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Faculty Of Sciences

Physics Department

ELECTRICITY

Lecture notes for Physics 2 (SM, ST MI Domains)

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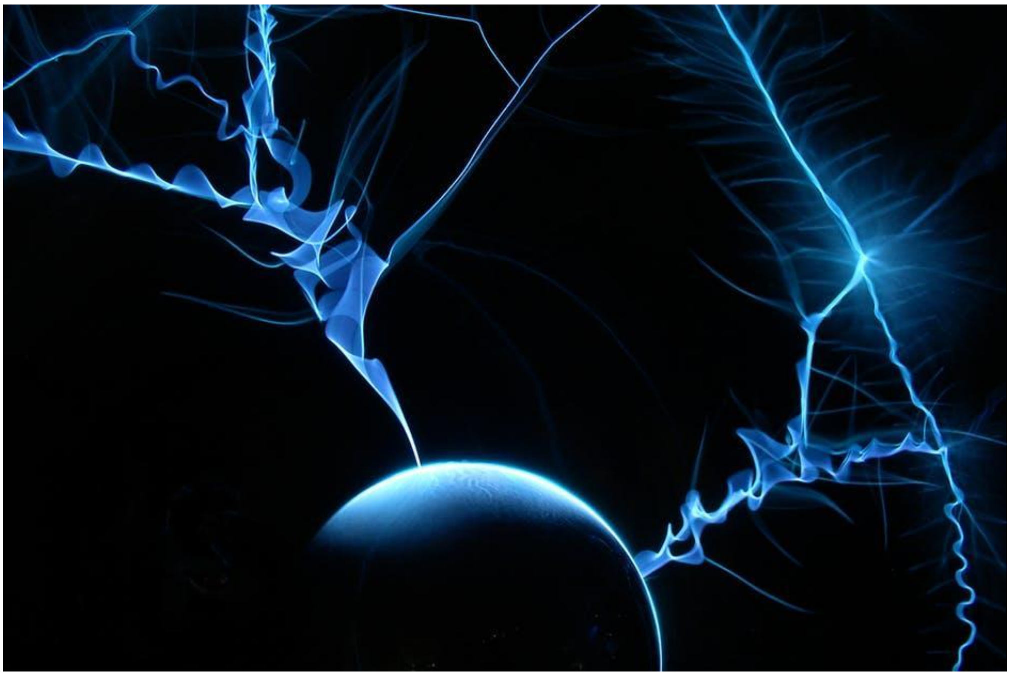

April 2024

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Chapter 1

Static electricity - General

1 Static electricity phenomenon

If two plastic rulers are rubbed vigorously with a piece of cloth, they will exhibit a repulsive force. This phenomenon can be observed by suspending one of the two rulers from a wire in the centre, allowing it to rotate freely. The end of the other ruler is then brought into proximity with the moving ruler by holding it in the hand (see Figure 1.1-a). Similarly, if two glass rods are rubbed in the same way, they will also repel each other (see Figure 1.1-b). Conversely, when the glass rod is rubbed against the plastic ruler, or vice versa, an attractive force is observed (see Figure 1.1-c).It is notable that no attraction or repulsion is evident when the rulers or rods are not rubbed..

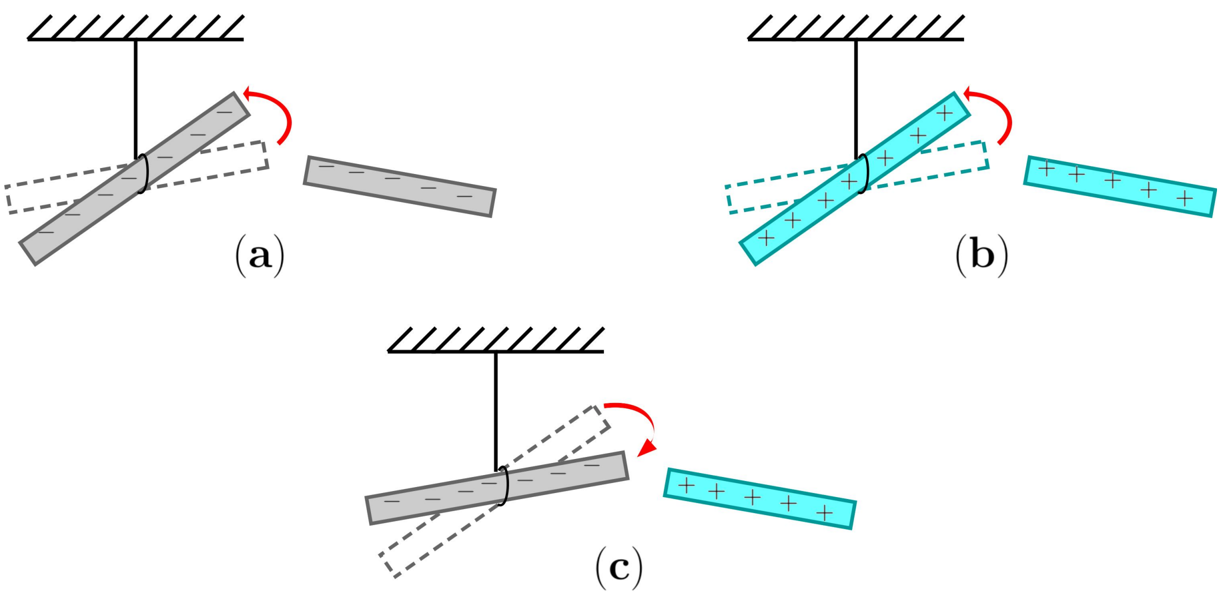


Figure 1.1: Static electricity phenomena.

The experiment described above involves a new type of force, designated as an ***electric force***. This force is distinct from the gravitational force in three fundamental aspects:

- The electric force may manifest as either an attractive or a repulsive force, whereas the gravitational force between two masses is exclusively attractive.
- The electric force can only occur between two objects that have been rubbed together, as the mass of the objects alone is insufficient to generate this force.
- The electric force is considerably stronger than the gravitational force. The gravitational force between rulers and/or rods is so weak that it cannot be observed.

2 Positive and negative electric charge

As illustrated in the preceding section, the electric force can only manifest between two objects that possess a specific attribute, designated as an ***electric charge***. This charge emerges when two objects are rubbed together. It is evident that two distinct forms of electricity exist: the *resinous state*, observed in the case of a rubbed plastic ruler, and the *vitreous state*, observed in the case of a rubbed glass rod.

One might be forgiven for assuming that, in the case of other materials, there would be other types of electricity attracted or repelled by the first two, but this is not the case. In fact, all materials can be classified into two categories. When rubbed together, they either attract a glass rod and repel a plastic ruler, or vice versa. Benjamin Franklin proposed that these two types of electric charge should be distinguished by their positive and negative signs. He arbitrarily selected the sign (+) to represent electric charges carried by a rubbed glass rod (vitreous state) and the sign (-) to represent charges carried by a rubbed plastic ruler (resinous state).

In accordance with the principles of electrostatics, like charges ((+) and (+) or (-) and (-)) are repelled from one another, whereas unlike charges ((+) and (-)) are attracted to one another.

The atomic structure of matter provides an explanation for the appearance of an electric charge on a rubbed object. Matter is composed of atoms, the radius of which is approximately 10^{-10} m. Each atom is comprised of a nucleus, with a radius of approximately 10^{-15} m, which contains positively charged particles, known as *protons*, and electrically

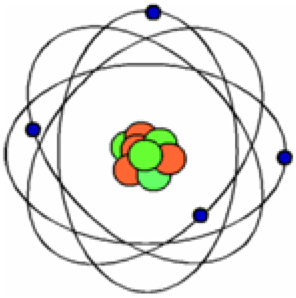


Figure 1.2: Atomic structure representation.

neutral particles, designated as *neutrons*. The negatively charged particles, known as *electrons*, which have the same absolute charge and are in the same number as the protons, form the outer structure of the atom. Consequently, the atom is electrically neutral, with the negative charges of the electrons compensating for the positive charges of the protons (see Figure 1.2). The generation of electric charges by rubbing can be explained by assuming that a few electrons can be lost or gained on the contact surface, which are transferred to (or torn from) the atoms of the piece of cloth. This results in an excess of electrons on the rubbed plastic rulers and a shortage of electrons on the glass rods. It should be noted that only the electrons are in motion, while the protons remain fixed in the nucleus. Atoms whose number of electrons is no longer equal to their proton number are referred to as *ions*. Ions are not electrically neutral; they are either positive or negative, depending on whether they have lost or gained electrons.

In the International System of Units (SI), the derived unit of electric charge is the ***coulomb*** (C). One coulomb is equivalent to a substantial quantity of charge. For instance, the charge that is observed on a rubbed object is typically in the range of 10^{-8} C, whereas a flash of lightning can reach up to 20 Cbetween a cloud and the Earth. The smallest electric charge that has been isolated thus far is that of a proton, which has the value of e. This was first measured by Millikan in 1909 and is approximately:

$$e \approx 1,602 \times 10^{-19}\,\mathrm{C}.$$

The electric charges of the proton and the electron are given by the expressions $q_p = e$ *and* $q_e = -e$*, respectively.*

3 Conservation of charge

The electric charge observed in the rubbing of a plastic ruler or glass rod is not, in fact, created. Only a specific number of electrons are transferred from the cloth to the ruler or from the rod to the cloth. The electric charge is transferred from one object to the other; if one object acquires a charge of $+Q$, the other object acquires a charge of $-Q$. The sum of the charges of the two objects remains equal to zero. This is an example of the conservation of electric charges.

> *The net amount of electric charge that is produced during each transformation is equal to zero.*

More generally, the charge conservation law can be stated as follows:

> *In an isolated system, the total electric charge is defined as the algebraic sum of the positive and negative charges present at any given time. It is a fundamental principle of electrostatics that this quantity is conserved, that is, it remains constant.*

The term "isolated" is used to describe a system in which there is no potential for charges to enter or leave the system, whether through an electrical wire or moist air.

4 Insulators and Conductors

When an iron rod is brought into contact with two metal spheres, one of which is highly charged and the other is neutral, the latter will rapidly become electrically charged (see Figure 1.3-a). Conversely, if the two spheres are connected by a wooden rod or a rubber band, the neutral sphere will remain neutral and the charged sphere will retain its charge (see Figure 1.3-b). Materials such as iron are designated as ***conductors***, whereas those such as wood or rubber are classified as ***insulators***. Therefore, we can conclude that:

- In the case of conductors, the electric charge is distributed throughout the material, with no particular concentration.
- In the case of insulators, the electric charge is localised at the point of generation, for example by rubbing.

An insulator, such as the plastic ruler and glass rod utilized in the preceding experiment, can be charged through friction. The resulting charge is static, meaning it cannot be transported away from the point of contact. Conversely, a conductor, such as the rubbed iron rod, will only become charged if it is held with an insulating handle. Otherwise,

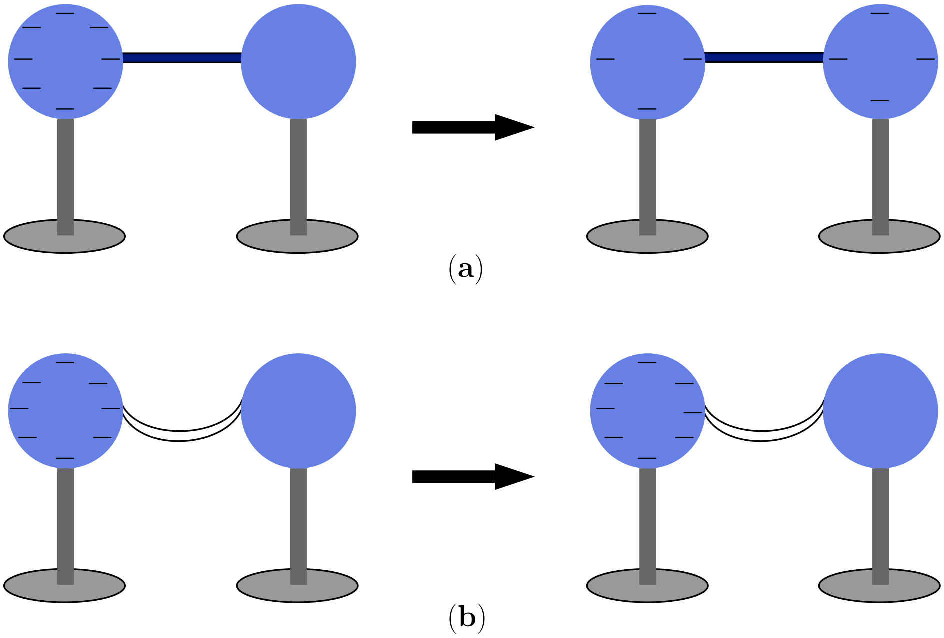


Figure 1.3: Example of metal spheres that are connected by a (a) metal rod and a (b) rubber band..

electrons will be transferred between the conductor and the ground via our body.

Plastics possess the property of acting as good insulators. All metals and carbon are classified as good conductors. In between these two categories are materials that are both poor conductors and poor insulators. This is due to the fact that they conduct to a certain extent. Examples of this include wood, paper, the human body and the Earth. Water is a good conductor when it contains a few impurities. Salt water, in particular, is a natural conductor of electricity. As it is free of impurities, pure water is an insulator. Dry air is a good insulator. Dry wood is a natural insulator that can be made conductive by moisture.

5 Electrostatic conduction and induction

An object may be charged by ***conduction***, that is to say, by being brought into contact with a charged object, either directly or via a conductor, as illustrated in 1.3-a. Furthermore, an insulated metallic object can also be charged without direct contact with the charged object. This process designated as ***induction***, whereby a charge is transferred without direct contact, is illustrated in Figure 1.4. Two metal spheres, designated A and B, are positioned on an insulating base and are in contact with one another, thereby forming a single conductor (see Figure 1.3-a). A positively charged rod is positioned in proximity to sphere A, without establishing contact. The free electrons of the conductor (A+B) are attracted by the rod's positive charge and tend to accumulate on the left of A, unable to reach the rod due to the lack of contact. The electrons leave the positive ions on the right-hand side of B, which is as far away from the rod as possible. This results in

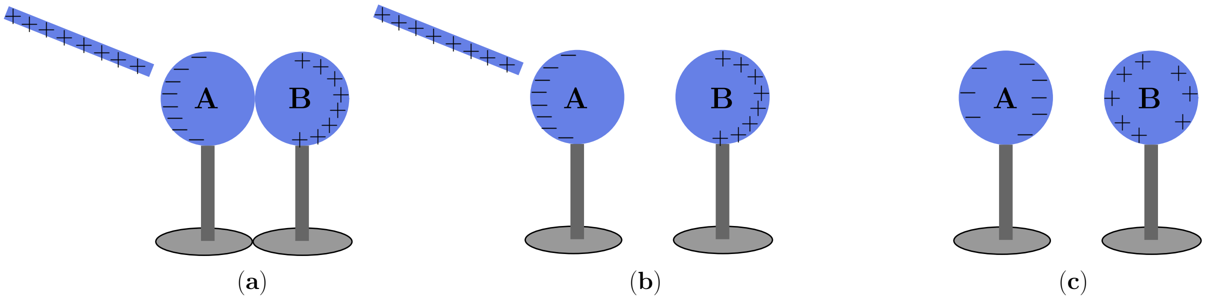


Figure 1.4: Induction charging of two metal spheres.

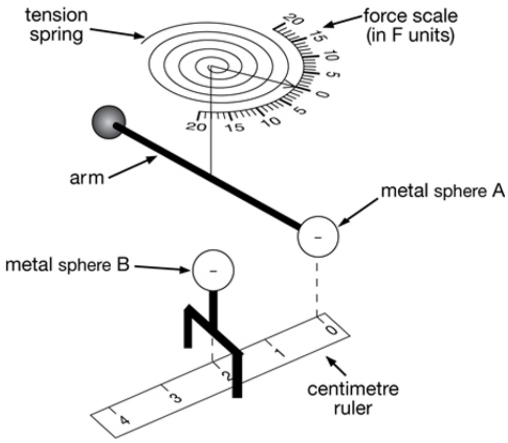


Figure 1.5: Modern illustration of Coulomb's experiment for the measurement of electrostatic force (credit).

a separation of charges, which has been caused or induced by the presence of the rod. If the two spheres are separated in the presence of the rod (Figure1.3-b) and then the rod is removed (Figure 1.3-c), it can be observed that the two spheres have acquired opposite charges by induction without any contact with the rod.

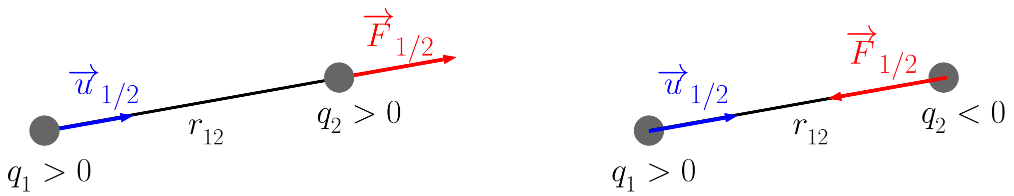
6 Electrostatic force - Coulomb's law

Charles-Augustin Coulomb (1736-1806) conducted a series of experiments employing a sensitive torsion balance (see Figure 1.5). This enabled him to determine, with a reasonable degree of accuracy, the properties of the electrostatic force exerted by a small charged sphere q_1 on another small charged sphere q_2. Coulomb's experiments yielded the following conclusions:

- The force in question is radial, acting along a line connecting the two charges.
- The force is proportional to the product of the magnitude of the charges, with an attractive force exerted between charges of opposite sign and a repulsive force between

charges of the same sign.

- The force is inversely proportional to the square of the distance between the two charges.



The vectorial form of Coulomb's law, which translates the aforementioned properties, is expressed as follows:

$$\vec{F}_{1/2} = K\frac{q_1 \cdot q_2}{r_{12}^2}\vec{u}_{1/2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 \cdot q_2}{r_{12}^2}\vec{u}_{1/2}. \quad (1.1)$$

The force exerted by the charge q_1 on the charge q_2 is represented by the vector $\vec{F}_{1/2}$. The unit vector in the direction from the charge q_1 to the charge q_2 is designated by $\vec{u}_{1/2}$. The distance between the two charges is denoted by r_{12}.

K is a constant. Its value is dependent upon the units in which r_{12}, $F_{1/2}$, and q are to be expressed. In the SI units $K = \dfrac{1}{4\pi\varepsilon_0} = 8.988 \times 10^9\,\mathrm{N \cdot m^2 \cdot C^{-2}} \approx 9 \times 10^9\,\mathrm{N \cdot m^2 \cdot C^{-2}}$. In the cgs system of units, short for centimetre-gram-second, the unit of electric charge is the "electrostatic unit" or esu. The constant K in equation 1.1 is equal to 1 (without units) if r_{12} is measured in cm, F in dynes and the values of q_1 and q_2 are in esu. Instead of K, it is common practise (due to historical reasons) to introduce the constant ε_0, otherwise known as the ***permittivity of free space*** or ***permittivity of vacuum***. This is given by $\varepsilon_0 = \frac{1}{4\pi K} \approx 8.854 \times 10^{-12}\,\mathrm{C^2 \cdot N^{-1} \cdot m^{-2}}$ and will feature in numerous expressions encountered throughout this course.

- For like electric charges q_1 and q_2, there is a repulsive force between them: The force vector $\vec{F}_{1/2}$ is oriented in a direction that is parallel to the unit vector $\vec{u}_{1/2}$.

- For unlike electric charges q_1 and q_2, an attractive force is observed. The force $\vec{F}_{1/2}$ is oriented in a direction opposite to that of $\vec{u}_{1/2}$.

Remarks

- In formulating the equation 1.1, it's essential to ensure that both charges are accurately localised, with each occupying a region that is relatively small compared to r_{12}. Without this, it would not be possible to define the distance r_{12} with precision. This is what is referred to as a point charge.

- Equation 1.1 applies only to static charges (electrostatic approximation) and vacuum. This law serves as the foundation for the entire field of electrostatics.

- The electrostatic force, as defined by equation 1.1, is in accordance with the third law of Newton, otherwise known as the law of action and reaction expressed as: $\overrightarrow{F}_{1/2} = -\overrightarrow{F}_{2/1}$.

- The electrostatic force, as defined by equation 1.1, exhibits vectorial properties analogous to those observed in the gravitational force, as described by Newton's law given by $\overrightarrow{F}_{g_{1/2}} = -G\,\frac{m_1 \cdot m_2}{r_{12}}\,\overrightarrow{u}_{1/2}$): The charges q_1 and q_2 are used to represent the masses m_1 and m_2, while the constant K is employed to represent the gravitational constant G. As previously noted, the primary distinction between these two forces is that the gravitational force is always attractive, whereas the electrostatic force can be either attractive or repulsive, depending on the sign of the electric charges.

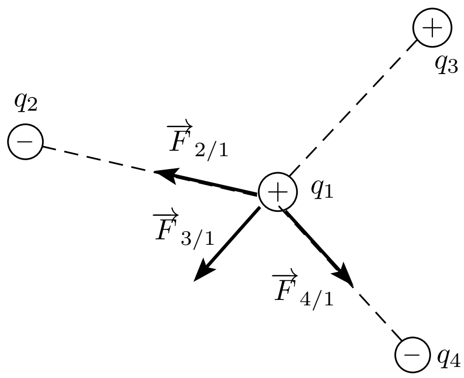
- By comparing the Coulomb repulsion and gravitational attraction between two electrons (of charge $q = -e = -1.6 \times 10^{-19}$ C and mass $m_e = 9.1 \times 10^{-31}$ kg):

$$\frac{F_e}{F_g} = \frac{K \cdot e^2}{G \cdot m_e^2} \approx 4 \times 10^{42},$$

  it can be deduced that the electrostatic force is dominant compared to the gravitational force.

7 Principle of superposition

The principle of superposition permits the calculation of the total electrostatic force exerted on a given point charge by any number of point charges acting upon it. Given that the electrostatic force, like all other forces, is a vectorial quantity, the electric forces exerted by different point charges $q_2, q_3, ..., Q_n$ on a charge q_1 are calculated independently of each other and added in a vectorial manner.

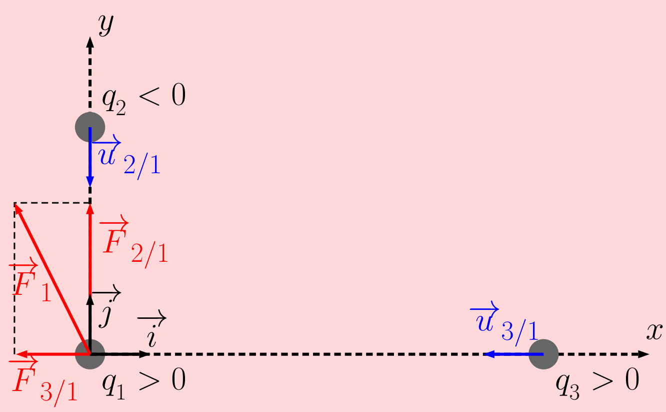


The total force exerted by the other charges on the charge q_1 is given by the following equation:

$$\vec{F}_1 = \vec{F}_{2/1} + \vec{F}_{3/1} + ... + \vec{F}_{n/1}.$$

Example

Consider a positive point charge $q_1 = 30\,\mu\text{C}$ and two other point charges $q_2 = -20\,\mu\text{C}$ and $q_3 = 40\,\mu\text{C}$, which exert an electrostatic force on it. The three point charges are located at the vertices of a rectangular triangle such that $r_{12} = 1\,\text{m}$ and $r_{13} = 2\,\text{m}$ (see figure below).



The resultant force acting on the charge q_1 can be written as follows, using the principle of superposition:

$$\vec{F}_1 = \vec{F}_{2/1} + \vec{F}_{3/1},$$

so that:

$$\overrightarrow{F}_{2/1} = K\,\frac{q_1 \cdot q_2}{r_{12}^2}\,\overrightarrow{u}_{2/1} \quad \text{and} \quad \overrightarrow{F}_{3/1} = K\,\frac{q_1 \cdot q_3}{r_{12}^2}\,\overrightarrow{u}_{3/1},$$

with :

$$\overrightarrow{u}_{2/1} = -\overrightarrow{j} \quad \text{and} \quad \overrightarrow{u}_{3/1} = -\overrightarrow{i}.$$

Then, we can write:

$$\overrightarrow{F}_{2/1} = -K\,\frac{q_1 \cdot q_2}{r_{12}^2}\,\overrightarrow{j} \quad \text{and} \quad \overrightarrow{F}_{3/1} = -K\,\frac{q_1 \cdot q_3}{r_{12}^2}\,\overrightarrow{i}.$$

Consequently, the resulting force is :

$$\overrightarrow{F}_1 = K \cdot q_1 \left[-\frac{q_3}{r_{13}^2}\,\overrightarrow{i} - \frac{q_2}{r_{12}^2}\,\overrightarrow{j} \right] \quad \Longrightarrow \quad \text{N.A.:} \quad \overrightarrow{F}_1 = -2,7\,\overrightarrow{i} + 5,4\,\overrightarrow{j}\ (\text{N}).$$

Chapter 2

Electrostatic field

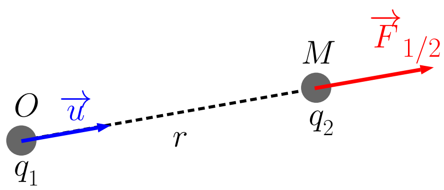
1 Introduction

Let us consider a source charge, designated as q_1, and a test charge, designated as q_2. The electrostatic force, as defined by Coulomb's law, is a force exerted by q_1 on q_2 that acts "at a distance". The question thus arises as to why q_1 exert this force on q_2 without making contact with it. In order to elucidate this phenomenon, Michael Faraday postulated the concept of the ***Electrostatic field***. If the source charge q_1 acts on the test charge q_2 at a distance, it is because q_1 induces a state in the surrounding space. In other words, at every point in the surrounding space, the source charge q_1 generates an Electrostatic field as a consequence of its mere presence. The Electrostatic field interacts with the test charge q_2 to produce the electrostatic force experienced by the latter. This notion of a field has been demonstrated yo be highly useful and practical. It has been employed to describe fundamental forces other than electrostatically (such as gravitational or magnetic forces). It enables phenomena to be described in an elegant manner.

The Electrostatic field describes the influence of a single (or multiple) perturbing charge(s) on the properties of space. In this context, the perturbing charge(s) is (are) assumed to be fixed.

2 Electrostatic field of a single charge

In order to illustrate this concept, we will consider the example of an Electrostatic source charge q_1, situated at a point O in space. This charge exerts an electrostatic force on a test Electrostatic charge q_2 of the same sign as q_1, located at a point M, such that $\overrightarrow{OM} = r\,\overrightarrow{u}$ (where r is the distance between these two points and $\overrightarrow{u}$ is the



unit vector along the line joining them). This force can be expressed as follows:

$$\overrightarrow{F}_{1/2} = K\frac{q_1 \cdot q_2}{r^2}\vec{u} = q_2\left[K\frac{q_1}{r^2}\vec{u}\right].$$

In the event of replacing the test charge q_2 by an alternative charge q_3, the force is then expressed as follows:

$$\overrightarrow{F}_{1/3} = K\frac{q_1 \cdot q_3}{r^2}\vec{u} = q_3\left[K\frac{q_1}{r^2}\vec{u}\right].$$

The term in square brackets represents the force per unit charge experienced by any test charge placed at point M, irrespective of what is tested. This quantity is referred to as the *Electrostatic field generated by the charge* q_1 *at the point* M.

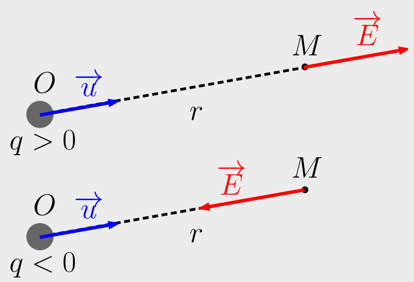
Definition 1

A particle of electric charge q at a point O produces a vector field at an arbitrary point M in space such that $\overrightarrow{OM} = r\,\overrightarrow{u}$ (where $\overrightarrow{u}$ is a unit vector directed from the particle of charge q to the point M). This vector field can be expressed as follows:

$$\boxed{\overrightarrow{E} = K\frac{q}{r^2}\overrightarrow{u} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\overrightarrow{u}.} \tag{2.1}$$

This vectorial field is referred to as the ***electrostatic field*** (E***-field***). Its unit in SI units are newton per coulomb (N/C). By convention, this electrostatic field :

- points toward negative source charge ($q < 0$).
- points away from positive source charge ($q > 0$).



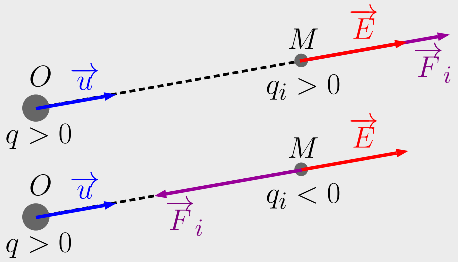
Definition 2

The application of an electric charge q_i at a specific point M, within an electrostatic field $\overrightarrow{E}$ generated by a source charge q, will result in the exertion of an electrostatic force:

$$\boxed{\overrightarrow{F}_i = q_i\overrightarrow{E}.} \tag{2.2}$$

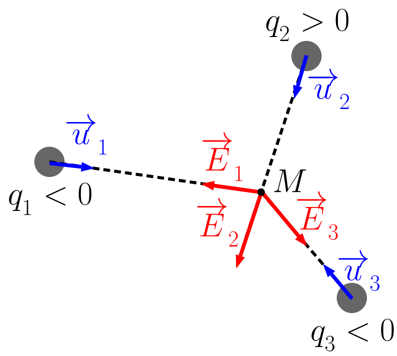
This electrostatic force is:

- in the same direction as $\vec{E}$ if $q_i > 0$.
- in the opposite direction to $\vec{E}$ if $q_i < 0$.



3 Electrostatic field of discrete charge distribution

Let us consider three electric charges, designated as q_1, q_2, and q_3, situated at points M_1, M_2, and M_3, respectively, within a three-dimensional space. Each of these charges generates an electrostatic field at a point M in the space. The E-field is subject to the same principle of superposition that applies to Coulomb's law (see Chapter 1, section 7).



It follows that the net E-field at point M is the vectorial sum of all the E-fields produced separately by q_1, q_2 and q_3 at that point.

More generally:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots + \vec{E}_n = \sum_{i=1}^{n} \vec{E}_i = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \left(\frac{q_i}{r_i^2}\right) \vec{u}_i . \tag{2.3}$$

Example

In this calculation, we consider the E-field of three charges, $q_1 = 30\,\mu\text{C}$, $q_2 = -20\,\mu\text{C}$, and $q_3 = 40\,\mu\text{C}$, generated at the point $M\,(2\,a, a)$. It is assumed that the three charges are at the vertices of a right triangle with a length of $a = 1\,\text{m}$, as illustrated in the figure below.

