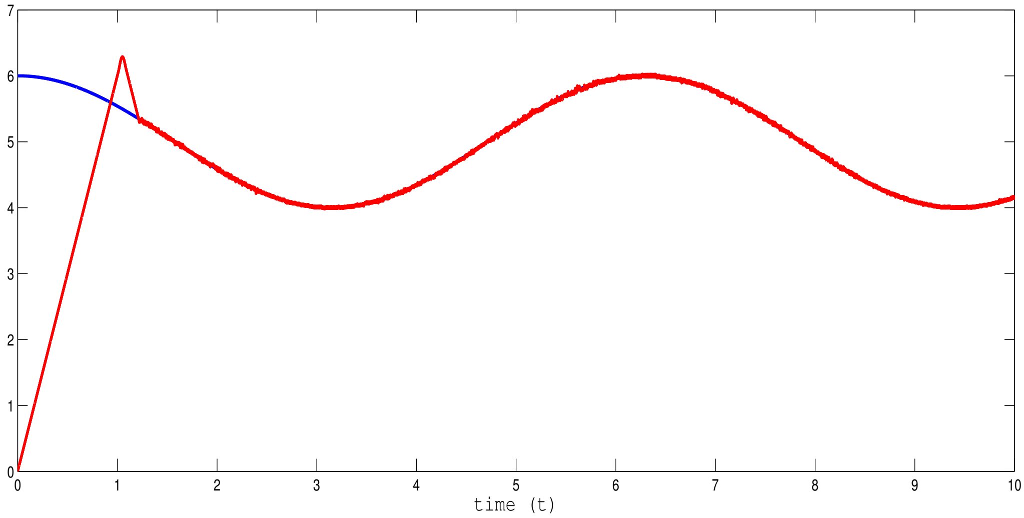


Figure 2.6: Simulation results of example 2.1: $f_0(t)$ z_0

(a) $\lambda_0 = 1.1 \quad \lambda_1 = 1.5$



(b) $\lambda_0 = 6 \quad \lambda_1 = 4$



(c) $\lambda_0 = 12 \quad \lambda_1 = 8$

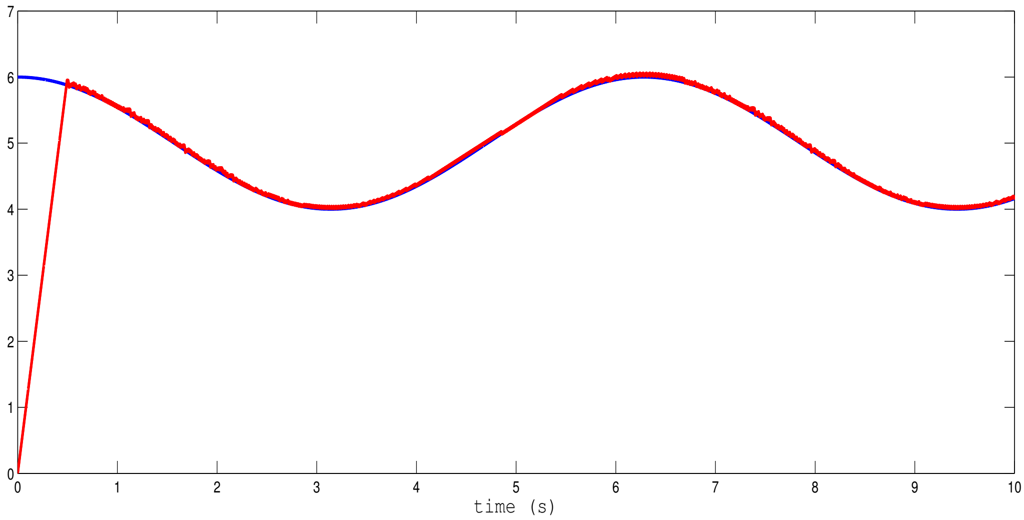
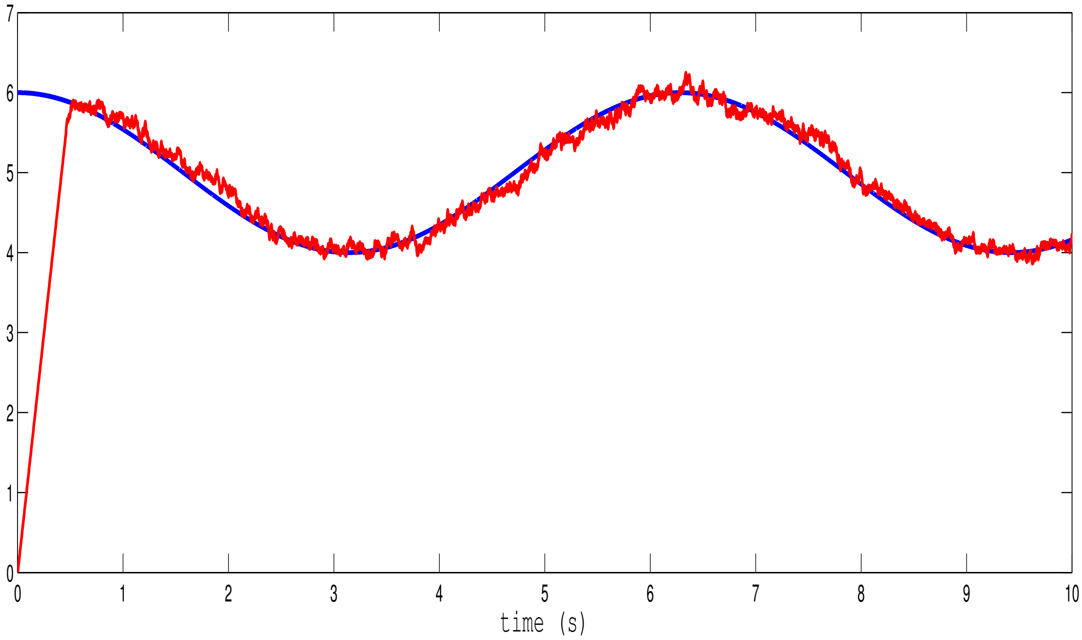


Figure 2.7: Simulation results of example 2.1: — $\dot{f}_0(t)$ — z_1

In noise free case, state variables z_0 and z_1 exactly estimate in finite time the derivatives f_0 and $\dot{f}_0$ respectively. The choice of λ_0 and λ_1 values is very important and needs to be high as high values of this gains speed-up the convergence of the estimates toward the real derivatives of the signal and make the finite time of convergence smaller.

In presence of noise, we consider the signal f_0 corrupted by a white gaussian noise. The noise effect on estimation is shown in the following figure.

(a) SNR= 40 dB



(b) SNR=10 dB

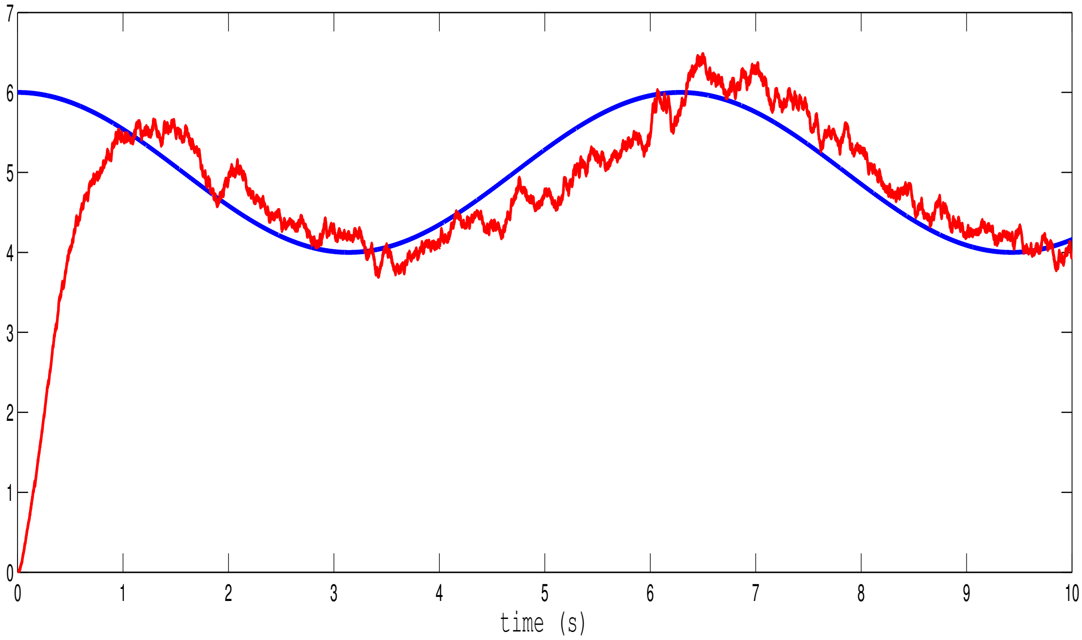


Figure 2.8: Simulation results of example 2.1: — $\dot{f}_0(t)$ — z_1 in presence of noise.

The above figure shows the noise effect on the estimation of the the first derivative $\dot{f}_0(t)$. We can notice that the differentiator performance is degraded and furthermore in accordance with noise magnitude a phase shift appears.

2.3.2 Arbitrary-Order Exact Robust Differentiator

The r-sliding mode realization provides for up to the rth order of sliding precision compared with the first order of the standard sliding mode. Such controllers require higher-order real-time derivatives of the outputs to be available. The lacking information is achieved by means of proposed arbitrary-order robust exact differentiators with finite-time convergence. These differentiators feature optimal asymptotics with respect to input noises and can be used for numerical differentiation as well.

