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The Foldy–Lax Approximation for the Full Electromagnetic waves and application to the electromagnetic metamaterials.

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Résumé

Nous nous intéressons à la diffusion d'ondes électromagnétiques en régime harmonique, à travers un fond homogène contenant des hétérogénéités anisotropes ou conductrices modélisées par des ouverts à frontières Lipschitziennes, nous en dérivons une approximation de type Foldy-Lax ("point-like interaction") où il est question d'approximer les inclusions par des pôles modulo des amplitudes constantes, données pour ces dernières, par un système linéaire. Nous établissons la validité de cette approximation lorsque le rapport entre le diamètre maximal ($\boldsymbol{a}$) et la distance minimale ($\boldsymbol{\delta}$) séparant les inhomogènes est fini i.e. $\boldsymbol{a} \sim \boldsymbol{\delta}$.

Mots-clés

Diffusion électromagnétique, Petites particules , Diffraction multiple, Approximation de Foldy-Lax.

Abstract

We deal with the scattering of time harmonic electromagnetic waves through a cluster of both anisotropic and perfectly conducting particles, embedded in homogeneous background, modeled by open bounded Lipschitz Domains, we derive the Foldy-Lax approximation, valid for the mesoscale regime that is $\boldsymbol{a} \sim \boldsymbol{\delta}$ where $\boldsymbol{a}$ the maximum diameter of the particles and $\boldsymbol{\delta}$ is the minimum distance separating them.

Key words:

Electromagnetic scattering, Small bodies, Multiple scattering, Foldy-Lax approximation.

Contents

Chapter 1

Introduction

1.1 Motivations

Multiple scattering theory could be described as a field of mathematical physics concerned by the phenomena of diffusion through a cluster of particles. Either at quantum or continuum levels, the main purpose is to account for the phenomenon of interaction in the most accurate and efficient way depending on the purpose. With an abundant literature on the subject, see [49], the domain continues to receive growing attention due to the increase of perspective in material science. The field covers many aspects of the modern sciences and at different regimes (small or large scales and resonating or not) from the scattering of light to seismic propagation passing by different imaging sciences, see [43], [49], [7] and [70] for instance.

The continuous model, if considered by the integral equation approach, due to the usual boundary conditions, would involve solving a system which is as large as the number of the scatterers. Naturally, one should avoid such approach due to its complexity, and rather relay on an approximation of the problem based in the obvious physical intuition, suggesting that, the more the diameter of the particle is small the more it should be considered as point-like scatterer. This is precisely what L. L. Foldy initiated in [28]. Indeed, considering the scalar wave equation, that is $\Delta u + k^2 u = 0$, defined "*the external field acting on the* j^{th} *scatterer*" to be the difference between the total field and the Green function evaluated in the position of the scatterer modulo a constant coefficient A_j (i.e a point-like scatterer of amplitude A_j). Then stating the assumption that "*the strength of the scattered wave from a scatterer is proportional to the external field acting on it*" that is $u_j(r_j) = g_j A_j$, deriving a consistent algebraic linear system involving his scattering coefficients; a physical argument yet not mathematically justified. Later on, M. Lax [46] generalized the approach for anisotropic cases, with the same hypothesis on the strength of the field, on the assumption that "*The single scattering problem has been solved either experimentally or theoretically*".

In the seventies, there were two methods proposed to justify, or precisely to give sense to these arguments. The first one is based the use of the Fourier transform, then regularize (or cut the singularity) in the Fourier domain and then comeback to the space variable. This is called a renormalization method, see [3]. The second method is based on self-adjoint extensions of symmetric operators and the Krein's resolvent representation. Both the two methods apply to a single particle but provide with the exact model.

Regarding multiple particles, there were several methods proposed in the literature. They deal with the case where these particles are stated in the small scaled regimes with different scales of the contrasts. To cite a few of them, we start with Mazy'a et al. see [52] where Laplace and Lamé related problems are considered. Their approach is heavily based on the maximum principles and hence limted to low frequency scattering (actually the stationary regime). Extending their method to Maxwell is then problematic. There were also a long list of works by A. Ramm, see [63]. However, the proposed approximations are mostly formal and their results related to Maxwell are quite questionable. Let us also cite Ammari, see [32] who proposed the full expansions at any order, but limited to single, or well separated, particles.

In the context of the previous intuition , and inspired by the approach in [19, 2], in the present work, we were able to derive and rigorously justify such an approximation for the time harmonic electromagnetic scattering. More precisely, we derived an approximation of the far field of the scattered wave, which uniquely characterizes it, as follows

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph} \Big(\frac{k^2}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \Big(Q_m \times \hat{x} \Big) + \frac{ik}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times R_m \Big) + O(Error), \tag{1.1.1}$$

where the amplitudes A_m and B_m can be evaluated using the following linear system,

$$\begin{aligned} [T^1]_m R_m &= \sum_{(j\geq 1)_{j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) R_j - ik\nabla\Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times Q_j \Big] + H^{\text{in}}(\boldsymbol{z}_m) \\ [T^2]_m Q_m &= \sum_{(j\geq 1)_{j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) Q_j + ik\nabla\Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times R_j \Big] + E^{\text{in}}(\boldsymbol{z}_m). \end{aligned} \tag{1.1.2}$$

Here Φ_k and $\Pi_k := (k^2\mathbb{I} + \nabla \otimes \nabla)\Phi_k$, are respectively the fundamental solution of the Helmholtz problem and the dyadic Green function. The Polarization tensor $([T^l]_m)_{m=1}^{\aleph}$, $l = 1,\ 2$, could be seen as the scattering coefficient which precisely are evaluated by solving single scattering problem.

The main focus of the thesis is to derive these approximations by considering general configurations on the contrast of the electromagnetic materials, the number of the particles $\aleph$, the minimum distance between them $\boldsymbol{\delta}$ and their maximum radius $\boldsymbol{a}$. The term $O(Error)$, appearing above, is provided in terms of these parameters as well. The thesis contains essentially two chapters. The first chapter concerns the scattering of electromagnetic wave from a cluster of perfect conductors, in which we derived the Foldy-Lax approximation which is valid under the condition that

$$\ln(\boldsymbol{a})\frac{\boldsymbol{a}^3}{\boldsymbol{\delta}^3} = O(1), \text{ as } \boldsymbol{a} << 1. \tag{1.1.3}$$

The only restriction in (1.1.3) is the appearance of the term $\ln(\boldsymbol{a})$. This is due to the use of Neumann series inversion criterion for bounded operators. We obtained expansion for both the far field and the near field of the scattered wave. The latter one is described as the sum of poles modulo constants that are described by a self containing linear system; which is invertible under a less restrictive condition, namely

$$\frac{\boldsymbol{a}}{\boldsymbol{\delta}} = O(1), \text{ as } \boldsymbol{a} << 1. \tag{1.1.4}$$

The second chapter comprises two parts. The first part, is devoted to the transmission problem, namely the anisotropic inhomogeneities case. We derived the related results under the general condition (1.1.4). We handled this case using Lippmann-Schwinger integral equation that we could invert under the condition (1.1.4). In the second section, we revisited the scattering by a cluster of perfect conductors and improved the results previously obtained by removing the $\ln(\boldsymbol{a})$ term in the condition (1.1.3). This was possible using more pde-techniques as a certain Rellich-type identity to estimate the densities. Precisely, we showed that both the exterior and the interior traces of the electromagnetic fields are equivalent in norm.

Observe that the condition (1.1.4) means that we can handle the meso-scale regime of the multiple scattering where $\boldsymbol{\delta} \sim \boldsymbol{a}$.

At the very technical level, based on the Stratton-Chu formula and taking advantage of the properties of the different used layer potentials, we could handle the approximations of both the two cases (perfect conductors and anisotropic inhomogeneities) in a similar way.

The two main chapters, chapter 2 and chapter 3, are written in an article style. They can be read separately and independently. The notations are not always the same as some of them are imposed to comply with the related literature. We are aware that this might create little confusion at some point. We do apologize if this is the case!

1.2 Main tools and the strategy of analysis

In this section, after stating the problem, we describe the main tools used in chapter 2 and chapter 3. Precisely, we first motivate the uniqueness issue of the solution and it natural relation to the radiation condition. The existence of the solution is related to inverting an equivalent integral equation. Regarding the perfectly conducting problem, we derive a surface integral system via the Maxwell layer potentials, while for the transmission problem, we derive the Maxwell Lippmann-Schwinger integral equation using Volume (or Newtonian) potentials. Both of these formulations are derived starting from the Stratton-Chu formula.

1.2.1 The Maxwell system and our models of interest

Maxwell's equation forms a particularly interesting model to describe the wave propagation of electromagnetic phenomena. Stated for the wave diffusion of the electric field $\boldsymbol{\mathcal{E}}$ and the magnetic field $\boldsymbol{\mathcal{H}}$ through a given medium, with electric permittivity ε, magnetic permeability μ, and electric conductivity σ. It reads as follows

$$\begin{aligned} \operatorname{curl} \boldsymbol{\mathcal{E}} - \mu \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} &= 0, \\ \operatorname{curl} \boldsymbol{\mathcal{H}} + \varepsilon \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} &= \sigma \boldsymbol{\mathcal{E}}, \end{aligned} \tag{1.2.1}$$

The parameters ε, σ and μ will be either complex/real tensor or scalar functions.

In particular, we are be interested by the diffusion of time-harmonic electromagnetic plane wave at a fixed frequency ω. This is stated through both isotropic and anisotropic materials, represented by multiply connected bounded Lipschitz domain $D^- = \cup_{m=1}^{\aleph} D_m$ where $\aleph$ is the number of connected component surrounded by a homogeneous medium. We recall, that a lipschitz domain is locally, i.e in some neighbor of each of its point, represented by the graph of a Lipschitz function.

For the anisotropic case, setting $k := w\sqrt{\epsilon_0\mu_0}$, $\mathcal{E} := \sqrt{\varepsilon_0{\mu_0}^{-1}}Ee^{-iwt}$ and $\mathcal{H} := \sqrt{\varepsilon_0{\mu_0}^{-1}}He^{-iwt}$ with $\mu_r := (\mu_0)^{-1}\mu$ and $\varepsilon_r := (\varepsilon_0)^{-1}(\varepsilon + i\sigma/\omega)$, the Maxwell equation reads

$$\begin{cases} \operatorname{curl} E - ik\mu_r H = 0, \\ \operatorname{curl} H + ik\varepsilon_r E = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{\partial D} := \cup_{m=1}^{\aleph} \partial D_m, \\ E = E^{\text{in}} + E^{\text{sc}}, \\ \nu \times E|_+ = \nu \times E|_- \quad \text{across} \quad \partial D, \\ \nu \times H|_+ = \nu \times H|_- \quad \text{across} \quad \partial D. \end{cases} \tag{1.2.2}$$

When ε, σ and μ are scalar constant we will consider the scattering from a perfect conductor, precisely, with $k = \sqrt{(\varepsilon + i\frac{i\sigma}{w})\mu}\ w$, $\Re\Big((\varepsilon + i\frac{i\sigma}{w})^{-1/2}Ee^{-iwt}\Big) = \boldsymbol{\mathcal{E}}$ and $\Re\Big((\mu)^{-1/2}He^{-iwt}\Big) = \boldsymbol{\mathcal{H}}$ the following problem

$$\begin{cases} \nabla \times E - ikH = 0, \\ \nabla \times H + ikE = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{D} := \cup_{m=1}^{\aleph} \overline{D}_m, \\ E = E^{\text{in}} + E^{\text{sc}}, \\ \nu \times E|_+ = 0 \quad \text{on} \quad \partial D. \end{cases} \tag{1.2.3}$$

The boundary condition $\nu \times (\ \cdot\ )|_+$ respectively $\nu \times (\ \cdot\ )|_-$ must be understood as the extension of exterior resp. interior trace operator, from the space of $\boldsymbol{C}^{\infty}$-regular functions to the working specified space, if it exist.