The principle of superposition permits the expression of the resultant E-field at point M to be written as follows:

$$\vec{E}(M) = \vec{E}_1 + \vec{E}_2 + \vec{E}_3,$$

such that:

$$\begin{aligned} \vec{E}_1 &= K\,\frac{q_1}{r^2}\,\vec{u}_1 \\ \vec{E}_2 &= K\,\frac{q_2}{(2\,a)^2}\,\vec{u}_2 \\ \vec{E}_3 &= K\,\frac{q_3}{a^2}\,\vec{u}_3, \end{aligned}$$

with :

$$\begin{cases} \vec{u}_1 &= \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j} \\ \vec{u}_2 &= \vec{i} \\ \vec{u}_3 &= \vec{j}, \end{cases}$$

and:

$$\begin{cases} r^2 &= a^2 + (2\,a)^2 = 5\,a^2 \\ \cos\alpha &= \frac{2\,a}{\sqrt{5}a} = \frac{2}{\sqrt{5}} \\ \sin\alpha &= \frac{a}{\sqrt{5}a} = \frac{1}{\sqrt{5}}. \end{cases}$$

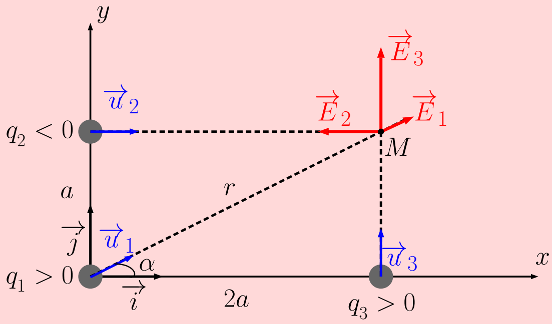
Then, we can write:

$$\begin{aligned} \vec{E}_1 &= K\,\frac{q_1}{5\,a^2}\left(\frac{2}{\sqrt{5}}\,\vec{i} + \frac{1}{\sqrt{5}}\,\vec{j}\right) \\ \vec{E}_2 &= K\,\frac{q_2}{(2\,a)^2}\,\vec{i} \\ \vec{E}_3 &= K\,\frac{q_3}{a^2}\,\vec{j}. \end{aligned}$$

Consequently, the resulting electrostatic field is :

$$\boxed{\vec{E}(M) = \frac{K}{a^2}\left[\left(\frac{2\,q_1}{5\sqrt{5}} + \frac{q_2}{4}\right)\vec{i} + \left(\frac{q_1}{5\sqrt{5}} + q_3\right)\vec{j}\right].}$$

N.A.: $\boxed{\vec{E}(M) \approx \left[4,1\times10^4\,\vec{i} + 38,2\times10^4\,\vec{j}\right]} \quad \Longrightarrow \quad \boxed{\left|\vec{E}(M)\right| \approx 38,4\times10^4\,\mathrm{C\cdot N^{-1}}.}$



4 Electrostatic field of continuous charge distribution

In practice, when a large number of particles are involved, the expression 2.3 for the electrostatic field resulting from a discrete charge distribution is rarely applicable. This is simply due to the fact that the spatial scales under consideration are very large in comparison to the distances between particles, which precludes the possibility of distinguishing one particle from another. In such cases, it is more appropriate to consider a continuous distribution of electric charges.

4.1 Charge distribution

Prior to defining the E-field resulting from a continuous charge distribution, it is necessary to define the charge density for that distribution within a volume, across a surface or along a line.

Volume charge density: If an object has an electric charge q that is uniformly distributed over its volume V (for example, a sphere), it can be considered as a set of small elements of elementary volume dV that carry an elementary charge dq. The ***charge per unit volume*** or ***volume charge density***, is then defined as:

$$\rho = \frac{\mathrm{d}q}{\mathrm{dV}}.$$

Its units are coulombs per cubic metre ($\mathrm{C} \cdot \mathrm{m}^{-3}$).

Surface charge density: In the event that one of the dimensions of the charge distribution is significantly smaller than the other two (for example, a hollow sphere or a plane), the electric charge q is distributed continuously over a surface S, which is considered as a set of small elementary surface elements dS carrying an elementary charge dq. The ***charge per unit area*** or the ***surface charge density***, is then defined as follows:

$$\sigma = \frac{\mathrm{d}q}{\mathrm{d}S}.$$

Its units are coulombs per square metre ($\mathrm{C} \cdot \mathrm{m}^{-2}$).

Linear charge density: In the case of a charge distribution comprising two negligible dimensions in comparison to the third (such as that of a wire), the electric charge q is distributed continuously along a (straight or curved) line of length, which is considered as a set of small elements of elementary length dℓ carrying an elementary charge dq. The ***charge per unit length*** or the ***linear charge density***, is then defined as follows:

$$\lambda = \frac{\mathrm{d}q}{\mathrm{d}\ell}.$$

Its units are coulombs per metre ($\mathrm{C} \cdot \mathrm{m}^{-1}$).

***Remark:** Mesoscopic scale - Local charge density*

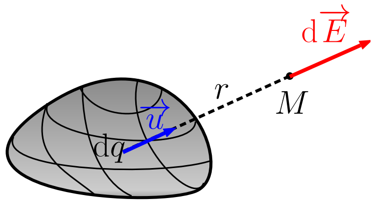
When an electrically charged body is studied in terms of volume, surface area or length, it is broken down into small elements that can be compared to point charges. These small elements must be :

- large on the microscopic scale of length, so that all the microscopic elementary charges can be averaged together.
- small with respect to the macroscopic scale, so that local behaviour can be defined.

A mesoscopic scale is therefore selected. Around each point, the local density of charges is very well defined.

4.2 Calculating the electrostatic field

The Calculation of the E-field produced at a point M in space, by a charge q distributed in an object over a certain region of space is performed as follows:



- The object is divided into infinitesimally small elements of charge $\mathrm{d}q$ and of distance r from the point M.
- Each element, considered as a point charge, will then generate a field $\mathrm{d}\vec{E}$ in accordance with the law:

$$\mathrm{d}\vec{E} = K\frac{\mathrm{d}q}{r^2}\vec{u} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}q}{r^2}\vec{u}, \qquad \vec{u} \text{ unit vector directed from } \mathrm{d}q \text{ to } M.$$

- The net electrostatic field is obtained by applying the principle of superposition and summing all the electrostatic fields $\mathrm{d}\vec{E}$ produced by all the charges $\mathrm{d}q$ contained in the space under consideration. the summation in Equation 2.3 becomes an integral. Furthermore, q_i is replaced by $\mathrm{d}q$. This results in the following equation:

$$\boxed{\vec{E} = \int \mathrm{d}\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathrm{d}q}{r^2}\vec{u}.} \tag{2.4}$$

- Depending on whether the charge is distributed by volume, area or length, the net field produced at a point M in space is expressed as follows:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\iiint_{\text{volume}}\frac{\mathrm{d}q}{r^2}\vec{u} = \frac{1}{4\pi\varepsilon_0}\iiint_{\text{volume}}\left(\frac{\rho\cdot\mathrm{dV}}{r^2}\right)\vec{u},$$

or:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\iint_{\text{surface}}\frac{\mathrm{d}q}{r^2}\vec{u} = \frac{1}{4\pi\varepsilon_0}\iint_{\text{surface}}\left(\frac{\sigma\cdot\mathrm{dS}}{r^2}\right)\vec{u},$$

or:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\frac{\mathrm{d}q}{r^2}\vec{u} = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\left(\frac{\lambda\cdot\mathrm{d}\ell}{r^2}\right)\vec{u}.$$

Example : *E-field of a line segment*

Consider a straight line segment, of length $\ell = 4\,\mathrm{m}$, that carries a charge $q = 80\,\mu\mathrm{C}$ with positive uniform linear charge density.

1. **Find the electrostatic field at point M, a distance $x = 6\,\mathrm{m}$ on its axis:**
   The line segment is situated on the x-axis between the abscissa points $x = 0\,\mathrm{m}$ and $x = 4\,\mathrm{m}$. Given that this is a continuous charge distribution, it is possible to conceptualise the charged segment as a series of differential segments of length $\mathrm{d}\ell = \mathrm{d}x$, with abscissa x, each of which carries a differential amount of charge $\mathrm{d}q = \lambda \mathrm{d}x$.
   Given that the linear charge density is constant and that the integration is performed along a line of charge situated on the x-axis, the following expression can be derived:

$$\mathrm{d}q = \lambda \mathrm{d}x \quad \Longrightarrow \quad q = \int \lambda \mathrm{d}x = \lambda \int \mathrm{d}x = \lambda \cdot \ell \quad \Longrightarrow \lambda = \frac{q}{\ell}\ (\lambda > 0)\,.$$

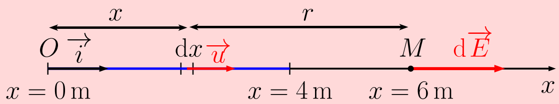
   The differential electrostatic field $\mathrm{d}\vec{E}$ generated by this length element at point M is oriented in the x-direction. Its expression is given by:

$$\mathrm{d}\vec{E} = K\left(\frac{\mathrm{d}q}{r^2}\right)\vec{u} = \mathrm{d}E_x\vec{i} = K\left(\frac{\lambda \mathrm{d}x}{r^2}\right)\vec{i}.$$

   r is the distance between M and the element $\mathrm{d}x$. So $r = 6 - x$.
   The electrostatic field is then calculated as follows:

$$E_x = \int \mathrm{d}E_x = \int_{x=0}^{x=4} K\left(\frac{\lambda \mathrm{d}x}{(6-x)^2}\right) = K\,\lambda\left[\frac{1}{6-x}\right]_0^4 = \frac{K\,\lambda}{3} \quad \Longrightarrow \quad \boxed{\vec{E} = \frac{K\,\lambda}{3}\vec{i}.}$$

   N.A.: $\lambda = \frac{q}{\ell} = 20\,\mu\mathrm{C}\cdot\mathrm{m}^{-1} = 2\times 10^{-5}\,\mathrm{C}\cdot\mathrm{m}^{-1} \quad \Longrightarrow \quad \boxed{E = 6\times 10^4\,\mathrm{N}\cdot\mathrm{C}^{-1}.}$



2. **Find the electrostatic field at point M, a distance $R = 2\,\mathrm{m}$ above the midpoint of the wire:**
   The line segment is situated on the x-axis between the abscissa points $x = -2\,\mathrm{m}$ and $x = 2\,\mathrm{m}$. Similarly, the segment is divided into differential line elements $\mathrm{d}\ell = \mathrm{d}x$. Let us consider one of the aforementioned line segments at abscissa x, carrying an elementary charge $\mathrm{d}q = \lambda \mathrm{d}x$. At the point M, the differential electrostatic field $\mathrm{d}\vec{E}$ is generated at an angle α (varying between $-\alpha_0$ and α_0) to the y-axis:

$$\mathrm{d}\vec{E} = K\left(\frac{\mathrm{d}q}{r^2}\right)\vec{u} = K\left(\frac{\lambda\,\mathrm{d}x}{r^2}\right)\vec{u} \quad ; \quad \vec{u} = -\sin\alpha\,\vec{i} + \cos\alpha\,\vec{j}\,.$$

   Then $\mathrm{d}\vec{E}$ can be expressed as follows:

$$\mathrm{d}\vec{E} = \mathrm{d}E_x\,\vec{i} + \mathrm{d}E_y\,\vec{j}\,,$$

   with:

$$\begin{cases} \mathrm{d}E_x & = -K\frac{\lambda\,\mathrm{d}x}{r^2}\sin\alpha \\ \mathrm{d}E_y & = K\frac{\lambda\,\mathrm{d}x}{r^2}\cos\alpha. \end{cases}$$

   The differential terms of the E-field comprise three variables, namely r, α and x. Two of these (r and x) are subsequently expressed in terms of the third (α):

$$\begin{cases} \tan\alpha = \frac{x}{R} & \Longrightarrow \quad x = R\,\tan\alpha \Longrightarrow \mathrm{d}x = R\frac{1}{\cos^2\alpha}\mathrm{d}\alpha \\ r = \frac{R}{\cos\alpha}. \end{cases}$$

Then :

$$\begin{cases} \mathrm{d}E_x = -K\frac{\lambda}{R}\sin\alpha\,\mathrm{d}\alpha \\ \mathrm{d}E_y = K\frac{\lambda}{R}\cos\alpha\,\mathrm{d}\alpha \end{cases}$$

The net electrostatic field created at M is then :

$$\overrightarrow{E} = \int \mathrm{d}\overrightarrow{E} = E_x\,\overrightarrow{i} + E_y\,\overrightarrow{j},$$

with :

$$\begin{cases} E_x = -\int_{-\alpha_0}^{+\alpha_0} K\frac{\lambda}{R}\sin\alpha\,\mathrm{d}\alpha = K\frac{\lambda}{R}\left[\cos\alpha\right]_{-\alpha_0}^{+\alpha_0} = K\frac{\lambda}{R}\left[\cos\alpha_0 - \cos(-\alpha_0)\right] = 0 \\ E_y = \int_{-\alpha_0}^{+\alpha_0} K\frac{\lambda}{R}\cos\alpha\,\mathrm{d}\alpha = K\frac{\lambda}{R}\left[\sin\alpha\right]_{-\alpha_0}^{+\alpha_0} = K\frac{\lambda}{R}\left[\sin\alpha_0 - \sin(-\alpha_0)\right] = 2K\frac{\lambda}{R}\sin\alpha_0. \end{cases}$$

So, we have:

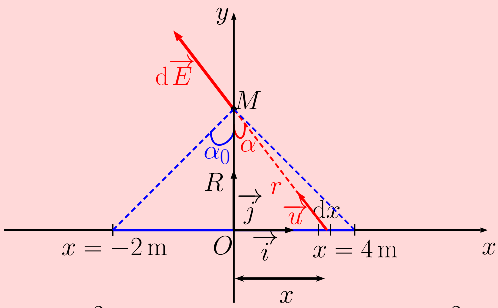
$$\boxed{\overrightarrow{E} = \frac{2K\lambda}{R}\sin\alpha_0\,\vec{\jmath}.}$$

with:

$$\sin\alpha_0 = \frac{\ell/2}{\sqrt{\left(\frac{\ell}{2}\right)^2 + R^2}},$$

and finally:

$$\boxed{\overrightarrow{E} = \frac{K\lambda}{R}\frac{\ell}{\sqrt{\left(\frac{\ell}{2}\right)^2 + R^2}}\vec{\jmath}.} \implies \textbf{N.A.}: \boxed{E = 1,3\times 10^5\,\mathrm{N\cdot C^{-1}}.}$$



- It can be observed that for $R \gg \ell$, R^2 has a greater influence than ℓ^2 on the result obtained previously. Consequently, the net electrostatic field may be simplified to:

$$\boxed{\overrightarrow{E} = \frac{K\lambda\ell}{R^2} = \frac{Kq}{R^2}\vec{\jmath}.}$$

We retrieve the expression for the field of a point charge. From far away, the finite line segment looks like a point charge.

- Furthermore, for an infinite wire (where $\ell \gg R$), ℓ^2dominates R^2 in the denominator of the aforementioned result. Consequently, the net electrostatic field may be simplified to:

$$\boxed{\overrightarrow{E} = \frac{2K\lambda}{R}\vec{\jmath}.}$$

5 Electrostatic field lines

Given the enhanced experience we have accrued in calculating electrostatic fields, it's now appropriate to direct our attention to their geometric representation. As previously stated, an electrostatic field describes a change in space that is caused by the mere presence

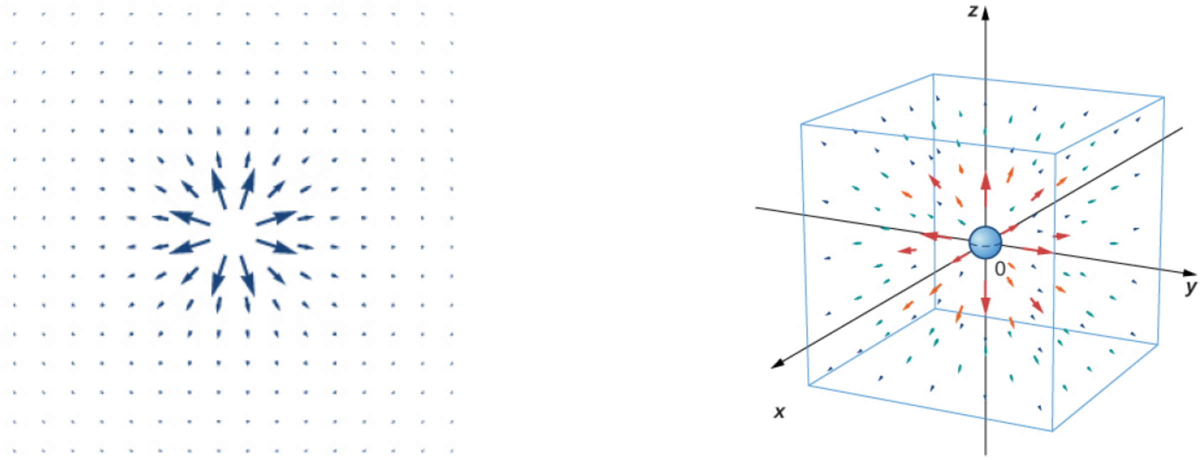


Figure 2.1: The electrostatic field vector distribution around a positive point charge in (a) two dimensions and (b) three dimensions (credit: D.Janzen).

of an electrically charged object (the source charge). This change is manifested in the form of an electrostatic force exerted on another electrically charged object (the test charge) placed in the same region. An illustration of electrostatic field vectors that are uniformly distributed around a positive point source charge is provided in the Figure 2.1. The arrows displayed at each point in space indicate to the direction of the electrostatic field vector relative to the source charge. Their lengths correspond to the magnitude of the electrostatic field vector, which decays as the square of the distance from the charge. When multiple source point charge are considered (see the example of two identical source point charges of opposite sign in the Figure 2.2), the electrostatic field vector distribution becomes more complex.

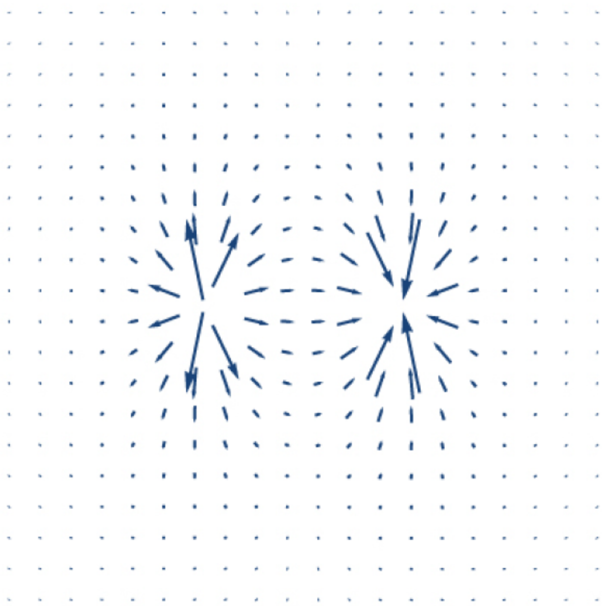


Figure 2.2: The electrostatic field vector distribution around two identical point charges of opposite signs (credit: D.Janzen).

Rather than depicting this electrostatic field vector distribution with a multitude of diminishing vector arrows, it is more efficacious to interconnect them to form uninterrupted lines and curves, designated as ***electrostatic field lines*** (alternatively termed ***force lines***). The notion of electrostatic field lines is invaluable for attaining a spatial representation of the electrostatic field and subsequently visualising the manner in which the surrounding space is altered by the source charge. The electrostatic field line can be defined as follows:

Definition

An electrostatic field line is defined as a curve whose tangent at any given point in space is aligned with the direction of the field vector $\vec{E}$. Thus, when an elementary displacement $\vec{d\ell}$ is considered along this curve, the electrostatic field $\vec{E}$, at any point on the curve is parallel to $\vec{d\ell}$. This can be expressed as:

$$\vec{E} \times \vec{d\ell} = \vec{0}.$$

The electrostatic field line diagrams for the two preceding cases are presented in Figure 2.3. In both diagrams, the field vector (not shown) is tangent to the field line at all points. The arrowhead placed on a field line indicates its direction.

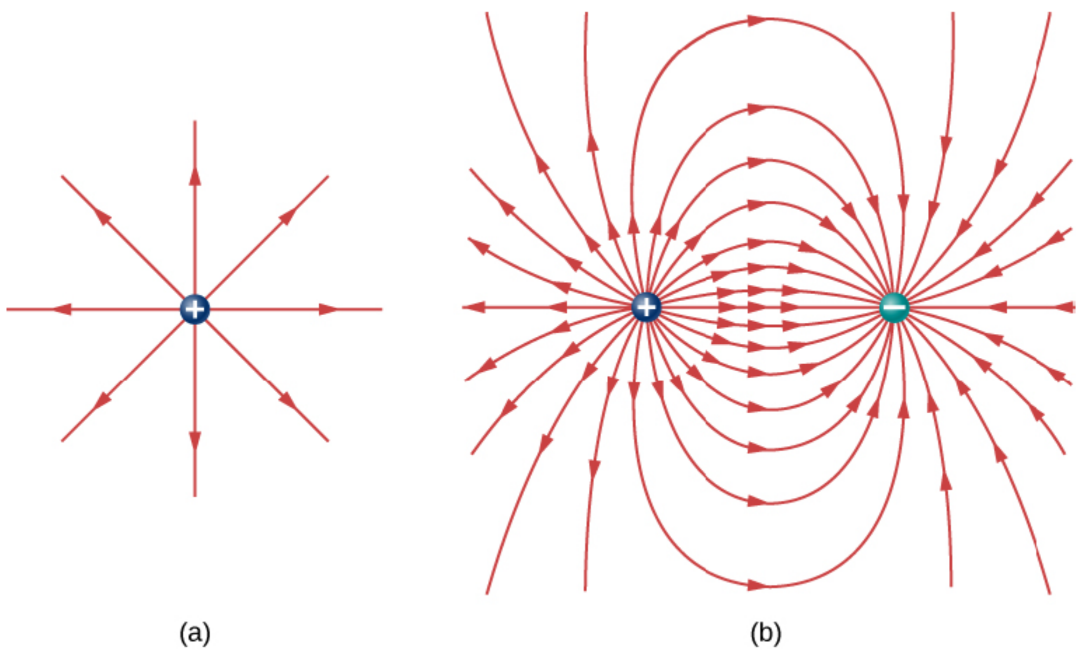


Figure 2.3: The electrostatic field lines of (a) a positive point charge and (b) two dissimilar point charges (credit: D.Janzen).

Field line properties

- The behaviour of electrostatic field lines can be described as either diverging away from positive source charges or converging towards negative source charges.

- It is not possible for two electrostatic field lines to cross each other. It is not possible for the electrostatic field to have two different directions at the same point.

- The magnitude of the electrostatic field is indicated by *field line density*, which is defined as the number of field lines per unit area passing through a small-cross-sectional area perpendicular to the electrostatic field. This is illustrated in Figure 2.4. It can be concluded from this property that the magnitude of $\overrightarrow{E}$ at the point of interest will be large if the field lines are close together (i.e. greater field line density) and small if the field lines are far apart in the cross section (i.e. lower field line density).

- The number of field lines emanating from or converging on a charge is directly proportional to the magnitude of that charge. As illustrated in Figure 2.5, a charge of $+3q$ will have three times as many field lines leaving it as entering the charge of $-q$.

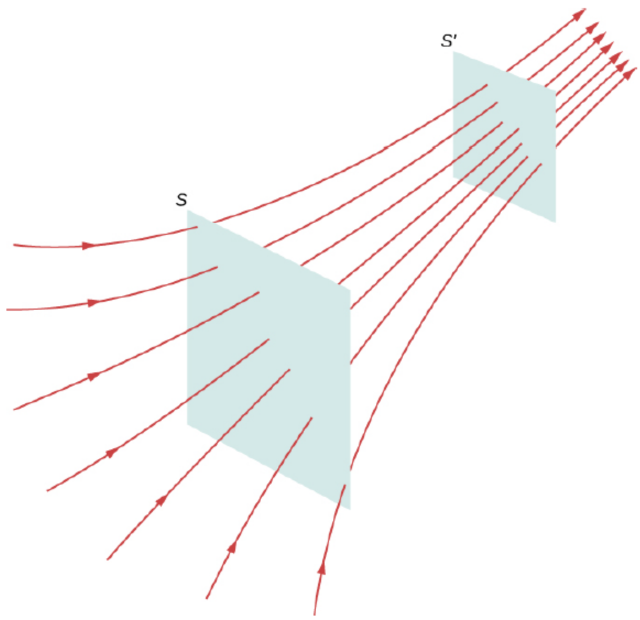


Figure 2.4: Illustration of the existence of disparate magnitudes of electrostatic field at disparate points, attributable to the variation in field density across the two regions. The identical number of field lines pass through the distinct imaginary areas designated as S and S' (credit: D.Janzen).

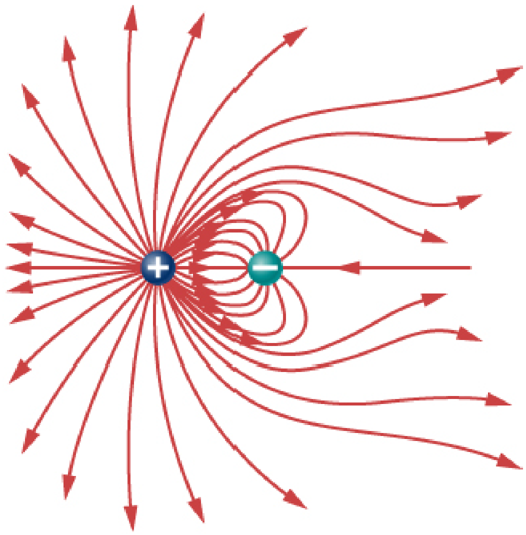


Figure 2.5: Illustration of the electrostatic field lines of two charges with opposite signs and different magnitudes (credit: D.Janzen).

Chapter 3

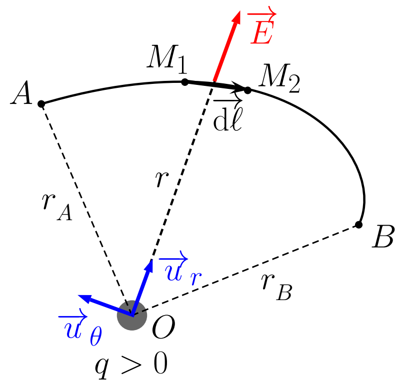
Electrostatic Potential and potential energy

1 Electrostatic potential

Prior to examining the concept of electrostatic potential, it's essential to introduce an integral mathematical attribute of electrostatic fields, namely the circulation of the electrostatic field vector. The concept of circulation is employed in a multitude of applications pertaining to electricity and magnetism.

Consequently, it is of paramount importance to establish a formal definition of circulation, and susequently determine the optimal methodology for utilising it throughtout the remainder of this chapter.

1.1 Circulation of the electrostatic field



Let us consider the case of an electric charge q at point O. This charge will produce an electrostatic vector field at any point M in space, at a distance r from O. The E-field is given by:

$$\overrightarrow{E} = K\,\frac{q}{r^2}\,\vec{u} \ ,\ \vec{u} = \frac{\overrightarrow{OM}}{r}.$$

Consider two points, designated M_1 and M_2, which are situated in close proximity and situated along a curve, designated (AB), within this field. It is defined that the ***elementary circulation of the electrostatic field*** between points M_1 and M_2 is defined by the quantity :

$$\mathrm{d}\mathscr{C} = \overrightarrow{E}\cdot\overrightarrow{M_1M_2} = \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell}.$$

In the polar basis $(\vec{u_r}, \vec{u_\theta})$, the elementary displacement $\overrightarrow{\mathrm{d}\ell}$ is expressed as:

$$\overrightarrow{\mathrm{d}\ell} = \mathrm{d}r\,\vec{u}_r + r\,\mathrm{d}\theta\,\vec{u}_\theta,$$

where $\mathrm{d}r$ and $\mathrm{d}\theta$ are elementary displacements along the radial and angular directions, respectively.

Given that the electrostatic field vector is radial, that is to say, that $\vec{u} \equiv \vec{u}_r$, then:

$$\mathrm{d}\mathscr{C} = \left(K\,\frac{q}{r^2}\,\vec{u}_r\right)\left(\mathrm{d}r\,\vec{u}_r + r\,\mathrm{d}\theta\,\vec{u}_\theta\right) = K\,\frac{q}{r^2}\,\mathrm{d}r.$$

Consequently, the ***circulation of the electrostatic field*** between points A and B can be expressed as follows:

$$\mathscr{C} = \int_A^B \mathrm{d}\mathscr{C} = \int_A^B \vec{E}\cdot\mathrm{d}\vec{\ell} = \int_{r_A}^{r_B} K\,\frac{q}{r^2}\,\mathrm{d}r = \left[-K\,\frac{q}{r}\right]_{r_A}^{r_B} \quad\Longrightarrow\quad \int_A^B \vec{E}\cdot\mathrm{d}\vec{\ell} = K\,\frac{q}{r_A} - K\,\frac{q}{r_B}.$$

Definition

The circulation of an electrostatic field vector $\vec{E}$ along a curve from point A to point B is given by the following equation:

$$\boxed{\int_A^B \vec{E}\cdot\mathrm{d}\vec{\ell} = K\,\frac{q}{r_A} - K\,\frac{q}{r_B}.} \tag{3.1}$$

- The circulation of the electrostatic field is independent of the path taken, only of the initial and final states. The electrostatic field is conservative.
- The circulation of the electrostatic field over a closed curve is equal to zero. This is of great importance in electrokinetics, as will be discussed subsequently.

1.2 Electrostatic potential of a single point charge

The Equation 3.1, which serves to translate the circulation of the electrostatic field generated by a point charge q can be written as follows:

$$\int_A^B \vec{E}\cdot\mathrm{d}\vec{\ell} = K\,\frac{q}{r_A} - K\,\frac{q}{r_B} = \left(K\,\frac{q}{r_A} + C\right) - \left(K\,\frac{q}{r_B} + C\right) = V(A) - V(B).$$

V is a scalar function called ***electrostatic potential***. It's defined by:

$$V(r) = K\,\frac{q}{r} + C = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r} + C.$$

The assumption that the electrostatic potential is zero at infinity ($V(r \longrightarrow \infty) = 0$) eliminates the constant C ($C = 0$).

Definition

The ***electrostatic potential*** generated by a point charge q at a distance r (in a vacuum) is a scalar function, expressed as follows:

$$\boxed{V(r) = K\,\frac{q}{r} = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r} + C.} \tag{3.2}$$

- Its units in SI units are volts (V).
- The electrostatic potential is defined with a certain degree of precision, within the limits set by a constant C. This constant can be determined by the boundary conditions $V(r \longrightarrow \infty) = 0$.

1.3 Electrostatic potential of discrete charge distribution

Let us consider a set of point charges, designated as q_i $(i = 1, 2, 3, \ldots)$, situated at points M_i $(i = 1, 2, 3, \ldots)$, respectively. Each of these charges generates an electrostatic potential at a point M in space, which we may write as V_1, V_2, $V_3, \ldots$.