Proposed by Levant in [27], higher order sliding mode differentiators are designed withing the framework of homogeneity and differential inclusion properties. An n-th order differentiator allows robust and finite time estimation of the n-th first derivatives of an input signal.

By considering again the differentiation problem (1.18), the corresponding homogeneous sliding mode differentiator is given in its following recursive form

$$\begin{aligned} \dot{z}_0 &= v_0 & \dot{z}_0 &= z_1 - \lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} \\ \dot{z}_1 &= v_1 & \dot{z}_1 &= z_2 - \lambda_{n-1} L^{\frac{1}{n}} \lfloor z_1 - v_0 \rceil^{\frac{n-1}{n}} \\ &\vdots & &\vdots \\ \dot{z}_{n-1} &= v_{n-1} & \dot{z}_{n-1} &= z_n - \lambda_1 L^{\frac{1}{2}} \lfloor z_{n-1} - v_{n-2} \rceil^{\frac{1}{2}} \\ \dot{z}_n &= v_n & \dot{z}_n &= -\lambda_0 L sign(z_n - v_{n-1}) \end{aligned} \tag{2.38}$$

where $\lfloor \sigma \rceil^p = |\sigma|^p sign(\sigma)$.

A non recursive form can be obtained [27], [3] as

$$\begin{aligned} \dot{z}_0 &= z_1 - \tilde{\lambda}_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} \\ \dot{z}_1 &= z_2 - \tilde{\lambda}_{n-1} L^{\frac{2}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n-1}{n+1}} \\ &\vdots \\ \dot{z}_{n-1} &= z_n - \lambda_1 L^{\frac{n}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{1}{n+1}} \\ \dot{z}_n &= -\tilde{\lambda}_0 L sign(z_0 - f(t)) \end{aligned} \tag{2.39}$$

with $\tilde{\lambda}_0 = \lambda_0,\ \tilde{\lambda}_n = \lambda_n$ and $\tilde{\lambda}_i = \lambda_i \, \tilde{\lambda}_{i+1}^{\frac{i}{i+1}}, \quad i = 1, \ldots, n-1.$

For any $\lambda_0 > 0$, an infinite sequence of positive parameters λ_i, $i = 1, \ldots, n$ can be properly built to ensure convergence for all natural n of solutions for (2.38) and (2.39) which are understood in the Filippov sens [3].

Assuming the input signal to be bounded $|f^{n+1}(t)| < L$. The particular choise of the sequence λ_i introduced in [27] as $\lambda_i = \lambda_{0i} \, L^{\frac{1}{n-i+1}}$ are valid for any $L > 0$. and provide convergence for differentiators (2.38), (2.39).

Based on the work of Levant [27], the choice of sequence λ_i for a 5th order differentiator for instance is as follows: $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$ and $\lambda_5 = 12$.

Another sequence can be found in [56] and goes until 7th order. It has been experimentally found and is listed in the following table:

order (n)	$\lambda_0, \ldots, \lambda_n$							
0	1.1							
1	1.1	1.5						
2	1.1	2.12	2					
3	1.1	3.06	4.16	3				
4	1.1	4.57	9.30	10.03	5			
5	1.1	6.75	20.26	32.24	23.72	7		
6	1.1	9.91	43.65	101.96	110.08	47.69	10	
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12

Table 2.1: Parameters $\lambda_0, \ldots, \lambda_n$ for $n = 1, \ldots, n$.

In absence of measurement noise, exact finite time estimation of $f_0(t)$ and its derivatives are provided by homogeneous sliding mode differentiators (2.38), (2.39) which means there exists a finite settling time $t_s > 0$ such that $z_i(t) = f_0^{(i)}(t)$, $\forall t \geq t_s$.

In presence of bounded noise, differentiators (2.38) and (2.39) provide for an asymptotic optimal error with accuracy of $|z_i - f_0^i(t)| \leq \lambda_i\,(2L)^{\frac{i}{(n+1)}}\,\epsilon^{\frac{(n-i+1)}{(n+1)}}\; for\, t \geq t_s$. The upper bound of the settling time depends on the initial differentiation errors.

Example 2.2

To show the performance of the higher order sliding mode differentiator discussed in this chapter, simulations are carried out by considering the base signal $f_0(t) = 5t + \sin(t)$.

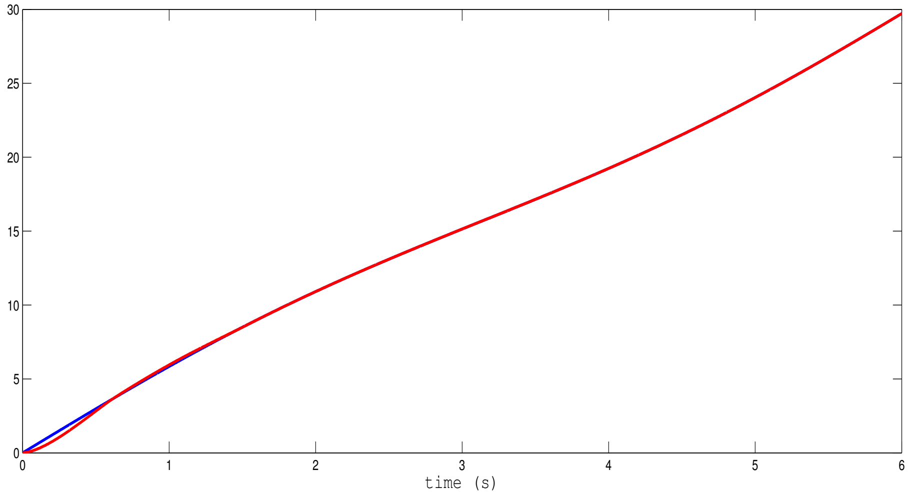
The main objective is to design a differentiator that is capable of estimating the first two derivatives $\dot{f}_0(t)$, $\ddot{f}_(0)_(t)$ respectively, of the considered base signal $f_0(t)$.

Simulations are carried out under MATLAB/Simulink by considering a second order sliding mode exact differentiator under its non-recursive form as in (2.39).

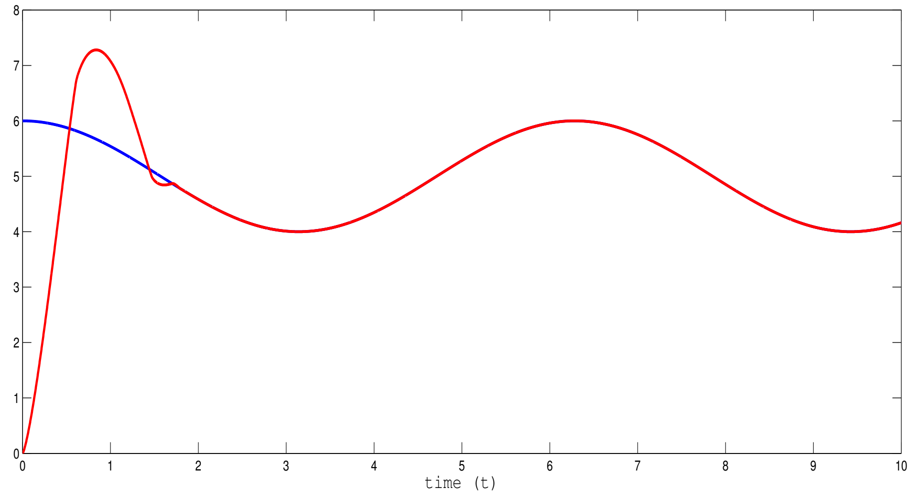
First of all, we consider the noise free case, we choose parameters of the differentiator to be $\lambda_0 = 1.1$, $\lambda_1 = 1.5\sqrt{2}$, $\lambda_2 = 3$, and constant $L = 12$, the initial conditions of the differentiator are taken null. Simulation results are shown in figure 2.9.

The noisy case is also of interest, for that we consider the same base signal as before corrupted by a white gaussian noise (we take SNR=12dB), the same differentiator as in noise free case is designed with the same parameters and null initial conditions. Simulation results in presence of noise are shown in figure 2.10.