1.2.2 The radiation conditions and the uniqueness of the solutions

The radiation condition describes the behavior of the scattered wave at infinity. In order to properly justify the adequacy of such a condition with the scattering problem, we need the following lemma due to Rellich.

Lemma 1. *(Lemma 2.12 [24]) Let D be a bounded open set. If E is a solution to the Helmholtz equation $\mathbb{R}^3 \setminus D$ (i.e $\Delta E + k^2 E = 0$ outside of D.) satisfying*

$$\lim_{r \longrightarrow \infty} \int_{\partial B(0,r)} |E|^2 ds = 0, \tag{$\mathcal{R}ellich$}$$

then $E \equiv 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Proof. The proof, which is standard, is based on spherical harmonic expansion (i.e Harmonic homogeneous, polynomials of degree n), which form a complete orthonormal system in $\mathbb{L}^2(\mathbb{S}^2)$. Precisely

$$u(|x|) = \sum_{n=0}^{\infty} \sum_{-n \leq q \leq n} \Big(C_n^q(|x|) := \int_{\partial B(0,1)} u(ry) Y_n^q(y) ds_y \Big) Y_n^q(\frac{x}{|x|}) \tag{1.2.4}$$

with

$$Y_n^q(\theta, \phi) = \Big(\frac{2n+1(n-|q|)!}{4\pi(n+|q|)!} \Big) P_n^{|q|}(cos(\theta)) e^{iq\phi}$$

and $(P_q^n)_{(n,q)\in \mathbb{N}\times\mathbb{Z};\ |q|\leq n}$ stands for the Legendre functions given by

$$P_n^q(t) := (1-t^2)^{\frac{q}{2}} \big(P_n(t)\big)^{(q)}.$$

Then after writing the Helmholtz equation in polar coordinates and differentiating under the integral (1.2.4), that the C_n^q satisfies the Bessel's differential equation

$$(C_n^q(r))^{(2)} + \frac{2}{r}(C_n^q)^{(1)} + (k^2 - \frac{n(n+1)}{r^2}) C_n^q = 0$$

whose solution are expressed in terms of Hankel functions, given by

$$C_n^q(r) = \alpha_n^q h_n^{(1)}(kr) + \beta_n^q h_n^{(2)}(kr), \tag{1.2.5}$$

with the following asymptotic behavior

$$h_n^{(l)}(r) = \frac{e^{\pm i(r - n\pi/2 - \pi/2)}}{r}(1 + O(r^{-1})),\ \ l = 1, 2.$$

Finally, Parseval's inequality gives

$$\int_{\partial B(0,r)} |u|^2 ds = r^2 \sum_{n=0}^{\infty} \sum_{-n\leq q\leq n} |C_n^q(|x|)|^2,$$

and helps us get

$$\lim_{r\longrightarrow\infty} r^2 |C_n^q(|x|)|^2 = 0.$$

Replacing in (1.2.5), we conclude that $\alpha_n^q = \beta_n^q = 0$. □

Remark 2. *One must notice that, no particular hypothesis has been made concerning the geometry of D. Further, due to the identity*

$$\mathrm{curl}^2 = -\Delta + \nabla\,\mathrm{div},$$

every solution to Maxwell equation is a solution to the Helmholtz. However, a solution to the vector Helmholtz problem is not necessarily solution to Maxwell equation unless it is divergence free.

Lemma 1 describes a sufficient condition in order to have an identically null solution. In what follows, we describe a way which helps links ($\mathcal{R}ellich$) with some general behavior of the scattered wave. Let $\overline{D}$ be a Lipschitz domain and $r > 0$ such that the ball $B(0, r)$, centered at the origin with radius r, contains D. An application of Green's formula gives

$$\begin{aligned} \int_{\partial B(0,r)} \nu \times E \cdot \overline{H} ds - & \int_{\partial D} \nu \times E \cdot \overline{H} ds \\ & = \int_{\partial D_r} \nu \times E \cdot \overline{H} ds, \\ & = \int_{D_r} (ikH \cdot \overline{H} - \overline{(-ik)}\ \overline{E} \cdot E) dv = ik \int_{D_r} (|H|^2 - \ |E|^2) dv. \end{aligned} \tag{1.2.6}$$

Taking the real part in the above identity, for $k \in \mathbb{R}^+$, we have

$$\Re \int_{\partial B(0,r)} \nu \times E \cdot \overline{H} ds = \Re \int_{\partial D} \nu \times E \cdot \overline{H} ds. \tag{1.2.7}$$

Replacing in the following development

$$\int_{\partial B(O,R)} |H \times \nu - E|^2 ds = \int_{\partial B(O,R)} \left(|\nu \times H|^2 + |E|^2\right) ds - 2\Re \int_{\partial B(0,R)} \nu \times E \cdot \overline{H} ds, \tag{1.2.8}$$

gives

$$\int_{\partial B(O,R)} |H \times \nu - E|^2 ds = \int_{\partial B(O,R)} \left(|\nu \times H|^2 + |E|^2\right) ds - 2\Re \int_{\partial D} \nu \times E \cdot \overline{H} ds. \tag{1.2.9}$$

Obviously, if $E = E_1 - E_2$, is the superposition of two solutions of (1.2.3) with a similar boundary condition that is $\nu \times E_1 = \nu \times E_2$, and subject to the same incident field (1.2.9) becomes

$$\int_{\partial B(O,R)} |H \times \nu - E|^2 ds = \int_{\partial B(O,R)} \left(|\nu \times H|^2 + |E|^2\right) ds \tag{1.2.10}$$

Together, this last equation and Lemma 1, motivate to impose the following, well known, Silver-Müller radiation condition

$$H(x) \times x/|x| - E(x) = O\left(1/|x|^2\right), \tag{1.2.11}$$

or its relaxed one

$$\lim_{R \longrightarrow 0} \int_{\partial B(O,R)} |H \times \nu - E|^2 ds = 0, \tag{1.2.12}$$

in order to get $E \equiv H \equiv 0$, outside of D. Making $E_1 \equiv E_2$ and $H_1 \equiv H_2$, provids the uniqueness of the solution.

Remark 3. *We add the following two remarks.*

- *When $k \in \mathbb{C}$ and $\Im k > 0$ it is easier to prove the uniqueness of the exterior problem, without relaying on the Lemma 1, Indeed, from 1.2.6, we have*

$$\int_{\partial B(0,r)} \nu \times E \cdot \overline{H} ds - \int_{\partial D} \nu \times E \cdot \overline{H} ds = \int_{D_r} (ik|H|^2 - \overline{(-ik)}|E|^2) dv.$$

Taking the real part, we get

$$\Re\Big(\int_{\partial B(0,r)} \nu \times E \cdot \overline{H} ds - \int_{\partial D} \nu \times E \cdot \overline{H} ds\Big) = -\Im(k) \int_{D_r} (|H|^2 + |E|^2) dv.$$

Obviously, as $\Im k > 0$, we have

$$\Re \int_{\partial B(0,r)} \nu \times E \cdot \overline{H} ds \to 0, r \to \infty$$

which guaranties that

$$\Re\Big(\int_{\partial D} \nu \times E \cdot \overline{H} ds\Big) \geq \Im(k) \int_{\mathbb{R}^3 \setminus D} (|H|^2 + |E|^2) dv.$$

- *The radiation condition for the Helmholtz equation was introduced by Sommerfeld in [67], explaining the physical intuition that "We call it the condition of radiation: the sources must be sources, not sinks, of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into the prescribed singularities of the field (plane waves are excluded since for them even the condition $u \longrightarrow 0$ fails to hold at infinity)."*

- *It is obvious that the condition (1.2.11) implies (1.2.12). We get the equivalence using the Stratton–Chu formula valid with the later condition, and the behavior of the Green's kernel (see Theorem 6.7 [24] for the Stratton–Chu formula or here after).*

1.2.3 The surface and volume potentials and the existence of the solutions

Recapitulating the previous section and the motivated radiation condition, we have both the following problems

- The time harmonic scattering from perfect conductors

$$\begin{cases} \nabla \times E - ikH = 0, \\ \nabla \times H + ikE = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{D} := \cup_{m=1}^{\aleph} \overline{D}_m. \\ E = E^{\text{in}} + E^{\text{sc}}, \\ \nu \times E|_+ = 0, \\ H(x) \times x/|x| - E(x) = O\Big(1/|x|^2\Big). \end{cases} \tag{$\mathcal{P}$.Perf-cond}$$

- The time harmonic scattering from anisotropic inhomogeneities

$$\begin{cases} \nabla \times E - ik\mu_r H = 0, \\ \nabla \times H + ik\varepsilon_r E = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{D} := \cup_{m=1}^{\aleph} \overline{D}_m, \\ E = E^{\text{in}} + E^{\text{sc}}, \\ \nu \times E|_+ = \nu \times E|_- \quad \text{across} \quad \partial D, \\ \nu \times H|_+ = \nu \times H|_- \quad \text{across} \quad \partial D. \\ H(x) \times x/|x| - E(x) = O\Big(1/|x|^2\Big). \end{cases} \tag{$\mathcal{P}$.Aniso}$$