As previously seen, the circulation of the electrostatic field and the principle of superposition of electrostatic fields allow us to express this as follows:

$$\vec{E} \cdot \vec{d\ell} = \sum_i \vec{E}_i \cdot \vec{d\ell} \quad \Longrightarrow \quad \int_A^B \vec{E} \cdot \vec{d\ell} = \sum_i \left(\int_A^B \vec{E}_i \cdot \vec{d\ell} \right) = \sum_i \left(V_{i_A} - V_{i_B} \right).$$

The net electrostatic potential at M is thus given by:

$$\boxed{V = V_1 + V_2 + V_3 + \ldots = \sum_i V_i.}$$

Example

Let's calculate the electrostatic potential created by three charges $q_1 = 30\,\mu\text{C}$, $q_2 = -20\,\mu\text{C}$ and $q_3 = 40\,\mu\text{C}$ at a point designated as $M\,(2\,a, a)$. The three charges are situated at the vertices of a right triangle such that $a = 1\,\text{m}$ (see figure below).
The net electrostatic potential at point M is:

$$V\,(M) = V_1 + V_2 + V_3,$$

with:

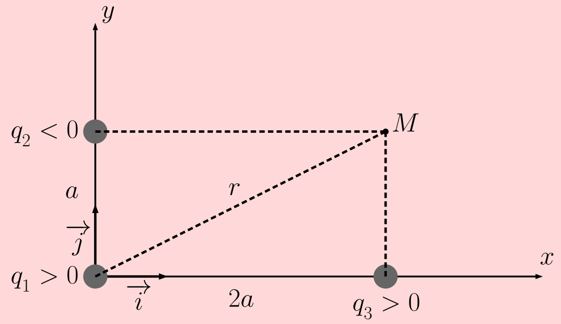
$$\begin{aligned} V_1 &= K\,\frac{q_1}{r} \\ V_2 &= K\,\frac{q_2}{2\,a} \\ V_3 &= K\,\frac{q_3}{a}. \end{aligned}$$

and:

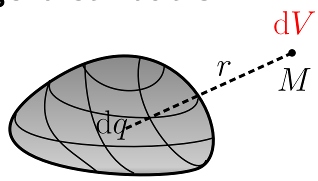
$$r^2 = a^2 + (2\,a)^2 = 5\,a^2 \quad \Longrightarrow \quad r = \sqrt{5}a.$$

Thus :

$$\boxed{V\,(M) = \frac{K}{a}\left[\frac{q_1}{\sqrt{5}} + \frac{q_2}{2} + q_3\right]} \quad \Longrightarrow \quad \textbf{A.N.}: \quad \boxed{V\,(M) \approx 3,9 \times 10^5\,\text{V}}$$



1.4 Electrostatic potential of continuous charge distribution



The electrostatic potential V at a point M in space, created by a charge q that is continuously distributed, along a line or over a surface or a volume, on a non-point object, is calculated in the same way as the electrostatic field:

- The object is divided into infinitesimally small elements of charge, designated as $\mathrm{d}q$, situated at a distance r from the point M.
- Each element is regarded as a point charge and generates an electrostatic differential potential $\mathrm{d}V$, in accordance with the law:

$$\mathrm{d}V = K\frac{\mathrm{d}q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}q}{r}.$$

- By applying the principle of superposition and summing up all the potentials $\mathrm{d}V$ created by all the charges $\mathrm{d}q$ contained in the space under consideration, the net electrostatic potential is obtained:

$$\boxed{V = \int \mathrm{d}V = \int \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}q}{r}.} \tag{3.3}$$

- The net electrostatic potential produced at a point M in space is expressed as follows, depending on whether the charge is distributed by volume, area or length, :

$$V = \frac{1}{4\pi\varepsilon_0}\iiint_{\text{volume}}\frac{\mathrm{d}q}{r} = \frac{1}{4\pi\varepsilon_0}\iiint_{\text{volume}}\left(\frac{\rho\cdot\mathrm{dV}}{r}\right),$$

or:

$$V = \frac{1}{4\pi\varepsilon_0}\iint_{\text{surface}}\frac{\mathrm{d}q}{r} = \frac{1}{4\pi\varepsilon_0}\iint_{\text{surface}}\left(\frac{\sigma\cdot\mathrm{dS}}{r}\right),$$

or:

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\frac{\mathrm{d}q}{r} = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}}\left(\frac{\lambda\cdot\mathrm{d}\ell}{r}\right).$$

Example

Consider a straight line segment, of length $\ell = 4\,\mathrm{m}$, that carries a charge $q = 80\,\mu\mathrm{C}$ with positive uniform linear charge density. The line segment is situated on the x-axis between the abscissa points $x = 0\,\mathrm{m}$ and $x = 4\,\mathrm{m}$. Given that this is a continuous charge distribution, it is possible to conceptualise the charged segment as a series of differential pieces of length $\mathrm{d}\ell = \mathrm{d}x$, with abscissa x, each of which carries a differential amount of charge $\mathrm{d}q = \lambda\mathrm{d}x$.

Given that the linear charge density is constant and that the line of charge in question lies on the x-axis, we can write:

$$\mathrm{d}q = \lambda\,\mathrm{d}x \quad \Longrightarrow \quad q = \int \lambda\,\mathrm{d}x = \lambda \int \mathrm{d}x = \lambda \cdot \ell \quad \Longrightarrow \lambda = \frac{q}{\ell}\ (\lambda > 0)\,.$$

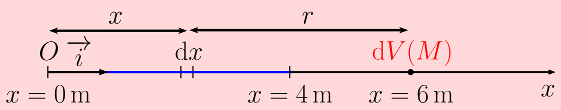
The differential electrostatic potential generated by this length element at point M is given by the following equation:

$$\mathrm{d}V = K\,\frac{\mathrm{d}q}{r} = K\,\frac{\lambda\,\mathrm{d}x}{r}.$$

r is the distance between M and the element $\mathrm{d}x$. Thus, $r = 6 - x$.
The net electrostatic potential is then calculated as follows:

$$V = \int \mathrm{d}V = \int_{x=0}^{x=4} K\,\frac{\lambda\,\mathrm{d}x}{(6-x)} = K\,\lambda\,[-\ln{(6-x)}]_0^4 \quad \Longrightarrow \quad \boxed{V = K\,\lambda\,\ln{(3)}\,.}$$

N.A.: $\lambda = \frac{q}{\ell} = 20\,\mu\mathrm{C}\cdot\mathrm{m}^{-1} = 2\times10^{-5}\,\mathrm{C}\cdot\mathrm{m}^{-1} \quad \Longrightarrow \quad \boxed{V = 1{,}98\times10^{5}\,\mathrm{V}.}$



1.5 Relationship with the electrostatic field

Having defined electrostatic potential, we now turn our attention to the relationship between electrostatic field and electrostatic potential. We have previously expressed the elementary circulation of the electrostatic field generated by a point charge q along an elementary displacement between two infinitely closed points M_1 and M_2 located along a curve (AB) situated in this field:

$$\mathrm{d}\mathscr{C} = \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell} = K\,\frac{q}{r^2}\,\mathrm{d}r \quad \Longrightarrow \quad \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell} = -\mathrm{d}\left(K\,\frac{q}{r}\right) = -\mathrm{d}V.$$

It can thus be inferred that a *local relationship* exists between the electrostatic field $\overrightarrow{E}$ and the electrostatic potential V, given as follows:

$$\boxed{\mathrm{d}V = -\overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell}.} \tag{3.4}$$

It is therefore possible to deduce the specific relationship between the electrostatic field $\overrightarrow{E}$ and the electrostatic potential difference between two positions, A and B, by calculating the circulation of the electrostatic field along the curve (AB):

$$\mathrm{d}V = -\overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell} \quad \Longrightarrow \quad \int_{V_A}^{V_B}\mathrm{d}V = -\int_A^B \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell} \quad \Longrightarrow \quad \boxed{\Delta V = V_B - V_A = -\int_A^B \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell}.}$$

If the value electrostatic field is known, the aforementioned relationship can be employed to ascertain the electrostatic potential difference, ΔV.

Conversely, the electrostatic field can be calculated by determining the scalar function of the electrostatic potential. The elementary circulation of the electrostatic field can be developed in Cartesian coordinates as follows:

$$\overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell}=\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}\begin{pmatrix}\mathrm{d}x\\\mathrm{d}y\\\mathrm{d}z\end{pmatrix}=E_x\,\mathrm{d}x+E_y\,\mathrm{d}y+E_z\,\mathrm{d}z,$$

while the differential electrostatic potential $\mathrm{d}V$ is written as :

$$\mathrm{d}V=\frac{\partial V}{\partial x}\mathrm{d}x+\frac{\partial V}{\partial y}\mathrm{d}y+\frac{\partial V}{\partial z}\mathrm{d}z.$$

By undertaking a term-by-term identification of the two preceding equations, we obtain the following result:

$$\begin{cases}E_x=-\dfrac{\partial V}{\partial x}\\E_y=-\dfrac{\partial V}{\partial y}\\E_z=-\dfrac{\partial V}{\partial z},\end{cases}$$

and then:

$$\boxed{\overrightarrow{E}=-\left[\frac{\partial V}{\partial x}\vec{\imath}+\frac{\partial V}{\partial y}\vec{\jmath}+\frac{\partial V}{\partial z}\vec{k}\right]}\quad\Longrightarrow\quad\boxed{\overrightarrow{E}=-\overrightarrow{\nabla}\,V,}$$

where $\overrightarrow{\nabla}$ is the gradient operator.

The electrostatic field component in a given direction $(Ox, Oy$ or $Oz)$ is defined as the change in electrostatic potential per unit length in the same direction, with a change in sign.

Example

Given the scalar function of the electrostatic potential, $V\,(x,\,y)=x^2y-5\,x$, and in consideration of the relationship $\overrightarrow{E}=-\overrightarrow{\mathrm{grad}}\,V$, the components of the electrostatic field are given by:

$$\begin{cases}E_x=-\dfrac{\partial V}{\partial x}=-2\,x\,y+5\\E_y=-\dfrac{\partial V}{\partial y}=-x^2.\end{cases}$$

The electrostatic field vector is therefore:

$$\boxed{\overrightarrow{E}=(5-2\,x\,y)\,\vec{\imath}-x^2\vec{\jmath}.}$$

Supplement: *Relationship between $\vec{E}$ and V in polar coordinates*

In polar coordinates, the electrostatic potential is expressed as a function of r and θ, that is, $V(r, \theta)$. In polar coordinates, the electrostatic field is written as :

$$\vec{E} = E_r\,\vec{u}_r + E_\theta\,\vec{u}_\theta.$$

The elementary displacement vector is subsequently expressed in the following form:

$$\vec{\mathrm{d}\ell} = \mathrm{d}\overrightarrow{OM} = \mathrm{d}\,(r\,\vec{u}_r) = \mathrm{d}r\,\vec{u}_r + r\,\mathrm{d}\vec{u}_r.$$

Since :

$$\frac{\mathrm{d}\vec{u}_r}{\mathrm{d}r} = \frac{\mathrm{d}\vec{u}_r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}r} \quad \Longrightarrow \quad \mathrm{d}\vec{u}_r = \vec{u}_\theta \mathrm{d}\theta,$$

then we can write:

$$\vec{\mathrm{d}\ell} = \mathrm{d}r\,\vec{u}_r + r\,\mathrm{d}\theta\,\vec{u}_\theta.$$

The differential electrostatic potential is:

$$\mathrm{d}V = \frac{\partial V}{\partial r}\,\mathrm{d}r + \frac{\partial V}{\partial \theta}\,\mathrm{d}\theta,$$

so that:

$$\mathrm{d}V = -\vec{E}\cdot\vec{\mathrm{d}\ell} \quad \Longrightarrow \quad \frac{\partial V}{\partial r}\,\mathrm{d}r + \frac{\partial V}{\partial \theta}\,\mathrm{d}\theta = -E_r\,\mathrm{d}r - E_\theta\,r\,\mathrm{d}\theta.$$

The radial and angular components of the electrostatic field can then be expressed as follows:

$$E_r = -\frac{\partial V}{\partial r}$$
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta},$$

and the electrostatic field vector is:

$$\boxed{\vec{E} = -\overrightarrow{\mathrm{grad}}\,V = -\frac{\partial V}{\partial r}\,\vec{u}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\,\vec{u}_\theta}$$

Summary

- ***The variation $\triangle V$ of the scalar electrostatic potential between two positions, designated A and B, in space is equal to the circulation of the electrostatic field vector along the curve (AB) connecting these two positions.***

- ***The magnitude of the electrostatic field is equal to the gradient of the scalar electrostatic potential, with the sign changed.***

1.6 Equipotential surfaces

In light of the interrelationship between the electrostatic field and the electrostatic potential, it's feasible to construct diagrams of electrostatic potentials in a manner analogous to the construction of diagrams of electrostatic fields. In the case of an isolated point charge, the electrostatic field lines are depicted as originating from the charge in the positive case or terminating at the charge in the negative case. Regions of constant electrostatic potential are represented by lines known as ***equipotential surfaces*** in three dimensions, or ***equipotential lines*** in two dimensions, as illustrated in the figure below.

Definition

An ***equipotential surface*** or ***equipotential*** is defined as a set of points that have the same electrostatic potential. It can thus be concluded that $V = C^{te}$ and $\mathrm{d}V = 0$ on an equipotential surface.
It can therefore be stated that:

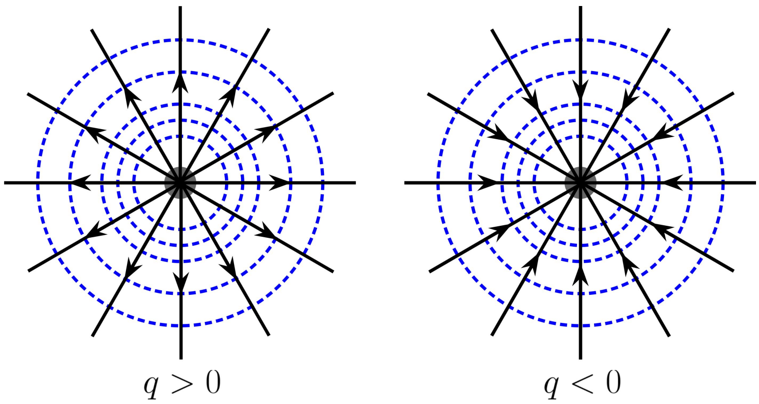
- In the case of a point charge: $V(r) = V_0 \implies r = \mathrm{C^{ste}}$. An equipotential surface may be conceptualised as a sphere with the charge situated at its centre.
- Equipotential surfaces are always perpendicular to electrostatic field lines :

$$\mathrm{d}V = 0 \implies \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}\ell} = 0 \implies \boxed{\overrightarrow{E} \perp \overrightarrow{\mathrm{d}\ell}.}$$

- The orientation of the electrostatic field lines is towards regions of decreasing electrostatic potential:

$$\mathrm{d}V < 0 \implies \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}\ell} > 0$$

- Two equipotential surfaces that correspond to different potentials cannot cross each other.
- In the case of an infinite straight wire with a uniform charge, $V(r) = V_0 \implies r = \mathrm{C^{ste}}$. This implies that the equipotential surfaces are cylinders with the charged wire as their axis.



2 Electrostatic potential energy

Let us consider a region of space with an electrostatic field $\overrightarrow{E}$. At a point M on a curve (AB), a charge q is placed within this region. The elementary circulation of the electric field during an elementary displacement $\overrightarrow{d\ell}$ is given by :

$$\overrightarrow{E} \cdot \overrightarrow{d\ell} = -dV.$$

The charge q is subject to an electrostatic force, $\overrightarrow{F} = q\,\overrightarrow{E}$, so that :

$$q\,\overrightarrow{E} \cdot \overrightarrow{d\ell} = -q\,dV \quad \Longrightarrow \quad \overrightarrow{F} \cdot \overrightarrow{d\ell} = -q\,dV.$$

The quantity $\overrightarrow{F} \cdot \overrightarrow{d\ell}$ represents the elementary work done by the electrostatic force $\overrightarrow{F}$ during the elementary displacement $\overrightarrow{d\ell}$. The total work on the path (AB) is therefore:

$$W_{A\longrightarrow B} = \int_A^B \overrightarrow{F} \cdot \overrightarrow{d\ell} = -q\int_{V_A}^{V_B} dV = q\,V_A - q\,V_B = E_p\,(A) - E_p\,(B)\,,$$

where the scalar function E_p is the ***electrostatic potential energy***.

Definition

- The electrostatic potential energy of an electric charge q subjected to an electrostatic potential V is given by the following equation :

$$\boxed{E_p = q\,V.} \tag{3.5}$$

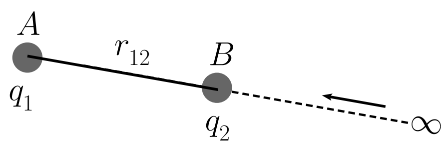
- The work of an electrostatic force $\overrightarrow{F}$ acting on an electric charge q along a path (AB) is expressed as follows:

$$\boxed{W_{A\longrightarrow B} = q\,V_A - q\,V_B = E_p\,(A) - E_p\,(B)\,.} \tag{3.6}$$

In light of the fact that the work done by the electrostatic force is independent of the path taken, the electrostatic force may be classified as a conservative force.

2.1 Interaction energy of two electric point charges

The interaction energy of a system comprising two electric charges q_1 and q_2, represents the work required to bring these two initially unrelated charges into proximity. In other words, it is the potential energy of the charge q_2 situated within the field of the charge q_1 (or conversely).



Let us consider a charge q_1 situated at point A, which creates an electrostatic potential V_1 at point B at a distance r_{12}. A second charge q_2, initially at infinity, is brought back to point B by the action of a force $\overrightarrow{f} = -\overrightarrow{F}$ (where $\overrightarrow{F}$ is the electrostatic force experienced by the charge q_2) to form a system of two charges. The interaction energy of the system is the work done by the force $\overrightarrow{f}$ from infinity to B:

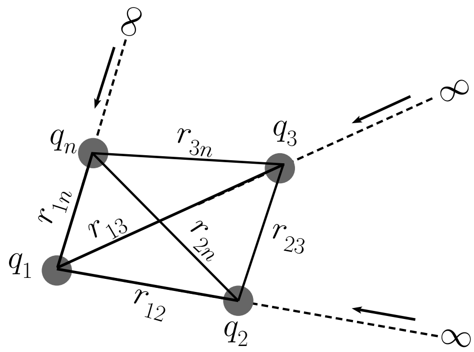
$$U = W_{\infty\longrightarrow B} = \int_\infty^B \overrightarrow{f} \cdot \overrightarrow{d\ell} = -\int_\infty^B \overrightarrow{F} \cdot \overrightarrow{d\ell} = q_2\int_{V_\infty}^{V_1} dV = q_2\,V_1 - q_2\,V_\infty.$$

given that we have established that the electrostatic potential is zero at infinity ($V_\infty = 0$), it follows that the interaction energy of the system of two charges formed in this way is equal to the potential energy of the charge q_2 in the field of q_1:

$$\boxed{U = E_{p_2} = q_2\,V_1 = q_2\left(K\,\frac{q_1}{r_{12}}\right) = K\,\frac{q_1\,q_2}{r_{12}}.} \tag{3.7}$$

2.2 Interaction energy of n electric point charges

The result for a system of two electric charges can be extended to a system of any number of electric charges. The interaction energy of a system of n electric charges q_1, q_2, , q_n can be expressed as the sum of the potential energies of each electric charge, initially situated infinite distance, brought successively into the fields of the charge already present:



$$\begin{aligned} U &= K\left[\frac{q_1\,q_2}{r_{12}} + \frac{q_1\,q_3}{r_{13}} + \frac{q_2\,q_3}{r_{23}} + \ldots + \frac{q_1\,q_n}{r_{1n}} + \frac{q_2\,q_n}{r_{2n}} + \frac{q_3\,q_n}{r_{3n}} + \frac{q_{n-1}\,q_n}{r_{n-1,n}}\right] \\ &= K\left[\frac{q_1\,q_2}{r_{12}} + \frac{q_1\,q_3}{r_{13}} + \ldots\frac{q_1\,q_n}{r_{1n}} + \frac{q_2\,q_3}{r_{23}} + \ldots\frac{q_2\,q_n}{r_{2n}} + \ldots\right]. \end{aligned}$$

In general:

$$\boxed{\begin{aligned} U &= \frac{1}{2}\sum_{\imath}\sum_{\jmath} K\,\frac{q_\imath\,q_\jmath}{r_{\imath\jmath}} \qquad (\imath \neq \jmath) \\ &= \frac{1}{2}\sum_{\imath} q_i\,V_i \\ &= \frac{1}{2}\sum_{\imath} E_{p_i}. \end{aligned}} \tag{3.8}$$

Example

In this example, we will calculate the interaction energy of a system of three electric point charges, $q_1 = q_2 = 60\,\mu$C and $q_3 = 120\,\mu$C, placed at the vertices of an equilateral triangle with side $a = 9\,$cm(see figure below).

$$U = \frac{1}{2}\sum_{\imath}\sum_{\jmath}\frac{Kq_\imath q_\jmath}{r_{\imath\jmath}} \qquad (\imath \neq \jmath)\,.$$

There are 3 point charges, so that : $i = 1, 2, 3$ and $j = 1, 2, 3$.
For $i = 1$, then $j = 2, 3$, we have:

$$U_1 = \frac{1}{2}\left(\frac{Kq_1q_2}{r_{12}} + \frac{Kq_1q_3}{r_{13}}\right) = \frac{1}{2}\left(\frac{Kq_1q_2}{a} + \frac{Kq_1q_3}{a}\right),$$

for $i = 2$, then $j = 1, 3$, we have:

$$U_2 = \frac{1}{2}\left(\frac{Kq_2q_1}{r_{21}} + \frac{Kq_2q_3}{r_{23}}\right) = \frac{1}{2}\left(\frac{Kq_2q_1}{a} + \frac{Kq_2q_3}{a}\right),$$

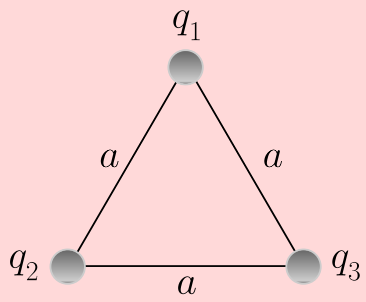
and for $i = 3$, then $j = 1, 2$, we have:

$$U_3 = \frac{1}{2}\left(\frac{Kq_3q_1}{r_{31}} + \frac{Kq_3q_2}{r_{32}}\right) = \frac{1}{2}\left(\frac{Kq_3q_1}{a} + \frac{Kq_3q_2}{a}\right).$$

The net interaction energy is therefore:

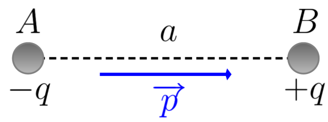
$$U = \frac{1}{2}\left(\frac{Kq_1q_2}{a} + \frac{Kq_1q_3}{a}\right) + \frac{1}{2}\left(\frac{Kq_2q_1}{a} + \frac{Kq_2q_3}{a}\right) + \frac{1}{2}\left(\frac{Kq_3q_1}{a} + \frac{Kq_3q_2}{a}\right)$$

$$U = \frac{Kq_1q_2}{a} + \frac{Kq_1q_3}{a} + \frac{Kq_2q_3}{a} \implies \boxed{U = \frac{K}{a}(q_1q_2 + q_1q_3 + q_2q_3)} \implies \textbf{A.N.} \boxed{U = 1800\,\text{J.}}$$



3 Application: electrostatic dipole

An intriguing configuration of electric charges is that which forms an ***electric dipole***. An electric dipole can be defined as a system comprising two equal but opposite charges, $-q$ and $+q$, separated by a a fixed distance a, which is very small.



Definition

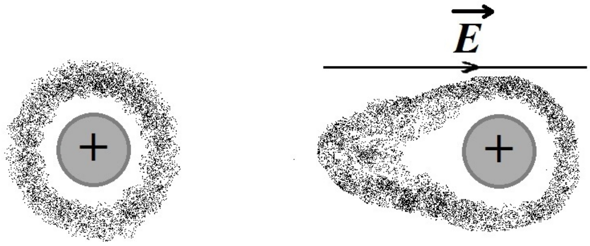
An electric dipole is defined as a vector quantity called the ***electric dipole moment***, which is expressed as follows:

$$\boxed{\vec{p} = q\,\overrightarrow{AB} = q\,\vec{a}.} \tag{3.9}$$

- The electric dipole moment is a vector pointing from the negative to the positive charge.
- Its unit in SI units is the coulomb-meter ($\text{C} \cdot \text{m}$).
- The Debye (D) is another unit of measurement used in atomic physics and chemistry: $1\,\text{D} = \frac{1}{3} \cdot 10^{-29}\,\text{C} \cdot \text{m}$.

This specific configuration is observed at both the atomic and molecular levels.

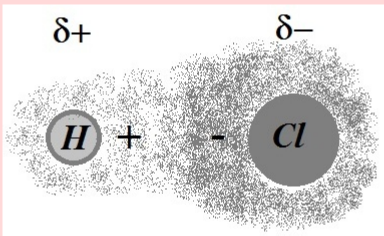
- In an atom, the centres of mass (barycentres) of the negative charges (the cloud of electrons) and the positive charges (the nucleus) coincide. The average dipole moment of the atom is equal to zero. When an external electric field is applied, the electron cloud is driven to move in the opposite direction to the electric field, resulting in the formation of an *induced electric dipole* with a dipole moment proportional to the electric field. This induced dipole is observed to disappear when the external electric field is removed.



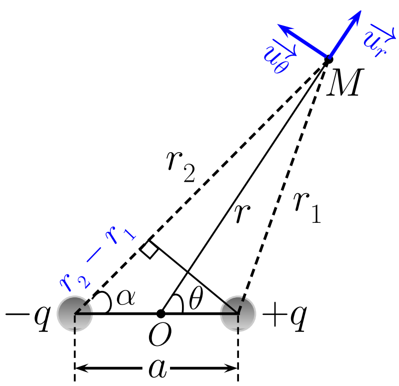
- In some molecules, the centre of mass of the positive charges does not coincide with that of the negative charges in the absence of an external electric field. These molecules exhibit the behaviour of *permanent electric dipoles* and are therefore classified as *polar molecules* (examples include HCl and H_2O). Molecules that do not possess a permanent electric dipole are designated as *non-polar*.

Example

In the hydrogen chloride (HCl) molecule, the electronegativity of the chlorine atom is greater than that of the hydrogen atom. The electron doublet in the covalent bond exhibits a greater degree of attraction towards chlorine than towards hydrogen. The chlorine atom exhibits a slight excess of negative charge (denoted as $\delta-$), which is less than a full negative charge as observed in the Cl^- ion. Additionally, a slight excess of positive charge (denoted $\delta+$) is present, which is less than a full positive charge as observed in the H^+ ion. The arrangement of charges, which are separated from each other by a very small distance, gives rise to the formation of an electric dipole. The molecule exhibits a dipole moment $\overrightarrow{p}$ with a magnitude $p \approx 3.4 \times 10^{-30}\,\mathrm{C\cdot m} \approx 1.08\,\mathrm{D}$.



3.1 Electrostatic potential of a dipole



Let us consider a point M situated at a distance r from the centre O of a dipole with a dipole moment $\overrightarrow{p} = q\,\overrightarrow{a}$. It is also assumed that M is situated at a considerable distance from $O : r \gg a$. The electrostatic potential generated in M by the two charges is:

$$V = K\left(\frac{q}{r_1} - \frac{q}{r_2}\right) = K\,\frac{q\,(r_2 - r_1)}{r_1 \cdot r_2}$$

Since M is situated at a considerable distance from O, we can assume that : $r_1 \gg a$ et $r_2 \gg a$. In this case, we can write:

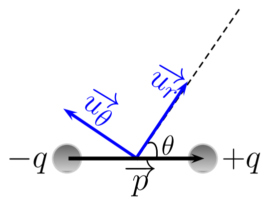
$$r_1 \simeq r_2 \simeq r \quad \Longrightarrow \quad r_1 \cdot r_2 \simeq r^2$$
$$r_2 - r_1 = a\,\cos\alpha \simeq a\,\cos\theta \qquad (\alpha \simeq \theta)$$

θ is the angle between $\vec{p}$ and $\overrightarrow{OM}$.

The electric potential of the dipole is therefore expressed as follows :

$$\boxed{V \simeq K\,\frac{q\,a\,\cos\theta}{r^2} = K\,\frac{p\,\cos\theta}{r^2}.} \tag{3.10}$$

The electric dipole moment in the polar base $(\vec{u}_r,\ \vec{u}_\theta)$ is given by the following equation:



$$\vec{p} = p\,\cos\theta\,\vec{u}_r - p\,\sin\theta\,\vec{u}_\theta.$$

Then, we can write :

$$\vec{p}\cdot\vec{u}_r = p\,\cos\theta.$$

Thus, the electric potential of the dipole can be expressed as follows:

$$\boxed{V = K\,\frac{\vec{p}\cdot\vec{u}_r}{r^2}.} \tag{3.11}$$

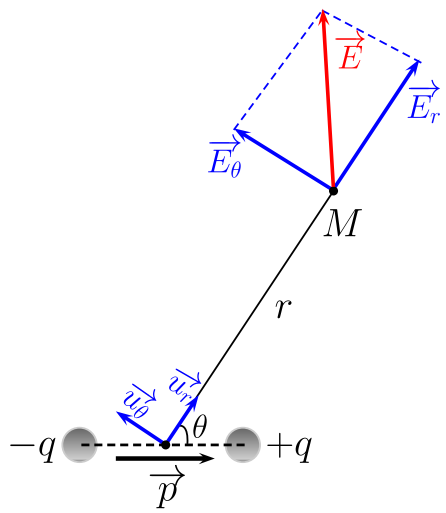
3.2 Electric field of a dipole

The electric field generated by an electrostatic dipole at a given point M can be obtained by means of the following relationship, derived from the expression for the electric potential generated by the dipole at the same point:

$$\vec{E} = -\overrightarrow{\text{grad}}\,V.$$

The electric field in the polar base $(\vec{u}_r,\ \vec{u}_\theta)$ is given by :

$$\vec{E} = E_r\,\vec{u}_r + E_\theta\,\vec{u}_\theta,$$



with :

$$\boxed{\begin{cases} E_r = -\dfrac{\partial V}{\partial r} = \dfrac{2\,K\,p\,\cos\theta}{r^3} \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta} = \dfrac{K\,p\,\sin\theta}{r^3}. \end{cases}} \tag{3.12}$$

The magnitude of the electric field at M is therefore :

$$\boxed{E = \sqrt{E_r^2 + E_\theta^2} = \frac{K\,p}{r^3}\left(1 + 3\,\cos^2\theta\right)^{1/2}.}$$

Supplement: *Intrinsic field vector expression*

in light of the preceding outcome, the field vector generated by an electric dipole at a point M, situated at a distance r from the centre of the dipole, can be expressed as follows:

$$\overrightarrow{E} = \frac{2\,K\,p\,\cos\theta}{r^3}\,\vec{u}_r + \frac{K\,p\,\sin\theta}{r^3}\,\vec{u}_\theta,$$

with: $\overrightarrow{p} = p\,\cos\theta\,\vec{u}_r - p\,\sin\theta\,\vec{u}_\theta$.
Then :

$$p\,\sin\theta\,\vec{u}_\theta = p\,\cos\theta\,\vec{u}_r - \overrightarrow{p},$$

and :

$$\overrightarrow{E} = \frac{K}{r^3}\left[3\,(p\,\cos\theta)\,\vec{u}_r - \overrightarrow{p}\right].$$

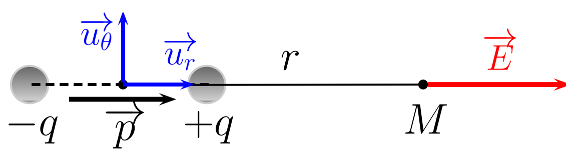
Knowing that $p\,\cos\theta = \overrightarrow{p}\cdot\vec{u}_r$, the final field vector expression is :

$$\boxed{\overrightarrow{E} = \frac{K}{r^3}\left[3\,(\overrightarrow{p}\cdot\vec{u}_r)\,\vec{u}_r - \overrightarrow{p}\right].}$$

The variation of the electrostatic potential in terms of $\frac{1}{r^2}$ and the electric field in terms of $\frac{1}{r^3}$ characterises the dipolar structure and conditions the interactions between dipoles and matter.