(a)



(b)



(c)

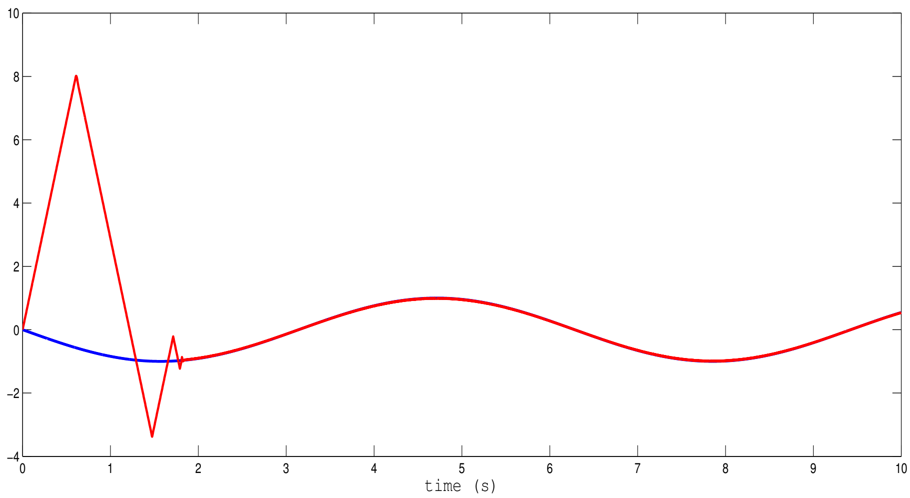
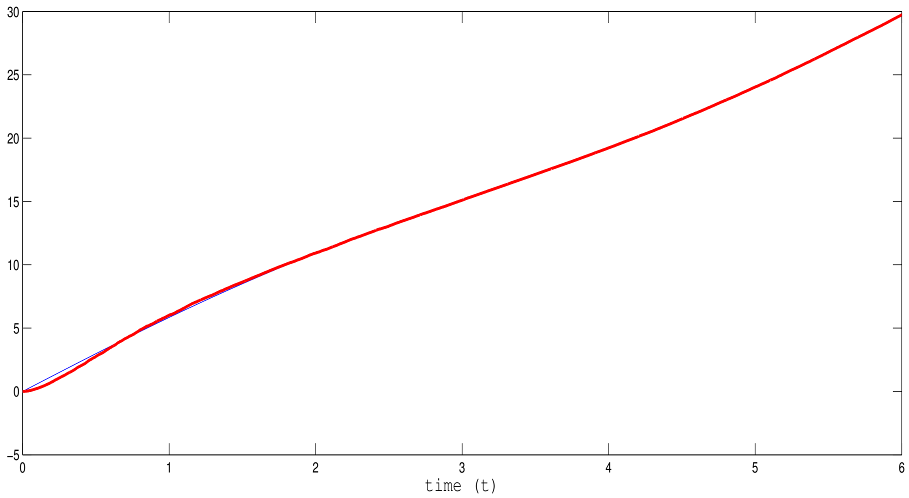
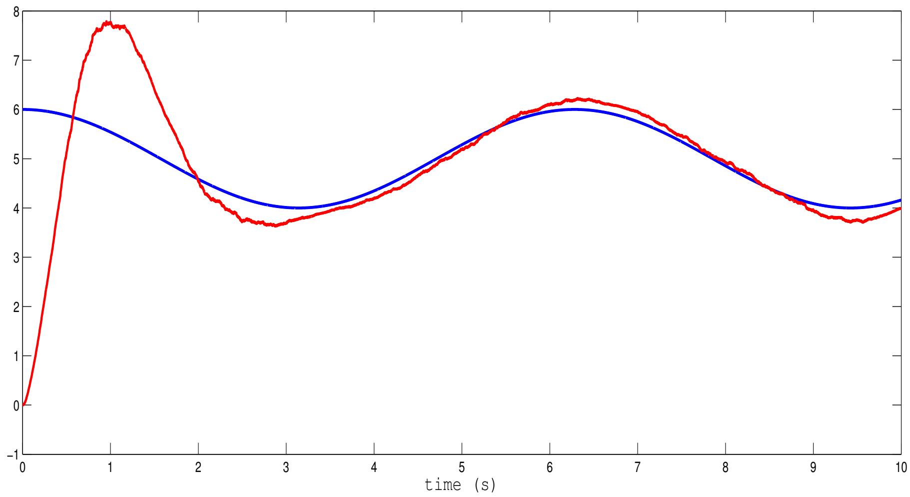


Figure 2.9: Simulation results of example 2.2: (a) — $f_0(t)$ — z_0 (b)— $\dot{f}_0(t)$ — z_1 (c)— $\ddot{f}_0(t)$ — z_2

(a)



(b)



(c)

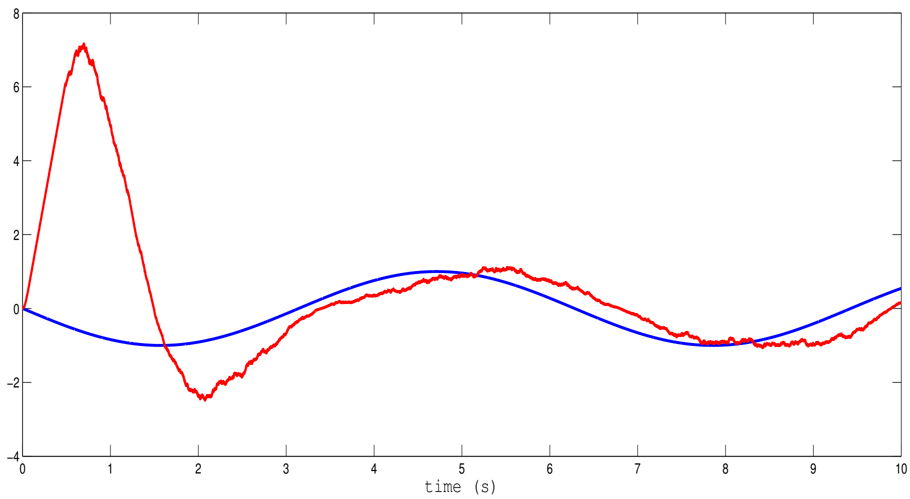


Figure 2.10: Simulation results of example 2.2: (a) — $f_0(t)$ — z_0 (b)— $\dot{f}_0(t)$ — z_1 (c)— $\ddot{f}_0(t)$ — z_2 in presence of noise

Figure 2.9 shows performance of the non-recursive form of the second order sliding mode differentiator in noise free case. We notice that the estimation is exacte and convergence of the estimates toward real signal derivatives is established in finite time which depends on the initial estimation error.

Figure 2.10 shows performance of the non-recursive form of the second order sliding mode differentiator in presence of noise. The noise influences the estimation and we can notice appearance of peaks in transient response but also phase shift especially as the order of derivative is more and more high.

2.4 Conclusion

Sliding mode differentiators represent a robust and efficient approach for estimating the derivatives of signals in the presence of noise and uncertainties. These differentiators leverage the principles of sliding mode control to ensure accurate and reliable differentiation even in non ideal conditions. Their inherent robustness to parameter variations and external disturbances makes them particularly valuable in applications where traditional differentiation methods may fail.

The primary advantages of sliding mode differentiators include their simplicity in implementation and finite time convergence properties. These features are especially beneficial in practical applications such as control systems, signal processing, and robotics, where precise and real-time differentiation is crucial.

Despite their advantages, the design and tunning of sliding mode differentiators require careful consideration of system dynamics and noise characteristics. The trade-offs between convergence speed and chattering must be managed to optimize performance. Furthermore, one may wish for a desired convergence settling time which is not achievable with the above discussed differentiators.

Chapter 3

Fixed Time Convergent Differentiators

The main drawback encountered with the previousely presented sliding mode differentiators design is the impact of initial estimation errors on the convergence time. Their settling time depend on initial estimation errors. It is important to reajust the gains of the differentiator to important values to speed up convergence which can lead to performance deterioration and increased noise sensitivity and high transient peaks.

A lot of research is going on the design of differentiators with exact convergence in finite time regardless of initial errors. In [4], the Lypunov analysis is used to design a modified Levant differentiator for which convergence time is arbitrarily assigned independently of initial estimation error within the framework of fixed-time stability. In many applications it is necessary to ensure an exact desired value of convergence time. Predefined-time stability and prescribed-time stability are introduced in this scope [32], [33], [34]. Homogeneity is employed to achieve prescribed performance as in [63], and a Lyapunov approach in [29], [30]. Methods based on time varying gains are also investigated in [33], [34], [64], these gains however diverge to infinity at the preselected settling time which renders these methods unrealizable in practice.

The upcoming presented sliding mode differentiators with prescribed-time convergence are presented in [20], where convergence time is chosen arbitrarily whatever large initial estimation errors are. It consists of a transformation using modulating functions which makes it possible to cancel the effect of initial conditions on convergence time. Lyapunov functions and homogeneity property tools are used to prove convergence of the proposed first order and arbitrary order differentiators.

3.1 Fixed-Time stability

Finite-time, fixed-time, predefined-time and prescribed time stability are very important concepts regarding stability achieved in some special time. These concepts are formulated under the following definitions.

We consider the autonomous nonlinear system defined by

$$\dot{X}(t) = f(X(t)) \tag{3.1}$$

where $X(t) \in R^n$ denotes the system state with the initial condition $X_0 = X(t_0)$, $(t_0 = 0)$. We assume the origin to be an equilibrium of 3.1 $(f(0) = 0)$. The set of all solutions $X(t, X_0, t_0)$ of (3.1) from the initial conditions (X_0, t_0) is denoted by $\mathcal{S}(X_0, t_0)$.

Definition 3.1 ***Finite-time stability*** *[28]*
The origin of (3.1) is globally finite-time stable if it is globally asymptotically stable and if, for every initial condition $X_0 \in R^n$, the solution $X(t, X_0, t_0)$ of (3.1) reaches the origin at some finite-time moment, i.e., $X(t, X_0, t_0) = 0, \quad \forall t \geq T_0(X(t, X_0, t_0))$
where

$$T_0(X(t, X_0, t_0)) = \inf\{\tau, \tau \geq t_0 : \lim_{t \to \tau} X(t, X_0, t_0) = 0\} - t_0$$

The settling time of system 3.1 is defined as

$$T(X_0) = \sup_{X(t,X_0,t_0) \in \mathcal{S}(X_0,t_0)} T_0(X(t, X_0, t_0))$$

Definition 3.2 ***Fixed-time stability*** *[29]*
The origin of (3.1) is globally fixed-time stable if it is globally finite-time stable and if the settling time function $T(X_0)$ is bounded, i.e. there exists $T_{max} > t_0$ such that, for every initial condition $X_0 \in R^n$, $T(X_0) \leq T_{max}$. T_{max} is designed as the upper bound of the settling time (UBST).