We recall the Green's function for the Helmholtz operator $(\Delta + k^2)$,

$$\phi_k(x,y) := \frac{1}{4\pi}\frac{e^{-ik|x-y|}}{|x-y|}.$$

1.2.3.1 Boundary layer potentials and the exterior problem ($\mathcal{P}$.Perf-cond)

Let D be a bounded $\boldsymbol{C}^1$ domain, and E, H two vector field in $\boldsymbol{C}^1(D) \cap \boldsymbol{C}(\overline{D})$ satisfying Maxwell equation. As a consequence of Green's formula, we have the following Stratton-Chu representation

$$\begin{aligned} &- \operatorname{curl} \int_{\partial D} \Phi_k(x,y) \nu \times E(y)|_- \, ds_y \\ &+ \frac{1}{ik} \operatorname{curl}^2 \int_{\partial D} \phi_k(x,y) \nu \times H(y)|_- \, ds_y = \begin{cases} 0, \text{ in } \mathbb{R}^3 \setminus \overline{D}, \\ E. \text{ if } x \in D. \end{cases} \end{aligned} \tag{1.2.13}$$

see [24] and Theorem 5.49 of [40] for the Lipschitz boundary case. When E and H are in $\boldsymbol{C}^1(\mathbb{R}^3 \setminus \overline{D}) \cap \boldsymbol{C}(\mathbb{R}^3 \setminus D)$, (i.e E, H are continuous up to the boundary.) and radiate to the infinity, we have a similar formula, namely

$$\begin{aligned} &\operatorname{curl} \int_{\partial D} \Phi_k(x,y) \nu \times E(y)|_+ \, ds_y \\ &- \frac{1}{ik} \operatorname{curl}^2 \int_{\partial D} \phi_k(x,y) \nu \times H(y)|_+ \, ds_y = \begin{cases} E(x), \text{ in } \mathbb{R}^3 \setminus \overline{D}, \\ 0, \text{ if } x \in D. \end{cases} \end{aligned} \tag{1.2.14}$$

The Green function satisfying, naturally, the Silver-Müller radiation condition, then also do both the potentials

$$\boldsymbol{M}^k_{\partial D}(\nu \times U)(x) = \operatorname{curl} \int_{\partial D} \Phi_k(x,y) \nu \times U(y) \, ds_y \tag{1.2.15}$$

and

$$\boldsymbol{N}^k_{\partial D}(\nu \times V)(x) = \operatorname{curl}^2 \int_{\partial D} \phi_k(x,y) \nu \times V(y) \, ds_y. \tag{1.2.16}$$

Furthermore, the above potentials are $C^\infty(\mathbb{R}^3 \setminus \partial D)$. If we consider the second potential, recalling that $\operatorname{curl}^2 = -\Delta + \nabla \operatorname{div}$, we get for a sufficiently smooth vector field V,

$$\begin{aligned}\boldsymbol{N}^k_{\partial D}(\nu \times V)(x) =&(k^2 + \nabla \operatorname{div}) \int_{\partial D} \phi_k(x,y) \nu \times V(y)\, ds_y,\\ =&k^2 \int_{\partial D} \phi_k(x,y)\nu \times V(y)\, ds_y - \nabla \int_{\partial D} \nabla_y \phi_k(x,y) \cdot \nu \times V(y)\, ds_y,\\ =&k^2 \int_{\partial D} \phi_k(x,y)\nu \times V(y)\, ds_y + \nabla \int_{\partial D} \phi_k(x,y)(-\nu \cdot \operatorname{curl} V(y))\, ds_y.\end{aligned} \tag{1.2.17}$$

The last identity motivates to introduce the surface divergence of the tangential trace of a smooth vector field as

$$\operatorname{Div}(\nu \times V) = -\nu \cdot \operatorname{curl} V, \tag{1.2.18}$$

to rewrite, with the above notation,

$$\boldsymbol{N}^k_{\partial D}(\nu \times U)(x) = k^2 \int_{\partial D} \phi_k(x,y)\nu \times V(y)\, ds_y + \nabla \int_{\partial D} \phi_k(x,y) \operatorname{Div}(\nu \times V(y))\, ds_y. \tag{1.2.19}$$

When D has a Lipschitz boundary, then both the potential are continuous, up to the boundary with the following trace, almost everywhere on ∂D, see [57],

$$\begin{aligned}\nu \times \boldsymbol{M}^k_{\partial D}(A)(x)|_\pm =&[\pm I/2 + M^k_{\partial D}](A)(x)\\ :=& \pm \frac{1}{2} A + \nu \times p.v. \operatorname{curl} \int_{\partial D} \Phi_k(x,y)\; A(y) ds_y,\end{aligned} \tag{1.2.20}$$

and

$$\begin{aligned}\nu \times \boldsymbol{N}^k_{\partial D}(A)(x)|_\pm = [N^k_{\partial D}](A)(x) :=& k^2 \nu \times \int_{\partial D} \Phi_k(x,y) A\\ &+ \nu \times\; p.v. \nabla \int_{\partial D} \Phi_k(x,y)\; \operatorname{Div} A(y) ds_y,\end{aligned} \tag{1.2.21}$$

for any tangential density A such that $\|A\|_{\mathbb{L}^2(\partial D)}$ and $\|\operatorname{Div} A\|_{\mathbb{L}^2(\partial D)}$ are finite. The space $\mathbb{L}^{2,Div}_t(\partial D)$ will signify the space of all such densities.

Similarly in the frame of the wave equation, we introduce, the single layer and the double layer potentials as radiating solutions to the Helmholtz equation defined respectively as

$$\begin{aligned}\boldsymbol{S}^k_{\partial D}\phi(x) :=& \int_{\partial D} \Phi_k(x,y)\phi(y) ds_y,\; x \in \mathbb{R}^3 \setminus \partial D,\\ \boldsymbol{K}^k_{\partial D}\phi(x) :=& \int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu_y} \phi(y) ds_y,\; x \in \mathbb{R}^3 \setminus \partial D,\end{aligned} \tag{1.2.22}$$

for $\phi \in \mathbb{L}^2(\partial D)$. We have the following boundary traces

$$\begin{aligned}\boldsymbol{S}^k_{\partial D}(\phi)|_\pm =&[S^k_{\partial D}](\phi) := \int_{\partial D} \Phi_k(x,y)\phi(y) ds_y,\\ \nu \cdot \nabla \boldsymbol{S}^k_{\partial D}(\phi)|_\pm =&[\mp I/2 + (K^k_{\partial D})^*](\phi) := \mp \frac{\phi}{2} + p.v. \int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu_x} \phi(y) ds_y,\end{aligned} \tag{1.2.23}$$

and

$$\boldsymbol{K}^k_{\partial D}(\phi)|_{\pm} = [\pm I/2 + K^k_{\partial D}] := \pm\frac{\phi}{2} + p.v.\int_{\partial D}\frac{\partial \Phi_k(x,y)}{\partial \nu_y}\phi(y)ds_y. \tag{1.2.24}$$

Here $(\cdot)^*$ stands for the adjoint. The above operators enjoy the following property,

1. $[I/2 + K^k_{\partial D}]^* = [I/2 + (K^k_{\partial D})^*]$,
2. $\operatorname{Div}[\pm I/2 + M^k_{\partial D}](A) = -[\mp I/2 + K^k_{\partial D}]^*(\operatorname{Div} A) - k^2\nu\cdot[S^k_{\partial D}](A)$,
3. $\nu\times\nabla[I/2 + K^k_{\partial D}](\phi) = [I/2 + M^k_{\partial D}](\nu\times\nabla\phi) + k^2\nu\times[S^k_{\partial D}](\nu\phi)$,
4. $\operatorname{Div}(\nu\times\nabla)\equiv 0$.

The first and the last one are obvious. The second and the third are consequences of $\operatorname{curl}^2 = -\Delta + \nabla\operatorname{div}$ and the identity $\nabla_x\Phi_k(x,y) = -\nabla_y\Phi_k(x,y)$ integrating by part outside of the boundary and using the traces (1.2.20), (1.2.21), (1.2.23) and (1.2.24).

It is clear that the field $E^{\mathrm{sc}} := \boldsymbol{M}^k_{\partial D}(A)$ solves the problem ($\mathcal{P}$.Perf-cond) whenever $A \in \mathbb{L}^{2,Div}_t(\partial D)$ is a solution of

$$[\frac{1}{2} + \boldsymbol{M}^k_{\partial D}]A = -\nu\times E^{in}, \quad \text{on } \partial D. \tag{1.2.25}$$

1.2.3.2 Volume Potentials and the transmission problem ($\mathcal{P}$.Aniso)

Let us consider the following formula, (see [24] Theorem 6.1) valid, for any vector fields E, H which are continuously differentiable in D and continuous up to ∂D,

$$\begin{aligned}
E(x) = &-\operatorname{curl}\int_{\partial D}\Phi_k(x,y)\nu\times E(y)ds_y + \frac{k^2}{ik}\int_{\partial D}\Phi_k(x,y)\nu\times H(y)ds_y\\
&+\nabla\int_{\partial D}\Phi_k(x,y)\nu\cdot E(y)ds_y - \nabla\int_D\Phi_k(x,y)\operatorname{div}E(y)dy\\
&+\operatorname{curl}\int_D\Phi_k(x,y)(\operatorname{curl}E - ikH)(y)dy + ik\int_D\Phi_k(x,y)(\operatorname{curl}H + ikE)(y)dy.
\end{aligned} \tag{1.2.26}$$

We have, integrating by part

$$\begin{aligned}
\int_D\Phi_k(x,y)\operatorname{div}E(y)dy &= \int_D\operatorname{div}(\Phi_k(x,y)E(y))dy - \int_D\nabla_y\Phi_k(x,y)\cdot E(y)dy,\\
&= \int_{\partial D}\Phi_k(x,y)\nu\cdot E(y)dy + \operatorname{div}\int_D\Phi_k(x,y)E(y)dy.
\end{aligned} \tag{1.2.27}$$

Taking the gradient of this last identity and injecting it in the second term of the right hand member of (1.2.26) gives

$$\begin{aligned}
E(x) = &-\operatorname{curl}\int_{\partial D}\Phi_k(x,y)\nu\times E(y)ds_y + \frac{k^2}{ik}\int_{\partial D}\Phi_k(x,y)\nu\times H(y)ds_y\\
&-\nabla\operatorname{div}\int_D\Phi_k(x,y)E(y)dy\\
&+\operatorname{curl}\int_D\Phi_k(x,y)(\operatorname{curl}E - ikH)(y)dy + ik\int_D\Phi_k(x,y)(\operatorname{curl}H + ikE)(y)dy.
\end{aligned} \tag{1.2.28}$$

Having, in one hand,

$$\frac{1}{ik}\operatorname{div}\int_{\partial D}\phi(x,y)\nu\times H(y)ds_y = \frac{1}{ik}\int_{\partial D}\phi(x,y)\operatorname{Div}(\nu\times H(y))ds_y \tag{1.2.29}$$

and, in another hand, integrating the left-hand side of the above, recalling that $\operatorname{div}\operatorname{curl}\equiv 0$, gives

$$\begin{aligned}\frac{1}{ik}\operatorname{div}\int_{\partial D}\phi(x,y)\nu\times H(y)ds_y &= \frac{1}{ik}\operatorname{div}\int_{D}\operatorname{curl}_y(\phi(x,y)H(y))ds_y,\\ &= \frac{1}{ik}\operatorname{div}\int_{D}\phi(x,y)\operatorname{curl}H(y))ds_y.\end{aligned} \tag{1.2.30}$$

We conclude that

$$\frac{1}{ik}\operatorname{div}\int_{D}\phi(x,y)\operatorname{curl}H(y))ds_y = \frac{1}{ik}\int_{\partial D}\phi(x,y)\operatorname{Div}(\nu\times H(y))ds_y. \tag{1.2.31}$$