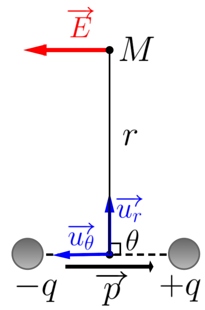
- In the particular case of $\theta = 0$ (M on the axis of the dipole on the positive pole side), the electric field is given by :

$$\overrightarrow{E} = \overrightarrow{E}_r = \frac{2\,K\,p\,\cos\theta}{r^3}\,\vec{u}_r = \frac{2\,K\,(\overrightarrow{p}\cdot\vec{u}_r)}{r^3}\,\vec{u}_r = \frac{2\,K\,\overrightarrow{p}}{r^3}.$$

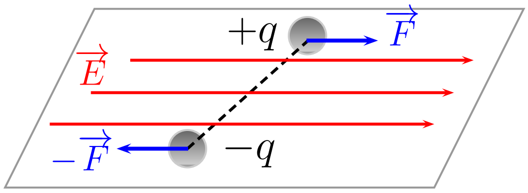


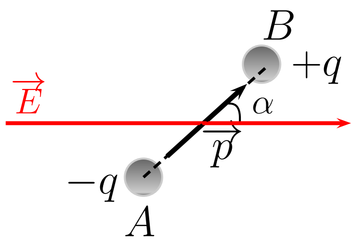
- In the particular case of $\theta = \frac{\pi}{2}$ (M on the perpendicular bisector of the dipole), the electric field is given by :

$$\overrightarrow{E} = \overrightarrow{E}_\theta = \frac{K\,p\,\sin\theta}{r^3}\,\vec{u}_\theta = \frac{K}{r^3}\,(p\,\cos\theta\,\vec{u}_r - \overrightarrow{p}) = -\frac{K\,\overrightarrow{p}}{r^3}.$$



3.3 Rotation of the dipole caused by an electric field





If an electric dipole with a dipole moment $\overrightarrow{p}$ is placed in an external uniform electric field $\overrightarrow{E}$, then the two electric charges that comprise the dipole will experience an electric force. The positive charge will experience an electrostatic force, $\overrightarrow{F} = -q\,\overrightarrow{E}$, and the negative charge will experience an electrostatic force, $-\overrightarrow{F} = -q\,\overrightarrow{E}$. The net force on the dipole is zero because the forces on the two charges are equal and opposite. However, there is a torque, represented by the vector $\overrightarrow{\Gamma}$, given by:

$$\overrightarrow{\Gamma} = \overrightarrow{a} \wedge \overrightarrow{F} = q\,\overrightarrow{a} \wedge \overrightarrow{E} = \overrightarrow{p} \wedge \overrightarrow{E}$$
$$\left\| \overrightarrow{\Gamma} \right\| = p \cdot E \cdot \sin\alpha, \qquad \alpha = \left(\overrightarrow{p}, \overrightarrow{E}\right).$$

The torque exerted by the external electric field has the effect of rotating the dipole so that its direction is parallel to that of the external field.

3.4 Potential energy of a dipole in an external electric field

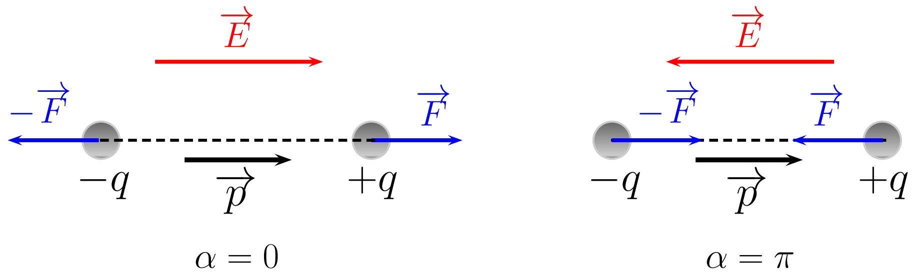
If a dipole with a dipole moment of $\overrightarrow{p}$ is placed in a uniform external electric field, then each of the charges in the dipole has a potential energy. As we have seen, the potential energy of the dipole is expressed as follows:

$$\begin{aligned} E_p &= q\,V_B - q\,V_A = q\,(V_B - V_A) = q\,[\Delta V]_A^B \\ &= q\left(-\overrightarrow{E} \cdot \overrightarrow{AB}\right) = -q\,\overrightarrow{E} \cdot \overrightarrow{a}. \end{aligned}$$

Therefore, the potential energy of a dipole in a uniform external electric field $\overrightarrow{E}$ is expressed as follows:

$$\boxed{E_p = -\overrightarrow{E} \cdot \overrightarrow{p} = -E \cdot p \cdot \cos\alpha.}$$

- For $\cos\alpha = 1$ ($\alpha = 0$), the potential energy is minimal. The dipole is in a stable state of equilibrium and oriented in the same direction as the external electric field. If the dipole is moved slightly away from its position of stable equilibrium, it will experience a pair of forces that will bring it back to its original position.
- For $\cos\alpha = -1(\alpha = \pi)$ the potential energy is maximal. The dipole is in an unstable equilibrium and is oriented in the opposite direction to the external electric field. If the dipole is moved slightly away from its position of unstable equilibrium, it will tilt towards the stable equilibrium state ($\alpha = 0$).



Chapter 4

Gauss Theorem

Introduction

This chapter presents a demonstration and application of a theorem of great importance in the theory of electromagnetism. The theorem was formulated in the 19th century by the German mathematician and astronomer Friedrich Gauss (1777-1855). This theorem, which bears his name, provides a highly useful and straightforward method for calculating the electric field $\overrightarrow{E}$ produced by specific distributions of charges with a high degree of symmetry (for instance, an infinitely charged wire, a sphere charged in volume or surface, and so forth).

As previously discussed, a qualitative representation of the electrostatic field can be constructed by depicting electric field lines without specifying the precise number of lines, with the field magnitude $\left|\overrightarrow{E}\right|$ proportional to the field line density. This latter quantity is defined as the number of lines N_{fl} per unit area. To illustrate, consider the case of a single charge generating the electric field given by:

$$\left|\overrightarrow{E}\right| = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}.$$

If the single charge is circumscribed by a sphere of radius r and area $4\pi r^2$, the field line density is given by:

$$\frac{N_{fl}}{4\pi r^2} \propto \left|\overrightarrow{E}\right|,$$

so that :

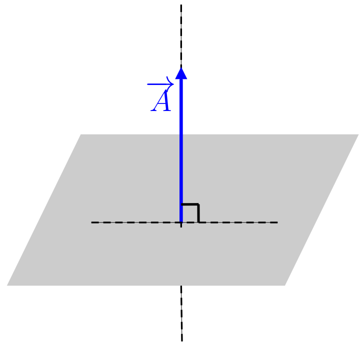
$$N_{fl} \propto \left|\overrightarrow{E}\right| 4\pi r^2 = \frac{q}{\varepsilon_0}.$$

The aforementioned sphere is to be referred to as the *Gaussian sphere* and the quantity represented by the expression $\frac{q}{\varepsilon_0}$ is to be understood as the *electric field flux.*

it should be noted that the number of electric field lines emerging from the single charge and crossing the Gaussian sphere is independent of the sphere's size, provided that the charge remains inside. With this simple situation in mind, we are now prepared to introduce and prove the Gauss theorem, after presenting some necessary definitions.

1 Electric field flux

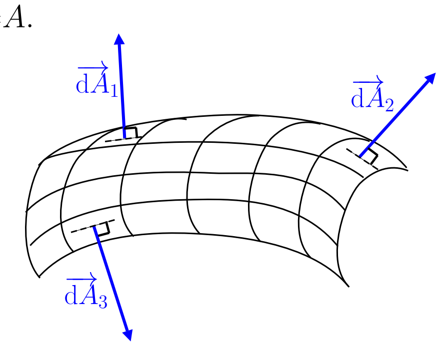
1.1 Area vector



Prior to defining the flux of a vector field, it is beneficial to introduce the area vector, which will facilitate the expression of the vector flux in a more concise manner.

in the case of a flat surface of area A, the area vector $\overrightarrow{A}$ is defined as follows:

- Its direction is aligned with the normal to the surface, that is, $\overrightarrow{A}$ is perpendicular to the surface.
- Its magnitude is equal to the area, that is, $\left\|\overrightarrow{A}\right\| = A$.



If the event that the surface in question is of an arbitrarily shape, it is divided into a multitude of tiny surfaces that are approximately flat, with areas designated as $\mathrm{d}A_i$. Each of theses elements is represented by a vector $\mathrm{d}\overrightarrow{A}_i$, such that:

- $\mathrm{d}\overrightarrow{A}_i$ is oriented perpendicular to the surface of the element.
- Its magnitude is $\left\|\mathrm{d}\overrightarrow{A}_i\right\| = \mathrm{d}A_i$.

Remarks

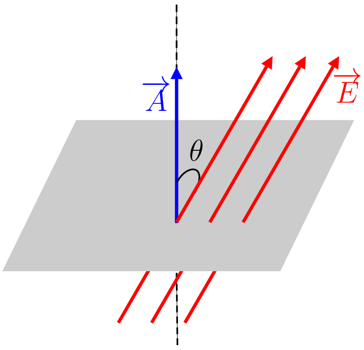
- The normal to a flat surface can point in either direction from the surface. Therefore, the direction of the area vector $\overrightarrow{A}$ of an open surface must be selected.
- In the cas of a surface that is closed, that is, a surface that encloses a volume, the direction of the normal area vector $\overrightarrow{A}$ or $\mathrm{d}\overrightarrow{A}$ at any point on the surface is chosen to point from the inside to the outside.

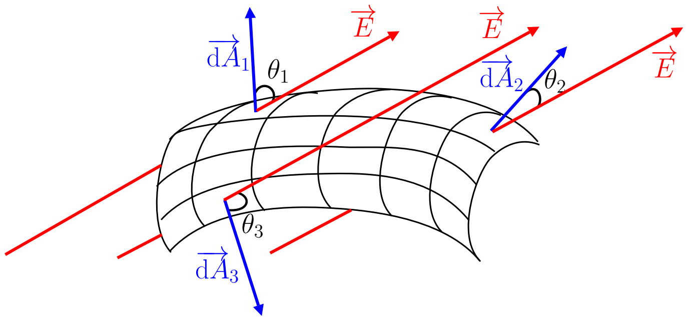
1.2 Electric field flux

Having defined the area vector of a surface, we may proceed to define the electric flux of a uniform electric field through a flat surface.

Let us consider a uniform electric field $\vec{E}$ and a flat surface of area A represented by an outgoing vector $\vec{A}$. The ***flux of the electric field*** $\vec{E}$ through this surface is defined as the scalar product of the electric field and area vectors :

$$\boxed{\Phi = \vec{E} \cdot \vec{A} = E \cdot A \cdot \cos\theta, \qquad \theta = \left(\vec{E}, \vec{A}\right).} \tag{4.1}$$





In the event that the surface is not flat, the elementary flux of the electric field $\vec{E}$ over the surface element $\mathrm{d}A$ is defined as follows :

$$\mathrm{d}\Phi = \vec{E} \cdot \vec{\mathrm{d}A}.$$

The net flux of the electric field $\vec{E}$ across the area A is given by:

$$\boxed{\Phi = \iint_S \vec{E} \cdot \vec{\mathrm{d}A} = \iint_S E \cdot \mathrm{d}A \cdot \cos\theta.} \tag{4.2}$$

In the case of a closed surface S, the net flux of the electric field $\vec{E}$ across this surface can be expressed as follows :

$$\Phi = \oiint_S \vec{E} \cdot \mathrm{d}\vec{A}.$$

- The units of the electric field flux in the SI units are newton-metres squared per coulomb $(\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{C}^{-1})$ or volt-meters $(\mathrm{V} \cdot \mathrm{m})$
- The electric field flux through a surface is proportional to the number of field lines passing through it.
- The electric flux is positive if the field lines are in the same direction as $\vec{A}$.

Example

In this example, we will calculate the flux of a uniform electric field, oriented in the positive direction of the (ox) axis of a Cartesian frame of reference, across the surfaces of a cube with an edge length of ℓ, whose axes correspond to the three axes of the Cartesian frame of reference.

The cube is characterised by 6 flat faces, each of which can be represented by an elementary area vectors $\overrightarrow{\mathrm{d}A_i}$. The electric field, which is perpendicular to faces (1) and (2), is therefore parallel to the vectors $\overrightarrow{\mathrm{d}A_1}$ and $\overrightarrow{\mathrm{d}A_2}$. Conversely, the electric field $\vec{E}$ is parallel to the remaining faces and therefore perpendicular to the vectors$\overrightarrow{\mathrm{d}A_3}$, $\overrightarrow{\mathrm{d}A_4}$, $\overrightarrow{\mathrm{d}A_5}$ and $\overrightarrow{\mathrm{d}A_6}$. consequently, the electric flux through these lateral surfaces is zero ($\Phi_{\text{lateral}} = 0$).

With regard to the face (1), the electric field exhibits a direction opposite to that of $\overrightarrow{\mathrm{d}A_1}$. Consequently, we may conclude that:

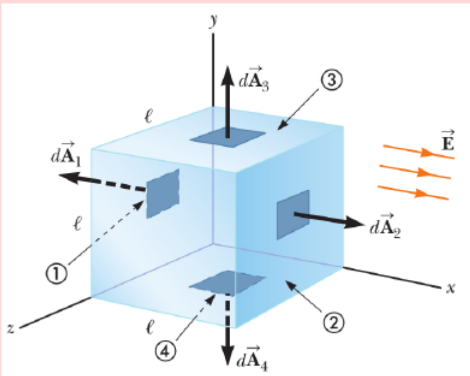
$$\Phi_1 = \iint \vec{E} \cdot \overrightarrow{\mathrm{d}A_1} = \iint E \cdot \mathrm{d}A_1 \cdot \cos\pi = -E \iint \mathrm{d}A_1 = -E \cdot A_1 = -E \cdot \ell^2.$$

With regard to the face (2), the direction of the electric field is identical to that of $\overrightarrow{\mathrm{d}A_2}$. Therefore, we may write that:

$$\Phi_2 = \iint \vec{E} \cdot \overrightarrow{\mathrm{d}A_2} = \iint E \cdot \mathrm{d}A_2 \cdot \cos 0 = E \iint \mathrm{d}A_2 = E \cdot A_2 = E \cdot \ell^2.$$

The net electric flux through the entire surface area of the cube can thus be expressed as follows:

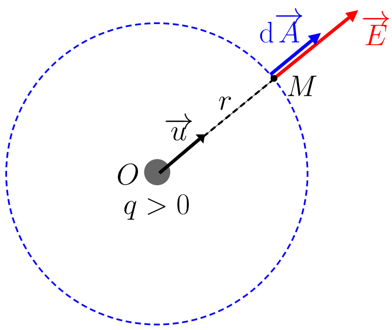
$$\Phi_{\text{net}} = \Phi_1 + \Phi_2 + \Phi_{\text{lateral}} = -E.\ell^2 + E.\ell^2 + 0 \quad \Longrightarrow \quad \boxed{\Phi = 0\,\mathrm{V \cdot m}.}$$



From this example we can derive a general rule, which is that:

If a closed surface has no charges within the enclosed volume, then the net electric flux through the surface is zero.

2 Gauss Theorem



in order to illustrate Gauss theorem, we will consider the example of a point charge $q > 0$ situated at a point O in space. The electric field generated by this charge at a point M situated at a distance r from O is given by:

$$\vec{E} = K\,\frac{q}{r^2}\,\vec{u}, \qquad \vec{u} = \frac{\overrightarrow{OM}}{r}.$$

Consider an imaginary sphere $(\mathscr{S})$ with radius r and centre O. At every point on the surface of this sphere, the magnitude of the electric field is constant and equal to :

$$E = K\,\frac{q}{r^2}.$$

At any point on the surface, the vector $\overrightarrow{\mathrm{d}A}$ is perpendicular to the sphere. As the surface is closed, the vector is an outgoing vector and is therefore:

$$\overrightarrow{\mathrm{d}A} \parallel \overrightarrow{u} \qquad \Longrightarrow \qquad \overrightarrow{\mathrm{d}A} \parallel \overrightarrow{E}.$$

The total electric field flux Φ across the surface of the sphere $(\mathscr{S})$ is therefore given by :

$$\Phi = \oiint \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}A} = \oiint E\cdot\mathrm{d}A\cdot\cos 0 = E\oiint \mathrm{d}A = E\cdot A.$$

The area of $(\mathscr{S})$ is $A = 4\pi\,r^2$. Consequently :

$$\Phi = E\,4\pi\,r^2 \qquad \Longrightarrow \qquad \Phi = K\,\frac{q}{r^2}\,4\pi\,r^2 = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2}\,4\pi\,r^2.$$

Finally, we get the electric field flux as:

$$\Phi = \frac{q}{\varepsilon_0}.$$

The aforementioned result can be readily extended to any set of charges through the application of the principle of superposition.

Gauss Theorem

The total electric field flux through a closed surface in a vacuum is equal to the ratio of the total electric charge Q_{in} contained in that surface to the permittivity of vacuum ε_0.

$$\boxed{\Phi = \oiint \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}A} = \frac{Q_{in}}{\varepsilon_0}.} \tag{4.3}$$

The shape of the surface is inconsequential, as is any external charge.

Remarks

- In the field of physics, Gauss Theorem serves to establish a connection between the flux of electric field and the sources of this field, namely, electric charges.
- The preceding demonstration employed Coulomb's law, which is experimentally verified phenomenon. Conversely, Coulomb's law can be derived from Gauss theorem. This is a process that occurs in the context of electromagnetism, where Gauss theorem is, in fact, a fundamental law that cannot be proven. It is one of the four Maxwell equations.

How to apply Gauss theorem for Calculating electric field

Gauss's theorem offers a highly effective method for calculating the magnitude of an electric field when it exhibits a special symmetry properties. These properties facilitate the calculation of the flux. Given that the Gauss theorem is applicable to any surface, it is sufficient to identify a suitable closed surface that respects the electric field's symmetry properties.

To calculate the electric field generated by a continuous distribution of charges at point M in space, the following steps are required:

1. A suitable closed surface $(\mathscr{S})$, designated as a Gaussian surface and consistent with the requisite symmetry, must be selected:

   a) This surface passes through the point M,

   b) the magnitude of the electric field is the same at every point on $(\mathscr{S})$,

   c) the electric field at any point is perpendicular or parallel to $(\mathscr{S})$, i.e. $\vec{E} \parallel \overrightarrow{\mathrm{d}A}$ or $\vec{E} \perp \overrightarrow{\mathrm{d}A}$ respectively.

2. Calculate the net flux Φ over the entire surface $(\mathscr{S})$.

3. Write the Gauss theorem : The flux Φ est equal to Q_{in}/ε_0, where Q_{in} is the sum of all charges inside the surface $(\mathscr{S})$.

3 Application examples

Gauss theorem has two principal applications. The first is to determine the field of a distribution of charges that is symmetric and, in general, continuous. The second is to identify the charge when the field is known, as in the case of a conductor. This section will focus on the first application, while the second will discussed in greater details in the following chapter. Two symmetric charge distributions will be introduced, and other types of symmetry will be discussed in greater depth in the susequent exercises.

3.1 Electric field created by a uniformly charged sphere

Let us consider a sphere with centre O and radius R, carrying a positive charge Q uniformly distributed throughout its volume with a volume charge density $\rho > 0$. We shall calculate the magnitude of the electric field generated by this sphere at any point M in space, r away from its centre O.

1. Given the spherical symmetry of the distribution in question, the resulting electrostatic field will also exhibits this symmetry and will be radial in nature: $\vec{E} = E(r)\,\vec{u}_r$.

a) in consideration of this symmetry, the sphere with a radius r and a centre O is selected as the adapted Gaussian surface.

b) It can be asserted that at any point on the Gaussian surface, where r is held constant, $E(r)$ is also constant.

c) At any point on the Gaussian surface, the outgoing area vector $\mathrm{d}\overrightarrow{A}$ is radial. Consequently, The electric field is parallel to $\mathrm{d}\overrightarrow{A}$: $\mathrm{d}\overrightarrow{A} \parallel \overrightarrow{u}_r$ and $\mathrm{d}\overrightarrow{A} \parallel \overrightarrow{E}$.

2. The Gauss theorem is employed to express the electric field flux as follows:

$$\Phi = \oiint \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} = \frac{Q_{in}}{\varepsilon_0},$$

with:

$$\oiint \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} = \oiint E(r)\ \overrightarrow{u}_r \cdot \mathrm{d}\overrightarrow{A} = \oiint E(r)\ \mathrm{d}S,$$

and:

$$Q_{in} = \iiint \rho\, \mathrm{d}V.$$

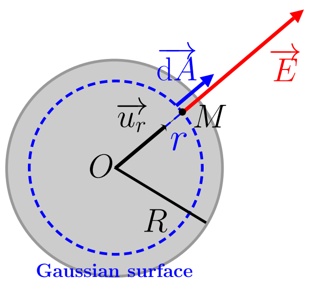
The constant value of $E(r)$ at any point on the Gaussian surface gives rise to the following result:

$$\oiint \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} = E(r) \oiint \mathrm{d}A = E(r)\ A_{\text{Gauss}} = E(r)\ 4\pi\, r^2.$$

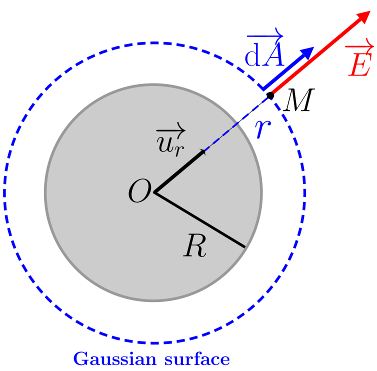
3. by applying Gauss theorem, we can express the following :

$$\oiint \overrightarrow{E} \cdot \mathrm{d}\overrightarrow{A} = \frac{Q_{in}}{\varepsilon_0} \quad \Longrightarrow \quad E(r)\ 4\pi\, r^2 = \frac{Q_{in}}{\varepsilon_0} \quad \Longrightarrow \quad E(r) = \frac{Q_{in}}{4\pi\varepsilon_0\, r^2}.$$

4. The calculation of the electric field is contingent upon whether it is being performed within the interior of the sphere ($r < R$) or outside the sphere ($r > R$).



$r < R$



$r > R$

- In the case where $r < R$, the Gaussian surface is located within the sphere. The charge inside the Gaussian sphere can then be expressed as follows:

$$Q_{in} = \iiint \rho\, \mathrm{d}V = \rho \iiint \mathrm{d}V = \rho\, V_{\text{Gauss sphere}} = \rho\, \frac{4\pi\, r^3}{3}.$$

therefore, the electric field can be expressed as :

$$E(r) = \frac{Q_{in}}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0 r^2}\left(\rho\frac{4\pi r^3}{3}\right) \implies \boxed{E(r) = \frac{\rho}{3\,\varepsilon_0}\, r.}$$

- For $r > R$: The Gaussian surface is observed to surround the sphere. The charge inside the Gaussian sphere is therefore the total charge of the sphere, given by the following equation:

$$Q_{in} = \iiint \rho\, \mathrm{d}V = \rho \iiint \mathrm{d}V = \rho\, V_{\text{Shell}} = \rho\frac{4\pi R^3}{3}.$$

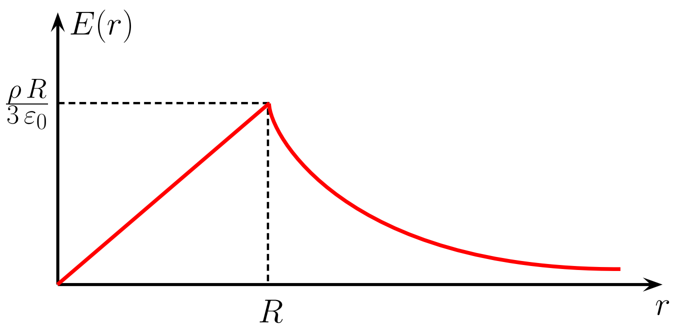
Consequently, the electric field can be expressed as follows:

$$E(r) = \frac{Q_{int}}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0 r^2}\left(\rho\frac{4\pi R^3}{3}\right) \implies \boxed{E(r) = \frac{\rho}{3\,\varepsilon_0}\frac{R^3}{r^2}.}$$

Upon substituting the expression of the e charge Q contained within in the sphere, we arrive at the final expression of the electric field magnitude, which is given by:

$$\boxed{E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}.}$$

This indicates that a spherically symmetric distribution of charges will generate an identical electric field magnitude at any point outside of it as a charge of of equivalent size situated at the centre point O.



The magnitude of the electric field is zero at the centre of a volume-charged sphere. It then increases linearly with distance inside the sphere until it reaches its maximum at the surface. Thereafter, it decreases in $\frac{1}{r^2}$, outside the sphere, reaching zero at infinity.

3.2 Electric field created by an infinite positively charged plane

Consider an infinite plane (Π) carrying a positive charge Q uniformly distributed over its surface with a surface charge density $\sigma > 0$. Let's calculate the magnitude of the electric field generated by this plane at any point M in space, z away from the infinite plane (Π).

1. All the planes that are perpendicular to the infinite plane (Π) are planes of symmetry of the latter. The electric field $\overrightarrow{E}$ belongs to these planes of symmetry and is therefore perpendicular to (Π). If this plane is generated by the vectors $\left(\overrightarrow{i}, \overrightarrow{j}\right)$, then $\overrightarrow{E} = E(z)\ \overrightarrow{k}$ above the plane and $\overrightarrow{E} = -E(-z)\ \overrightarrow{k}$ below the plane. Furthermore, since the plane (Π) is itself a plane of symmetry, $E(z)$ is odd ($E(-z) = -E(z)$).

   a) Because of these symmetry properties, the most suitable Gaussian surface is a cylinder of height $h = 2\,z$ with sections perpendicular to the plane and at symmetrical heights; the 2 bases of the cylinder with radius r are symmetrical with respect to (Π).

   b) At any point on the surface of the 2 bases of the Gaussian cylinder ($z = \frac{h}{2}$), $E(z)$ is constant.

   c) At any point on the lateral surface of the Gaussian cylinder, the outgoing vector $\overrightarrow{\mathrm{d}A}_L$ is radial. The electric field is then perpendicular to $\overrightarrow{\mathrm{d}A}_L$): $\overrightarrow{\mathrm{d}A}_L \perp \overrightarrow{E}$
   At any point on the surface of the 2 bases of the Gaussian cylinder, the outgoing vectors $\overrightarrow{\mathrm{d}A}_1$ and $\overrightarrow{\mathrm{d}A}_2$are parallel to the electric field : $\overrightarrow{\mathrm{d}S}_1 \parallel \overrightarrow{E}$ et $\overrightarrow{\mathrm{d}S}_2 \parallel \overrightarrow{E}$.

2. We write the Gauss theorem expressing the electric field flux as:

$$\Phi = \oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} = \frac{Q_{in}}{\varepsilon_0},$$

with:

$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} = \oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_L + \oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_1 + \oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_2,$$

and:

$$Q_{in} = \iint \sigma\, \mathrm{d}A.$$

Since the electric field $E(z)$ is constant at any point of the Gaussian surface, so that $E(-z) = -E(z)$, and since $\overrightarrow{\mathrm{d}A}_L \perp \overrightarrow{E}$, $\overrightarrow{\mathrm{d}A}_1 \parallel \overrightarrow{E}$, and $\overrightarrow{\mathrm{d}A}_2 \parallel \overrightarrow{E}$, we have :

$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_L = 0$$
$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_1 = \oiint E(z) \cdot \mathrm{d}A_1 = E(z) \oiint \mathrm{d}A_1 = E(z) \cdot A_1$$
$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_2 = \oiint -E(-z) \cdot \mathrm{d}A_2 = -E(-z) \oiint \mathrm{d}A_2 = E(z) \cdot A_2.$$

Since $A_1 = A_2 = \pi\, r^2$:

$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} = 2\,E(z) \cdot A_1 = E(z)\ 2\pi\, r^2.$$

3. Applying Gauss theorem, we can write :

$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} = \frac{Q_{in}}{\varepsilon_0} \quad \Longrightarrow \quad E(z)\ 2\pi\, r^2 = \frac{Q_{in}}{\varepsilon_0} \quad \Longrightarrow \quad E(z) = \frac{Q_{in}}{2\pi\varepsilon_0\, r^2},$$

with:

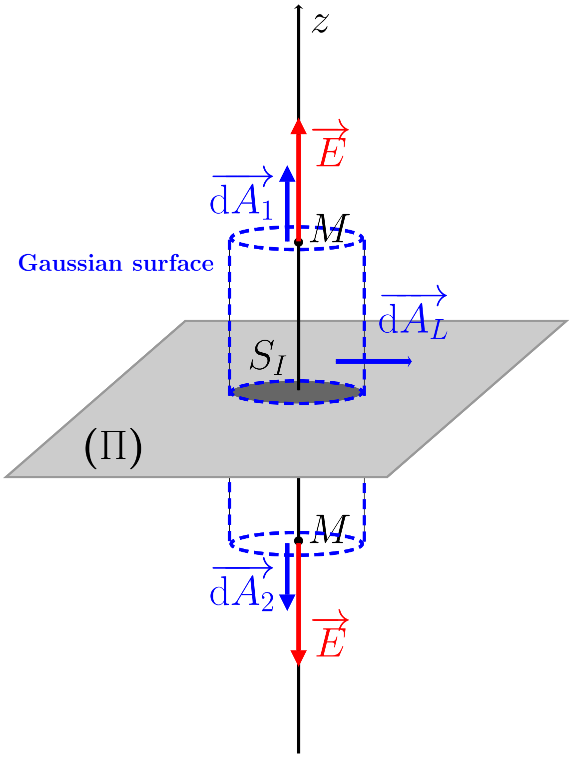
$$Q_{in} = \iint \sigma \, \mathrm{d}A_I = \sigma \iint \mathrm{d}A_I = \sigma \, A_I$$

$A_I = \pi \, r^2$ is the area occupied by the charge inside the Gauss cylinder, we have:

$$E\,(z) = \frac{Q_{in}}{2\pi\varepsilon_0 \, r^2} = \frac{\sigma \, \pi \, r^2}{2\pi\varepsilon_0 \, r^2} \implies \boxed{E\,(z) = \frac{\sigma}{2\,\varepsilon_0}.}$$

a) For any point M above the plane : $\overrightarrow{E} = \dfrac{\sigma}{2\,\varepsilon_0}\,\overrightarrow{k}$.

b) For any point M below the plane : $\overrightarrow{E} = -\dfrac{\sigma}{2\,\varepsilon_0}\,\overrightarrow{k}$.



Remarks

- The electric field generated by an infinitely charged plane changes direction as it passes through.
- This result can also be applied to any uniformly charged surface. It is sufficient to interpret E as the electric field immediately around the surface, close enough to the surface to be assimilated to an infinite plane.

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Chapter 5

Conductors in Electrostatic Equilibrium

Introduction

Thus far, our focus has been on electric charges and their effects in space, specifically the electrostatic field and potential. The objective of this chapter is to examine the behaviour of electric charges when they are placed on a conductor. It should be recalled that an ***electrical conductor*** is a body in which some of the electrons are capable of moving freely within the body. These electrons are referred to as ***conduction electrons***. A material is defined as a ***perfect conductor*** if, upon being electrified, the excess charge carriers are able to move freely throughout the volume occupied by the material. If the uncompensated carriers are unable to move freely and remain localised, the material will be a ***perfect insulator*** (or dielectric). In the presence of an electric field (usually external), the free charges of a conductor redistribute and reach an ***electrostatic equilibrium*** very quickly. The resultant charge distribution and electric field have many interesting properties that can be studied with the help of the Gauss theorem and the concept of electric potential.