Definition 3.3 ***Predefined-time stability*** *[32]*
Denote by ρ the system parameters of (3.1) and let $T_c = T_c(\rho)$ a design parameter.

- *The origin of (3.1) is globally weakly predefined-time stable if it is globally fixed-time stable and if the settling time finction satisfies*

$$T(X_0) \leq T_c, \ \forall X_0 \in R^n \tag{3.2}$$

- *The origin of (3.1) is globally strongly predefined-time stable if it is globally fixed-time stable and if the settling time function satisfies*

$$\sup_{X_0 \in R^N} T(X_0) = T_c, \quad \forall X_0 \in R^n \tag{3.3}$$

Definition 3.4 ***Prescribed-time stability*** *[33], [34]*
The origin of (3.1) is globally prescribed-time stable if it is globally fixed-time stable and if every nonzero trajectory reaches the origin at exactly a desired user defined finite constant T_p after t_0, i.e. $T(X_0) = T_p$, for all $X_0 \neq 0$.

3.2 Modulating Annihilator Functions

Introduced by Shinbrot in 1954 for parameter identification of dynamical systems [17], Modulating functions have been applied in signal processing and control and are extensively used in estimation which is an important part of control. The use of modulating functions is extended to fractional differential equations for parameters estimations [35], [36], and for state estimation as in [35], [37], [39].

Robust differentiator design based on modulating functions is developed in [20]. A fractional order differentiator is designed in [38] using generalized modulating functions.

Definition 3.5 ***Uniform prescribed-time exact differentiator*** *[20]*
Consider a signal $f(t)$ defined as in (1.17). Let $z_i(t)$ be an estimate of $f_0^i(t) \quad i = 0, 1, \ldots, n$ provided by an n-th order differentitor. Then, this differentiator is said to be uniformly prescribed-time exact with an arbitrary user-defined convergence finite time T_d, if, for any initial differentiation errors $\sigma_i(0) = z_i(0) - f_0^{(i)}(0)$, i=0,1,...,n, conditions $z_i = f_0^{(i)}(t)$, $i = 0, 1, \ldots, n$, are satisfied for all $t \geq T_d$.

Definition 3.6 ***Modulating functions*** *[20]*
Consider the positive real-valued increasing monotonic scalar function $\mu(t) \in \mathcal{C}^n : R_+ \to R_+$. Assume that $\mu(t)$ and its derivatives $\mu^{(j)}(t) = \dfrac{d^j\mu(t)}{dt^j}$, $j = 0, 1, \ldots, n-1$ satisfy the vanishing conditions

$$\begin{aligned} &\mu^{(j)}(0) = 0 \quad \forall j = 0, 1, \ldots, n-1 \\ &\mu^{(j)}(t) \neq 0 \quad for \quad t > 0, \quad \forall j = 0, 1, \ldots, n-1 \end{aligned} \tag{3.4}$$

Moreover, assume that $\mu(t)$ and its derivatives $\mu^{(j)}(t)$, $j = 0, 1, \ldots, n$ are bounded, i.e. there exists positive constants M_j, $j = 0, 1, \ldots, n$ such that $|\mu^{(j)}(t)| \leq M_j$, $j = 0, 1, \ldots, n \quad \forall t \geq 0$, then $\mu(t)$ is called an n-th order modulating function.

We generally choose the modulating function $\mu(t)$ to be a positive increasing function such that $\mu(0) = 0$ and $\underset{t\to\infty}{\mu(t)} = A$ where A denotes a finite constant. Besides, for all $t > 0$, modulating functions $\mu(t)$ have no zero-crossing. An example of such functions is $\mu(t) = A(1 - \mathrm{e}^{-\lambda t})^i$.

In order to use modulating functions as initial conditions annihilators and eliminate the influence of initial conditions on the convergence of the estimation error, we impose the n-th differentiable modulating function and its $(n-1)$ first derivatives to vanish at $t = 0$. Necessarily, modulating functions $\mu(t)$ have no zero-crossing for all $t > 0$.

3.3 Prescribed-Time Sliding Mode Differentiator Based on Modulating Functions

Proposed in [20], the prescribed-time n-th order modulating function sliding mode (MFSM) differentiator whose convergence time does not depend in any way on initial estimation errors and furthermore can be arbitrarily chosen in advance. The key point is to use modulating functions as annihilators of the initial conditions.

- In noise free case, the proposed differentiator exactly and instantaneously converge within a prescribed small finite time.

- In the presence of noise, the estimation error remains bounded.

The convergence settling time is chosen only according to the used modulating function.

We consider again the system (1.18)

$$\begin{gathered} \dot{x}_i(t) = x_{i+1}(t); \quad i = 1, n \\ \dot{x}_{n+1}(t) = u(t) \\ y = x_1(t) + w(t) = f(t) \end{gathered} \tag{3.5}$$

where the state variable $x_{i+1}(t)$ is the i-th derivative $f_0^{(i)}(t)$, $i = 0, 1, \ldots, n$ to be estimated and $x_1(t) = f_0(t)$. Given an n-order modulatiing function $\mu(t)$ and let the coordinate change be defined as follows

$$\xi_j(t) = \sum_{i=0}^{j-1} \alpha_{j,i}\, \mu^{(i)}(t)\, x_{j-i}(t) + \tilde{z}_{j-1}^0; \quad j = 1, 2, \ldots, n+1 \tag{3.6}$$

where $\tilde{z}_j^0 = \xi_{j+1}(0)$, $j = 0, 1, \ldots, n$, are arbitrary fixed known initial conditions.

This coordinates change can be expressed as

$$\xi(t) = \Pi_{n+1}(\mu(t))\, x(t) - z^0 \tag{3.7}$$

with $z^0 = \begin{bmatrix} z_0^0 & \ldots & z_n^0 \end{bmatrix}^T$ and the $(n+1) \times (n+1)$ time-dependent transformation matrix $\Pi_{n+1}(\mu(t))$ given by

$$\Pi_{n+1}(\mu) = \begin{bmatrix} \alpha_{1,0}\,\mu & 0 & 0 & \ldots & 0 \\ \alpha_{2,1}\,\dot{\mu} & \alpha_{2,0}\,\mu & 0 & \ldots & 0 \\ \alpha_{3,2}\,\ddot{\mu} & \alpha_{3,1}\,\dot{\mu} & \alpha_{3,0}\,\mu & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & 0 \\ \alpha_{n+1,n}\,\ddot{\mu}^{(n)} & \alpha_{n+1,n-1}\,\ddot{\mu}^{(n-1)} & \alpha_{n+1,n-2}\,\ddot{\mu}^{(n-2)} & \ldots & \alpha_{n+1,0}\,\mu \end{bmatrix} \tag{3.8}$$

3.3.1 MFSM Arbitrary Order Diffrentiator Design in Noise Free Case

In noise free signal case, the output is $y(t) = x_1 = f_0(t)$. Let the coordinate change be given by (3.6) with $\alpha_{j,0} = 1$, $j = 1, 2, \ldots, n+1$ and assuming the following equalities

$$\begin{gathered} \alpha_{j,i} + \alpha_{j,i-1} - \alpha_{j+1,i} = 0; \quad j = 2, 3, \ldots, n; \quad i = 1, 2, \ldots, j-1 \\ \alpha_{n+1,\;\; i} + \alpha_{n+1,i-1} = 0; \quad i = 1, 2, \ldots, n \end{gathered} \tag{3.9}$$

hold. Then in the new coordinates, system (3.5) takes the following form

$$\begin{gathered} \dot{\xi}_j(t) = \xi_{j+1}(t) + (\alpha_{j,j-1} - \alpha_{j+1,j})\mu^j(t) f_0(t) - \tilde{z}_j^0; \quad j = 1, 2, \ldots, n \\ \dot{\xi}_{n+1}(t) = \mu(t) u(t) + \alpha_{n+1,n}\mu^{n+1}(t) f_0(t) \end{gathered} \tag{3.10}$$

The proposed n-th order MFSM diffrentiator in the case of noise free signal is presented in the following theorem

Theorem 3.1 *Let the $(n+1)$th drivative of a base signal $f_0(t)$ be bounded. Consider the $(n+1)-$order modulatig function $\mu(t)$ given by definition 3.6. Let the MFSM n-th order differentiator be given by the following non recursive form*

$$\begin{aligned}
\dot{\tilde{z}}_0(t) &= \tilde{z}_1(t) - \tilde{\lambda}_n L^{\frac{1}{n+1}} \lfloor \tilde{z}_0(t) - \mu(t) f_0(t) - \tilde{z}_0^0 \rceil^{\frac{n}{n+1}} + (\alpha_{1,0} - \alpha_{2,1})\mu^{(1)}(t) f_0(t) - \tilde{z}_1^0 \\
\dot{\tilde{z}}_1(t) &= \tilde{z}_2(t) - \tilde{\lambda}_{n-1} L^{\frac{2}{n+1}} \lfloor \tilde{z}_0(t) - \mu(t) f_0(t) - \tilde{z}_0^0 \rceil^{\frac{n-1}{n+1}} + (\alpha_{2,1} - \alpha_{3,2})\mu^{(2)}(t) f_0(t) - \tilde{z}_2^0 \\
&\vdots \\
\dot{\tilde{z}}_{n-1}(t) &= \tilde{z}_n(t) - \tilde{\lambda}_1 L^{\frac{n}{n+1}} \lfloor \tilde{z}_0(t) - \mu(t) f_0(t) - \tilde{z}_0^0 \rceil^{\frac{1}{n+1}} + (\alpha_{n,n-1} - \alpha_{n+1,n})\mu^{(n)}(t) f_0(t) - \tilde{z}_n^0 \\
\dot{\tilde{z}}_n(t) &= -\tilde{\lambda}_0 L sign(\tilde{z}_0(t) - \mu(t) f_0(t) - \tilde{z}_0^0) + \alpha_{n+1,n} \mu^{(n+1)} f_0(t)
\end{aligned} \tag{3.11}$$