Now, we take the gradient of the precedent identity, adding its left-hand member and subtracting the right-hand one to (1.2.26), with the notations (1.2.15) and (1.2.19), gives

$$\begin{aligned}E(x) =& -\boldsymbol{M}^k_{\partial D}(\nu\times E(y))(x) + \frac{1}{ik}\boldsymbol{N}^k_{\partial D}(\nu\times H(y))(x)\\ &- \nabla\operatorname{div}\int_D \Phi_k(x,y)E(y)dy - \frac{1}{ik}\nabla\operatorname{div}\int_D\phi(x,y)\operatorname{curl}H(y))ds\\ &+\operatorname{curl}\int_D\Phi_k(x,y)(\operatorname{curl}E - ikH)(y)dy + ik\int_D\Phi_k(x,y)(\operatorname{curl}H + ikE)(y)dy.\end{aligned} \tag{1.2.32}$$

If $E = E^{\mathrm{sc}} + E^{\mathrm{in}}$, $H = H^{\mathrm{sc}} + H^{\mathrm{in}}$ are solution of ($\mathcal{P}$.Aniso) we have,

$$(\operatorname{curl}E - ikH)(y) = ik(\mu_r - \mathbb{I})H, \quad (\operatorname{curl}H + ikE) = -ik(\varepsilon_r - \mathbb{I})E. \tag{1.2.33}$$

Due to the continuity of the field across the boundary, and E^{sc} radiating solution of the exterior Maxwell problem, we have in D, (cf. eqs. (1.2.13) and (1.2.14))

$$-\boldsymbol{M}^k_{\partial D}(\nu\times E^{\mathrm{sc}}(y))(x) + \frac{1}{ik}\boldsymbol{N}^k_{\partial D}(\nu\times H^{\mathrm{sc}}(y))(x) = 0, \tag{1.2.34}$$

and

$$-\boldsymbol{M}^k_{\partial D}(\nu\times E^{\mathrm{in}}(y))(x) + \frac{1}{ik}\boldsymbol{N}^k_{\partial D}(\nu\times H^{\mathrm{in}}(y))(x) = E^{\mathrm{in}}, \tag{1.2.35}$$

summing the last two equation and replacing the result in (1.2.32), motivates the following representation,

$$\begin{aligned}E(x) =& E^{\mathrm{in}}(x) + ik\operatorname{curl}\int_D\Phi_k(x,y)\boldsymbol{\mathcal{C}}_{\mu_r}H(y)dy\\ &+ (k^2 + \nabla\operatorname{div})\int_D\Phi_k(x,y)\boldsymbol{\mathcal{C}}_{\varepsilon_r}E(y)dy,\end{aligned} \tag{1.2.36}$$

where $\boldsymbol{\mathcal{C}}_{\varepsilon_r} := \varepsilon_r - \mathbb{I}$ and $\boldsymbol{\mathcal{C}}_{\mu_r} := \mu_r - \mathbb{I}$. Setting

$$\mathcal{M}^{k,\boldsymbol{\mathcal{C}}_A}_D(V) := \operatorname{curl}\int_D\Phi_k(x,y)\boldsymbol{\mathcal{C}}_A V(y)dy$$

and

$$\mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_A}(V) := (k^2 + \nabla\,\mathrm{div}) \int_D \Phi_k(x,y) \boldsymbol{\mathcal{C}}_{\varepsilon_r} E(y) dy$$

gives us the following integral formulation

$$\begin{pmatrix} I - \mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}} & -ik\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_{\mu_r}} \\ ik\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}} & I - \mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_{\mu_r}} \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} E^{\mathrm{in}} \\ H^{\mathrm{in}} \end{pmatrix} \tag{1.2.37}$$

that we call the Lipmann-Schwinger equation.

It is clear that if (E, H) is solution of (1.2.37) then it is also solution of the transmission problem ($\mathcal{P}$.Aniso).

Chapter 2

The Foldy-Lax Approximation for the Full Electromagnetic Scattering by Small Conductive Bodies of Arbitrary Shapes

2.1 Introduction and main results

The interest of estimating the waves propagating through a cluster of small inhomogeneities, embedded in a homogeneous background, goes back at least to Rayleigh in the 19th century. This interest is motivated by numerous applications in physics as in acoustics, electromagnetism and elasticity where the inhomogeneities model acoustic bubbles, electromagnetic particles and elastic deffects, respectively. Rayleigh already understood that as the size of the inhomogeneities is small, when compared to the used wavelength, the dominating reflected fields are mainly poles and he derived, formally, explicit approximations of these fields for special geometries, see [25] for an account on these situations. Then Mie [54] proposed a general method for estimating the electromagnetic waves, not necessarily in the Rayleigh regimes, but for special shapes as spheres, see also [33] for more information. These approaches apply for well separated inhomogeneities, i.e. when the distance between them is large as compared to their relative size. To handle clusters allowing very close inhomogeneities, Foldy, see [29], proposed an original method in the framework of acoustic waves where the refraction indices are concentrated on the centers of the inhomogeneities, that means there is a huge contrast between the inhomogeneities (the bubbles in this case) and the background. Briefly, Foldy's idea goes as follows. Motivated by physical considerations, and as the refraction is concentrated on 'centers' z_i, $i = 1, ..., m$ of the inhomogeneities, he represents the wave as a sum of poles,

$$U(x) = U^i(x) + \sum_{i=1,...,m} A_i^a \Phi_k(x, z_i) \tag{2.1.1}$$

where $A_i^a, i = 1, .., m$, are unknown constants and Φ_k is the fundamental solution of the Helmholtz propagator $\Delta + k^2$, i.e. the pole itself. The term U^i is the incident acoustic wave. The key concept in his approach is the introduction of the field

$$U_j(x) := U^i(x) + \sum_{i \neq j} A_i^a \Phi_k(x, z_i) \tag{2.1.2}$$

which is regarded as the external field incident on the j^{th} inhomogeneity in the presence of all the other inhomogeneities. The physical assumptions in Foldy's method is that the strength of the scattered wave from the inhomogeneity z_j is proportional to the external field on it, i.e.

$$A_j^a = g^e U_j(z_j). \tag{2.1.3}$$

Evaluating (2.1.2) at the centers $z_i, i = 1, ..., m$, we see that the coefficients $A_i^a, i = 1, ..., m$, are solution of the following algebraic system, called also self-consistent system,

$$A_i^a - \sum_{j \neq i} g_i^a \Phi_k(x, z_j) A_j^a = g_i^a U^i(z_i) \tag{2.1.4}$$

and the coefficients g_i^a, $i = 1, ..., m$, are called the corresponding scattering coefficients of the model. We can see this system as boundary conditions on the inhomogeneities. The novelty here is that the representation (2.1.1) coupled with self-consistent system (2.1.4) take into account the multiple interactions between the inhomogeneities. Assuming the inhomogenerities to be far away from each other, we end up with waves reflected by single inhomogeneities. Since this work of Foldy, there was a huge effort in understanding more its implications and extending it to other types of waves. The reader is oriented to Martin's book [49] form more insight in this direction.

As we can see, the derivation of the model (2.1.1)-(2.1.4) is formal even though it is driven by physical considerations. Fortunately, these formal calculations were mathematically justified by Berezin and Faddeev in their seminal work [12] where they use Krein theory of selfadjoint extensions, see also [4, 3] for further developments in this direction. This is done in the framework of the Schroedinger equation with point-like potentials. Another way used to give sense to Foldy's computations is the so-called regularization approach, see [3]. The difference between these two approaches is that the latter handles only isotropic interaction while the former allows for anisotropic interactions. But, to our understanding, the latter is easier to implement for other types of waves as electromagnetic and elastic waves, see [35, 17]. The Foldy method was also extended to multiple scaled objects in [36, 37, 34] with applications to inverse scattering theory.

With this in mind, it is natural to expect that regarding small acoustic inhomogeneities (but not necessarily highly concentrated ones as point-like scatterers), the dominated field would be the one corresponding to the Foldy field given in (2.1.1)-(2.1.4) with appropriate scattering coefficients $g_i^a, i = 1, ..., m$. This was confirmed by several authors, see for instance ([27, 64, 53, 15, 18]) for several related models.

Mimicking Foldy's approach for the electromagntic waves propagating through small inhomogeneities with highly concentrated index of refraction, we obtain the system

$$E(x) = E^i(x) + \sum_{i=1,...,m} A_i^e \Pi_k(x, z_i) \tag{2.1.5}$$

where Π_k is the dyadic Green's tensor, i.e. the fundamental solution of the Maxwell propagator $\operatorname{curl}\operatorname{curl} - k^2$. Here, the term E^i is the incident electromagnetic wave. The coefficients $A_i, i = 1, ..., m$, are now solution of the following algebraic system, which we can call the electric self-consistent system,

$$A_i^e - \sum_{j \neq i} g_i^e \Pi_k(x, z_j) A_j^e = g_i^e U^i(z_i) \tag{2.1.6}$$

and $g_i^e, i = 1, ..., m$ are the corresponding electric scattering coefficients.

As we can see, for waves generated by higly concentrated indices of refractions, i.e. point-like refractive indices, the Foldy method provides the representation of the electric field as (solely) electric dipoles, see [26] for a formal derivation and related issues. A mathematical justification of this model can be found in [17]. However, this is not a surprise if the contrast comes from the index of refraction via the electric permittivity keeping the magnetic permeabilitty constant and the same as the one of the background. Regarding small inhomogeneities (but not highly concentrated indices of refraction as point-like scatterers), the dominating part will be the Foldy field given in (2.1.5)-(2.1.6) with appropriate scattering coefficients $g_i^e, i = 1, ..., m$.

In the present work, we show that when there is a contrast for both the electric permittivity and magnetic permeability, then the dominating field is a combination of electric as well as magnetic dipoles. This is already observed in the literature, see for instance [44, 45, 72] and the references therein where formal computations are proposed and discussed related to the moment method. In [16, 10] a mathematical justification is provided for the case of well separated inhomogeneities. We refer the interested readers to these works for more insight and related issues.

In the next section, we discuss briefly the results we derived for the case of small conductive inhomogeneities and show the natural corresponding Foldy system which is the dominating field. Let us emphasize that the approximation is derived taking into account all the parameters modeling the inhomogeneities as the size, minimum distance, their number but also the used frequency.

Before closing this introduction, let us stress that in the recent twenty years or so, there were different and highly important fields where such kind of approximations are key tools. Let us mention few of them which are of particular interest to us:

1. Let us start with the mathematical imaging field where the small bodies can model impurities or small tumors, for instance, that one should localize and estimate the sizes from the measured fields (either near fields or far-fields), see [8]. In this case, based on our approximations, we can indeed localize the small bodies by reconstructing the points z_i, via MUSIC type algorithms [8, 17], and then estimate the polarization tensors. From these polarization tensors, one can derive lower and upper estimates of the small bodies' sizes, see [1]. The small bodies can also model electromagnetic nanoparticles. Imaging using electromagnetic nanoparticles as contrast agent is a recent and highly attractive imaging modality that uses special properties of the nanoparticles to create high contrasts in the tissue and then enhance the resolution of permittivity reconstruction for instance. There are at least two types of such special properties:one is related to the plasmonic nanoparticles (which are nearly resonant nanoparticles but might create high dissipation) and the other one related to the all dielectric nanoparticles (which are characterized by their high refraction indices), see [42], for instance.