1 Conductor in Electrostatic Equilibrium

A conductor is defined as being in *electrostatic equilibrium* when no electric charge is present within its interior.

1.1 Properties of a conductor in electrostatic equilibrium

1. ***In an electrostatically equilibrium state, the electric field $\overrightarrow{E}$ inside a conductor is identically zero.***
   If the electric field inside a conductor were not zero, an electric force $\overrightarrow{F} = q\,\overrightarrow{E}$ would be exerted on the free electrons. this would result in acceleration of the electrons,

indicating a departure from the equilibrium state. The charge is therefore distributed so that the electric field inside the conductor disappears when electrostatic equilibrium is reached.

2. ***In electrostatic equilibrium, the outer electric field near the surface of a conductor in electrostatic equilibrium is everywhere perpendicular to the surface.***
   If the electric field had a tangential component parallel to the surface, this would indicate that the free electric charges on the surface would also move and generate a surface current. This would be in contradiction with the condition of the electrostatic equilibrium of the conductor.

3. ***A conductor in electrostatic equilibrium exhibits a consistent electrostatic potential at all points within its interior and along its surface. Such a conductor can be described as an equipotential surface.***
   Indeed, the variation of the electrostatic potential between two arbitrary points M_1 et M_2, can be expressed as follows:

   $$\mathrm{d}V = -\overrightarrow{E}\cdot\overrightarrow{M_1M_2} = 0 \qquad \left(\text{because } \overrightarrow{E} = \overrightarrow{0} \text{ inside and } \overrightarrow{E} \perp \overrightarrow{M_1M_2} \text{ near the surface}\right).$$

4. ***It is no possible for an electric field line to "return" to the conductor in a state of electrostatic equilibrium***
   Indeed, the circulation of the electric field along a field line gives rise to the following equation:

   $$V(A) - V(B) = \int_A^B \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}\ell}$$

   In the event that points A and B are situated within the same conductor (such that the field line returns to the conductor), the circulation must be deemed to be zero (due to the fact that $V(A) = V(B)$). This is not feasible along a field line (where, by definition, $\overrightarrow{E} \parallel \overrightarrow{\mathrm{d}\ell}$).

5. ***In a conductor in electrostatic equilibrium, the total charge is zero*** $(\rho = 0)$***, indicating that the number of positive and negative charges is equal. When an additional charge is introduced to this conductor, it is distributed uniformly across the surface, with no net accumulation within the conductor.***
   Inside a conductor in electrostatic equilibrium, the electric field flux over a Gaussian surface is given by:

   $$\Phi = \oiint_S \overrightarrow{E}\cdot\overrightarrow{\mathrm{d}A} = 0 \qquad \left(\text{because } \overrightarrow{E} = \overrightarrow{0} \text{ inside}\right).$$

   Therefore, from the Gauss theorem, the net charge inside the conductor is:

   $$Q_{in} = \iiint \rho\,\mathrm{d}V = 0.$$

At the outer surface of a conductor in electrostatic equilibrium, the electric field flux over a Gaussian surface is given by:

$$\Phi = \oiint_S \vec{E} \cdot \mathrm{d}\vec{A} = \oiint_S E \cdot \mathrm{d}A = E\,A \qquad \left(\text{because } \vec{E} \parallel \mathrm{d}\vec{A} \text{ at the outer surface}\right).$$

Subsequently, the Gauss theorem can be applied to yield the following result:

$$Q_{in} = Q_{\text{Surface}} = \iint \sigma\,\mathrm{d}S = E\,S\,\varepsilon_0.$$

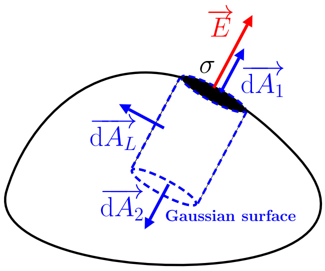
The free electric charges are distributed over the surface of a conductor in electrostatic equilibrium, which in reality has a thickness of a few atomic layers. This is due to the repulsive nature of the charges, which cause them to repel each other and therefore tend to be as far apart as possible in equilibrium.

Remarks

- The properties of a solid charged conductor in electrostatic equilibrium are applicable to a hollow conductor.
- When a charged conductor is connected to another conductor (e.g. the earth), an exchange of electric charges occurs in such a way that, once the charges have been transported, the entire system forms a single equipotential.

1.2 Coulomb's Theorem

The electrostatic field on the outer surface of a conductor in electrostatic equilibrium can be determined by applying the Gauss theorem.



- At any point M close to the surface S of a conductor in electrostatic equilibrium, the electrostatic field $\vec{E}$ is constant and perpendicular to S (one of the properties of a conductor in electrostatic equilibrium).

- We can therefore select a Gaussian surface in the form of a cylinder passing through the conductor surface, with $\vec{E}$ parallel to the outer base vector of the Gaussian cylinder $\mathrm{d}\vec{A}_1$.

- As the total charge is distributed over the surface (a property of a conductor in electrostatic equilibrium), the Gauss theorem can be written as follows:

$$\Phi = \oiint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{in}}{\varepsilon_0},$$

with :

$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A} = \oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_L + \oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_1 + \oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_2,$$

and:

$$Q_{in} = \iint \sigma \, \mathrm{d}A = \iint \sigma \, \mathrm{d}A_1; \quad (\sigma \text{ is the surface charge density at the considered point})$$

Given that $\overrightarrow{\mathrm{d}A}_L \perp \overrightarrow{E}$, and that $\overrightarrow{\mathrm{d}A}_1 \parallel \overrightarrow{E}$, and $\overrightarrow{E} = \overrightarrow{0}$ inside the conductor (a property of a conductor in electrostatic equilibrium) , we conclude that:

$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_L = 0$$
$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_1 = \oiint E \cdot \mathrm{d}A_1$$
$$\oiint \overrightarrow{E} \cdot \overrightarrow{\mathrm{d}A}_2 = 0.$$

Therefore, we obtain:

$$\oiint E \cdot \mathrm{d}A_1 = \frac{Q_{in}}{\varepsilon_0} = \frac{\iint \sigma \, \mathrm{d}A_1}{\varepsilon_0} \quad \Longrightarrow \quad \boxed{E = \frac{\sigma}{\varepsilon_0}.}$$

Coulomb's Theorem

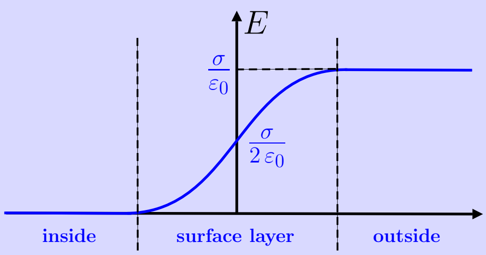
The electrostatic field in the immediate external vicinity of a conductor in electrostatic equilibrium with a surface charge density σ is given by the following equation :

$$\boxed{\overrightarrow{E} = \frac{\sigma}{\varepsilon_0} \overrightarrow{n}.} \tag{5.1}$$

the unit vector $\overrightarrow{n}$ is defined as outgoing and normal to the conductor surface.

Remark

Indeed, as it traverses the surface layer of the conductor where its electric charge is situated, the electric field undergoes a continuous variation from zero (within this layer) to $\dfrac{\sigma}{\varepsilon_0}$ (outside this layer), as illustrated in the figure below.



It can be deduced that the average electric field generated by all charges on the conductor at point M inside this surface layer is:

$$\overrightarrow{E}_m = \frac{\sigma}{2\,\varepsilon_0} \overrightarrow{n}.$$

1.3 Electrostatic pressure

The experiment indicates that when a bubble is charged, it expands irrespective of the sign of the charge. This phenomenon can be attributed to the exertion of repulsive forces between the charges present on the surface of the conducting bubble. This property is characteristic of the electrostatic pressure p, a force per unit area, that exists at any point on a conductor, which is expressed as follows:

$$p = \frac{\mathrm{d}F}{\mathrm{d}A},$$

with:

$$\overrightarrow{\mathrm{d}F} = \mathrm{d}q \cdot \overrightarrow{E}_m = \sigma \cdot \mathrm{d}A \, \frac{\sigma}{2\,\varepsilon_0} \, \overrightarrow{n}.$$

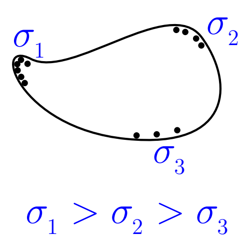
Then:

$$\mathrm{d}F = \frac{\sigma^2}{2\,\varepsilon_0} \, \mathrm{d}S \quad \Longrightarrow \quad \boxed{p = \frac{\sigma^2}{2\,\varepsilon_0}.}$$

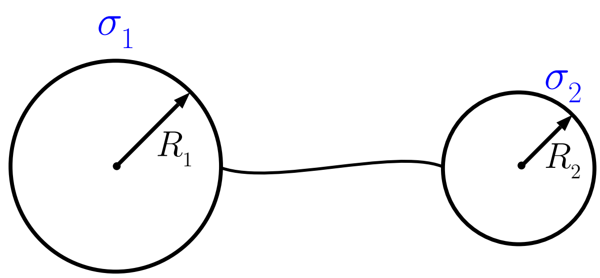
In general, the pressure exerted is insufficient to facilitate the removal of charges from the surface of the conductor. Nevertheless, the transfer of electrostatic force may result in deformation or displacement of the conductor.

1.4 Tip effect

It is a well-established phenomenon that the distribution of the charges on the surface of a conductor is not uniform. This implies that the charge on the surface is not a constant value. Parts of the surface with a small radius of curvature exhibit a tendency to accumulate charges. Consequently, the charge density is greater at the extremity of the tip. This phenomenon is known as the ***tip effect***.



To illustrate this phenomenon, consider two charged spheres of different radii R_1 and R_2, with different surface charge densities σ_1 and σ_2, connected by a conductor wire and sufficiently far apart. in this configuration, we can consider that each sphere is isolated, but that they share the same electric potential (one of the properties of a conductor in electrostatic equilibrium).



In accordance with Gauss theorem, the electric potential at the surface of a sphere conductor of charge Q is given by the expression $V = \dfrac{Q}{4\pi\varepsilon_0\, R}$. Consequently, the property of an equipotential volume imposes that:

$$V_1 = V_2 \quad \Longrightarrow \quad \frac{\sigma_1 \cdot A_1}{4\pi\varepsilon_0\, R_1} = \frac{\sigma_2 \cdot A_2}{4\pi\varepsilon_0\, R_2} \quad \Longrightarrow \quad \frac{\sigma_1 \cdot 4\pi R_1^2}{4\pi\varepsilon_0\, R_1} = \frac{\sigma_2 \cdot 4\pi R_2^2}{4\pi\varepsilon_0\, R_2},$$

and finally that:

$$\boxed{\sigma_1 \cdot R_1 = \sigma_2 \cdot R_2.}$$

It can be concluded that a greater surface charge density is associated with a reduction in the radius of curvature of the surface under consideration. It can therefore be deduced that a tip, with a relatively low radius of curvature, will carry a high surface charge density.

1.5 Capacitance of an isolated conductor

Consider an isolated conductor on which a quantity of charge Q has been deposited. The conductor generates, at any point in space, an electric field $\overrightarrow{E}$ and an electric potential V proportional to this charge. The charge Q and the potential V of a conductor in electrostatic equilibrium are proportional to each other and are related by the following equation:

$$\boxed{Q = C \cdot V.} \tag{5.2}$$

- C represents a positive constant, specifically the ***electrostatic capacitance*** of a conductor at equilibrium. Its SI unit is the farad (F), however, the units most commonly utilised in electrokinetics are the nanofarad (nF) or the picoFarad (pF), given that the capacitance of a single conductor is typically very small.

- The electrostatic capacitance C is contingent upon the geometric characteristics of the conductor in question. It is defined as the ability of a conductor, when raised to a given electrical potential, to store the electrical charge that it has received.

Example

The electrostatic capacitance of a conducting sphere of radius R, charged with a surface density σ, can be expressed as follows:

$$C = \frac{Q}{V},$$

with:

$$V = \frac{Q}{4\pi\varepsilon_0 R}.$$

Thus:

$$\boxed{C = 4\pi\varepsilon_0 R.}$$

- A conducting sphere with a radius of $R = 10\,\text{cm}$ has an electrostatic capacity of $C \simeq 10\,\text{pF}$.

- The Earth, with a radius of $R = 6400\,\text{km}$ has an electrostatic capacity of $C_{Earth} \simeq 710\,\mu\text{F}$.

1.6 Energy of a charged conductor in electrostatic equilibrium

Consider an isolated conductor of capacitance C carrying a charge Q. The electrostatic energy of this conductor can be defined as the work required to charge it, that is to say, to bring it to the electrostatic potential $V = \dfrac{Q}{C}$:

$$W_{q=0\longrightarrow Q} = E_p(q = Q) - E_p\,(q = 0) = \int_{q=0}^{q=Q} \mathrm{d}E_p = \int_{q=0}^{q=Q} V\,\mathrm{d}Q.$$

Given that $E_p\,(q = 0) = 0$, we can conclude that:

$$E_p = E_p(q = Q) = \int_{q=0}^{q=Q} V\,\mathrm{d}Q = \int_{q=0}^{q=Q} \frac{Q}{C}\,\mathrm{d}Q = \frac{1}{2}\frac{Q^2}{C}.$$

The electrostatic energy of the conductor can then be expressed as follows:

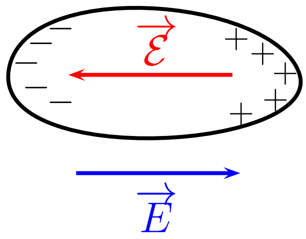
$$\boxed{E_p = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\,Q\,V = \frac{1}{2}\,C\,V^2.} \tag{5.3}$$

- The electrostatic energy of a charged conductor in electrostatic equilibrium is always positive.

- If the conductor is charged by a generator with a constant electromotive force (e.m.f.) of V, the energy supplied by the generator is $Q\,V$. This energy is twice the energy stored in the conductor ($\frac{1}{2}\,Q\,V$). The other half has been converted into heat during the charge transport, which is known as the Joule effect.

2 Electrostatic inductions

2.1 Neutral conductor exposed to an external electric field



A uniform external electrostatic field $\overrightarrow{E}$ is applied to a neutral conductor of zero total charge. As the electric charges are free to move within the conductor, the positive charges will move in the direction of $\overrightarrow{E}$, while the negative charges will move in the opposite direction to $\overrightarrow{E}$. The phenomenon whereby positive and negative charges are separated and positive and negative poles are created is referred to as ***polarisation***. This results in a non-uniform surface distribution, whereby the total charge remains zero due to the absence of an external charge addition. This new charge distribution gives rise to a new electric field $\overrightarrow{\mathcal{E}}$, known as the ***polarisation field***, which is opposite to the applied field $\overrightarrow{E}$. As the charges move, this polarisation field increases until it reaches the same magnitude as the applied field:

$$\left\|\overrightarrow{\mathcal{E}}\right\| = \left\|\overrightarrow{E}\right\|.$$

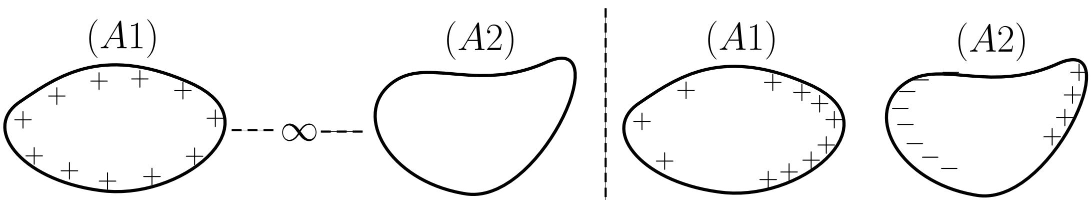
Consequently, the conductor will once again be in electrostatic equilibrium, as the sum of the two fields is equal to zero:

$$\overrightarrow{E}_{in} = \overrightarrow{E} + \overrightarrow{\mathcal{E}} = \overrightarrow{0}.$$

In contrast to its initial state, the conductor is polarised when in electrostatic equilibrium.

2.2 Mutual electrostatic induction

We may now consider the case of a conductor $(A1)$ of charge $Q_1 > 0$ with a surface charge density σ_1 placed in the vicinity of a neutral conductor $(A2)$. As a consequence of the electrostatic field generated by the charges on the conductor $(A1)$, a non-uniform surface charge density σ_2 is observed on the conductor $(A2)$. Conversely, the presence of charges in the vicinity of $(A1)$ modifies the charge distribution of $(A1)$ due to the electric field they create in the vicinity of $(A1)$.



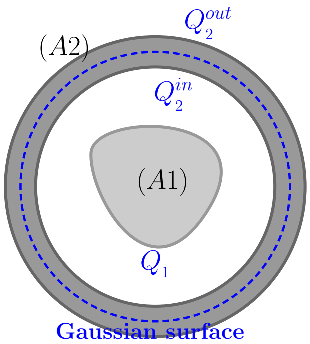
(A1) and (A2) isolated from each other | **(A1) in the vicinity of (A2)**

In an equilibrium state, the two charge distributions σ_1 and σ_2 are mutually dependent. This interdependence is referred to as ***mutual electrostatic induction***.

In this example, some of the electrostatic field lines originating from $(A1)$ do not reach $(A2)$, indicating a ***partial influence***.

2.3 Total electrostatic induction

The placement of a conductor $(A1)$ of charge $Q_1 > 0$ within a hollow conductor $(A2)$ allows for the creation of conditions conducive to ***total electrostatic influence***. In such a configuration, the electric field within $(A2)$ is observed to be zero in electrostatic equilibrium. Consequently, the electrostatic field flux through a closed Gaussian surface $(\mathscr{S})$ selected within $(A2)$ is also found to be zero, given that $E = 0$:



$$\Phi = \oiint_{(S)} \vec{E} \cdot \mathrm{d}\vec{A} = 0$$

The Gauss theorem is thus formulated as follows:

$$\Phi = \frac{Q_{in}}{\varepsilon_0} = \frac{Q_2^{in} + Q_1}{\varepsilon_0} = 0 \quad \Longrightarrow \quad \boxed{Q_2^{in} = -Q_1.}$$

A chargeQ_2^{in} of the same sign and opposite to that of $(A1)$ is present on the inner surface of the conductor $(A2)$. This phenomenon is referred to as ***Faraday's law***. It is postulated that electricity condenses on the inner surface of the hollow conductor due to the appearance of charges on the inner surface of $(A2)$.

If Q_2 is defined as the total charge of the hollow conductor $(A2)$, then the charge Q_2^{out}, which is carried by the outer surface of the conductor $(A2)$, can be expressed as follows:

$$Q_2 = Q_2^{out} + Q_2^{in}.$$

Susequently, we can deduce that:

$$Q_2^{out} = Q_2 - Q_2^{in}.$$

Since $Q_2^{in} = -Q_1$, we can therefore conclude that:

$$\boxed{Q_2^{out} = Q_2 - Q_2^{in} = Q_2 + Q_1.}$$

In the specific instance where the conductor $(A2)$ is initially neutral $(Q_2 = 0)$, we get:

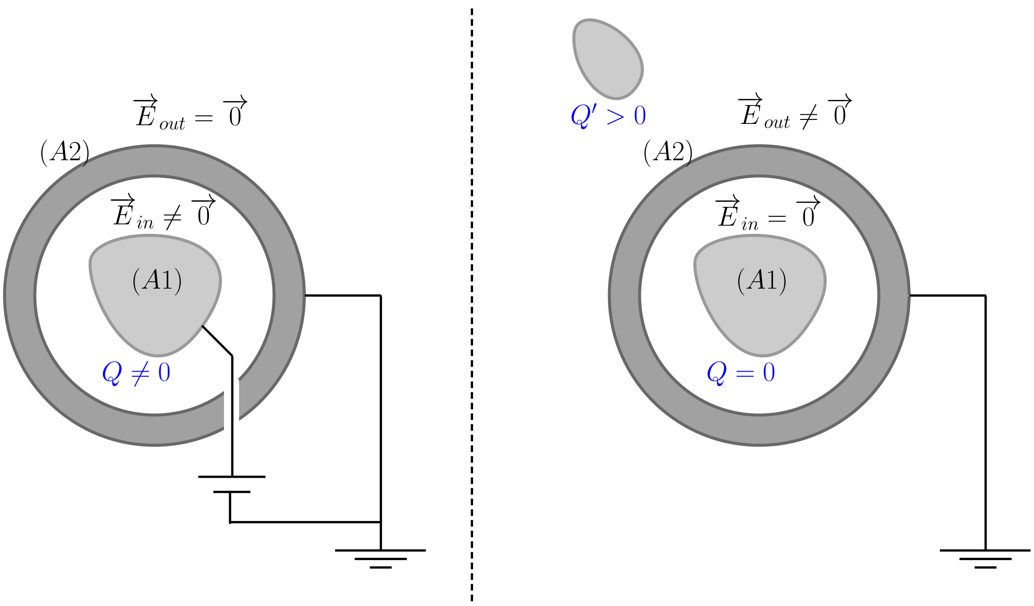
$$\boxed{Q_2^{out} = -Q_2^{in} = Q_1.}$$

In this example, the influence is classified as total, as all the electrostatic field lines originating from $(A1)$ converge on $(A2)$.

2.4 Faraday shield - Screening effect

It is now widely accepted common knowledge that a conductor in electrostatic equilibrium has an internal electric field of zero. Consequently, in the event that the conductor is of a hollow nature, the cavity within it is inherently electrostatically isolated. In the event that a hollow conductor is subjected to a charge, and is is situated in an environment devoid of any neutral or charged conductors, the charges will disperse throughout the conductor's outer surface, thereby verifying the property of a zero electric field within the cavity. The cavity is thus rendered immune to any external influence. The conductor thus serves to form an electrostatic screen in relation to its internal cavity.

In the aforementioned example of total influence, if the exterior of $(A2)$ is connected to the earth, we have $Q_2^{out} = 0$ (charges flowing to or from the earth). Consequently, the electrostatic field measured outside $(A2)$ is zero, despite the presence of charged $(A1)$ inside $(A2)$. This indicates that the space outside $(A2)$ is safeguarded from any electrostatic influence originating from the cavity. The converse is also true.



We may consider the case where $(A1)$ carries a zero charge and $(A2)$, still connected to the earth by its outer surface, is placed in proximity to other conductors.

In equilibrium, $Q_2^{in} = 0$, but a non-zero electrostatic field is present outside $(A2)$, which depends on the distribution of charge on the outer surface of $(A2)$. therefore, despite the charge carried by the outer surface of $(A2)$, the inner cavity has a zero electrostatic field. It can thus be concluded that the electrostatic field inside $(A2)$ is completely independent of that outside.

It has been shown that any hollow conductor kept at a constant potential will form an electrostatic screen in both directions. Such a device is referred to as a ***Faraday shield***.

3 Capacitor

3.1 Electricity condensation

A capacitor is defined as a system that is capable of storing potential electrical energy by maintaining the separation of positive and negative charges. It is composed of two conductors, referred to as armatures, which are separated by an insulating or dielectric material, such as air. The armatures are enabled to carry charges of equal magnitude but opposite sign.

A capacitor is represented by the symbol : ⊣⊢

Definition

A ***capacitor*** is defined as any system of two conductors under the electrostatic influence. There are two principal types of capacitor:

- Those with close armatures,
- and those with total induction.

These devices are designated as capacitors due to their ability to exhibit the phenomenon of ***electrical condensation***, which is the accumulation of electrical charges in a confined space. As a result, the construction of capacitors with high capacitance allows for the attainment of high electrical charges at low voltages.

In order to return to the case of two mutually inductive conductors, it is necessary to consider the case where the conductor $(A1)$ is maintained at a constant electric potential V_1 (for instance positive) by connecting it to a generator, and the conductor $(A2)$ is kept at an electric potential V_2 (for instance zero) by connecting it to earth. As a consequence of the negative charges on $(A2)$, additional positive charges emerge on $(A1)$. The generator will facilitate the transfer of these additional positive charges to maintain the constant electric potential V_1. In equilibrium, the charge on $(A1)$ has increased, as has that on $(A2)$. The phenomenon of *condensation* of electricity on the two opposite surfaces is observed, particularly given that the distance between the two conductors is smaller than their size. The combination of the conductors $(A1)$ and $(A2)$ forms a ***close-armature capacitor***.

To return to the example of the two conductors in total induction, let us consider the case where the conductor $(A1)$ is held at a constant electric potential V_1 and the outer surface of the hollow conductor $(A2)$ is connected to earth ($V_2 = 0$ and $Q_2^{out} = 0$). In this case $Q_2 = Q_2^{in} = -Q_1$, resulting in electrostatic condensation on the surface of $(A1)$ and on the inside of $(A2)$. The two conductors, $(A1)$ and $(A2)$, can therefore be considered to form a ***total induction capacitor***.

3.2 Capacitance of a capacitor

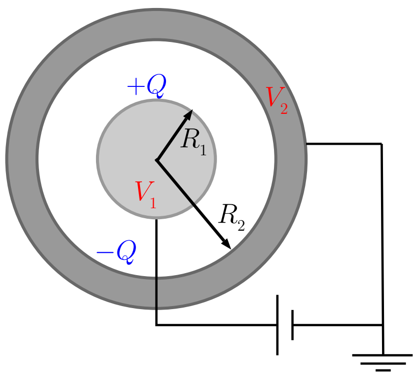
$(A1)$ and $(A2)$ are two conductors which form a capacitor of charge Q. The potential difference, or voltage, between$(A1)$ and $(A2)$ is represented by the symbol ΔV. The capacitance of the capacitor is therefore given by the equation:

$$C = \frac{Q}{\Delta V}.$$

Since $\Delta V = -\int \vec{E} \cdot \vec{d\ell}$, the capacitance of a capacitor is determined by calculating the circulation of the electric field between the two armatures and their charge.

3.2.1 Example 1 : Spherical capacitor

Consider a capacitor comprising two spherical conductors$(A1)$ and $(A2)$, with a common centre O. The conductor $(A1)$ has a radius R_1, while the hollow conductor $(A2)$ has an inner radius $R_2 > R_1$. These two conductors are separated by a vacuum. The conductor of radius R_1 is positively charged with the charge $+Q$, while the inner surface of the conductor of radius R_2 is characterised by a negative charge $-Q$. The outer surface of the latter is connected to the earth.



by applying the Gauss theorem with a spherical Gaussian surface of radius r, it can be demonstrated that the electrostatic field at a point M between the two armatures of the capacitor $(R_1 < r < R_2)$ is such that:

$$\oiint \vec{E} \cdot \vec{dA} = \frac{Q_{in}}{\varepsilon_0},$$

and then:

$$E \cdot 4\pi\, r^2 = \frac{Q}{\varepsilon_0} \implies E = \frac{Q}{4\pi\varepsilon_0\, r^2}.$$

The voltage between the two armatures (with $V_1 > V_2$) is :

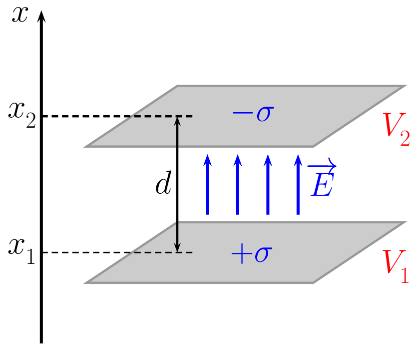
$$\Delta V = -\int \vec{E} \cdot \vec{d\ell} \implies V_1 - V_2 = -\int_{R_2}^{R_1} E \cdot \mathrm{d}r \implies V_1 - V_2 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

The capacitance of the spherical capacitor is then given by:

$$C = \frac{Q}{V_1 - V_2} \implies \boxed{C = 4\pi\varepsilon_0 \frac{R_1 \cdot R_2}{R_2 - R_1}.}$$

3.2.2 Example 2 : Flat capacitor

A flat capacitor is composed of two parallel planar armatures, or plates, $(A1)$ and $(A2)$, of equal surface area A, positioned orthogonally to an axis (Ox) and separated by a vacuum of thickness, $d = x_2 - x_1$, less than the dimensions of the plates themselves. the surface charge density of plate $(A1)$ is $+\sigma$, while the surface charge density of plate $(A2)$ is $-\sigma$.



The electric field between the armatures is uniform. by applying Gauss theorem, it can be demonstrated that the electric field generated by the positive armature, for example, is that of an infinite plane (as previously discussed in the preceding chapter):

$$\vec{E} = \frac{\sigma}{2\,\varepsilon_0}\,\vec{i}$$

The total electric field produced by the two plates can thus be considered as the superposition of the electric fields produced by the two infinite planes:

$$\vec{E} = \frac{\sigma}{2\,\varepsilon_0}\vec{i} + \left(\frac{-\sigma}{2\,\varepsilon_0}\right)\left(-\vec{i}\right) = \frac{\sigma}{\varepsilon_0}\,\vec{i}.$$

The voltage between the two plates (with $V_1 > V_2$) is :

$$\Delta V = V_1 - V_2 = -\int_{x_2}^{x_1} E \cdot \mathrm{d}x \quad \Longrightarrow \quad V_1 - V_2 = -E\,(x_1 - x_2) = \frac{\sigma}{\varepsilon_0}\,d.$$

The capacitance of the flat capacitor is then given by:

$$C_0 = \frac{Q}{V_1 - V_2} = \frac{\sigma \cdot S}{V_1 - V_2} \quad \Longrightarrow \quad \boxed{C_0 = \frac{S \cdot \varepsilon_0}{d}.}$$

in general, if the plates are separated by a dielectric material of permittivity ε and thickness d, the capacitance is given by the following equation:

$$\boxed{C = \frac{S \cdot \varepsilon}{d}.}$$

3.3 Energy storage in a capacitor

The electrical energy stored in a capacitor can be defined as the work required to charge the two plates of the capacitor. If the conductor $(A1)$ is positively charged with the charge $+Q$ and $(A2)$ is negatively charged with the charge $-Q$, the electrical energy stored in the capacitor can be expressed as:

$$E_p = E_{P_1} + E_{P_2} = \frac{1}{2}Q_1V_1 + \frac{1}{2}Q_2V_2,$$

and finally as:

$$E_p = \frac{1}{2}Q\,(V_1 - V_2) \quad \Longrightarrow \quad \boxed{E_p = \frac{1}{2}Q \cdot \Delta V.}$$

The electrical energy stored in a capacitor can be expressed in a number of way, including the following :

$$\boxed{E_p = \frac{1}{2}Q \cdot \Delta V = \frac{1}{2}C \cdot \Delta V^2 = \frac{1}{2}\frac{Q^2}{C}.}$$

3.4 Connection of several capacitors

Capacitors can be connected in two distinct ways:

1. **Series connection :** The positive plate of one capacitor is connected to the negative plate of the next capacitor. This configuration allows all the capacitors to carry the same charge.

2. **Parallel connection** : All the positive plates are connected to the same pole of the generator, and the negative plates to the other pole. The voltage of all capacitors is identical.

A single ***equivalent capacitor*** with the same capacitance as all n capacitors can be used in place of a set of n capacitors.

Equivalent capacitor

A single equivalent capacitor can be employed to replace a set of n capacitors such that the following conditions are met:

- The equivalent capacitor has the same voltage as all n capacitors.

- The equivalent capacitor stores the same energy as all n connected capacitors.