$$\begin{cases} z_j(t) = 0, \quad j = 0, 1, \ldots, n, \quad 0 < t < T_d \\ \begin{bmatrix} z_0(t) \\ z_1(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \Pi_{n+1}^{-1}(\mu(t)) \left(\begin{bmatrix} \tilde{z}_0(t) \\ \tilde{z}_1(t) \\ \vdots \\ \tilde{z}_n(t) \end{bmatrix} - \begin{bmatrix} \tilde{z}_0^0 \\ \tilde{z}_1^0 \\ \vdots \\ \tilde{z}_n^0 \end{bmatrix} \right), \quad t \geq T_d \end{cases} \tag{3.12}$$

where T_d is the activation time chosen so that the matrix $\Pi_{n+1}(\mu(t))$ is far from sigularity near the initial time $t = 0$.
There exist positive constants $\tilde{\lambda}_k$, $k = 0, 1, \ldots, n$ being properly chosen such that that the MFSM based differentiator (3.11), (3.15) is uniformly exactly prescribed-time convergent with respect to any initial conditions $\tilde{z}_j^0$, $j = 1, 2, \ldots, n$, that is $z_j(t) = f^{(j)}(t)$, $j = 0, 1, \ldots, n$ for all $t \geq T_d$.

Proof of Theorem 3.1

Let $\tilde{\sigma}_j(t) = \tilde{z}_j(t) - \xi_{j+1}(t)$, $j = 0, 1, \ldots, n$ be the estimation errors in the new coordinates. Then, subtracting (3.10) from (3.11) yields to

$$\begin{aligned}
\dot{\tilde{\sigma}}_0(t) &= \tilde{\sigma}_1(t) - \tilde{\lambda}_n L^{\frac{1}{n+1}} \lfloor \tilde{\sigma}_0(t) \rceil^{\frac{n}{n+1}} \\
\dot{\tilde{\sigma}}_1(t) &= \tilde{\sigma}_2(t) - \tilde{\lambda}_{n-1} L^{\frac{2}{n+1}} \lfloor \tilde{\sigma}_0(t) \rceil^{\frac{n-1}{n+1}} \\
&\vdots \\
\dot{\tilde{\sigma}}_{n-1}(t) &= \tilde{\sigma}_n(t) - \tilde{\lambda}_1 L^{\frac{n}{n+1}} \lfloor \tilde{\sigma}_0(t) \rceil^{\frac{1}{n+1}} \\
\dot{\tilde{\sigma}}_n(t) &= -\tilde{\lambda}_0 L\, sign(\tilde{\sigma}_0(t)) - \mu(t)\, u(t)
\end{aligned} \tag{3.13}$$

$u(t)$ is bounded and so is $\mu(t)$, then the last equation of (3.13) can be written in the Filipov sense as

$$\dot{\tilde{\sigma}}_n(t) \in -\tilde{\lambda}_0 L\, sign(\tilde{\sigma}_0(t)) + [-M_0 L \ \ + M_0 L] \tag{3.14}$$

The equations (3.13) and (3.14) are those of the standard Levant's higher order sliding mode differentiator in noise free case signal, this differentiator is homogeneous with $deg\, t = -1$, $deg\, \tilde{\sigma}_j = n + 1 - j$. It exactly converges in finite time for some constants $\tilde{\lambda}_j$, $j = 0, 1, \ldots, n$. Since $\tilde{\sigma}_j(0) = 0$, $j = 0, 1, \ldots, n$ then the differentiator converges instantneously at initial time $t = 0$ which means that the error $\tilde{\sigma}(t) = 0$, $j = 0, 1, \ldots, n$ for all $t \geq 0$. The computation of the estimates $z_j(t)$, $j = 0, 1, \ldots, n$ of the original signal $f_0(t)$ and its derivatives $f_0^j(t)$, $j = 1, \ldots, n$ respectively, requires the inversion of the matrix $\Pi_{n+1}(\mu(t))$. The determinant of this matrix is equal to $\mu^{n+1}(t)$. However in order to avoid overshoot due to nearly singularity of $\Pi^{n+1}(\mu(t))$ at the first instants near zero, a threshold ε is settled and the inversion is only executed whenever $det(\Pi_{n+1}(\mu(t))) > \varepsilon$. Td is the chosen activation time such that $det(\Pi_{n+1}(\mu(t))) > \varepsilon$, $t \geq T_d$.

Before T_d the differentiator is frozen, i.e. $z_0(t) = 0$. For $t \geq T_d$, the relation (3.15) and the following relation

$$\left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n+1}(t) \end{bmatrix} = \Pi_{n+1}^{-1}(\mu(t)) \left(\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_{n+1}(t) \end{bmatrix} - \begin{bmatrix} \tilde{z}_0^0 \\ \tilde{z}_1^0 \\ \vdots \\ \tilde{z}_n^0 \end{bmatrix} \right), \quad t \geq T_d \right. \tag{3.15}$$

hold, which means $z_j(t) = x_{j+1}$, $j = 0, 1, \ldots, n$ for all $t \geq T_d$. We conclude then that the MFSM differentiator converges exactly in the prescribed time T_d.

Remark 3.1 *Theoritically, the estimation errors instantaneously converge to zero at the time instant T_d without any overshoot. However, for small T_d some peaks and oscillations were observed in the simulations around T_d. These distortions, due only to the numerical singularities in the inversion of $\Pi_{n+1}(\mu(t))$ disappear for large values of T_d. In practical implementation, the threshold ε and then T_d can be tuned in accordance to the computational software and hardware capabilities using the heuristic try and error approach.*

Remark 3.2 *Equations (3.11) and (3.15) can be applied to the particular case of the well known super-twisting algorithm based first order differentiator.*

With $f_0(t)$ being a base signal and $|f_0^{(2)}(t)| \leq L,\ for\, t \geq 0$, where L is a known positive finite Lipschitz constant. Given $\mu(t)$ a second order modulating function defined as in (3.6) and the time dependant transformation

$$\Pi_2(\mu(t)) = \begin{bmatrix} \mu(t) & 0 \\ -\dot{\mu}(t) & \mu(t) \end{bmatrix} \tag{3.16}$$

which is singular only at $t = 0$. Let T_d be a prescribed time, chosen so that $\Pi_2(\mu(t))$ is numerically far from singularity. Consider the following modulating function super twisting algorithm based (MFSTA) differentiator

$$\begin{gathered} \dot{\tilde{z}}_0 = \tilde{z}_1(t) - \lambda_1 |\tilde{z}_0(t) - \mu(t) f_0(t) - \tilde{z}_0^0|^{\frac{1}{2}} sign(\tilde{z}_0(t) - \mu(t) f_0(t) - \tilde{z}_0^0) + 2\dot{\mu}(t) f_0(t) - \tilde{z}_1^0 \\ \tilde{z}_1(t) = -\lambda_0 sign(\tilde{z}_0(t) - \mu(t) f_0(t) - \tilde{z}_0^0) - \ddot{\mu}(t) f_0(t) \\ \tilde{z}_0^0 = \tilde{z}_0(0), \quad \tilde{z}_1^0 = \tilde{z}_1(0) \end{gathered} \tag{3.17}$$

$$z_0(t) = \frac{(\tilde{z}_0(t) - \tilde{z}_0^0)}{\mu(t)}, \quad z_1(t) = \frac{\tilde{z}_1(t) - \tilde{z}_1^0}{\mu(t)} + \frac{\dot{\mu}(t)(\tilde{z}_0(t) - \tilde{z}_0^0)}{\mu^2(t)}; \ t \geq T_d$$
$$z_0(t) = 0, \quad z_1(t) = 0; \quad 0 < t < T_d \tag{3.18}$$

with $\tilde{z}_0^0$ and $\tilde{z}_1^0$ are known initial conditions.
There exist positive constants λ_1 and λ_2 such that (3.17) and (3.18) is a uniformly fixed-time exact differentiator with respect to every initial conditions $\tilde{z}_0^0$, $\tilde{z}_1^0$ with a prescribed finite convergence time T_d, i.e. $z_0(t) = f_0(t)$, $z_1(t) = \dot{f}_0(t)$ for all $t \geq T_d$.

3.3.2 MFSM Arbitrary Order Diffrentiator Design in Presence of Noise

In the following, we consider the same previousely presented differentiator described by equations (3.17) and (3.18) in which the base signal $f_0(t)$ is replaced by the noisy measurement signal $f(t)$.

Theorem 3.2 *Consider the MFSM differentiator described by (3.17) and (3.18) and in which the base signal $f_0(t)$ is corrupted by additive bounded noise as $f(t) = f_0(t) + w(t)$. We assume the $(n+1)th$ derivative of $f(t)$ is bounded. We suppose also that the $(n+1)th$ order modulating function $\mu(t)$ obeys the conditions stated in definition 3.6. Given the activation time T_d chosen so that the matrix $\Pi_{n+1}(\mu(t))$ in (3.8) is far from singularity near the initial time $t = 0$. Then, there exist positive parameters $\tilde{\lambda}_j$, $j = 0, 1, \ldots, n$, such that this differentiator provides the following accuracy in user defined fixed time T_d and uniformly with respect to any initial estimation errors*

$$|z_j(t) - f_0^{(j)}(t)| \leq \beta_j \, L \, \rho^{n+1-j}; \quad \rho = \max\left\{ \left(\frac{M_j \, \varepsilon_w}{L}\right)^{\frac{j}{n+1-j}}, \quad j = 0, 1, \ldots, n \right\} \tag{3.19}$$

for some parameters $\beta_j > 0$ which only depend on $\tilde{\lambda}_j$, $j = 0, 1, \ldots, n$.