2. A second field where this kind of approximations are useful is the material sciences. Indeed, arranging appropriately the small bodies in a given bounded domain, the whole cluster will generate electromagnetic fields which are close to the fields generated by related indices of refraction (or permittivities and permeabilities). These indices of refraction are dependent on the properties of the small bodies, as the size, geometry and their own possible contrasts in addition to the used frequencies. This opens the door to the possibility of creating desired and new materials. Such ideas are already tested and justified to some extent mathematically in the framework of the homogenization theory. However, this theory is based on the periodicity (or randomness) in distributing the small bodies. As we can see it from the approximations we provide above, we can achieve similar goals but without assuming the periodicity. In addition, and as far the electromagnetic waves are concerned, we can handle in a unified way, the generation of volumetric metamaterials, Gradient metasurfaces and also metawires, [71, 68]. These properties will be quantified and justified in a future work were we plan to handle more general type of inhomogeneities than the conductive ones described in this work.

To describe precisely the inhomogeneities, let $(B_i)_{i=1}^m$ be m open, bounded and simply connected sets containing the origin, with Lipschitz boundaries. To these sets, we correspond the small bodies $(D_i)_{i=1}^m$, i.e. the inhomogeneities, which are defined as the translations and contractions of the m bodies $(B_i)_{i=1}^m$, that is

$$D_i = \boldsymbol{a} B_i + z_i, i = 1, ..., m \tag{2.1.7}$$

where $z_i, i = 1, ..., m$ are given positions in $\mathbb{R}^3$ and $\boldsymbol{a}$ a small parameter.

We consider the scattering of a time-harmonic electromagnetic plane wave by the perfectly conducting small bodies $(D_i)_{i=1}^m$ formulated as follows (see [24])

$$\begin{aligned} &\nabla \times E - ikH = 0 \ \text{ in } D^+ := \mathbb{R}^3 \setminus \cup_{i=1}^m \overline{D_i}, \\ &\nabla \times H + ikE = 0 \ \text{ in } D^+, \end{aligned} \tag{2.1.8}$$

with the boundary condition

$$\nu \times E = 0 \quad \text{on } \partial D = \cup_{i=1}^m \partial D_i. \tag{2.1.9}$$

The total electromagnetic fields are expressed as $E = E^{\text{in}} + E^{\text{sc}}$, $H = H^{\text{in}} + H^{\text{sc}}$ where the indices “in” and “sc” indicate the incident wave and the scattered wave, respectively. The condition (2.1.9) corresponds to the case of perfectly conducting obstacle and ν expresses the unit outward normal vector to the boundary of $D = \cup_{i=1}^m D_i$. Here the wave number k is defined through the relation $k^2 = (\xi\omega + i\sigma)\mu\omega$ where ξ, σ, μ are respectively the electric permittivity, electric conductivity and the magnetic permeability. In the case where $\sigma = 0$, and then k is real, the scattered wave $(E^{\text{sc}}, H^{\text{sc}})$ must satisfies the outgoing radiation condition

$$\lim_{|x| \to \infty} (H^{\text{sc}}(x) \times x - |x| E^{\text{sc}}(x)) = 0. \tag{2.1.10}$$

Motivated by applications, we restrict ourselves to incident waves of the form $E^{\text{in}}(x) = Pe^{ikx\cdot\theta}$, where θ is the incident direction, $P \in \mathbb{R}^3$ is the polarization that is orthogonal to θ.
We introduce the diameters $\boldsymbol{a}_i = \max_{x,y \in D_i} d(x, y), \ i \in \{1, ..., m\}$, and the distance between two

bodies $D_i, D_j, i \neq j$, as $\boldsymbol{\delta}_{ij} = \min_{x \in D_i, y \in D_j} d(x,y)$, for every $i,j \in \{1,...,m\}; i \neq j$ where $d(\cdot,\cdot)$ stands for the Euclidean distance. We set

$$\boldsymbol{a} := \max_{i \in \{1,\ldots,m\}} \boldsymbol{a}_i, \qquad \boldsymbol{\delta} := \min_{i \neq j \in \{1,\ldots,m\}} \boldsymbol{\delta}_{ij}. \tag{2.1.11}$$

We suppose in addition that $\cup_{i=1}^m \overline{D_i} \subset \Omega$, where Ω is a bounded Lipschitz domain such that

$$d(\partial\Omega, \cup_{i=1}^m \overline{D_i}) \geq \boldsymbol{\delta}.$$

The scattering problem (2.1.8) under the boundary condition (2.1.9) and the radiating condition (2.1.10) is well posed in appropriate spaces under appropriate conditions (see [60, 24]) which will described later. In addition, when $\Im k$ is different from zero, the scattered electromagnetic fields are fastly decaying at infinity as we have attenuation. But when $\Im k = 0$, i.e. in the absence of attenuation, we have the following behavior (as spherical-waves) of the scattered electric fields far away from the sources D_i's

$$E^{\mathrm{sc}}(x) = \frac{e^{ik|x|}}{|x|}\{E^\infty(\tau) + O(|x|^{-1})\}, \quad |x| \longmapsto \infty, \tag{2.1.12}$$

and we have a similar behavior for the scattered magnetic field as well:

$$H^{\mathrm{sc}}(x) = \frac{e^{ik|x|}}{|x|}\{H^\infty(\tau) + O(|x|^{-1})\}, \quad |x| \longmapsto \infty. \tag{2.1.13}$$

where $(E^\infty(\tau), H^\infty(\tau))$ is the electromagnetic far field pattern in the direction of propagation $\tau := \frac{x}{|x|}$.

To describe our results, let us recall the dyadic Green's function of the Maxwell propagator $\Pi_k(x,y) := k^2\Phi_k(x,y)I + \nabla_x\nabla_x\Phi_k(x,y)$ where $\Phi_k(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y$, is the outgoing fundamental solution of the Helmlholtz propagator, i.e. $\Delta + k^2$. In addition, we introduce generic functions $\epsilon(\boldsymbol{\delta}^s, |k|^l)$ and $\epsilon_{k,\boldsymbol{\delta},m}$ which express error functions as follows

$$\begin{aligned} \epsilon(\boldsymbol{\delta}^s, |k|^l) &= O(\frac{\max(1,|k|^l)}{\boldsymbol{\delta}^s}) \\ \epsilon_{k,\boldsymbol{\delta},m} &:= O\Big(\frac{(|k|+1)\ \ln(m^{\frac{1}{3}})}{\boldsymbol{\delta}^3} + \frac{(|k|+1)^2\ m^{\frac{1}{3}}}{\boldsymbol{\delta}^2} + \frac{(|k|+1)^3\ m^{\frac{2}{3}}}{\boldsymbol{\delta}}\Big). \end{aligned} \tag{2.1.14}$$

Now, we are ready to state the main result of this work.

Theorem 4. *Let the small inhomogeneities be of the form $D_i := \boldsymbol{a}B_i + z_i, i = 1,...,m$, and the total electromagnetic fields (E,H) satisfy (2.1.8) and (2.1.9) such that the corresponding scattered fields satisfy the outgoing radiation condition (2.1.10). There exists a constant C depending only on the Lipschitz character of the B_i's such that if*

$$|k|^2\boldsymbol{a} \ + (1+|k|^2)\frac{\boldsymbol{a}^3}{\boldsymbol{\delta}^3} + \ \left(\frac{\ln(m^{\frac{1}{3}})}{\boldsymbol{\delta}^3} + \frac{2|k|m^{\frac{1}{3}}}{\boldsymbol{\delta}^2} + \frac{m^{\frac{2}{3}}}{2\boldsymbol{\delta}}|k|^2\right)\boldsymbol{a}^3 < C, \tag{2.1.15}$$

then

1. *the scattered electric field has the following expansion, for $\Im k \geq 0$, for $x \in \mathbb{R}^3 \setminus (\cup_{i=1}^m D_i)$ such that $\min_{1\leq i\leq m} d(x, D_i) \geq \boldsymbol{\delta}$,*

$$\begin{aligned} E^{sc}(x) = &\sum_{i=1}^{m} \left(\nabla\Phi_k(x,z_i)\times\mathcal{R}_i + \operatorname{curl}\operatorname{curl}(\Phi_k(x,z_i)\mathcal{Q}_i)\right) \\ &+ O^{\epsilon}(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}) + O^{\epsilon}(\frac{\boldsymbol{a}^7}{\boldsymbol{\delta}^7}), \end{aligned} \tag{2.1.16}$$

2. *the far field pattern has the following expansion for $\Im k = 0$*

$$\begin{aligned} E^{\infty}(\tau) = &\frac{ik}{4\pi}\sum_{i=1}^{m} e^{-ik\tau.z_i}\tau\times\big(\mathcal{R}_i - ik\tau\times\mathcal{Q}_i\big) + O\Big(\ (|k|^3+|k|^2)\ m\boldsymbol{a}^4\Big) \\ &+ |k|\ \max(1,|k|)\ O^{\varepsilon}\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}\Big)m\boldsymbol{a}^3, \end{aligned} \tag{2.1.17}$$

and the errors in (2.1.16) and (2.1.17) correspond to

$$O^{\epsilon}(\frac{\boldsymbol{a}^7}{\boldsymbol{\delta}^7}) := O\Big(\frac{\boldsymbol{a}^7}{\boldsymbol{\delta}^7} + \epsilon(\boldsymbol{\delta}^6, |k|+|k|^2+|k|^3)\boldsymbol{a}^7 + \epsilon(\boldsymbol{\delta}^5, |k|^2)\boldsymbol{a}^7\Big), \tag{2.1.18}$$

$$O^{\epsilon}(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}) := O\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4} + (1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4 + \max(1+|k|,|k|^2)\boldsymbol{a}\Big). \tag{2.1.19}$$

The coefficients $(\mathcal{R}_i)_{i=1}^m$ and $(\mathcal{Q}_i)_{i=1}^m$ are the solutions of the following invertible linear system

$$\begin{aligned} &\mathcal{R}_i + \big[\mathcal{P}_{\partial D_i}\big]\sum_{(j\neq i)\geq 1}^{m}\big(\Pi_k(z_i,z_j)\mathcal{R}_j - k^2\nabla\Phi_k(z_i,z_j)\times\mathcal{Q}_j\big) = -\big[\mathcal{P}_{\partial D_i}\big]\operatorname{curl}E^{in}(z_i), \\ &\mathcal{Q}_i - \big[\mathcal{T}_{\partial D_i}\big]\sum_{(j\neq i)\geq 1}^{m}\big(\ -\nabla_x\Phi_k(z_i,z_j)\times\mathcal{R}_j\ +\Pi_k(z_i,z_j)\mathcal{Q}_j\big) = -\big[\mathcal{T}_{\partial D_i}\big]E^{in}(z_i). \end{aligned} \tag{2.1.20}$$

The coefficients $\big[\mathcal{P}_{\partial D_i}\big]$ and $\big[\mathcal{T}_{\partial D_i}\big]$ are respectively the magnetic and electric polarizability tensors. They are characterized by the shapes of the conductivities, see (2.2.7) and (2.2.8) in subsection 2.2.1 for their explicit definitions. The explicit conditions under which the algebraic system (2.1.20) is invertible as well the dependency of the upper bounds in approximations (2.1.16) and (2.1.17) on the conductivities are provided in section 2.4 and section 2.5 respectively.