3.4.1 Series connection

let us consider a set of n capacitors with capacitances C_i connected in series between points A, at potential V_A, and B, at potential V_B. A charge Q is applied to the initial capacitor. It is assumed that all the capacitors are initially at the same potential, and that the charge $\pm Q$ is established on the plates of the adjacent capacitors by induction. The total voltage of the series of capacitors is then simply written as follows:

$$\Delta V = V_A - V_B = \Delta V_1 + \Delta V_2 + \cdots + \Delta V_n$$

All of the capacitors carry the same charge Q, so we write:

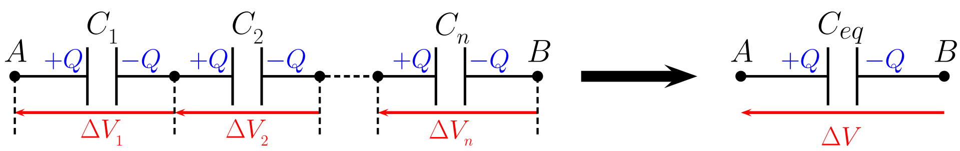
$$\Delta V_1 = \frac{Q}{C_1},\ \Delta V_2 = \frac{Q}{C_2},\ \cdots, \Delta V_n = \frac{Q}{C_n},$$

and then:

$$\Delta V = \Delta V_1 + \Delta V_2 + \cdots + \Delta V_n \quad \Longrightarrow \quad \frac{Q}{C_{eq}} = Q\left[\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}\right].$$

The single equivalent capacitor has the following characteristics:

$$\boxed{\begin{aligned} \frac{1}{C_{eq}} &= \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n} = \sum_{i=1}^{n} \frac{1}{C_i} \\ Q_{eq} &= Q. \end{aligned}} \tag{5.4}$$



3.4.2 Parallel connection

Now consider now a set of n capacitors with capacitances C_i connected in parallel between points A, at potential V_A, and B, at potential V_B. A charge Q_i is applied to each capacitor of capacitance C_i. The total electric charge is then:

$$Q_{eq} = Q_1 + Q_2 + \cdots + Q_n.$$

All of the capacitors have the same voltage, so we write:

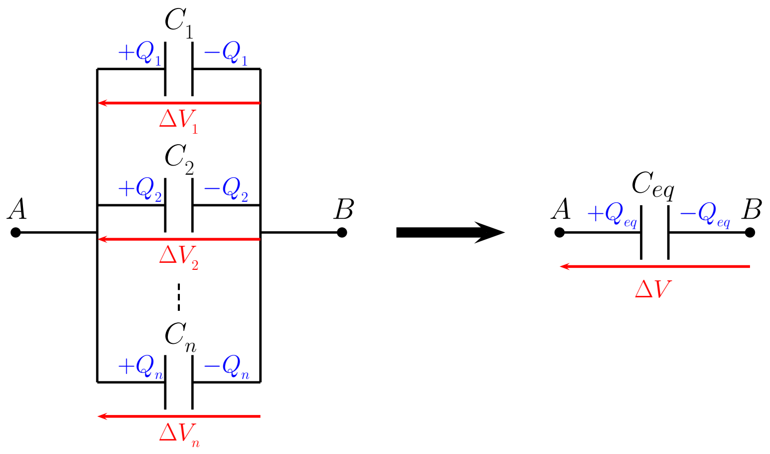
$$\Delta V = V_A - V_B = \Delta V_1 = \Delta V_2 = \cdots = \Delta V_n,$$

and then:

$$Q_{eq} = Q_1 + Q_2 + \cdots + Q_n \quad \Longrightarrow \quad \Delta V \cdot C_{eq} = \Delta V\left(C_1 + C_2 + \cdots + C_n\right).$$

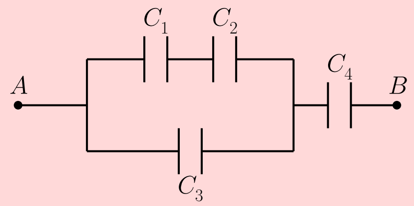
The single equivalent capacitor has the following characteristics:

$$\boxed{\begin{aligned} C_{eq} &= C_1 + C_2 + \cdots + C_n = \sum_{i=1}^{n} C_i \\ Q_{eq} &= Q_1 + Q_2 + \cdots + Q_n. \end{aligned}} \tag{5.5}$$



Example

Calculate the capacitance of the single equivalent capacitor of the combination of capacitors shown in the diagram below. We get : $C_1 = C_3 = 2\,\mu\text{F}$ et $C_2 = C_4 = 4\,\mu\text{F}$.



Capacitors C_1 et C_2 are connected in in series. Their equivalent capacitance is :

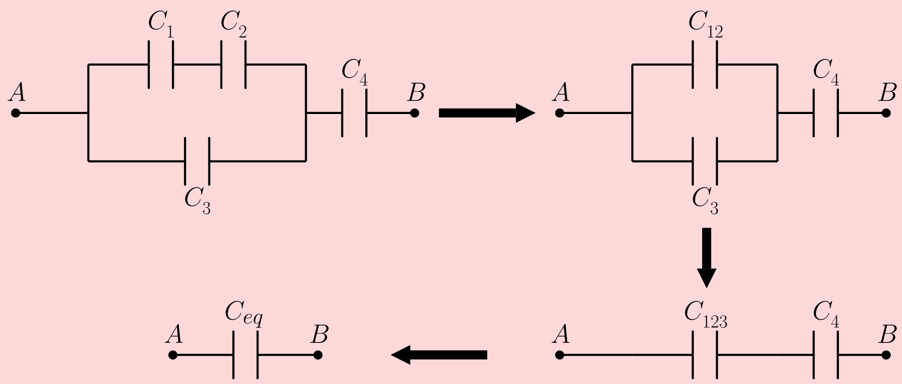
$$\frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \Longrightarrow \quad C_{12} = \frac{C_1 \cdot C_2}{C_1 + C_2} = \frac{4}{3}\,\mu\text{F}.$$

Capacitors C_{12} et C_3 are connected in parallel. Their equivalent capacitance is:

$$C_{123} = C_{12} + C_3 = \frac{10}{3}\,\mu\text{F}.$$

Capacitors C_{123} et C_4 are connected in series. The final single equivalent capacitance is therefore :

$$\frac{1}{C_{eq}} = \frac{1}{C_{123}} + \frac{1}{C_4} \quad \Longrightarrow \quad \boxed{C_{eq} = \frac{C_{123} \cdot C_4}{C_{123} + C_4} = \frac{20}{11}\,\mu\text{F}.}$$



Chapter 6

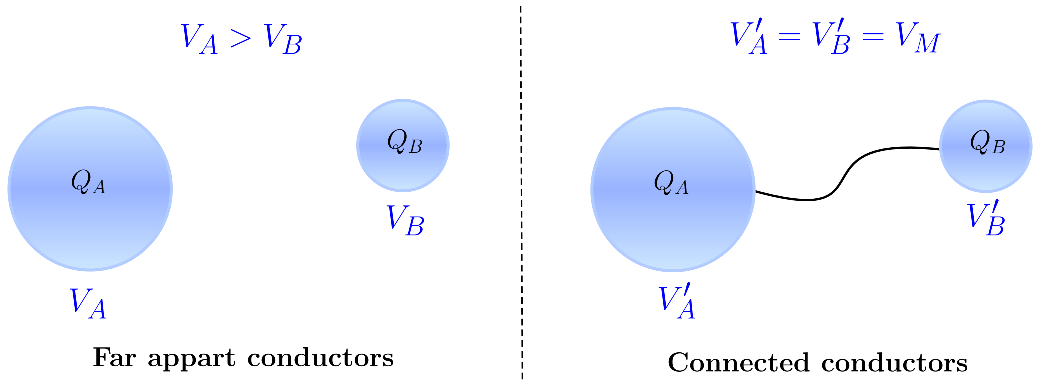
Electrokinetics

1 Electric current

In the first part of this course we looked at electrostatic phenomena caused by electric charges that are immobile in space. In this second part we will have a look at ***electrokinetic*** phenomena due to the displacement of free electric charges in a conductor by an electric current. In the following, we will only consider only the direct electric current.

1.1 Origin of electric current

Consider two conductors A and B in electrostatic equilibrium. They are so far apart that no mutual induction can be considered. These two conductors carry different charges, Q_A and Q_B, and have different electrostatic potentials, V_A and V_B, respectively. We will consider the case where $V_A > V_B$.



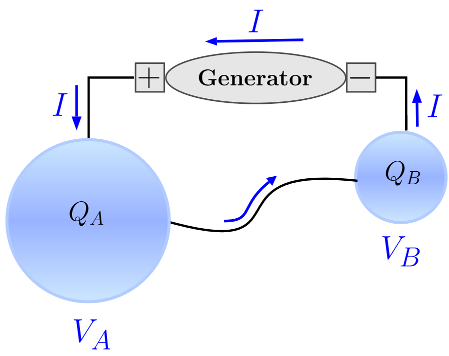
When these two conductors are connected by a conducting wire, a single conductor is formed (consisting of the two conductors and the wire). The electrostatic equilibrium of this conductor is broken (different electrostatic potential at each point). This leads to a displacement of electric charges along the connecting wire. Conductor A (the area with the highest electrostatic potential) is partially discharged, while conductor B (the area with the lowest electrostatic potential) is discharged. This displacement of electric charges is due to the action of an electric field directed along the decreasing potentials (hence an electric force $\overrightarrow{F} = q \cdot \overrightarrow{E}$) and will continue until a new state of electrostatic equilibrium is established with a new distribution of the total electric charge $Q = Q_A + Q_B$ between conductors A and B.

This charge displacement corresponds to a ***transient electric current*** passing through.

Remarks

- In the new state of equilibrium, the new electric charges of conductors A and B are $Q_A^{'}$ and $Q_B^{'}$ respectively. The charge q that has moved between the 2 states of equilibrium is:
$$q = Q_A - Q_A^{'} = Q_B^{'} - Q_B$$
- We can't tell whether the charge q represents the decreasing positive charge of A or its increasing negative charge. It is as if there were a migration of a positive charge $q = Q_A - Q_A^{'}$ from the highest electrostatic potential V_A to the lowest electrostatic potential V_B.

The state of non-equilibrium between conductors A and B, connected by a conducting wire, can be maintained by continuously adding electric charges to one of the two conductors. This is the role of the voltage source (generator), which collects the charges arriving on B and returns them to A, maintaining a constant potential difference $V_A - V_B$.



This continuous movement of electric charges corresponds to the passage of a ***permanent electric current***.

- In metals, electric current results from the movement of free electrons (negatively charged).
- In other materials, such as gases, semiconductors, electrolytes, current can result from the movement of negative charges (electrons, negative ions) or positive charges (positive ions).

Only electrical conduction in metals is considered below.

1.2 Some characteristics

Electric current intensity

Current intensity is defined as the amount of electric charge dQ carried per unit time dt:

$$\boxed{I = \frac{\mathrm{d}Q}{\mathrm{d}t}.} \tag{6.1}$$

- Its SI unit is the ampere (A), so that $1\,\mathrm{A} = 1\,\mathrm{C}\cdot\mathrm{s}^{-1}$.

- An electric current is called direct if its intensity is constant over time.

Conventional direction of electric current

Since electric current in metals results from the movement of electrons, the conventional direction of current, chosen by Ampere in the early 19th century, is opposite to that in which electrons move.

By convention, when a metal conductor is connected to the terminals of a generator, a continuous electric current flows from the positive pole to the negative pole outside the generator and from the negative pole to the positive pole inside the generator (see figure above).

Current density vector

In a conductor, electrons are driven by an electric potential difference and, consequently, an electric field $\overrightarrow{E}$. This phenomenon gives rise to the electric force $\overrightarrow{F} = q\,\overrightarrow{E}$. All electrons acquire an average velocity $\overrightarrow{v}$, which is referred to as the ***drift velocity***.

The ***current density*** vector $\overrightarrow{j}$ is then defined as follows:

$$\overrightarrow{j} = \rho\,\overrightarrow{v}.$$

The volume charge density in the conducting medium, denoted by the symbol ρ, can be expressed as a function of the number density n of free charges, which are electrons with a charge of $-e$, per unit volume:

$$\rho = -n\,e$$

It follows that the current density vector can also be written as follows:

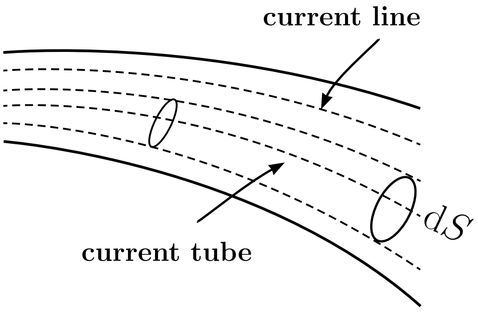
$$\boxed{\overrightarrow{j} = -n\,e\,\overrightarrow{v}.} \tag{6.2}$$

The value of the constant n is dependent upon the specific material under consideration.

current line and current tube

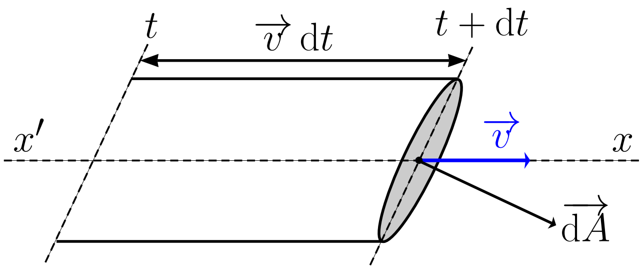
- A ***current line*** or a ***streamline*** is defined as a line that is tangent at any point in space to the current density vector $\overrightarrow{j}$.

- A ***current tube*** is formed by all the current lines that lie on an arbitrary closed contour $(\mathcal{C})$.



1.2.1 Relating current intensity to current density vector

let us consider a cylindrical metal conductor with cross-section A and an axis (xx') through which an electric current of intensity I flows. Inside this conductor, let us imagine a cylindrical current tube with an axis (xx') and section $\mathrm{d}A$ (represented by an area vector $\overrightarrow{\mathrm{d}A}$ perpendicular to it), through which an electric charge $\mathrm{d}q$ passes at a velocity $\overrightarrow{v}$ (see the adjacent figure).



Over the specified time interval $\mathrm{d}t$, this quantity of charge will assume a cylindrical volume:

$$\mathrm{d}V = \overrightarrow{\mathrm{d}x} \cdot \overrightarrow{\mathrm{d}A} = (\overrightarrow{v}\,\mathrm{d}t) \cdot \overrightarrow{\mathrm{d}A}.$$

The value of this charge quantity is:

$$\mathrm{d}q = \rho \cdot dV = \rho\,(\overrightarrow{v}\,\mathrm{d}t) \cdot \overrightarrow{\mathrm{d}A},$$

so that:

$$\frac{\mathrm{d}q}{dt} = \rho\,\overrightarrow{v} \cdot \overrightarrow{\mathrm{d}A} = \overrightarrow{j} \cdot \overrightarrow{\mathrm{d}A}.$$

When considering the entirety of cross-sectional area S of the metal conductor, it can be determined that the total charge passing through is:

$$\frac{\mathrm{d}Q}{dt} = \iint_S \overrightarrow{j} \cdot \overrightarrow{\mathrm{d}A}.$$

Furthermore, the electric current intensity can be expressed as follows:

$$\boxed{I = \iint_S \vec{j} \cdot \overrightarrow{dA}.} \tag{6.3}$$

- The intensity of an electric current can be defined as the current density vector flux through a given surface.
- The SI unit of current density vector is the ampere per square metre ($A \cdot m^{-2}$).

2 Ohm's law

2.1 Macroscopic Ohm's law- Electric resistance

The movement of free electrons in a metal is impeded by their interactions with the positive ions that constitute the metal's crystal lattice. In the absence of an external electric field, the free charges move randomly in all directions, resulting in a net zero total motion. In the the absence of an an external electric field, there is no electric current.

However, when an external electric field (i.e. an electrostatic potential difference) is applied, the random motion of the electrons is superimposed by a drift motion, resulting in the formation of an electric current. The intensity of the current is dependent on the magnitude of the applied electric field and the internal structure of the metal in question. This relationship has been established through experimental observation, as expressed by Ohm's law:

> ***In a metallic conductor maintained at a constant temperature, the ratio of the voltage ΔV between two points to the electric current I remains constant.***

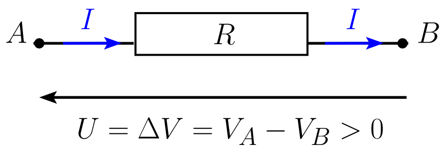
This constant, denoted R, is defined as the ***electrical resistance*** of the conductor between the two points. The conductor is referred to as an ***ohmic conductor*** or ***resistor***.

$$\boxed{\frac{\Delta V}{I} = R.} \tag{6.4}$$

- The SI units of R are Ohms (Ω), with the definition of $1\,\Omega = 1\,V \cdot A^{-1}$.
- The Electrical resistance of a metal is dependent upon the internal structure of the material.

- In an electric circuit, the electrical resistance is represented by the diagram shown in the adjacent figure. The ***receptor convention***, which is a standard practice in this field, indicates that that the potential difference ΔV (or electric voltage U) across the resistance is represented by an arrow pointing towards the higher potential, in the opposite direction to the electric current.

2.2 Microscopic origin of Ohm's law - Electric conductivity



Let us consider a cylindrical metal conductor of length ℓ and cross-section S which is subjected to a potential difference ΔV between its two ends. The resulting electric field is given by:

$$E = \frac{\Delta V}{\ell}.$$

The result is a permanent electric current in accordance with the relation 6.3:

$$I = \iint_S \vec{j} \cdot \overrightarrow{\mathrm{d}A} = j\,A.$$

Ohm's law permits the following equation to be written:

$$R = \frac{\Delta V}{I} = \frac{E\,\ell}{j\,A}.$$

A relationship is then derived between the current density and the applied electric field:

$$j = \frac{\ell}{R\,A}\,E = \sigma\,E.$$

It can be demonstrated from the receptor convention of electric current that the vectors $\vec{j}$ and $\vec{E}$ have the same direction. The microscopic Ohm's law can therefore be written as follows:

$$\boxed{\vec{j} = \sigma\,\vec{E}.} \tag{6.5}$$

- The constant $\sigma = \dfrac{\ell}{R\,A}$ is called the ***electrical conductivity*** of the metal. It depends on the geometric parameters of the conductor and its internal structure.
- The SI unit of the electrical conductivity is $\Omega^{-1} \cdot \mathrm{m}^{-1}$.

Supplement*: *electrons velocity

From equations 6.2 and 6.5, we can derive the following relationship:

$$\vec{j} = -n\,e\,\vec{v} = \sigma\,\vec{E}.$$

It can thus be deduced that:

$$\vec{v} = -\frac{\sigma}{n\,e}\,\vec{E} = \mu\,\vec{E}.$$

- The constant $\mu = -\dfrac{\sigma}{n\,e}$ is referred to as ***electron mobility***. Its SI units are square metres per volt second ($\mathrm{m^2 \cdot V^{-1} \cdot s^{-1}}$).
- In a metal, conduction electrons reach a constant drift velocity under the action of an external electric field. This is contrast to vacuum, where, according to the Newton's law of motion, the electrons velocity, increases continuously with time:

$$\vec{F} = -e\,\vec{E} = m\,\frac{\mathrm{d}\vec{v}}{dt} \quad \Longrightarrow \quad \vec{v} = -\frac{e\,t}{m}\,\vec{E}$$

- In addition to the electric force exerted by the external electric field, the effect of the crystal lattice within the metal can be considered as a frictional force, $\vec{f} = -k\,\vec{v}$ (where k being a constant depending on the nature of the metal), which would cancel out the electric force:

$$\vec{F} + \vec{f} = \vec{0} \quad \Longrightarrow \quad -e\,\vec{E} = k\,\vec{v} \quad \Longrightarrow \quad \vec{v} = -\frac{e}{k}\,\vec{E} = \mu\,\vec{E}.$$

- In the transient regime between the moment the electric field is applied and the moment the electron drift velocity becomes constant (steady state), the drift velocity increases during this time interval. This can be expressed by:

$$-e\,\vec{E} - k\,\vec{v} = m\,\frac{\mathrm{d}\vec{v}}{dt}.$$

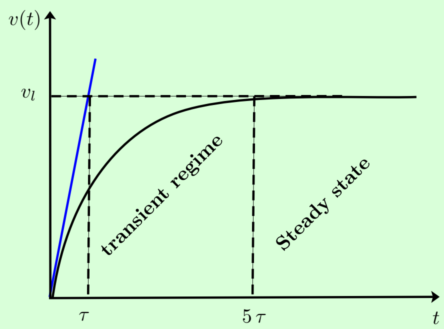
This leads to a differential equation of the form:

$$\frac{\mathrm{d}\vec{v}}{dt} + k\,\vec{v} = -\frac{e}{m}\,\vec{E}.$$

The solution of the differential equation is given by:

$$\boxed{\vec{v}\,(t) = -\frac{e}{k}\,\vec{E}\left(1 - e^{-\frac{k}{m}t}\right).}$$

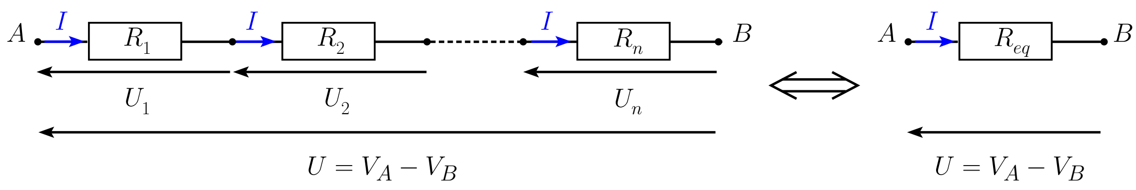
The average limiting velocity, $\vec{v}_l = -\dfrac{e}{k}\,\vec{E}$, which characterises the steady state, is reached after a time $t = 5\,\tau$ ($\vec{v}\,(t = 5\,\tau) \simeq 0,99\,\vec{v}_l$), where $\tau = \dfrac{m}{k}$ is the characteristic time constant.



2.3 Connection of several resistors

2.3.1 Series connection

Consider a series of n resistors, each with a resistance R_i $(i = 1, \ldots, n)$, connected in such a way that the same current I flows through them (see figure below). This combination of resistors is referred to as a ***series connection***.



The total electric potential difference across the resistors is expressed as follows:

$$\begin{aligned}\Delta V &= U_1 + U_2 + \ldots + U_n \\ &= R_1\, I + R_2\, I + \ldots + R_n\, I \\ &= (R_1 + R_2 + \ldots + R_n)\ I.\end{aligned}$$

This potential difference is analogous to that observed in a single ohmic conductor through which the same current flows and which has an equivalent resistance such that:

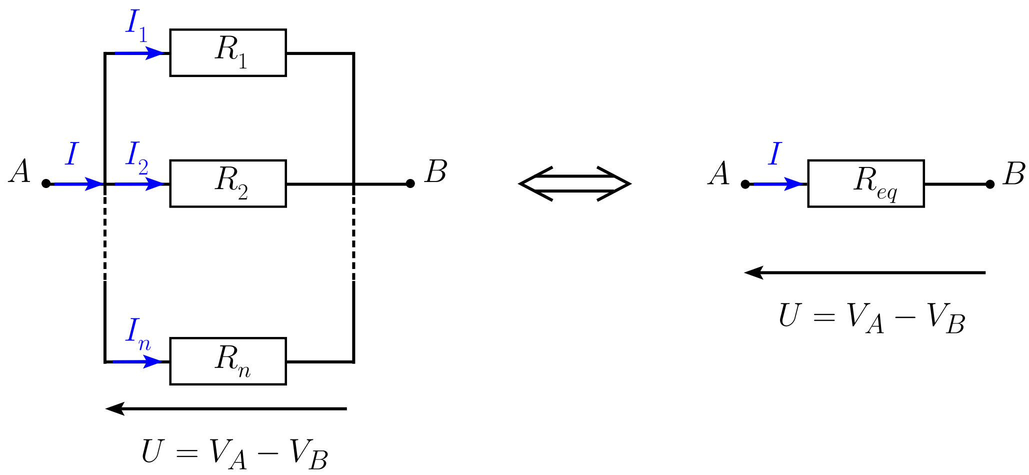
$$\Delta V = R_{eq}\, I,$$

with:

$$\boxed{R_{eq} = R_1 + R_2 + \ldots + R_n = \sum_{i=1}^{i=n} R_i.} \tag{6.6}$$

2.3.2 Parallel connection

We may now consider a series of n resistors, each with a resistance R_i $(i = 1, \ldots, n)$, connected in such a way that they are exposed to the same potential difference ΔV (see figure below). This combination of resistors is referred to as a ***parallel connection***.



The net electric current through all the resistors can be expressed as follows:

$$\begin{aligned} I &= I_1 + I_2 + \ldots + I_n \\ &= \frac{U}{R_1} + \frac{U}{R_2} + \ldots + \frac{U}{R_n} \\ &= U\left(\frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}\right). \end{aligned}$$

This current is analogous to that which would be observed with a single ohmic conductor subjected to the same potential difference and having an equivalent resistance such that:

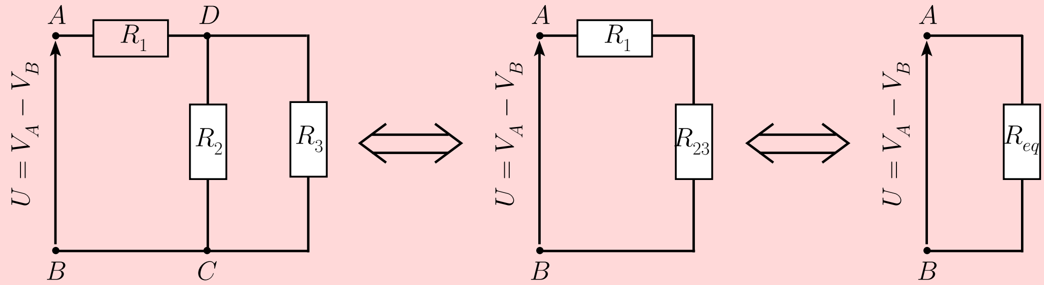
$$I = \frac{U}{R_{eq}},$$

with:

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} = \sum_{i=1}^{i=n} \frac{1}{R_i}.} \tag{6.7}$$

Example

In the electric circuit depicted below, the objective is to ascertain the equivalent resistance between terminals A and B. For the purposes of this analysis, the following values have been assumed: $R_1 = 5\,\Omega$, $R_2 = 10\,\Omega$, $R_3 = 15\,\Omega$



- the resistors with resistances R_2 and R_3 are connected in parallel de to their subjection to the same potential difference between nodes C and D. The resistance R_{23}, which represents this association is such that:

$$\frac{1}{R_{23}} = \frac{1}{R_2} + \frac{1}{R_3},$$

and then:

$$R_{23} = \frac{R_2\, R_3}{R_2 + R_3} = 6\,\Omega.$$

- The resistors with resistances R_1 and R_{23} are associated in series, given that the same electric current flows between nodes A and B. The resistance R_{eq} equivalent to this association is such that:

$$R_{eq} = R_1 + R_{23} = 11\,\Omega.$$

3 Joule's Law

It has been established that in order to sustain a constant electric current in a conductor, energy must be supplied to the conduction electrons in the metal. Some of this energy is dissipated in collisions with the crystal lattice, which increases the vibrational energy and the temperature of the material. This thermal effect of the electric current is referred to as the ***Joule effect***.

Let us consider the quantity $\mathrm{d}q$ of charge passing through a conductor from point A to point B under the action of an electric field. The work done by the electric forces is written as:

$$\mathrm{d}W = \mathrm{d}q\,(V_A - V_B) = I\,\mathrm{d}t\,(V_A - V_B)\,.$$

We write the Ohm's law as :

$$\Delta V = V_A - V_B = R\,I.$$

Subsequently, the expression for the electric force work is derived as follows:

$$\mathrm{d}W = \Delta V\, I\,\mathrm{d}t = R\,I^2\,\mathrm{d}t.$$

This energy is dissipated in the form of heat, and its value is equivalent to the electric electrical power such that:

$$\boxed{P = \frac{\mathrm{d}W}{\mathrm{d}t} = \Delta V\, I = R\,I^2.} \tag{6.8}$$

The electrical power dissipated by the Joule effect is constant over time, as ΔV and I remain constant.

In the case of a cylindrical metal conductor of length ℓ and cross-section S subjected to a potential difference ΔV between its two ends, and introducing current density and electrical conductivity, the expression for electrical power is given by:

$$P = \frac{\ell}{\sigma A} (j\,A)^2 = \frac{j^2}{\sigma} \ell\,A.$$

The power density dissipated by the Joule effect at any point on the conductor is then expressed as follows:

$$p = \frac{P}{\mathrm{V}} = \frac{P}{\ell\,A} = \frac{j^2}{\sigma}.$$

Given that the current density vector is proportional to the applied electric field, the power density can then be written as:

$$p = \frac{j^2}{\sigma} = \frac{j\,(\sigma\,E)}{\sigma} = j\,E.$$

The general formulation of the power density dissipated by the Joule effect is :

$$\boxed{p = \overrightarrow{j}\,\overrightarrow{E}.} \tag{6.9}$$

4 Electrical circuit components

4.1 Definition

- An electric circuit can be defined as a closed structure comprising a series of devices, known as ***dipoles***, connected by conducting wires that permit the flow an electric current. The resistance of the connecting wires is typically considered to be negligible in comparison to that of the dipoles.

- A dipole is defined as any device that is connected to an electrical circuit via two poles: the positive pole, through which the electric current enters, and the negative pole, through which the electric current leaves.

- Each dipole is characterised by a curve, known as the characteristic $I = f(U)$, which describes the response of the dipole (the electric current I flowing through it) to the application of a potential difference or electric voltage U between its two poles.

- If the characteristic curve of the dipole is a straight line, the dipole is said to be ***linear***. Should the characteristic curve pass through the origin, the dipole is classified as ***passive*** (resistors, coils, capacitors, etc.). In contrast, it is designed to be ***active*** (presence of an electric current even in the absence of an electric voltage).

- In an electrical circuit, a ***node*** represents a point of connection where three or more wires converge.
- In an electrical circuit, a ***branch*** is defined as a section of the circuit situated between two nodes.
- In an electrical circuit, a ***loop*** is a configuration of branches forming a closed loop.

4.2 Electromotive force - Electric generator

It has been demonstrated that a constant electric current can be sustained in an electric circuit by maintaining a potential difference between the terminals of the various dipoles that comprise the circuit. This is the role of the electric generator, which moves mobile electric charges and transports electric energy through the circuit.

There are two types of generators:

- The voltage generator, which maintains a constant potential difference at its terminals regardless of the external circuit.
- The current generator, which delivers a constant current regardless of the external circuit.

the subsequent discussion will focus on voltage generators.

Remark

It is a common misconception that an electrical generator produces energy. In fact, it is merely transforms one form of energy (such as mechanical, chemical or light) into electrical energy.

- A battery, or accumulator, when discharged, converts chemical energy into electrical energy. It is an electrochemical generator.
- A dynamo or alternator converts mechanical energy into electrical energy. It is an electromechanical generator.

4.2.1 Electromotive force and internal resistance

The experimental characteristic curve $U = f(I)$ of a voltage generator is a decreasing straight line that does not pass through the origin:

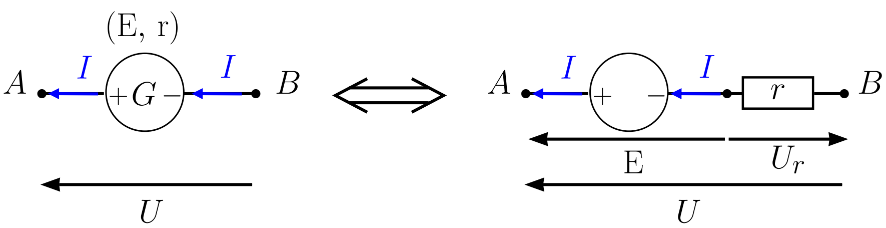
$$U = -a\,I + b \qquad ; \quad a \text{ and } b \text{ positive constants}$$

- The coefficient b has the dimension of an electrical voltage. It represents the no-charge (open circuit) voltage of the generator ($I = 0$). This voltage is referred to as the ***electromotive force***, or ***emf*** of the generator and is represented by the symbol E.