Proof of Theorem 3.2

The estimation error in presence of noise satisfies the relations

$$\begin{aligned}
\dot{\tilde{\sigma}}_0(t) &= \tilde{\sigma}_1(t) - \tilde{\lambda}_n \, L^{\frac{1}{n+1}} \lfloor \sigma_0(t) - \mu(t)w(t) \rceil^{\frac{n}{n+1}} + \pi_0(t) \\
\dot{\tilde{\sigma}}_1(t) &= \tilde{\sigma}_2(t) - \tilde{\lambda}_{n-1} \, L^{\frac{2}{n+1}} \lfloor \sigma_0(t) - \mu(t)w(t) \rceil^{\frac{n-1}{n+1}} + \pi_1(t) \\
&\vdots \\
\dot{\tilde{\sigma}}_{n-1}(t) &= \tilde{\sigma}_n(t) - \tilde{\lambda}_1 \, L^{\frac{n}{n+1}} \lfloor \sigma_0(t) - \mu(t)w(t) \rceil^{\frac{1}{n+1}} + \pi_{n-1}(t) \\
\dot{\tilde{\sigma}}_n(t) &= -\tilde{\lambda}_0 \, L \, sign(\tilde{\sigma}_0(t) - \mu(t)w(t)) + \pi_n(t)
\end{aligned} \tag{3.20}$$

where $\pi_j(t) = (\alpha_{j,j-1} - \alpha_{j+1,j})\mu^{(j+1)}(t)w(t), \quad j = 0, 1, \ldots, n$. Accoding to homogeneity property of differential inclusions of (3.20), as in[56] Lemma 1 , it is homogeneous of the degree -1 with $deg\tilde{\sigma}_j = n + 1 - j$. It follows that there exist some parameters $\tilde{\lambda}_j$, $j = 0, 1, \ldots, n$ such that the errors $\tilde{\sigma}_j(t)$, $j = 0, 1, \ldots, n$ are stabilized in the region $|\tilde{\sigma}_j(t)| \leq \beta_j \, L \, \rho^{n+1-j}$ where $\rho = \max\left\{ \left(\frac{M_j \, \varepsilon_w}{L}\right)^{\frac{j}{n+1-j}}, j = 0, 1, \ldots, n \right\}$. Since $\tilde{\sigma}_j(0) = 0$, $j = 0, 1, \ldots, n$ and from the fact that $\sigma(t) = \Pi_{n+1}^{-1}(\mu(t))\tilde{\sigma}(t)$ for $t \geq T_d$ where $\sigma(t) = \begin{bmatrix} \sigma_0(t) & \sigma_1(t) & \ldots & \sigma_n(t) \end{bmatrix}^T$ and $\tilde{\sigma}(t) = \begin{bmatrix} \tilde{\sigma}_0(t) & \tilde{\sigma}_1(t) & \ldots & \tilde{\sigma}_n(t) \end{bmatrix}^T$, it follows that the estimation errors $\sigma_j(t)$ are immediately confined in the ball whose radius is bounded by $\beta_j L \rho^{n+1-j}$ for $t \geq T_d$. This completes the proof.

Remark 3.3 *In the presence of noise, the MFSTA differentiator in (3.17) and (3.17) can be applied to estimate the derivatives of the noisy signal $f(t)$.*

For that we consider the MFSTA based differentiator in (3.17) and (3.18) and by replacing the base signal $f_0(t)$ by the noisy signal $f(t) = f_0(t) + w(t)$. we assume the noise is bounded and so is $\ddot{f}_0(t)$ and let z_0 and z_1 be the estimates for $f_0(t)$ and $\dot{f}_0(t)$ respectively, Then there exist positive constants λ_0 and λ_1 such that the estimation errors $\sigma_0(t) = z_0 - f_0$, $\sigma_1(t) = z_1 - \dot{f}_0$ remain confined indefinitely in a ball whose the radius is a function of $\mu(t)$, $w(t)$, $andL$ for all $t \geq T_d$ where T_d is an arbitrary prescribed activation time chosen on the basis of the modulating function $\mu(t)$.

Example 3.1 ***First order differentiator***

In this example, differentiation of the base signal $f_0(t) = 5t + \sin(t)$ is tested.

We start by considering the noise free scenario, the differentiator is designed as in 3.17 and 3.18 with the following parameters:
$\lambda_0 = 6$ and $\lambda_1 = 1$, the activation time is $T_d = 0.5\,s$ and initial conditions $f_0(0) = 0$ and $\dot{f}_0(0) = 6$ and those of the differentiator are taken as zero.

The choice of the modulating function is as follow $\mu(t) = a\,(1 - e^{-\lambda t})^2$ where $a = 1$ and $\lambda = 1$. Simulation results are shown in figure 3.1

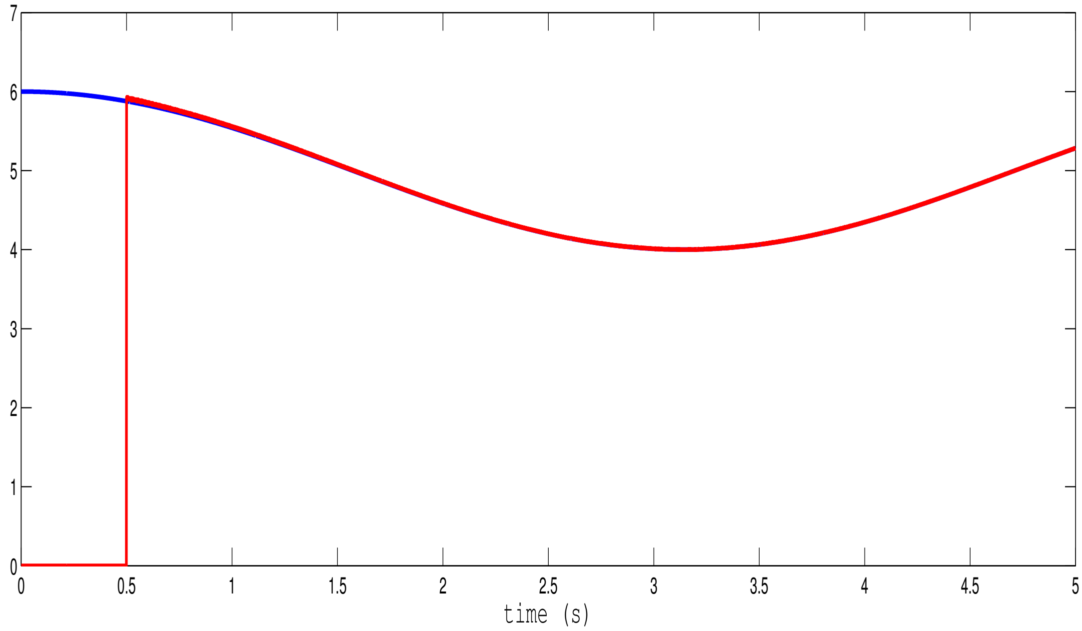


Figure 3.1: Simulation results of example 3.1: —— $\dot{f}_0(t)$ —— z_1

We can notice that the introduced first order differentiator gives better performance and we have exact estimation of the first derivative in a prescribed time $T_d = 0.5\,s$.

The estimation error is shown in the following figure 3.2

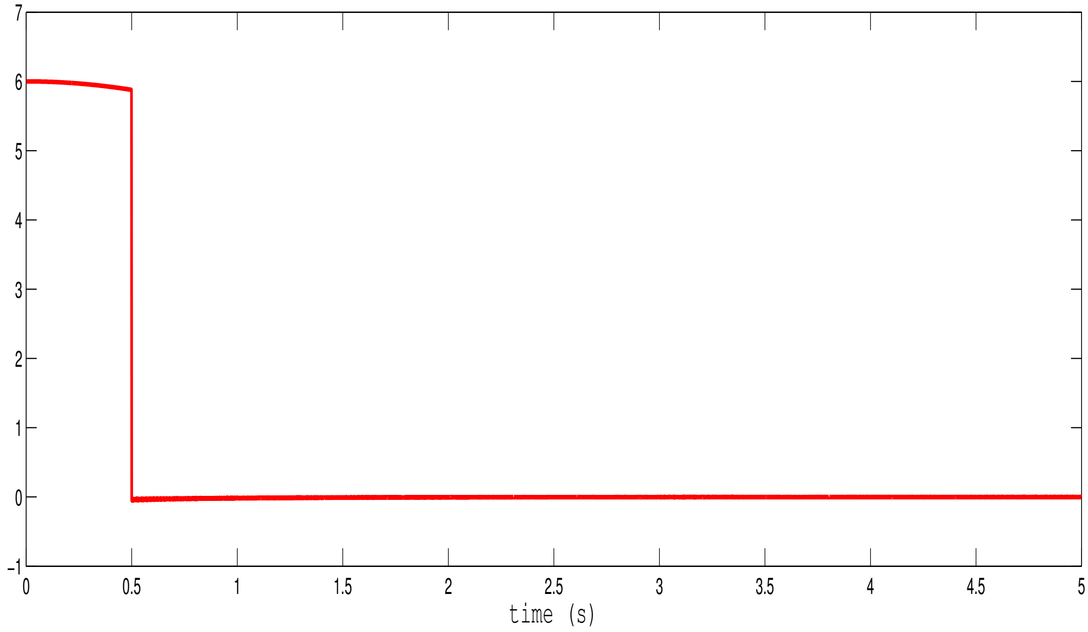


Figure 3.2: Simulation results of example 3.1: estimation error $z_1 - \dot{f}_0(t)$