Remark 5. *Based on the algebraic system (2.1.20), we can compute approximations of the electric (and also the magnetic) fields generated by inhomogeneities with boundary conditions of the type $\nu\times E = 0$. Let us now introduce the following change of parameters:$\tilde{\mathcal{R}}_j := ik\mathcal{Q}_j$, $\tilde{\mathcal{Q}}_j := \frac{i}{k}\mathcal{R}_j$, $\widetilde{\big[\mathcal{P}_{\partial D_i}\big]} := -\big[\mathcal{T}_{\partial D_i}\big]$ and $\widetilde{\big[\mathcal{T}_{\partial D_i}\big]} := -\big[\mathcal{P}_{\partial D_i}\big]$. Then the system (2.1.20) becomes, remembering that $\operatorname{curl}E^{in} =$*

ikH^{in} and $E^{in} = \frac{i}{k}$ curl H^{in},

$$\tilde{\mathcal{R}}_i + [\tilde{\mathcal{P}_{\partial D_i}}] \sum_{(j\neq i)\geq 1}^{m} \Big(\Pi_k(z_i,z_j)\tilde{\mathcal{R}}_j - k^2 \nabla\Phi_k(z_i,z_j) \times \tilde{\mathcal{Q}}_j \Big) = -[\tilde{\mathcal{P}_{\partial D_i}}] \operatorname{curl} H^{in}(z_i),$$
$$\tilde{\mathcal{Q}}_i - [\tilde{\mathcal{T}_{\partial D_i}}] \sum_{(j\neq i)\geq 1}^{m} \Big(-\nabla_x \Phi_k(z_i,z_j) \times \tilde{\mathcal{R}}_j + \Pi_k(z_i,z_j)\tilde{\mathcal{Q}}_j \Big) = -[\tilde{\mathcal{T}_{\partial D_i}}] H^{in}(z_i). \tag{2.1.21}$$

This is the algebraic system corresponding to the inhomogeneities with boundary conditions of the type $\nu \times \operatorname{curl} H = 0$. Hence, under the introduced change of parameters, our approximation formulas in Theorem 4 provide the magnetic (and also the electric) fields generated by the inhomogeneities with boundary conditions of the type $\nu \times \operatorname{curl} H = 0$.

We call the approximation in (2.1.16) and (2.1.17) the point-interaction approximations or the Foldy-Lax approximations as the dominant field is reminiscent to the field describing the interactions between the points $z_i, i = 1, .., m$, with the scattering coefficients given by the polarization tensors $[\mathcal{P}_{\partial D_i}]$ and $[\mathcal{T}_{\partial D_i}]$, see [49, 36, 37, 17, 34] for particular situations. As we can see, the dominating field is a combination of both electric and magnetic dipoles. The algebraic system above encodes the multiple electromagnetic interactions.

As we mentioned above, since the pioneering works of Rayleigh till Foldy, the original goal of such approximations was to reduce the computation of the fields generated by a cluster of small bodies to inverting an algebraic system (called the Foldy linear algebraic system). With our approximations above, and regarding the full Maxwell system, such a goal is reached with high generality as we take into account all the parameters, $m, \boldsymbol{a}, \boldsymbol{\delta}$ and k, modeling the scattering by the cluster of small conductors D_i's. To our best knowledge, there is no result in the literature where such approximations are provided with such generality and precision. At the mathematical analysis level, and as we are using integral equations methods, we needed to derive an a priori estimate of the related densities. The first key observation here is to derive it in the $L_t^{2,\mathrm{Div}}$ spaces instead of the usual L^2 spaces. As a second observation, to derive such estimates, we used a particular decomposition of the densities, see Proposition 6 or Theorem 8, which allows to obtain the needed qualitative as well as quantitative estimates while refining the approximation. Finally, to prove the invertibility of the algebraic system (2.1.20) under the general condition (2.1.20), the key point is to have reduced the coercivity inequality to the one related to scalar Helmholtz model. Let us emphasize here that as this linear algebraic system comes form the boundary conditions, such a reduction of Maxwell to Helmholtz is not a trivial one (even, a priori, not a natural one).

The only restriction we have in our condition (2.1.15) is the appearance of the factor $\ln(m)$. At the technical level, see the last part of the proof of Theorem 8, its appearance is due to the fact the singularity of the dyadic Green's function is of the order 3 (in contrast to the ones of the Green's functions for the Laplace or Lamé related models). We believe that this factor can be removed using more pde tools to invert the Calderon operator, see (2.2.20), and hope to report on this in the near future.

The closest published works (i.e. deriving the Foldy-Lax type of approximations) are those by A. Ramm in one side and those by V. Maz'ya, A. Movchan and M. Nieves in another side. The several works by A. Ramm on Maxwell are derived more in a formal way, see [65, 66]. In addition, condition of the type $\frac{\boldsymbol{a}}{\boldsymbol{\delta}} << 1$ are used (meaning at least that the mesoscale regimes where $\boldsymbol{a} \sim \boldsymbol{\delta}$ are not handled) and without clarifying the rates. Finally, and unfortunately, to our opinion the form of the derived algebraic system is unclear and questionable. In a series of works dedicated to the Laplace and Lamé models (assuming $k = 0$), V. Maz'ya, A. Movchan and M. Nieves proposed a method which indeed takes into account the parameter and state the results in the mesoscale regimes too, see [51, 50, 53]. One possible limit to their approach is the need of the maximum principle in handling the link between the system on the boundary to fields outside. This might be a handicap for tackling the Maxwell system for which such maximum principles are not at our hands.

2.1.1 The formal computations

The remaining part of the paper is dedicated to the proof of Theorem 4. But before going into the details of the proof, let us briefly describe its main formal steps that we shall justify in the next sections. Our proof is based on integral equation methods.

In Section 2.2, we show that the scattered wave, admitting the following representation

$$E^{\mathrm{sc}}(x) = \operatorname{curl} \int_{\cup_{i=1}^m \partial D_i} \Phi_k(x,y) A(y)\, ds(y),\;\; x \in \mathbb{R}^3 \setminus \cup_{i=1}^m \partial D_i,$$

is the unique radiating solution of the equation (2.1.8) and the boundary condition (2.1.9), with a particular density A that belongs to $L_t^{2,\mathrm{Div}}(\cup_{i=1}^m \partial D_i) = \prod_{i=1}^m L_t^{2,\mathrm{Div}}(\partial D_i)$.

In addition, and this is a key observation, we show that each restriction of the density A, $a_i := a/\partial D_i$, can be decomposed as sum of the trace of a divergence free $A_i^{[1]}$ and curl free $A_i^{[2]} = \nu \times \nabla u_i$ vector-fields, and provide the corresponding a priori estimate of the decomposition (see Proposition 6), as summarized in Theorem 8.

In Section 3.3.3, we first derive the expansion

$$E^\infty(\tau) = \frac{ik}{4\pi} \sum_{i=1}^m e^{-ik\tau . z_i} \tau \times \left\{ \int_{\partial D_i} A_i\, ds - \int_{\partial D_i} (ik\tau.(y-z_i))\, A_i(y)\, ds(y) \right\} + O(Error).$$

which is due to the following simple decomposition

$$E^\infty(\tau) = \sum_{i=1}^m \frac{ik}{4\pi} \tau \times \Big(\int_{\partial D_i} A(y) e^{-ik\tau . z_i} ds(y) + \int_{\partial D_i} A(y) (e^{-ik\tau . y} - e^{-ik\tau . z_i}) ds(y) \Big), \tag{2.1.22}$$

and the approximation, which is due to the above mentioned decomposition of the densities,

$$\int_{\partial D_i} (ik\tau.(y-z_i))\, A_i(y)\, ds(y) = \int_{\partial D_i} (ik\tau.(y-z_i))\, A_i^{[2]}(y)\, ds(y) + O(Error_i).$$

Integrating by part, using the fact that $A_i^{[2]} = \nu \times \nabla u_i$, and replacing in the previous expansion, we obtain

$$\begin{aligned} E^\infty(\tau) = & \frac{ik}{4\pi} \sum_{i=1}^m e^{-ik\tau . z_i} \tau \times \left\{ \mathcal{R}_i := \int_{\partial D_i} A_i\, ds - \mathcal{Q}_i := ik\tau \times \int_{\partial D_i} \nu u_i(y)\, ds(y) \right\} \\ & + O(Error). \end{aligned} \tag{2.1.23}$$

With similar but trickier calculations, we derive the corresponding approximation for the near fields.

Hence, we are left with the two total charges, namely $(\mathcal{R}_i)_{i=1}^m$ and $(\mathcal{Q}_i)_{i=1}^m$, to estimate. The main idea is to multiply the boundary condition (2.1.9) by particular curl free matrices, namely $(\nabla\phi_i)_{i=1}^m$, and a divergence free matrices $([\Psi_i] = [\psi_1^i, \psi_2^i, \psi_3^i])_{i=1}^m$, to get rid of the influence of, respectively, the divergence free part of the density A and its curl free part. Precisely, we derive the expansions

$$\begin{aligned}\int_{\partial D_i} \nabla\phi_i \cdot \nu \times E^{\text{in}} ds =& \mathcal{R}_i + \left[\mathcal{P}_{\partial D_i}\right] \sum_{(j\neq i)\geq 1}^{m} \left(\Pi_k(z_i,z_j)\mathcal{R}_j - k^2\nabla\Phi_k(z_i,z_j)\times \mathcal{Q}_j\right) \\ &+ (Error_i)\end{aligned} \tag{2.1.24}$$

and

$$\begin{aligned}\int_{\partial D_i} [\Psi_i] \cdot \nu \times E^{\text{in}} ds =& \mathcal{Q}_i + \left[\mathcal{T}_{\partial D_i}\right] \sum_{(j\neq i)\geq 1}^{m} \left(\nabla_x\Phi_k(z_i,z_j)\times \mathcal{R}_j - \Pi_k(z_i,z_j)\mathcal{Q}_j\right) \\ &+ (Error_i),\end{aligned} \tag{2.1.25}$$

see Proposition 11.