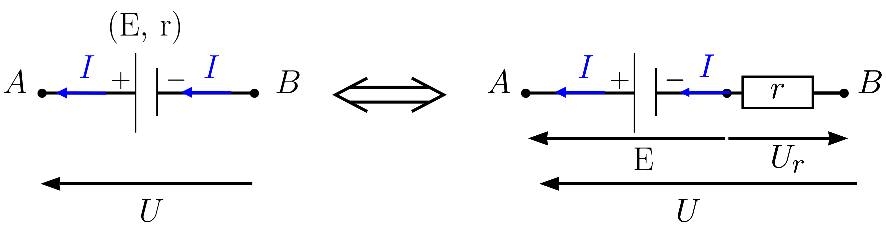
- The coefficient a is a quantity of dimension of resistance. This is the ***internal resistance*** of the generator, which is denoted r.

In an electrical circuit, a voltage generator, denoted (E, r), is modelled by the equivalent diagram shown in the figure below. It consists of an *emf* (E) in series with a resistor of resistance r, representing its internal resistance. in accordance with the convention, which is also known as the ***generator convention***, the potential difference, $\Delta V = \mathrm{E}$, is represented by an arrow pointing towards the higher potential (from the negative pole to the positive pole) in the same direction as the electric current.

If the generator has no internal resistance, it is said to be ***ideal***. In the following sections, the focus will be on the battery, for which an equivalent diagram is provided in the figure below.



Voltage Generator



Battery

The voltage at the terminals A (positive) and B (negative) of a voltage generator can thus be expressed as follows:

$$U = V_A - V_B = \mathrm{E} - r\,I.$$

Thus, the *emf* of a generator is given by:

$$\boxed{\mathrm{E} = U + r\,I.} \tag{6.10}$$

In a closed electrical circuit with a resistor of resistance R, the *emf* can be deduced from the following expression:

$$U = V_A - V_B = R\,I = \mathrm{E} - r\,I,$$

so that:

$$\mathrm{E} = (R + r)\ I.$$

4.2.2 Energy balance

The expression for the mechanical or chemical power received and transformed by a generator is given by:

$$P = \mathrm{E}\, I = (U + r\, I)\ I = U\, I + r\, I^2$$

- The initial term, $U\, I$, represents the power supplied by the alternator to the external circuit.
- - The second term, $r\, I^2$, represents the power dissipated in the generator by the Joule effect.

The efficiency of a generator, denoted by the symbol η, is defined as the ratio between the usable power at its terminals (i.e., the power supplied to the external circuit) and the power received and transformed by the generator. Its expression is given by:

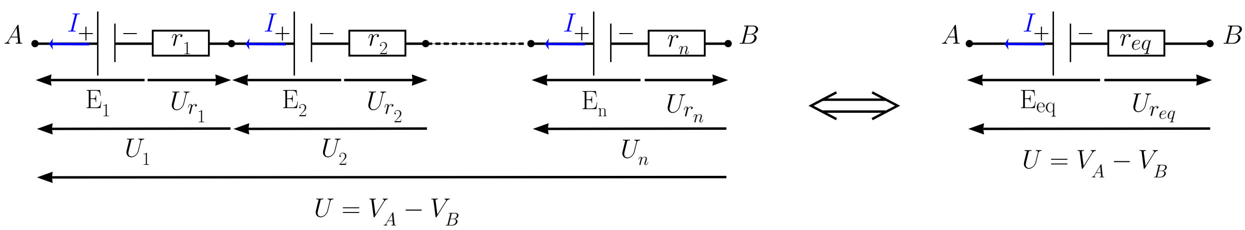
$$\eta = \frac{U\, I}{\mathrm{E}\, I} = \frac{V_A - V_B}{\mathrm{E}}.$$

- In the case of an ideal generator ($r = 0$), the usable voltage at its terminals is equal to its *emf.*, and the efficiency is therefore equal to 1.
- In the case of a real generator ($r \neq 0$), the usable voltage at its terminals is always less than its *emf*, and the efficiency is always less than 1.

4.2.3 Connection of several generators

Series connection

Let us establish a connection between n voltage generators (E_i, r_i) ($i = 1, \ldots, n$) in manner that ensure that the positive pole of each generator is linked to the negative pole of the subsequent one (see figure below). The same electric current flows through the generators connected in this manner. This configuration of generators is referred to as a ***series connection***.



The net potential difference across the generators is equal to the sum of the voltages across each generator:

$$\begin{aligned} U = V_A - V_B &= U_1 + U_2 + \ldots + U_n \\ &= (\mathrm{E}_1 - r_1 I) + (\mathrm{E}_2 - r_2 I) + \ldots + (\mathrm{E}_n - r_n I) \\ &= (\mathrm{E}_1 + \mathrm{E}_2 + \ldots + \mathrm{E}_n) - (r_1 + r_2 + \ldots + r_n)\ I. \end{aligned}$$

This potential difference is analogous to that obtained with a single voltage generator(E_{eq}, r_{eq}), through which the same current flows, such as:

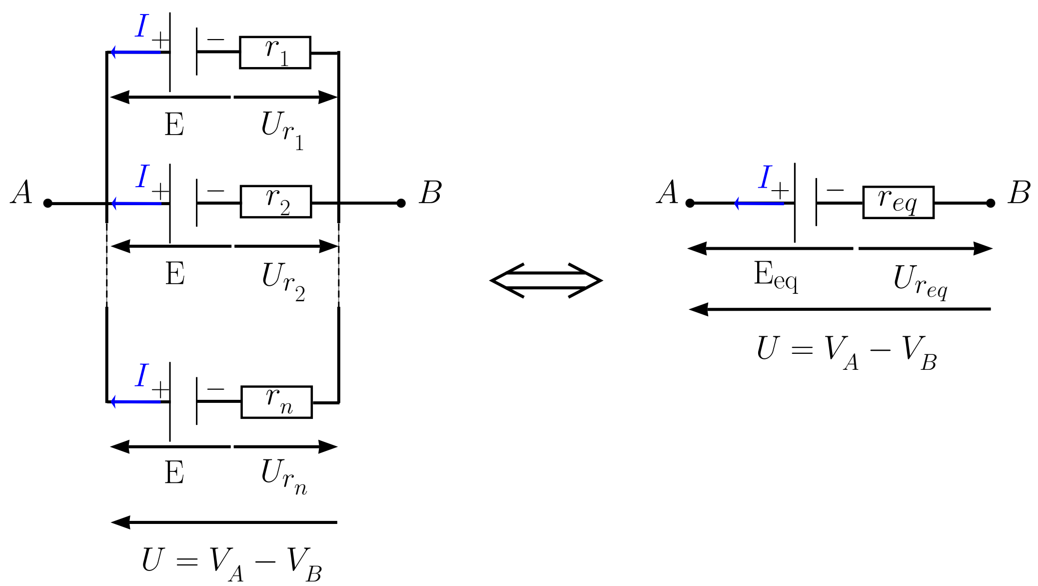
$$U = V_A - V_B = \mathrm{E}_{eq} - r_{eq} I,$$

with:

$$\boxed{\begin{cases} \mathrm{E}_{eq} = \mathrm{E}_1 + \mathrm{E}_2 + \ldots + \mathrm{E}_n = \sum_{i=1}^{i=n} \mathrm{E}_i \\ r_{eq} = r_1 + r_2 + \ldots + r_n = \sum_{i=1}^{i=n} r_i. \end{cases}} \tag{6.11}$$

4.2.4 Parallel connection

Let there be n generators (E_i, r_i) ($i = 1, \ldots, n$), of identical *emf* ($\mathrm{E}_i = \mathrm{E}$), connected in such a way that they are subjected to the same potential difference $U = V_A - V_B$ (see figure below). This connection of generators is known as a ***parallel connection***.



The net electrical current delivered by all these generators can be expressed as follows:

$$\begin{aligned} I &= I_1 + I_2 + \ldots + I_n \\ &= \frac{\mathrm{E} - U}{r_1} + \frac{\mathrm{E} - U}{r_2} + \ldots + \frac{\mathrm{E} - U}{r_n} \\ &= (\mathrm{E} - U)\left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\right). \end{aligned}$$

This current can be considered analogous to that delivered by a single voltage generator (E_{eq}, r_{eq}), subjected to the same potential difference, such that:

$$I = \frac{\mathrm{E}_{eq} - U}{r_{eq}},$$

with:

$$\boxed{\begin{cases} \mathrm{E}_{eq} = \mathrm{E}_1 = \mathrm{E}_2 = \ldots = \mathrm{E}_n = \mathrm{E} \\ \frac{1}{r_{eq}} = \frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n} = \sum_{i=1}^{i=n} \frac{1}{r_i}. \end{cases}} \tag{6.12}$$

4.3 Back electromotive force - Electrical receptor

An ***electrical receptor*** is defined as a dipole that, when an electric current flows through it, transforms the received electrical energy into another form of energy. There are two main types of electrical receptors:

- ***Active receptors*** that convert the electrical energy received into mechanical energy (for example, electric motors), chemical energy (for example, charged batteries) or light (for example, fluorescent tubes).

- ***Passive receptors***, such as resistors, which dissipate the electrical energy they are exposed to as heat.

4.3.1 Back electromotive force and internal resistance

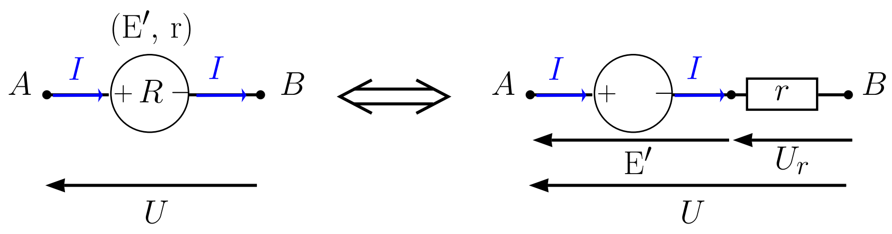
The experimental characteristic curve $U = f(I)$ of an active receiver is an increasing straight line that does not pass through the origin:

$$U = a\,I + b \qquad ; \quad a \text{ and } b \text{ positive constants}$$

- The coefficient b has the dimension of an electric voltage, and represents the back electromotive force, or ***bemf***, of the receptor. It is commonly denoted E'.

- The coefficient a is expressed in units of resistance. This is the internal resistance of the receptor, denoted r.

In an electrical circuit, a receptor (E', r) is modelled by the equivalent diagram shown in the figure below. If the receptor has no internal resistance, it is said to be ***ideal***. In the receptor convention, the potential difference $\Delta V = E'$ across the *bemf* is represented by an arrow pointing towards the higher potential (from the negative pole to the positive pole) in the opposite direction to the electric current.



The voltage at terminals A (positive) and B (negative) of an active receptor is therefore expressed as follows:

$$U = V_A - V_B = E' + r\,I,$$

and the expression for the *bemf* of a receptor is therefore:

$$E' = U - r\,I.$$

The *bemf* is therefore the minimum voltage that must be applied to an active receptor in order for it to convert the electrical energy it receives into energy other than heat.

4.3.2 Energy balance

The electrical power received by the receptor is expressed as follows:

$$P = U\,I = (E' + r\,I)\;I = E'\,I + r\,I^2.$$

- The initial term, $E'\,I$, represents the power supplied by the generator to the external circuit, which may be of a mechanical, chemical, or other nature.
- The second term, $r\,I^2$, represents the power dissipated by the Joule effect in the receptor.

The efficiency η of a receptor is defined as the ratio between the usable power at its terminals to the power received and transformed by the receptor. it can be expressed as follows:

$$\eta = \frac{E'\,I}{U\,I} = \frac{E'}{V_A - V_B}.$$

- In the case of an ideal receptor ($r = 0$), the usable voltage at its terminals is identical to its *bemf* . Consequently, the efficiency is equal to 1.

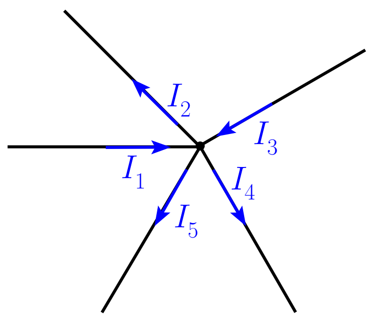
- In contrast, in the case of a real receptor ($r \neq 0$), the usable voltage at its terminals is always less than its *bemf.* Therefore, the efficiency is always less than 1.

5 Kirchhoff's laws

the principles of electrokinetics are founded upon the laws of conservation of current and electrical energy, which are collectively known as the ***Kirchhoff Laws***.

5.1 Conservation of the electrical current - Kirchhoff's current law

Consider any node in an electric circuit comprising a number n of electric wires carrying different electric currents $I_i(i = 1, \cdots, n)$ (see adjacent figure).



In the steady state, the principle of conservation of electric charge dictates that:

No accumulation of electric charge can occur at any point in an electric circuit.

This provides the first Kirchhoff's law, or ***Kirchhoff's law of conservation of electric current***, which is more commonly referred to as the ***Kirchhoff's current law***. This states that:

The total current entering a node is equal to the total current leaving it.

the law can be written as follows:

$$\sum I_{enter} = \sum I_{exit}.$$

In the case of the adjacent diagram :

$$I_1 + I_3 = I_2 + I_4 + I_5.$$

in other words:

The algebraic sum of all currents entering and leaving the node must be zero.

The law can be written as follows:

$$\sum_{i=1}^{n} I_i = 0.$$

I_i represents the algebraic value of the intensity of the current. in accordance with this definition, a plus sign (+) is assigned to this value when the current enters the node, and a minus sign (−) is assigned when the current leaves the node.

5.2 Conservation of the electrical energy - Kirchhoff's voltage law

In an electrical circuit, consider a loop consisting of n branches containing resistors, *emfs* and *bemfs*.

In the receiver convention, the voltage across each branch i is given by the general expression

$$U_i = R_i I_i + \mathrm{E}_i^{'} - \mathrm{E}_i,$$

where R_i, I_i, E_i, and $\mathrm{E}_i^{'}$ are, respectively the total resistance, current, *emf* and *bemf* contained in this branch.

the principle of conservation of electric energy states that:

The algebraic sum of the voltages occurring in a loop is equal to zero.

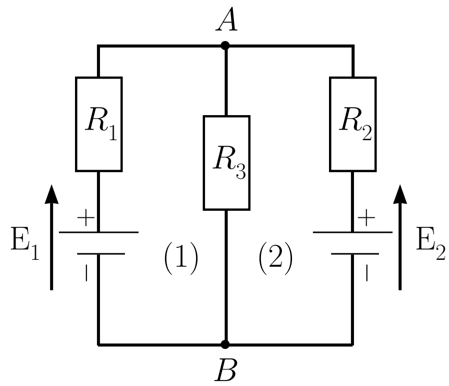
Indeed, the total potential difference in a loop is zero, as the same point of electrical potential is traversed when the loop is both initiated and completed. Thus, we can express the following:

$$\sum_{i=1}^{n} U_i = \sum_{i=1}^{n} R_i I_i + \mathrm{E}_i^{'} - \mathrm{E}_i = 0.$$

This yields the second Kirchhoff's law, or ***Kirchhoff's law of conservation of electric energy***, more commonly known as the ***Kirchhoff's voltage law***.

5.3 Application example

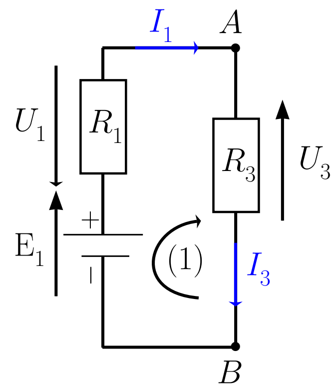
The adjacent electrical circuit, comprising two nodes (A and B) and three branches, is considered.



- Loop 1 contains the following components: E_1, R_1 and R_3.
- Loop 2 contains the following components: R_3, R_2 and E_2.
- Loop 3 contains the following components: E_1, R_1, R_2 and E_2.

Although the direction of the *emf* arrows is known (coming from the positive pole), the direction of the various currents is not always known.

Let us consider loop 1, for which we will apply the second Kirchhoff's law:



- The electrical currents (I_1 et I_3) in each branch are assigned an arbitrary direction and the voltages across each resistor are represented using the receptor convention.
- An arbitrary direction is selected for the loop.
- The sign convention is employed whereby voltages whose representative arrow is in the direction of travel of the loop are assigned a sign of (+), and voltages whose representative arrow is in the opposite direction to the direction of travel of the loop are assigned a sign of (−).
- The Kirchhoff's voltage law is then expressed as follows: $E_1 - U_1 - U_3 = E_1 - R_1 I_1 - R_3 I_3 = 0$.

Subsequently, the Kirchhoff's voltage law is written for each of the other two loops, following the same procedure. This yields a system of threee equations with three variables, which enables the calculation of the three current intensities I_1, I_2 and I_3.

it should be noted that if the calculation of a current intensity yields a positive value, then the arbitrary direction chosen for this current is correct. Otherwise, the real current is in the opposite direction to the one chosen initially.

5.4 General method for applying Kirchhoff's laws

Let us consider the general case of an electric circuit comprising N nodes and B branches. The objective is to calculate the different current intensities I_i $(i = 1, \cdots, B)$ in each of the B branches of the circuit. This necessitates the resolution of a system of B equations with B variables.

once an arbitrary direction has been selected for each of the B currents in each branch, the subsequent procedure is as follows:

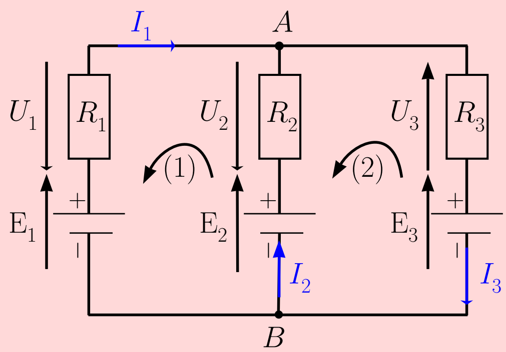
- The Kirchhoff's current law is written for the remaining $N-1$ nodes, resulting in $N-1$ equations.
- The Kirchhoff's voltage law is then written for the remaining $B-(N-1)$ loops, resulting in the necessary $B-(N-1)$ equations.
- This process yields the system of B equations required to determine the B current intensities. Solving this system will yield the required solution.

Example

The aim of this calculation is to determine the different currents flowing in the electrical circuit shown below with:

$E_1 = 110\,V$, $E_2 = 105\,V$, $E_3 = 90\,V$

$R_1 = 0,5\,\Omega$, $R_2 = 0,25\,\Omega$, $R_3 = 0,5\,\Omega$



1. Let us select an arbitrary direction for the three currents I_1, I_2, and I_3, which flow in the three branches of the circuit ($B = 3$). The differing voltages across the resistors are represented by arrows, in accordance with the receptor convention.
2. The circuit contains two nodes ($N = 2$). Kirchhoff's current law is expressed for a single node ($N - 1 = 1$). To illustrate, for node A:

$$I_1 + I_2 = I_3.$$

3. The number of loops is three, yet the Kirchhoff's voltage law is written for only two loops ($B - (N - 1) = 3 - 1 = 2$), with any direction chosen for these two loops. To illustrate:
   a) For Loop (1) containing E_1, R_1, R_2, and E_2, we can write that:

$$E_2 - U_2 + U_1 - E_1 = E_2 - R_2 I_2 + R_1 I_1 - E_1 = 0.$$

For Loop (2) containing E_2, R_2, R_3, and E_3, we can write:

$$\mathrm{E}_3 + U_3 + U_2 - \mathrm{E}_2 = \mathrm{E}_3 + R_3 I_3 + R_2 I_2 - \mathrm{E}_2 = 0.$$

4. This gives us the following system of three equations with three variables to solve:

$$\begin{cases} R_1 I_1 - R_2 I_2 + 0 & = \mathrm{E}_1 - \mathrm{E}_2 \\ 0 \ + R_2 I_2 + R_3 I_3 & = \mathrm{E}_2 - \mathrm{E}_3 \\ I_1 \ + I_2 \ - \ I_3 & = 0, \end{cases}$$

and then:

$$\begin{cases} 0,5\, I_1 - 0,25\, I_2 + 0 & = 5 \\ 0 \ + 0,25\, I_2 + 0,5\, I_3 & = 15 \\ I_1 \ + I_2 \ - \ I_3 & = 0. \end{cases}$$

This system can be solved using the Cramer method.
The determinant of the system is given by:

$$D = \begin{vmatrix} 0,5 & -0,25 & 0 \\ 0 & 0,25 & 0,5 \\ 1 & 1 & -1 \end{vmatrix} = -0.5$$

The system of equations possesses a unique solution $(I_1,\, I_2,\, I_3)$ due to the non-zero value of D $(D \neq 0)$. This solution is:

$$I_1 = \frac{\begin{vmatrix} 5 & -0,25 & 0 \\ 15 & 0,25 & 0,5 \\ 0 & 1 & -1 \end{vmatrix}}{-0,5} = \frac{-7,5}{-0,5} = 15\,\mathrm{A},$$

$$I_2 = \frac{\begin{vmatrix} 0,5 & 5 & 0 \\ 0 & 15 & 0,5 \\ 1 & 0 & -1 \end{vmatrix}}{-0,5} = \frac{-5}{-0,5} = 10\,\mathrm{A},$$

$$I_3 = \frac{\begin{vmatrix} 0,5 & -0,25 & 5 \\ 0 & 0,25 & 15 \\ 1 & 1 & 0 \end{vmatrix}}{-0,5} = \frac{-12,5}{-0,5} = 25\,\mathrm{A}.$$

Chapter 7

Magnetic force and field

1 Introduction

In order to illustrate the concept of magnetic force, let us consider the example of a metal wire surrounded by iron filings. When an intense electric current is flowing through the wire, the small iron particles are deposited in a circular network around the wire (see Figure 7.1). It is postulated that the electric current has exerted a magnetic force on the other moving charges or currents inside the iron particles. As with the electric force (Coulomb's Law), which is explained by the presence of an electric field induced by a source charge, the magnetic force can also be described in terms of the magnetic field produced by this source current.

Figure 7.1: Imprinting the magnetic field of a current-carrying wire using iron filings (credit).

Historically, magnetic interaction is the oldest known interaction when compared with the other interactions (gravitational and electrical). It has long been established that certain ores, such as magnetite (iron oxide, Fe_3O_4), possess the innate capacity to attract small fragments of iron. This natural property is also exhibited by various pure metals, including iron, cobalt, manganese, and alloys composed of these same metals. These materials are known as magnetic. A magnetic body is also referred to as a magnet. In its natural state, the magnet is said to be permanent. The magnetic force is thus defined as the force exerted by one magnet on another. This force, similar to electrical force, can exhibit either an attractive or repulsive nature. When two bar magnets are brought into close proximity, one pole of the first bar will attract one pole of the second bar and will repel the other pole. Consequently, each bar magnet possesses two distinct poles, designated as the north pole and the south pole.

This attraction or repulsion between the poles of a permanent magnet is responsible for the behaviour of the magnetised needle of a compass, which is free to rotate around a fixed pivot. When positioned in close proximity to a bar magnet, the needle undergoes a rotational movement until it attains a state of equilibrium, wherein its north pole is oriented towards the south pole of the bar magnet (see opposite figure). Once the influence of the bar magnet has been removed, the compass needle is subject to the exclusive influence of the magnetic force exerted by the Earth. The Earth behaves as a permanent magnet, with the geographical poles of the Earth aligning approximately with the magnetic poles. The magnetic force exerted by the Earth will cause the needle to rotate towards an equilibrium configuration in which its north pole is oriented towards the Earth's magnetic south pole, close to its geographic north pole. Its south pole is thus oriented towards the Earth's north magnetic pole, in close proximity to its south geographic pole. The phenomenon of attraction or repulsion between two permanent magnets can be summarised as follows:

Magnetic forces push like magnetic poles (north-north or south-south) apart and pull unlike magnetic poles together (north-south or south-north).

Although the existence of a magnetic force between two natural magnets had been known for centuries, it is only in the 19th century that experiments, performed by Hans Christian Oersted (1777-1851), demonstrated that an electric current flowing through a metal wire caused the magnetised needle to deviate towards an equilibrium configuration, resulting in it being oriented perpendicular to the metal wire. The experiment also demonstrated the existence of a magnetic force between two metal wires carrying electric currents. It has been demonstrated that metal wires carrying an electric current exhibit behaviour similar to that of non-permanent magnets, due to the fact that the magnetic force exerted disappears when the electric current does. In accordance with the principle of electrical force (Coulomb's law), it can thus be concluded that:

The magnetic force is exerted between electric currents, or more generally, between moving electric charges.

The magnetic forces observed between permanent magnets can be explained by the same mechanism as that involving electric currents or moving electric charges. Indeed, the phenomenon of magnetic forces between two permanent bar magnets has its origins in microscopic currents. These currents consist of the movement of charges between the atoms that constitute the magnet. Given the substantial number of atoms constituting the bar and the combination of these microscopic currents, the macroscopically induced magnetic force is significant enough to be perceived by another magnet, another current, or other nearby moving charges.

In a manner analogous to the conceptualisation of an electric charge, where the surrounding space is characterised by the concept of an electric field, the concept of a magnetic field facilitates the characterisation of the space surrounding a permanent magnet, an electric current, or a moving electric charge. In this chapter, the magnetic force exerted on a moving charge is expressed in terms of a magnetic field, in a manner analogous to the expression of the electric force in terms of an electric field. In addition, the generation of magnetic fields in response to currents will be investigated, encompassing various methodologies and established principles for the calculation of magnetic fields produced by specified current distributions.

2 Magnetic force

It is conceivable to conceptualise the formulation of magnetic forces between two moving electric point charges in a manner analogous to that of Coulomb's law between two static electric point charges. However, the strong dependence of the magnitude and direction of this magnetic force on the intensities and directions of the velocity vectors of these charges renders this formulation very complicated. It can thus be concluded that the magnetic force exerted on a moving point electric charge by an electric current flowing through a long metal wire, at a distance r from this charge, would be more readily apparent. This formulation will be regarded as a fundamental law of physics, substantiated by empirical evidence. This formulation bears a certain resemblance to the electric force exerted by a long charged wire on a point charge. The similarity with the electric force lies in the inverse proportionality to the distance r of the magnetic force. The distinction can be attributed to the dependence of the magnetic force, both in magnitude and direction, on the intensity and direction of the velocity vector $\overrightarrow{v}$ of the moving charge. In order to illustrate the aforementioned dependence, three distinct situations will be considered. These situations will involve an electric current flowing through a metal wire along the Ox axis in the direction $x > 0$ (see figure).

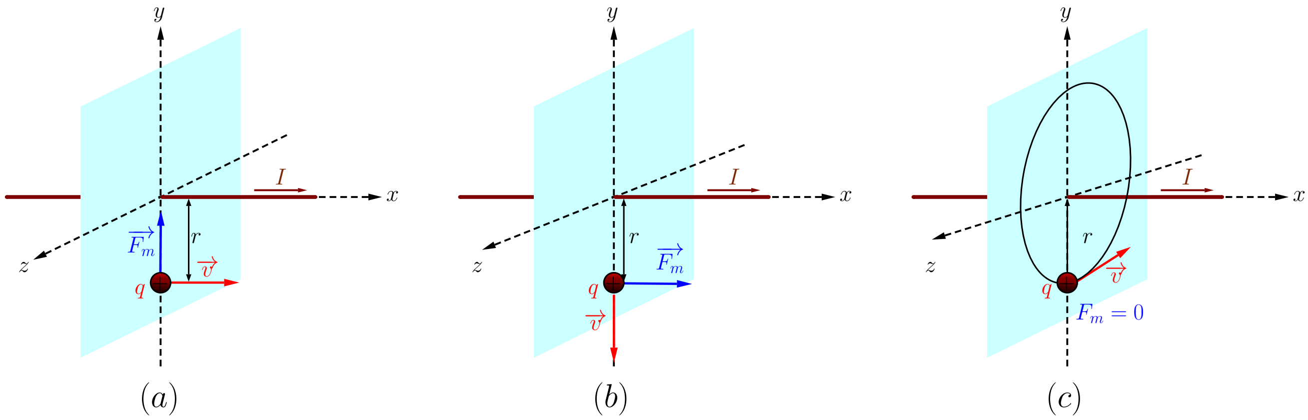


Figure 7.2: The schematic illustrate the direction of the magnetic force exerted, by a current flowing in the x direction, on a charge q (assumed to be positive) moving with a velocity in three possible directions.

1. $\vec{v}$ is oriented along the x-axis, , in parallel to the current I flowing through the wire (see Figure 7.2-a):
   - The magnetic force $\vec{F_m}$ is oriented radially (perpendicular to the direction of the electric current) along the y-axis.
   - In the case of a moving charge $q > 0$, $\vec{F_m}$ is attractive ($\vec{F_m} = F_m \vec{j}$) if $\vec{v}$ is directed in the same direction as I ($\vec{v} = v\vec{i}$), and repulsive ($\vec{F_m} = -F_m \vec{j}$) if $\vec{v}$ is directed in the opposite direction to I ($\vec{v} = -v\vec{i}$).
   - In the case of a moving charge $q < 0$, $\vec{F_m}$ is attractive ($\vec{F_m} = F_m \vec{j}$) if $\vec{v}$ is in the opposite direction to I ($\vec{v} = -v\vec{i}$), and repulsive ($\vec{F_m} = -F_m \vec{j}$) if $\vec{v}$ is in the same direction as I ($\vec{v} = v\vec{i}$).
   - The magnitude of the magnetic force is hereby determined as follows :

$$F_m = \frac{\mu_0}{2\pi}\frac{|q| \cdot |\vec{v}| \cdot I}{r}. \tag{7.1}$$

     μ_0 is the *magnetic permeability of free space*. Its value in the SI units is: $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Kg \cdot m \cdot A^{-2} \cdot s^{-2}}$.

2. $\vec{v}$ is oriented along the y-axis, perpendicularly to the direction of the current I (see Figure 7.2-b):
   - The magnetic force $\vec{F_m}$ is oriented in the same direction as the electric current, along the x-axis.
   - In the case of a moving charge $q > 0$, $\vec{F_m}$ is oriented in the same direction as I ($\vec{F_m} = F_m \vec{i}$) if $\vec{v}$ is radially outward from the wire ($\vec{v} = -v\vec{j}$), and in the opposite direction to I ($\vec{F_m} = -F_m \vec{i}$) if $\vec{v}$ is radially inward towards the wire ($\vec{v} = v\vec{j}$).
   - In the case of a moving charge $q < 0$, $\vec{F_m}$ is in the same direction as I ($\vec{F_m} = F_m \vec{i}$) if $\vec{v}$ is directed into the wire ($\vec{v} = v\vec{j}$), and in the opposite direction of I ($\vec{F_m} = -F_m \vec{i}$) if $\vec{v}$ is directed out of the wire ($\vec{v} = -v\vec{j}$).

- The magnitude of magnetic force is hereby expressed as follows: $F_m = \frac{\mu_0}{2\pi} \frac{|q| \cdot |\overrightarrow{v}| \cdot I}{r}$.