In presence of measurement noise, the performance of the first order differentiator is shown in the following figure 3.3

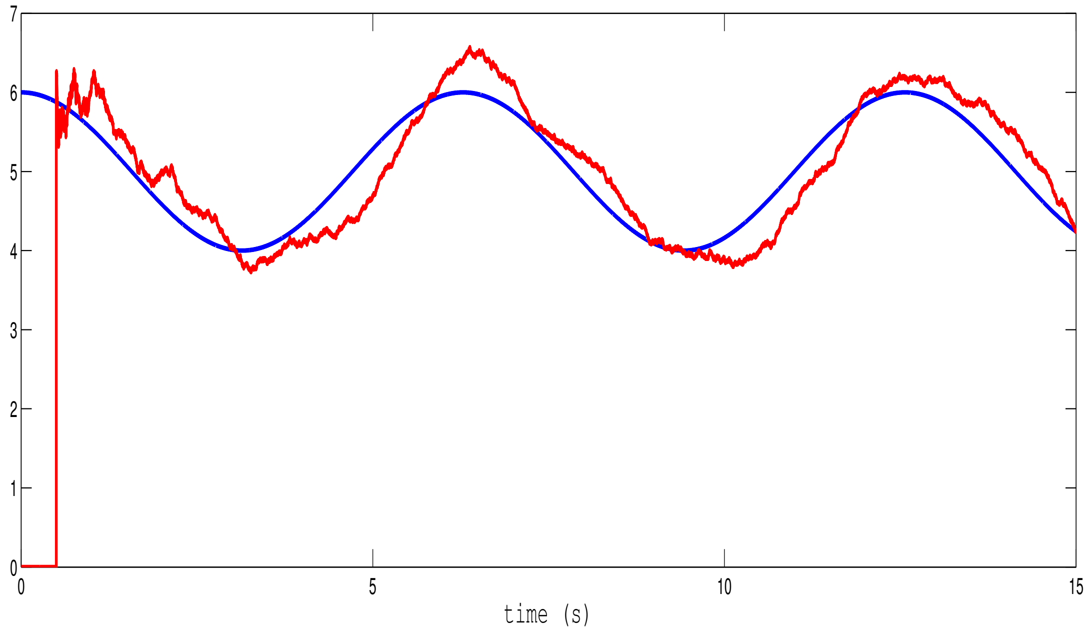


Figure 3.3: Simulation results of example 3.1: $\dot{f}_0(t)$ z_1 in presence of noise

The first order differentiator converge even in the presence of noise, and it offers a bounded estimation error 3.4 even it do not have the filtering capability. We can notice also the absence of ransient response.

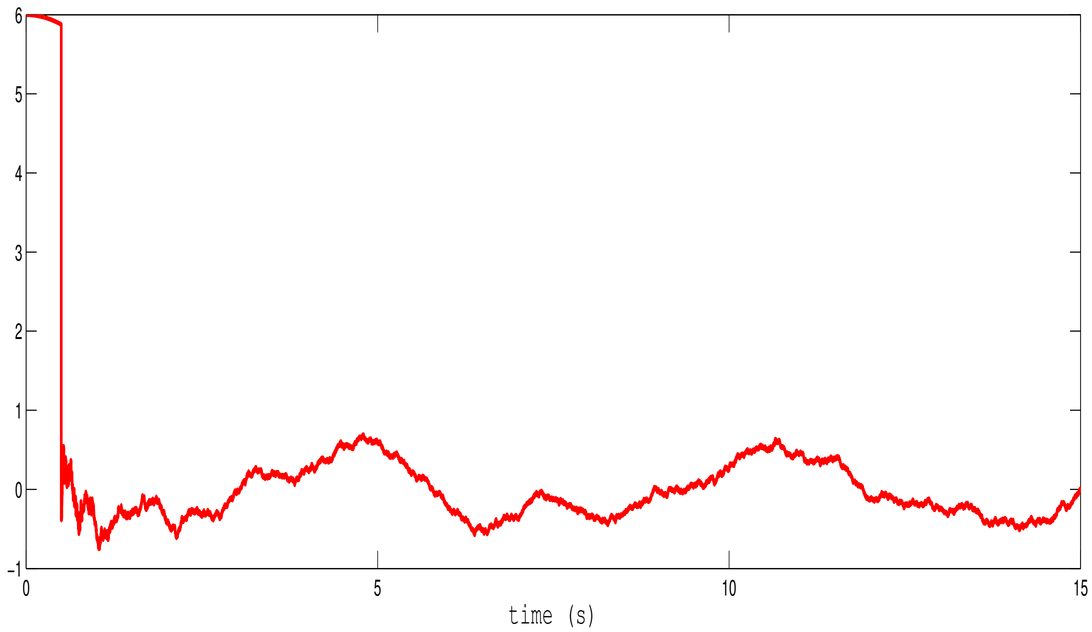


Figure 3.4: Simulation results of example 3.1: estimation error — $z_1 - \dot{f}_0(t)$

Example 3.2 ***Second order differentiator***

To illustrate the results of Theorem 3.1, we consider the second order differentiator for noise free signal. Simulations are performed on the base signal $f_0(t) = 5t + \sin(t)$. The modulating function is given by $\mu(t) = a(1 - e^{-\lambda t})^3$ with $a = 1$ and $\lambda = 1$. The activation time is taken as $T_d = 0.5\,s$.