Finally, in Section 2.4, we show that the above linear system is invertible and the total charges, that we denote by $(\widehat{\mathcal{R}}_i)_{i=1}^m$ and $(\widehat{\mathcal{Q}}_i)_{i=1}^m$, corresponding to the case without the error of approximation $Error_i$'s and replacing the left-hand sides with $-\left[\mathcal{P}_{\partial D_i}\right] \operatorname{curl} E^{inc}(z_i)$ and $-\left[\mathcal{T}_{\partial D_i}\right] E^{inc}(z_i)$ respectively in (2.1.24) and (2.1.25), satisfy the following estimate

$$\left(\sum_{i=1}^m \left(|\widehat{\mathcal{R}}_i|^2 + |\widehat{\mathcal{Q}}_i|^2\right)\right)^{\frac{1}{2}} \leq \frac{1}{C_{Li}\mu^-}\boldsymbol{a}^3\Big(\sum_{i=1}^m\left(\left|E^{inc}(z_i)\right|^2 + \left|\operatorname{curl} E^{inc}(z_i)\right|^2\right)\Big)^{\frac{1}{2}}. \tag{2.1.26}$$

With this kind of estimate, we show that

$$\left(\sum_{i=1}^m \left(|\widehat{\mathcal{R}}_i - \mathcal{R}_i|^2 + |\widehat{\mathcal{Q}}_i - \mathcal{Q}_i|^2\right)\right)^{\frac{1}{2}} \leq \frac{1}{C_{Li}\mu^-}\boldsymbol{a}^3\Big(\sum_{i=1}^m (Error_i)\Big)^{\frac{1}{2}} \tag{2.1.27}$$

and replacing it in the obtained near and far fields approximations, as in (2.1.23), we derive the estimates of Theorem 4.

2.2 Existence, unique solvability and an a priori estimation of the density

2.2.1 Preliminaries

Let us recall few properties of the surface divergence which will be important in our later analysis, see (Section 4 in [57] and Chapter 2 in [23]) for more details. First, we recall the surface gradient of a smooth function ϕ on ∂D, $\nabla_{\rm t}$, as $\nabla_{\rm t}\phi := \nabla\phi - (\nu\cdot\nabla\phi)\nu$ where ν is the exterior unit normal to ∂D. Then the (weak) surface divergence for a tangential field A is defined using the duality

$$\int_{\partial D} \phi \,\mathrm{Div}\, A \; ds = -\int_{\partial D} \nabla_{\rm t}\phi \cdot A \; ds, \tag{2.2.1}$$

for every $\phi \in \mathcal{C}^{\infty}(\partial D)$.

If A is a tangential field for which $\mathrm{Div}\, A$ exists in the sense above, and it is in $L^1(\partial D)$ for instance, then, taking $\phi(x) = x_i$ in (2.2.1), we have

$$\int_{\partial D} x \,\mathrm{Div}\, A(x) \; ds(x) = -\int_{\partial D} A(x) \; ds(x) \tag{2.2.2}$$

and taking $\phi(x) = 1$, we have

$$\int_{\partial D} \mathrm{Div}\, A(x) \; ds(x) = 0. \tag{2.2.3}$$

When $a := \nu \times u$ for a certain sufficiently smooth vector field u, we get

$$\mathrm{Div}\, A = -\nu \cdot \mathrm{curl}\, u. \tag{2.2.4}$$

Further, for a scalar function ψ, being $\psi \; A$ tangential (i.e. $\psi \; A \cdot \nu = \psi \; (A \cdot \nu) = 0$), the following identity holds, $\mathrm{Div}(\psi A) = \nabla_t \psi \cdot A + \psi \,\mathrm{Div}\, A$, and hence

$$\int_{\partial D_i} \psi \; A \; ds = \int_{\partial D_i} (x - z_i)(\nabla\psi(x) \cdot A(x) + \psi(x) \,\mathrm{Div}\; A(x)) \; ds(x). \tag{2.2.5}$$

Indeed, $\int_{\partial D_i} \psi \; A \; ds = \int_{\partial D_i} \nabla(x - z_i) \; \big(\psi \; A\big)(x) \; ds(x) = \int_{\partial D_i} (x - z_i) \,\mathrm{Div}(\psi \; A)(x) \; ds(x)$, here $\nabla(x - z_i) = I$ where I stands for the identity matrix of $\mathbb{R}^3 \times \mathbb{R}^3$.

The spaces $L^2_{\rm t}(\partial D)$ and $L^{2,Div}(\partial D)$ denote respectively the space of all tangential fields of $L^2(\partial D)$; and the subspace of $L^2_{\rm t}(\partial D)$ that have an L^2 weak surface divergence, precisely

$$L^2_{\rm t}(\partial D) = \{A \in L^2(\partial D);\; A \cdot \nu = 0\},$$

$$L^{2,Div}_{\rm t}(\partial D) = \{A \in L^2_{\rm t}(\partial D);\; \mathrm{Div}\, A \in L^2_0(\partial D)\}$$

where

$$L^2_0(\partial D) := \{u \in L^p(\partial D),\ \text{such that} \int_{\partial D} u \; ds = 0\}$$

finally $L^{2,0}_t(\partial D) = \{A \in L^{2,Div}_{\rm t}(\partial D);\, \mathrm{Div}\, A = 0\}$.

Let us now recall some properties on the needed polarization tensors. For $i = 1, ..., m$, we recall the single layer operator $[S^k_{ii,D}] : L^2(\partial D_i) \to H^1(\partial D_i)$, defined as

$$[S^k_{ii,D}](\psi)(x) := \int_{\partial D_i} \Phi_k(x, y)\psi(y) \; ds(y), \;\; x \in \partial D_i, \tag{2.2.6}$$

and the double layer operator $[K^k_{ii,D}] : L^2(\partial D_i) \to L^2(\partial D_i)$,

$$[K^k_{ii,D}](\psi)(x) = \int_{\partial D_i} \frac{\partial \Phi_k}{\partial \nu_y}(x,y)\psi(y)\; ds(y), \;\; x \in \partial D_i,$$

with its adjoint

$$[(K^k_{ii,D})^*](\psi)(x) = \int_{\partial D_i} \frac{\partial \Phi_k}{\partial \nu_x}(x,y)\psi(y)\; ds(y), \;\; x \in \partial D_i,$$

recalling that $\Phi_k(x,y) = \frac{1}{4\pi}\dfrac{e^{ik|x-y|}}{|x-y|}$ is the Green function for the Helmholtz equation at the wave number k.

The operator $[\lambda I + (K^0_{ii,D})^*]$ is invertible from $L^2_0(\partial D_i) := \{f \in L^2(\partial D_i)/ \; \int_{\partial D_i} f(y)ds(y) = 0\}$ onto itself for any complex number λ such that $|\lambda| \geq \frac{1}{2}$, see [9, 56] for instance. The following two quantities will play an important role in the sequel:

$$[\mathcal{P}_{\partial D_i}] := \int_{\partial D_i} [-I/2 + (K^0_{ii,D})^*]^{-1}(\nu)(y) y^T ds(y), \tag{2.2.7}$$

and

$$\left[\mathcal{T}_{\partial D_i}\right] := \int_{\partial D_i} [I/2 + (K^0_{ii,D})^*]^{-1}(\nu)(y) y^T ds(y). \tag{2.2.8}$$

The tensor $[\mathcal{P}_{\partial D_i}]$ is negative-definite symmetric matrix and $\left[\mathcal{T}_{\partial D_i}\right]$ is positive-definite symmetric matrix, (see Lemma 5 and Lemma 6 in [16] or Theorem 4.11 in [9]). Further, we have the following scales:

$$[\mathcal{P}_{\partial D_i}] = \boldsymbol{a}^3[\mathcal{P}_{\partial B_i}], \text{ and } \; [\mathcal{T}_{\partial D_i}] = \boldsymbol{a}^3[\mathcal{T}_{\partial B_i}]. \tag{2.2.9}$$

Indeed, [1]

$$\begin{aligned}
[\mathcal{P}_{\partial D_i}] &= \int_{\partial D_i} [-I/2 + (K^0_{ii,D})^*]^{-1}(\nu)(y) \; (y - z_i)^T ds(y),\\
&= \int_{\partial B_i} [-I/2 + (K^0_{ii,B})^*]^{-1}(\nu)(s_i) \; (\boldsymbol{a}s + z_i - z_i)^T \boldsymbol{a}^2 \; ds(y),\\
&= \boldsymbol{a}^3[\mathcal{P}_{\partial B_i}],
\end{aligned}$$

and we get it in the same way for the second identity, after noticing that[2]

$$\left[\mathcal{T}_{\partial D_i}\right] := \int_{\partial D_i} \nu_y \left[[I/2 + K^0_{ii,D}]^{-1} \; (x)\right]^T ds(y) = \int_{\partial D_i} \nu_y \left[[I/2 + K^0_{ii,D}]^{-1} \; (x - z_i)\right]^T ds(y).$$

For $i \in \{1, ..., m\}$ let $(\mu_i^{\mathcal{T}})^+$, $(\mu_i^{\mathcal{P}})^+$ be the respective maximal eigenvalues of $[\mathcal{T}_{\partial B_i}]$, $-[\mathcal{P}_{\partial B_i}]$, and let $(\mu_i^{\mathcal{T}})^-$, $(\mu_i^{\mathcal{P}})^-$ be their minimal ones. We define

$$\mu^+ = \max_{i\in\{1,\dots,m\}}((\mu_i^{\mathcal{T}})^+, (\mu_i^{\mathcal{P}})^+), \;\; \text{and } \mu^- = \min_{i\in\{1,\dots,m\}}((\mu_i^{\mathcal{T}})^-, (\mu_i^{\mathcal{P}})^-). \tag{2.2.10}$$

Hence for every vector $\mathcal{C}$, we get

$$\mu^-|\mathcal{C}|^2\boldsymbol{a}^3 \leq [\mathcal{T}_{\partial D_i}]\; \mathcal{C}\cdot\mathcal{C} \leq \boldsymbol{a}^3\mu^+|\mathcal{C}|^2, \;\; \text{and } \; \mu^-|\mathcal{C}|^2\boldsymbol{a}^3 \leq [-\mathcal{P}_{\partial D_i}]\; \mathcal{C}\cdot\mathcal{C} \leq \boldsymbol{a}^3\mu^+|\mathcal{C}|^2. \tag{2.2.11}$$

[1] Recall that $\int_{\partial D_i}[-I/2 + (K^0_{ii,D})^*]^{-1}(\nu)(y)\; z_i^T ds(y) = \int_{\partial D_i}[-I/2 + (K^0_{ii,D})^*]^{-1}(\nu)(y)\;\; ds(y)\; z_i^T \;\; = 0.$