3. $\overrightarrow{v}$ is directed tangentially to the circumference of a circle that is concentric with the wire, in the (yz) plane (see Figure 7.2-c):

   - The magnetic force is zero : $F_m = 0$.

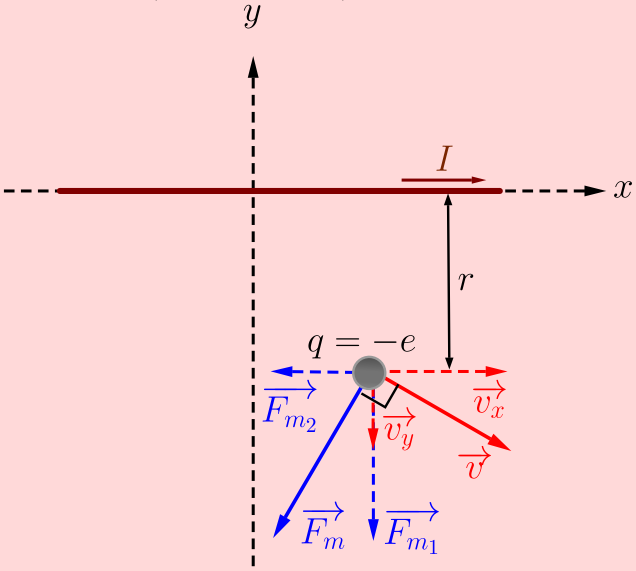
In the more general case where the velocity is oriented in an arbitrary direction, that is outside the three basic directions previously mentioned, the velocity vector can be divided into three components along these directions. The magnetic force is then obtained by vectorial summation of the contributions in each direction.

It is imperative to acknowledge that, in all instances:

> *The magnetic force vector is perpendicular to the velocity vector of the moving point charge.*

Example

Let us consider a long, straight wire carrying a current $I = 10\,\mathrm{A}$ along the x-axis and oriented in $x > 0$. An electron is moving in the vicinity of the wire with a velocity $\overrightarrow{v} = 8.7 \times 10^5 \overrightarrow{i} - 5.0 \times 10^5 \overrightarrow{j} \left(\mathrm{m \cdot s^{-1}}\right)$.
We are calculating the magnetic force exerted by the wire on the mobile charge when it attains the point M at a distance of $r = 2\,\mathrm{cm}$ (see figure below).



The velocity vector is not oriented in one of the three basic cases previously studied. Thus we split the velocity into two components along the parallel and radial directions respectively.

- For the parallel velocity contribution $\overrightarrow{v_x} = 8.7 \times 10^5 \overrightarrow{i}$, the magnetic force $\overrightarrow{F_{m_1}}$ is radial and repulsive because $q = -e < 0$. Therefore, we have:

$$\overrightarrow{F_{m_1}}(M) = -\frac{\mu_0}{2\pi} \frac{|e| \cdot |\overrightarrow{v_x}| \cdot I}{r} \overrightarrow{j}.$$

- For the radial velocity contribution $\overrightarrow{v_y} = -5.0 \times 10^5 \overrightarrow{i}$, the magnetic force $\overrightarrow{F_{m_2}}$ is along the x-axis, in the opposite direction of the current flow because $q = -e < 0$. Therefore, we have:

$$\overrightarrow{F_{m_2}}(M) = -\frac{\mu_0}{2\pi} \frac{|e| \cdot |\overrightarrow{v_y}| \cdot I}{r} \overrightarrow{i}.$$

Consequently, the total magnetic field exerted on the moving electron is :

$$\boxed{\overrightarrow{F_m}(M) = -\frac{\mu_0}{2\pi} \frac{|e| \cdot I}{r} \left[|\overrightarrow{v_y}| \overrightarrow{i} + |\overrightarrow{v_x}| \overrightarrow{j} \right].}$$

N.A.: $\boxed{\overrightarrow{F_m}(M) = -1.6 \times 10^{-24} \left[5.0 \times 10^5 \overrightarrow{i} + 8.7 \times 10^5 \overrightarrow{j} \right]}$ $\Longrightarrow$ $\boxed{\left| \overrightarrow{F_m}(M) \right| \approx 1.6 \times 10^{-18} \, \text{N}.}$

3 Magnetic field

As demonstrated previously in the case of static electric charges, the electric force exerted at a distance between two charges is communicated via the electrostatic field created by one of the charges and felt by the other. Similarly, the magnetic force exerted at a distance by a moving electric charge (or an electric current) on another moving electric charge is also communicated via a magnetic field created by one of the two moving charges and felt by the second.

In the preceding example concerning a point charge moving in the vicinity of a long, straight wire through which an electric current flows (see Figure 7.2), the expression 7.1 for the magnitude of the magnetic force in the first two cases where the speed is parallel or perpendicular to the direction of the electric current (see Figure 7.2-a and Figure 7.2-b) should be considered. The quantity associated with the moving charge should be isolated. The aforementioned assertion may then be expressed in the following terms:

$$F_m = \frac{\mu_0}{2\pi} \frac{|q| \cdot |\overrightarrow{v}| \cdot I}{r} = |q| \cdot |\overrightarrow{v}| \left[\frac{\mu_0 I}{2\pi r} \right].$$

The term enclosed in square brackets denotes the *Magnetic field generated by the electric current I at a distance r*. This is symbolised by B.

The magnitude of the magnetic force can then be expressed as follows:

$$F_m = |q| \cdot |\overrightarrow{v}| \cdot B.$$

Definition 1

An electric current I along a long, straight wire produces a vectorial ***magnetic field*** $\overrightarrow{B}$ at an arbitrary distance r.

- Its magnitude is given by:

$$\boxed{B = \frac{\mu_0 I}{2\pi r}} \tag{7.2}$$

- In a manner akin to the electric field, the magnetic field is inversely proportional to the distance r.

- In contrast to the electric field, which is radial, the magnetic field is tangential. Its direction is tangent to a circle, with the centre of the circle coinciding with the current. The determination of the magnetic field direction can be achieved through the implementation of the *right-hand rule*: In the event of the thumb of the right hand being placed in the direction of the current, the fingers will curl around the wire in the direction of the magnetic field (see Figure 7.3).

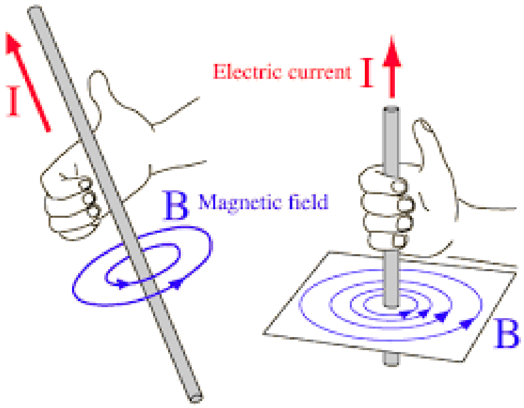
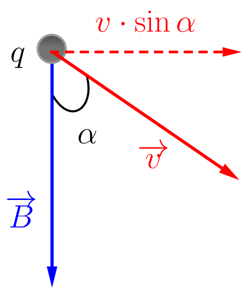


Figure 7.3: The right-hand rule is applied to determine the direction of the magnetic field exerted by an electric current through a wire.



A re-examination of the three preceding cases of Figure 7.2 reveals that, in cases 1 and 2, the velocity is oriented in a direction parallel and perpendicular, respectively, to the direction of the electric current. Consequently, it is perpendicular to the tangential direction of the magnetic field. In the event of the direction of velocity being parallel to the direction of the magnetic field (case 3 of Figure 7.2), the magnetic force is zero. The absence of magnetic force in the latter case signifies that, for the more general case of a charge moving in a direction that makes a given angle alpha with the direction of the magnetic field, only the component of the velocity perpendicular to the magnetic field ($v \cdot \sin\alpha$) contributes to the magnetic force. It can thus be concluded that the general expression for the magnitude of the magnetic force is as follows:

$$F_m = |q| \cdot |\overrightarrow{v}| \cdot B \cdot \sin\alpha.$$

- The magnetic force is zero when movement occurs in the direction parallel to the direction of the magnetic field $\overrightarrow{B}$ ($\alpha = 0$ ou $\alpha = \pi$).

- The magnitude of the magnetic force is at its maximum when movement occurs perpendicular to the direction of the magnetic field $\overrightarrow{B}$ ($\alpha = \frac{\pi}{2}$ ou $\alpha = \frac{3\pi}{2}$).

The vector expression of the magnetic force, taking into account its amplitude and direction, is expressed as follows:

$$\overrightarrow{F_m} = q\,\overrightarrow{v} \times \overrightarrow{B} \tag{7.3}$$

In summary, it can be stated that:

- The magnetic force exerted on a moving charge is directly proportional to its charge, its velocity, and the magnetic field to which it is subjected.
- The magnetic force is perpendicular to the plane formed by the velocity vector $\overrightarrow{v}$ and the magnetic field vector $\overrightarrow{B}$, such that the trihedron ($\overrightarrow{v}$, $\overrightarrow{B}$,$\overrightarrow{F}$) is direct.
- The work of the magnetic force is zero ($W\left(\overrightarrow{F_m}\right) = 0$), as its direction is perpendicular to the trajectory of the moving charge. Consequently, there is an absence of any variation in kinetic energy.
- In the more general case where the moving charge is subjected to an electric force and a magnetic force, the resultant of these two forces is the ***Lorentz force***, given by the following expression:

$$\overrightarrow{F} = \overrightarrow{F_e} + \overrightarrow{F_m} = q\,\overrightarrow{E} + q\,\overrightarrow{v} \times \overrightarrow{B} = q\left(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}\right) \tag{7.4}$$

Having established the manner by which the magnetic field resulting from a current flowing along a straight wire can be calculated, it is now possible to define the magnetic field produced by a distribution of moving charges or currents at a given point in space. This can be achieved by introducing test charges moving at different velocities and passing through that point.

Definition 2

The direction and magnitude of a magnetic field $\overrightarrow{B}$ at a given point in space are determined by the following set of rules:

- The magnitude is derived in instances where the test charge is moving perpendicular to its direction ($\alpha = 0$) and is subjected to maximum magnetic force:

$$\boxed{B = \frac{F_m}{q\,v}} \tag{7.5}$$

- The direction of the magnetic field must align either parallel or antiparallel to the direction motion resulting in zero magnetic force.
- The direction of the magnetic field can be determined by the right-hand rule, which is a method of identifying the direction of the magnetic force. In the event of orienting the fingers

of the right hand along the direction of $\overrightarrow{v}$ ad curling them towards the direction of $\overrightarrow{B}$ through the smallest angle between $\overrightarrow{v}$ and $\overrightarrow{B}$, the thumb will indicate the direction of the magnetic force $\overrightarrow{F_m}$ exerted by a positive test charge. In the event of a negative test charge, the direction of $\overrightarrow{F_m}$ is opposite to that obtained for a positive test charge.

- The magnetic field is defined as the force per unit charge and velocity. Its SI unit is the **tesla** (T) : $1\,\mathrm{T} = 1\,\mathrm{N} \cdot \mathrm{C}^{-1} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}$. Its cgs unit is the **gauss** (G) : $1\,\mathrm{G} = 10^{-4}\,\mathrm{T}$. For instance, the magnetic field at the surface of earth is $5 \times 10^{-5}\,\mathrm{T}$ and in MRI (Magnetic resonance Imaging) magnet is $1.5\,\mathrm{T}$.

4 Magnetic field lines

With regard to the electric field, it would be worthwhile to direct our attention to the geometric representation of the magnetic field. The magnetic field can be represented by field lines for which the tangent indicates its direction and the density indicates its relative magnitude. As illustrated in Figure 7.4, magnetic field lines are produced by a current flowing along a long, straight wire in two directions. It can be observed that as distance from the wire increases, there is a decrease in both the density of field lines, and the magnitude of the magnetic field. The direction of the magnetic field at each point in space is tangent to the field line at that particular point. The right-hand rule can be used to identify the direction of the magnetic field along these field lines as follows:

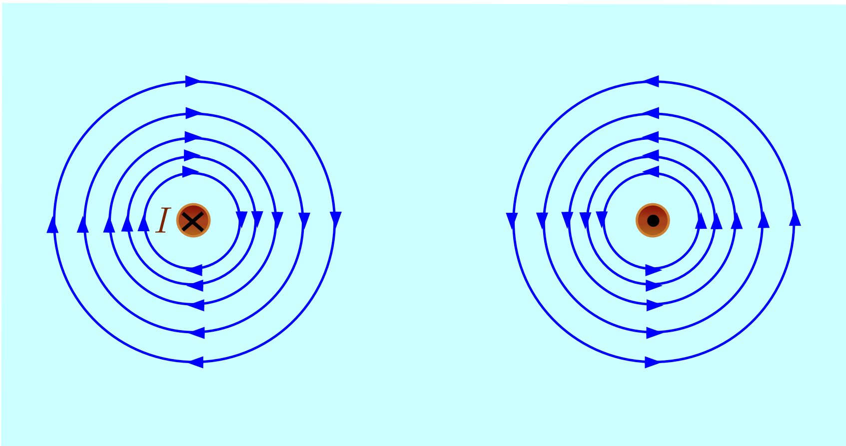


Figure 7.4: Magnetic field lines in the vicinity of an electric current flowing along a straight wire.

- Firstly, the right hand should be positioned in such a manner that the thumb is pointing in the direction of the current I.

- The next step is to close the right hand and curl the fingers around the axis of the wire. The direction of curvature of the fingers is thus found to correspond to the direction of the magnetic field along the field line.

Another illustration of magnetic field lines produced by a bar magnet is plotted in Figure 7.5. The field lines emerge from the north pole and enter into the south pole. The magnitude of the magnetic field is relatively stronger at the vicinity of the two poles.

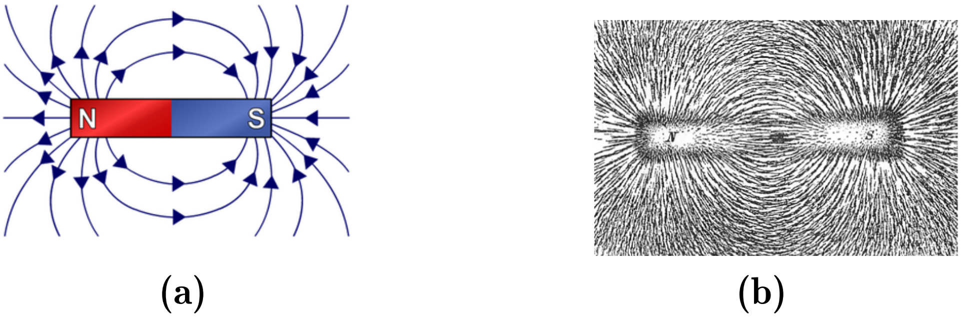


Figure 7.5: (a) Magnetic field lines in the vicinity of a bar magnet. (b) Magnetic field lines of a bar magnet revealed through the use of iron fillings, which are dispersed onto a sheet of paper positioned over the magnet.

It is important to note that, in contrast to the electric field, where field lines originate or terminate at positive or negative charges, magnetic field lines are closed loops and do have any defined starting or ending points. This property can be expressed by the total magnetic flux through a closed surface, of area A, which is:

$$\Phi_B = \oint \overrightarrow{B} \cdot \overrightarrow{A} = 0 \tag{7.6}$$

Equation 7.6 is the ***Gauss's Law for the magnetic field***. It states that the number of magnetic field lines entering any closed surface is equal to the number leaving it.

5 Some common laws of magnetism

5.1 Ampère's law

André-Marie Ampère (1775–1836) was the first to formulate the theorem that bears his name (Ampère's theorem or Ampère's law, 1822) and thereby establish a direct relationship between the circulation of the magnetic field over a closed path and the total electric current flowing through the surface bounded by that path (see Figure 7.6.

Ampère's Theorem

The circulation of the magnetic field along a closed, oriented path $(\mathcal{C})$ is proportional to the total electric current flowing through the area bounded by $(\mathcal{C})$:

$$\boxed{\oint \overrightarrow{B} \cdot \overrightarrow{\mathrm{d}\ell} = \mu_0 I_{\text{in}}.} \tag{7.7}$$

$\overrightarrow{\mathrm{d}\ell}$ is defined as the infinitesimal displacement vector.

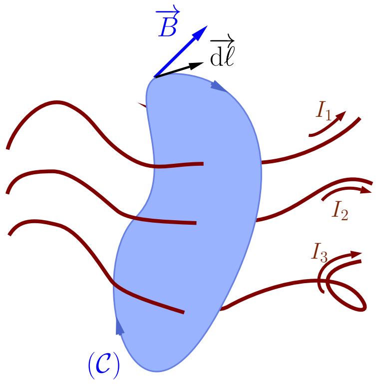


Figure 7.6: Arbitrary area bounded by a closed path, and intercepting some electric currents flowing on wires.

Remarks

- The Ampère's theorem constitutes a fundamental principle within the domain of magnetism, with the restriction that the currents must be characterised by steadiness. This equation is one of four Maxwell equations, which form the foundation of electromagnetism.
- It is imperative to note that solely the electric currents that traverse the surface, and are confined to the designated path, are to be considered. It is evident that external currents do not contribute to the circulation of the magnetic field.
- Ampere's theorem can be considered as a fundamental principle in magnetostatics, akin to Gauss's theorem in electrostatics. This theorem is employed to ascertain the magnetic field generated by an electric current or by a distribution (whether continuous or discontinuous) of electric currents.
- It is evident that the length and shape of the closed path $(\mathcal{C})$ are inconsequential to the result of the circulation of the magnetic field along this particular path. It is imperative to note that the sole factor that exerted an influence on the resultant value of the integral was the total quantity of current that passed through the surface delimited by each path.

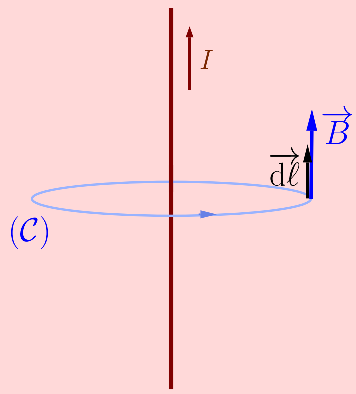
Ampère's theorem provides a highly effective analytic method for calculating the magnitude of a magnetic field of a given distribution of currents, assuming that this distribution possesses a high degree of symmetry.

In order to calculate the magnetic field at a point M in space, the following steps must be taken:

1. The direction of the magnetic field is determined by invoking symmetry arguments.
2. A suitable oriented closed path $(\mathcal{C})$ is selected :
   a) This path passes through the point M,
   b) the magnetic field at any given point is perpendicular or parallel to $(\mathcal{C})$, i.e. $\vec{B} \parallel \vec{d\ell}$ or $\vec{B} \perp \vec{d\ell}$ respectively.
3. The circulation of the magnetic field around this path is calculated.

Example 1: long, straight wire

In this instance, Ampère's theorem is to be applied in order to express the magnitude of the magnetic field produced by a current I, flowing in a long, straight wire, at a point M in space, distant by r from the wire.



We know that the magnetic field lines generated by this current are concentric circles. The magnitude of the magnetic field is constant along each of theses circles. The direction of the magnetic field, at each point M of space, is tangent to these circles and given by the right hand rule (see figure below).

By choosing a closed circular path $(\mathcal{C})$ of radius r,around the wire, following the circular field line at a distance of r, we have $\vec{B} \parallel \vec{d\ell}$ and B constant.
The Ampère's law can be written as:

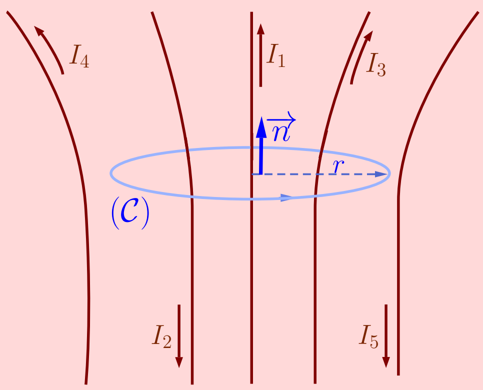
$$\begin{aligned} \oint \vec{B} \cdot \vec{d\ell} &= \mu_0 I_{\text{in}} \\ \oint B \cdot d\ell &= \mu_0 I \\ B \oint d\ell &= \mu_0 I \\ B \cdot 2\pi r &= \mu_0 I. \end{aligned}$$

As expected, this yields the magnetic field expression given by equation 7.2:

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}.$$

Example 2: discrete long, straight wires distribution

In this example, we have to ascertain the expression for the magnitude of the magnetic field generated by a discrete distribution of four long, straight wires, positioned in parallel and carrying electric currents I_1, I_2, I_3, I_4 and I_5, at a point M in space, at a distance r from the first wire carrying I_1(see figure below).



It is evident that by selecting a closed circular path $(\mathcal{C})$ of radius r, surrounding the wire that is flowed by I_1, and following its circular field line at a distance of r, we can ascertain that $\vec{B} \parallel \vec{d\ell}$, and B remains constant.

The Ampère's law can be written as:

$$\begin{aligned}\oint \vec{B} \cdot \vec{d\ell} &= \mu_0 I_{\text{in}} \\ B \cdot 2\pi r &= \mu_0 \sum_i I_{i_{\text{in}}},\end{aligned}$$

where $\sum_i I_{i_{\text{in}}}$ is the algebraic sum of the electric current intensities passing through the area delimited by the path $(\mathcal{C})$, i.e. electric currents I_1, I_2 et I_3. Depending on the orientation of $(\mathcal{C})$, electric currents I_1 et I_3 will be counted positively and electric current I_2 will be counted negatively. Thus we can write:

$$B \cdot 2\pi r = \mu_0 \left[I_1 - I_2 + I_3\right].$$

The magnitude of the magnetic field expression for this distribution of electric currents is therefore:

$$\boxed{B = \frac{\mu_0 \mu_0 \left[I_1 - I_2 + I_3\right]}{2\pi r}}.$$

Example 3: long, straight thick conducting wire with a cross section

We are now going to determine the expression for the magnitude of the magnetic field induced by a long, thick, straight conduction wire of cross-section A and radius R, through which a current I flows, uniformly distributed over its cross-section. (see figure below).

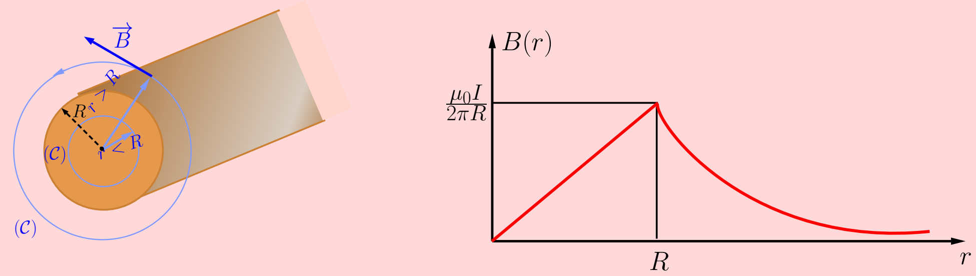
As the symmetry of this thick wire is identical to that of the thin wire treated in *Example 1*, the magnetic field lines are also concentric circles, both inside and outside the wire.

1. We determine the magnitude of the magnetic field at a point M inside the wire (at a distance $r < R$).
   By choosing a closed circular path $(\mathcal{C})$ of radius $r < R$, that follows its circular field line at a distance of r, we know that $\overrightarrow{B} \parallel \overrightarrow{\mathrm{d}\ell}$ and B constant.
   The Ampère's law can be written as:
   $$\begin{aligned} \oint \overrightarrow{B} \cdot \overrightarrow{\mathrm{d}\ell} &= \mu_0 I_{\text{in}} \\ B \cdot 2\pi r &= \mu_0 \iint j \cdot \mathrm{d}A_{\text{in}}, \end{aligned}$$
   where j is the electric current density given by:
   $$I = \iint j \cdot \mathrm{d}A = j \iint \mathrm{d}A = j \cdot \pi R^2.$$
   The magnetic field circulation is then written as:
   $$B \cdot 2\pi r = \mu_0 j \iint \mathrm{d}A_{\text{in}} = \mu_0 j \cdot \pi r^2 = \mu_0 \frac{I}{\pi R^2} \cdot \pi r^2.$$
   This provide an expression for the magnitude of the magnetic field at distance $r < R$:
   $$\boxed{B = \frac{\mu_0 I}{2\pi R^2}\, r}.$$

2. We determine the magnitude of the magnetic field at a point M outside the wire (at a distance $r > R$).
   By choosing a closed circular path $(\mathcal{C})$ of radius $r > R$, that follows its circular field line at a distance of r, we know that $\overrightarrow{B} \parallel \overrightarrow{\mathrm{d}\ell}$ and B constant.
   The Ampère's law can be written as:
   $$\begin{aligned} \oint \overrightarrow{B} \cdot \overrightarrow{\mathrm{d}\ell} &= \mu_0 I_{\text{in}} \\ B \cdot 2\pi r &= \mu_0 \iint j \cdot \mathrm{d}A \\ B \cdot 2\pi r &= \mu_0 j \cdot \pi R^2, \end{aligned}$$
   Knowimng that $j = \dfrac{I}{\pi R^2}$, we deduce an expression for the magnitude of the magnetic field at distance $r > R$:
   $$\boxed{B = \frac{\mu_0 I}{2\pi r}.}$$

   This results in the expression for the magnetic field being equivalent to that of a long, straight thin wire, as demonstrated in equation 7.2.

We can notice that the magnitude of the magnetic field increases linearly with r from $r = 0$ to a maximum value of $\dfrac{\mu_0 I}{2\pi R}$ at the wire's surface $(r = R)$, and decreases in proportion to $\dfrac{1}{r}$outside the conducting wire.



5.2 Biot-Savart law

Although Ampère's law facilitates the calculation of the magnetic field resulting from a highly symmetrical distribution of electric currents, there are instances where this is not applicable and the magnetic field resulting from a distribution of electric current other than this one must be calculated. The law governing the magnetic field created by this distribution of currents was formulated by Jean Baptiste Biot (1774-1862) and Félix Savart (1791-1841) by analogy with Coulomb's law for determining the electric field created by a charged wire.

We recall that the electric field created at a point M in space, distant from a charge element of linear density λ (located at a point P such that $\overrightarrow{PM} = \vec{r}$), has the following expression:

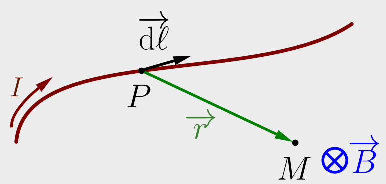
$$\mathrm{d}\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda \mathrm{d}\ell\,\vec{r}}{r^3},$$

Biot-Savart law is a generalisation of the aforementioned law, which accounts for the pseudovectorial nature of the magnetic field. It is expressed by substituting the charge density by electric current, the simple product by a vectorial product, and the dielectric permeability of a vacuum by its magnetic permeability.

Biot-Savart law

The magnetic field created at a point M in space, distant from a segment $\mathrm{d}\ell$ of wire through which an electric current I flows (located at a point P such that $\overrightarrow{PM} = \vec{r}$ as shown in the figure below), is given by the expression:

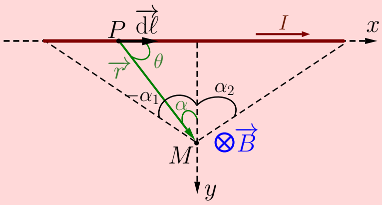
$$\boxed{\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\vec{\ell} \times \vec{r}}{r^3}.} \tag{7.8}$$



While the application of this law generally necessitates complex calculation methods, some elementary applications facilitate the straightforward determination of the resultant magnetic field.

Example 1: finite strength wire

Biot-Savart law is to be applied in determining the expression for the magnetic field produced at a point M located at a distance R on the y-axis, which is perpendicular to a finite straight wire located on the x-axis and through which a current I flows in the direction $Ox > 0$ (see figure below).



It is evident from the expression of the law that the magnetic field vector $\vec{B}$ is perpendicular to the plane that contains both the wire and the point M.
The magnitude of the magnetic field is expressed as follows:

$$B = \int \mathrm{d}B = \int \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}x\,\sin\theta}{r^2}.$$

θ denotes the angle between the direction of the electric current (x-axis) and the vector $\overrightarrow{PM} = \vec{r}$, which links the electric current element to the point at which the magnetic field is calculated.
θ, r et x are dependent variables such that:

$$\tan\theta = \frac{R}{-x} \implies x = \frac{-R}{\tan\theta}$$
$$r = \frac{R}{\sin\theta}.$$

Then:

$$\mathrm{d}x = \frac{R}{\sin^2\theta}\mathrm{d}\theta$$
$$\frac{1}{r^2} = \frac{\sin^2\theta}{R^2},$$

and the magnitude of the magnetic field expresses as follows:

$$B = \frac{\mu_0 I}{4\pi}\int \frac{R}{\sin^2\theta}\sin\theta\frac{\sin^2\theta}{R^2}\mathrm{d}\theta = \frac{\mu_0 I}{4\pi R}\int \sin\theta\mathrm{d}\theta.$$

The angle α is defined as the complement of the θ, such that $\theta = \frac{\pi}{2} - \alpha$. It is evident that the variation in the angle θ along the finite wire is directly proportional to the variation in the angle α between two limit values, $-\alpha_1$ and α_2, so that:

$$B = \frac{\mu_0 I}{4\pi R}\int_{-\alpha_1}^{\alpha_2}\cos\alpha\mathrm{d}\theta.$$

Finally, we can write that:

$$\boxed{B = \frac{\mu_0 I}{4\pi R}\left(\sin\alpha_2 + \sin\alpha_1\right).}$$

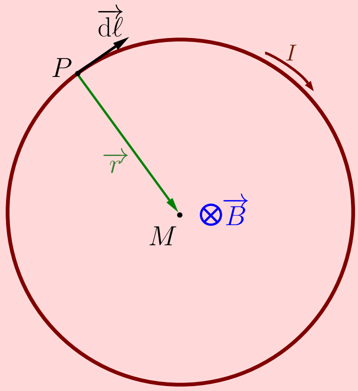
This result can be generalised to the case of an infinite wire by assuming $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, for which the expression of the magnetic field is given by:

$$\boxed{B = \frac{\mu_0 I}{2\pi R}.}$$

This result is consistent with the expression obtained through Ampère's theorem.

Example 2: circular ring of current

In order to determine the expression for the magnetic field produced by a current I, flowing in a circular loop of radius R, at its centre point M (see figure below), it is necessary to apply Biot-Savart law..



The magnitude of the magnetic field is given by :

$$B = \int dB = \int \frac{\mu_0 I}{4\pi} \frac{d\ell \sin\theta}{r^2}.$$

θ denote the angle between the direction of the electric current (as specified by $\overrightarrow{dl}$) and the vector $\overrightarrow{PM} = \overrightarrow{r}$, which links the current segment to the point where the magnetic field is calculated.
For the circular loop flowing current, $\left|\overrightarrow{PM}\right| = r = R$ and $\theta = 90°$. Thus the magnitude of the magnetic field expresses as follows:

$$B = \frac{\mu_0 I}{4\pi R^2} \int d\ell = \frac{\mu_0 I}{4\pi R^2} 2\pi R,$$

and finally:

$$\boxed{B = \frac{\mu_0 I}{2R}.}$$

This result can be extended to the case of a circular arc seen at an angle $\Delta\theta$, the magnetic field amplitude of which is determined at its centre of curvature by the following expression:

$$B = \frac{\mu_0 I}{4\pi R^2} \int d\ell = \frac{\mu_0 I}{4\pi R^2} R\,\Delta\theta,$$

and finally:

$$\boxed{B = \frac{\mu_0 I\,\Delta\theta}{4\pi R}.}$$

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