Simulation results are shown in the following figures 3.5, 3.6

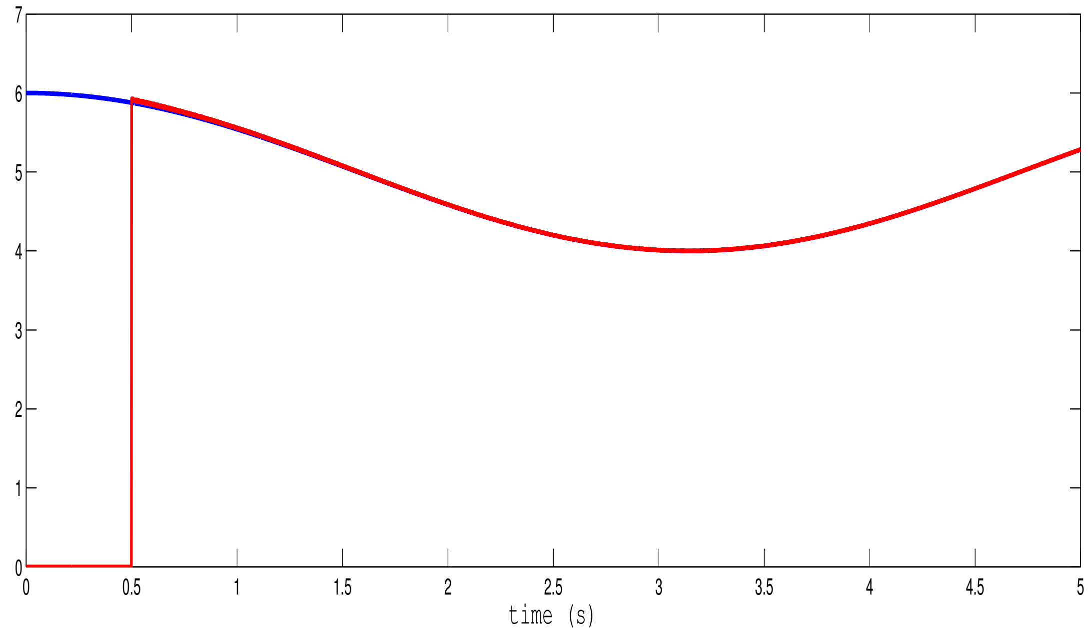


Figure 3.5: Simulation results of example 3.2: — $\dot{f}_0(t)$ — z_1

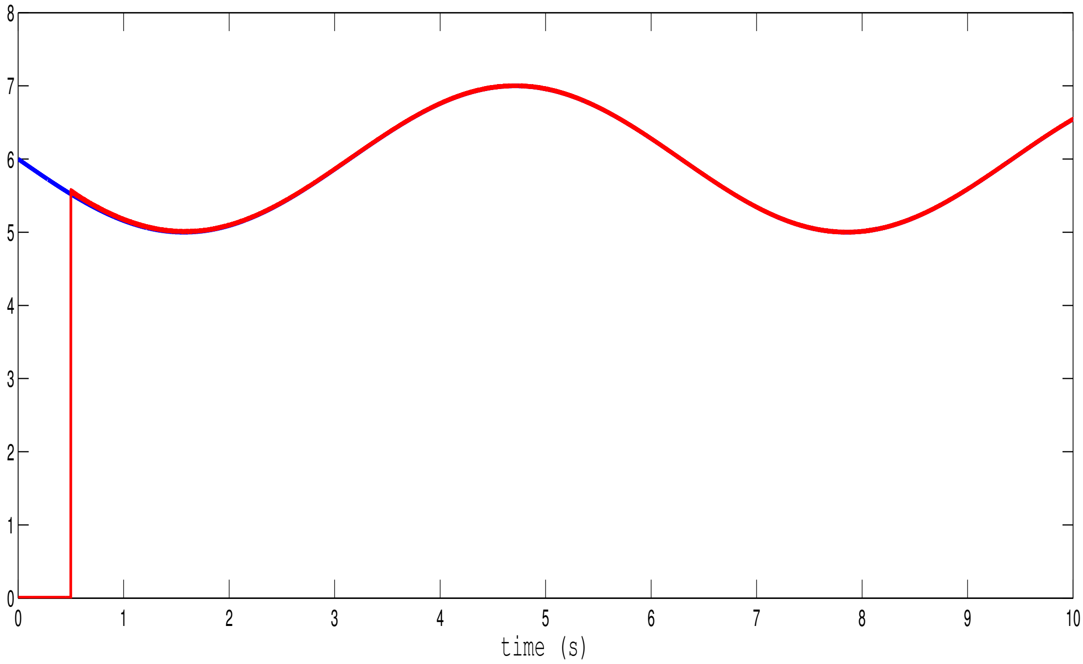


Figure 3.6: Simulation results of example 3.2: — $\ddot{f}_0(t)$ — z_2

In the absence of noise, the convergence of the MFSM differentiator is exact and uniform with respect to initial conditions, high transient peaks are also avoided.
Simulations in noisy case are presented in figures 3.7, 3.8

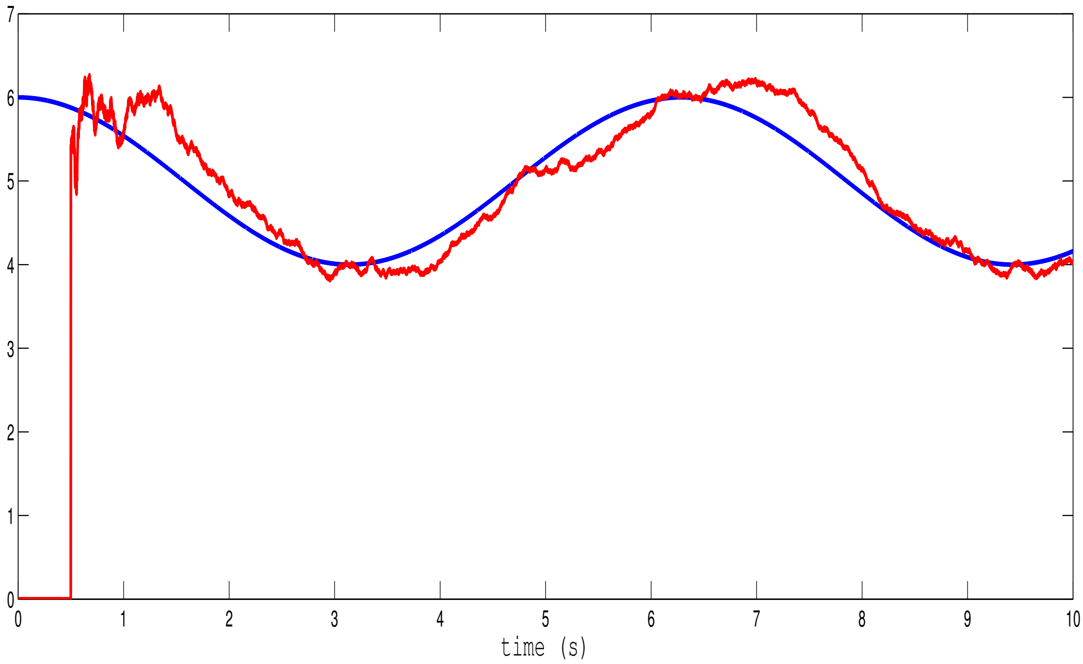


Figure 3.7: Simulation results of example 3.2: — $\dot{f}_0(t)$ — z_1 in presence of noise

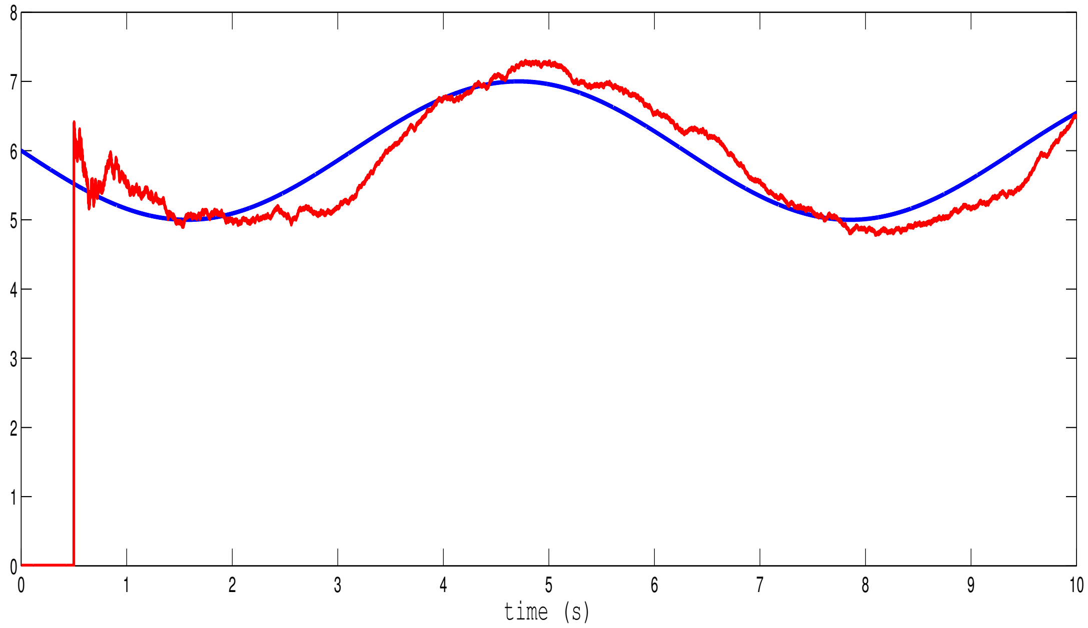


Figure 3.8: Simulation results of example 3.2: — $\ddot{f}_0(t)$ — z_2 in presence of noise

With noisy measurements, the MFSM second order differentiator converges. We can notice that the estimation errors remains bounded.

3.4 Conclusion

In the above chapter, a sliding mode based arbitrary order differentiator of time varying signal with predefined convergence time is presented. The predefined convergence time can be chosen arbitrarily and is independent of initial estimation errors. The differentiator is designed by mean of a time based transformation using modulating functions as annihilators of initial estimation errors. The convergence of first order differentiators is demonstrated by Lyapunov functions and homogeneity theory is used to proove convergence of arbitrary order diferentiator in both noise free case where the estimation is exact and in noisy signal case where the estimation error remains bounded. Throughout simulations one can notice the effeciency of such differentiators and furthermore their superiority compared with the previously introduced differentiators.

Conclusion

Summary

Observer-based differentiators are effective virtual sensors that play an important role in the estimation of the derivatives of a time varying signal.
Throughout this thesis, three recent differentiators are explored with the related design techniques: the high-gain differentiator, the sliding mode differentiators and a prescribed-time differentiator. Advantages and drawbacks of each are also discussed.

In the first chapter, notions of observability of nonlinear systems are introduced which are important to consider before starting the design process of any observer. The high-gain differentiators design method is presented. High-gain differentiators use a high-gain observer to achieve asymptotic estimation of the signal derivatives. Its gain is chosen sufficiently large in order to compensate for Lipschitz constant of the nonlinearities which are seen here as disturbances. This observer provides an exponential convergence which becomes faster with higher gains. The issue of the peaking phenomenon also become important with higher gains. It is also shown that the larger the gains, the larger the impact of disturbances and measurement noise is. A compromise has to be found before choosing these gains. The peaking phenomenon can destabilize the closed loop system for which solutions like saturating the control may be considered.

In the second chapter we introduced sliding mode differentiators as robust and efficient approach for estimating the derivatives of signals in the presence of noise and uncertainties. Their robustness to parameter variations and external disturbances makes them particularly valuable in applications where traditional differentiation methods may fail. The main features of sliding mode differentiators is their finite-time convergence. This is beneficial in practical applications such as control systems, signal processing, and robotics, where precise and real-time differentiation is crucial. However sliding mode differentiators have some drawbacks including their design and tunning which requires careful consideration of system dynamics and noise characteristics. The trade-offs between convergence speed and chattering must be managed to optimize performance.

The last category of presented differentiators is a sliding mode based arbitrary order differentiator of time varying signal with prescribed convergence time. The prescribed convergence time can be chosen arbitrarily and is independent of initial estimation errors. The differentiator is designed by mean of a time based transformation using modulating functions as annihilators of initial estimation errors. The convergence of first order differentiators is demonstrated by Lyapunov functions. To proove convergence of arbitrary order differentiator, homogeneity theory is applied in both noise free case, where the estimation is exact and in noisy signal case where the estimation error remains bounded. The choice of the modulating function is crucial and is related to noise attenuation.

Future Work

Future research should, in particular, focus on the following aspects

- Investigation of the role of the modulating function choice with respect to noise attenuation.

- Consideration of descretization problem of observer-based differentiator toward their numerical implementation in embeded systems.

- Implementation of the presented modulating-function based differentiator in practical applications.

- Combination of Kalman filter and the presented modulating function based differentiator to ensure better noise-robustness capabilities.

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