[2] We have $[I/2 + K^0_{ii,D}](z_i) = z_i[I/2 + K^0_{ii,D}](1) = 0.$

2.2.2 Existence and uniqueness of the solution

The solution to the problem (2.1.8) under the boundary condition (2.1.9) and the radiating conditions (2.1.10) can be expressed in terms of boundary integral equation (see [24, 60]), under certain conditions in appropriate spaces that will be specified later, using either one of the representations

$$E(x) = E^i(x) + \operatorname{curl} \int_{\partial D} \Phi_k(x,y) A(y)\, ds(y), \qquad x \in \mathbb{R}^3 \setminus (\bar{D} := \cup_{i=1}^m \bar{D}_i) \tag{2.2.12}$$

$$E(x) = E^i(x) + \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi_k(x,y) A(y)\, ds(y), \qquad x \in \mathbb{R}^3 \setminus \bar{D} \tag{2.2.13}$$

or a linear combination of the two, where A is the unknown vector density to be found to solve the problem.
Let us consider the representation (2.2.12), i.e. for a tangential field A and $s \in D^+$

$$E^{\mathrm{sc}}(s) = \operatorname{curl} \int_{\partial D} \Phi_k(s,y) A(y)\, ds(y), \tag{2.2.14}$$

and let be $\left\{\Gamma_+(x), \Gamma_-(x),\ \ x \in \cup_{i=1}^m \partial D_i\right\}$ a family of doubly truncated cones with a vertex at x such that $\Gamma_\pm(x) \cap D^\mp = \emptyset$, we have for almost every $x \in \cup_{i=1}^m \partial D_i$ (see [57])

$$\lim_{\substack{s \to x \\ s \in \Gamma_\pm(x)}} E^{\mathrm{sc}}(s) = \mp \frac{1}{2} \nu \times a + \operatorname{curl} \int_{\partial D} \Phi_k(x,y) A(y)\, ds(y) \tag{2.2.15}$$

where the integral is taken in the principal value of Cauchy sense, and the identity must be understood in the sens of trace operator, then using the condition (2.1.9), we get

$$\nu \times \lim_{\substack{s \to x \\ s \in \Gamma_\pm(x)}} E^{\mathrm{sc}}(s) = [\pm I/2 + M^k_{\partial D}](A)(x) := \pm \frac{1}{2} A + \nu \times \operatorname{curl} \int_{\partial D} \Phi_k(x,y) A(y)\, ds(y) \tag{2.2.16}$$

where $[\pm I/2 + M^k_{\partial D}]$ is called the magnetic dipole operator. Consequently, to solve the scattering problem we need to solve the integral equation

$$[I/2 + M^k_{\partial D}](A)(x) = -\nu \times E^{\mathrm{in}}, \quad \text{on } \cup_{i=1}^m \partial D_i \tag{2.2.17}$$

or,

$$[I/2 + M^k_{ii,D}](A) + [\sum_{(j \neq i) \geq 1}^{m} M^k_{ij,D}](A) = -\nu \times E^{\mathrm{in}}, \quad x_i \in \partial D_i. \tag{2.2.18}$$

where

$$M^k_{ij,D} A(x) := \nu \times \int_{\partial D_j} \nabla_x \Phi_k(x_i, y) \times A(y)\, ds(y).$$

In this case the magnetic field is represented by

$$H^{\mathrm{sc}}(x) = -ik \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi_k(x,y) A(y)\, ds(y).$$

Further it satisfies

$$\lim_{\substack{s\to x \\ s\in\Gamma_\pm(x)}} \nu \times H^{\rm sc}(s) = [N^k_{\partial D}](A) = k^2[\nu \times S^k_{\partial D}](A) + [\nu \times \nabla S^k_{\partial D}](\mathrm{Div}\, A), \tag{2.2.19}$$

which means that $\nu \times (H^{\rm sc})^+ = (\nu \times H^{\rm sc})^-$ i.e H is continuous across the boundary, the operator N_k is called electric dipole operator. We can write the equations in (2.2.18) in a compact form as follows

$$(I/2 + \mathcal{M}_D + \mathcal{M}_N)A = -\nu \times E^I, \tag{2.2.20}$$

where $\mathcal{R} = (A_1, ..., A_m)^T$ is a column matrix which vectorial components are $a_i := a/\partial D_i$. Similarly, $E^I = (E^{\rm in}_1, ..., E^{\rm in}_m)$ with $E^{\rm in}_i = E^{\rm in}/\partial D_i$ and $\mathcal{M}_D$ is the diagonal matrix operator given by

$$\mathcal{M}_D := \begin{cases} M^k_{ij,D} \text{ if } \ i = j \\ 0 \ \ \text{otherwise} \end{cases}$$

and finally $\mathcal{M}_N$ is the matrix operator with null diagonal

$$\mathcal{M}_N = \begin{cases} 0 \ \text{ if } \ i = j \\ M^k_{ij,D} \ \text{ if } \ i \neq j \end{cases}$$

For $k \in \mathbb{C} \setminus \{0\}$ such that $\Im k \geq 0$, the operators $(\pm I/2 + M^k_{ii,D})$ are Fredholm with index zero from $L^{2,\rm Div}_{\rm t}(\partial D_i)$ into it self, furthermore $(\pm I/2 + M^k_{ii,D})$ are Fredholm from $L^{2,\rm Div}_{\rm t}(\partial D_i)/L^{2,0}_{\rm t}(\partial D_i)$ into it self, likewise for $L^2_{\rm t}(\partial D_i)$. Moreover if k is not a Maxwell eigenvalue for D_i then the operators are in fact isomorphisms, see Theorem 5.3 [57] and the remark after, and [59].

Let us notice that when $\Im k > 0$, then k is not a Maxwell eigenvalue. In addition, when $\Im k = 0$ and as the radius of D_i is small, by a scaling argument, this condition on k is obviously fulfilled. As $\pm I/2 + \mathcal{M}_D$ is an isomorphism and $\mathcal{M}_N$ is compact (since the kernel of each component is of class $\mathcal{C}^\infty$), the operators

$$\pm\, I/2 + \mathcal{M}_D + \mathcal{M}_N : \prod_{i=1}^m \mathcal{E}(\partial D_i) \longrightarrow \prod_{i=1}^m \mathcal{E}(\partial D_i), \tag{2.2.21}$$

where $\mathcal{E} := L^2_{\rm t}, \ L^{2,Div}_{\rm t}$, are Fredholm with zero index, so to show that the operator above is in fact an isomorphism it is enough to show that the homogeneous problem (i.e $E^{\rm in} = 0$), has the unique identically null solution density that is $\mathcal{R} = 0$. We derive from (2.2.17), for $E^{\rm in} \equiv 0$ and the uniqueness to the exterior boundary problem that

$$E^{\rm sc}(x) = \mathrm{curl} \int_{\partial D} \Phi_k(x,y)A(y)\ ds(y) \equiv 0, \ \ x \in \mathbb{R}^3 \setminus \overline{D}, \tag{2.2.22}$$

taking the rotational, we obtain

$$H^{\rm sc}(x) = -ik\, \mathrm{curl}\, \mathrm{curl} \int_{\partial D} \Phi_k(x,y)A(y)\ ds(y) \equiv 0, \ \ x \in \mathbb{R}^3 \setminus \overline{D}, \tag{2.2.23}$$

then going to the boundary using (2.2.19) we get

$$\nu \times H^{\rm sc-} = 0, \ \ x \in \partial D = \cup_{i=1}^m \partial D_i.$$

As the homogeneous interior boundary problem admits the unique identically null solution, we get $H^{\mathrm{sc}}(x) = 0,\ x \in D$ and hence taking the rotational gives us $E^{\mathrm{sc}}(x) = 0,\ x \in D$ and ultimately $(\nu \times E^{\mathrm{sc}})^- = 0$ on ∂D. Finally we have

$$[\pm I/2 + M^k_{ii,D} + \sum_{(j\neq i)\geq 1}^{m} M^k_{ij,D}](A) = 0 \tag{2.2.24}$$

and taking the difference of the two identities we get $a_i = 0$ for every $i \in \{1, ..., m\}$, and yield that $a \equiv 0$ on $\partial D = \cup_{i=1}^m \partial D_i$. This shows that we have existence of the solution of our original scattering problem and it can be represented as (2.2.12) with a unique tangential density A. The uniqueness of the solution for the original scattering problem is deduced in the same way as it is done in Theorem 6.10 in [24] for instance.

2.2.3 A priori estimates of the densities

In order to derive suitable estimates of the densities $a_i, i = 1, ..., m$, we need to use the Helmholtz decomposition based on the following operators, which are isomorphism (see Theorem 5.1 and Theorem 5.3 in [57]),

$$\begin{aligned} \nu \cdot \operatorname{curl} S^0_{ii,D} &: L^{2,0}_{\mathrm{t}}(\partial D_i) \longrightarrow L^2_0(\partial D_i), \\ \nu \times \nabla S^0_{ii,D} &: L^2_0(\partial D_i) \longrightarrow L^{2,0}_t(\partial D_i), \\ \nu \times S^0_{ii,D} &: L^{2,0}_t(\partial D_i) \longrightarrow (L^{2,Div}_{\mathrm{t}} \setminus L^{2,0}_{\mathrm{t}})(\partial D_i) := L^{2,Div}_t(\partial D_i) \setminus L^{2,0}_t(\partial D_i). \end{aligned} \tag{2.2.25}$$

The following decomposition holds,

Proposition 6. *Each element V of $L^{p,Div}_t(\partial D_i)$ can be decomposed as*

$$V = \mathfrak{V} + \nu \times \nabla \mathfrak{v} \tag{2.2.26}$$

where

$$\begin{aligned} \mathfrak{V} &\in L^{p,Div}_t(\partial D_i) \setminus L^{p,0}_t(\partial D_i), \\ \nu \times \nabla \mathfrak{v} &\in L^{p,0}_t(\partial D_i),\ \mathfrak{v} \in H^1(\partial D_i) \setminus C; \end{aligned} \tag{2.2.27}$$

which satisfies

$$\|\mathfrak{V}\|_{L^2(\partial D_i)} \leq C_1 \boldsymbol{a} \|V\|_{L^{2,\mathrm{Div}}_t(\partial D_i)}, \tag{2.2.28}$$

$$\|\mathfrak{V}\|_{L^{2,\mathrm{Div}}_t(\partial D_i)} \leq C_2 \|V\|_{L^{2,\mathrm{Div}}_t(\partial D_i)}, \tag{2.2.29}$$

$$\|\nu \times \nabla \mathfrak{v}\|_{L^{2,0}_t(\partial D_i)} \leq C_3 \|V\|_{L^{2,\mathrm{Div}}_t(\partial D_i)}, \tag{2.2.30}$$

and

$$\|\mathfrak{v}\|_{L^2(\partial D_i)} \leq C_4 \boldsymbol{a} \|A\|_{L^{2,\mathrm{Div}}_t(\partial D_i)} \tag{2.2.31}$$

where $(C_i)_{i=1,2,3,4.}$ are constants which depend only on $B_i's$,

To prove Proposition 6, we need the following identities.

Lemma 7. *For $x_i \in \partial D_i$, $s_i \in \partial B_i$, with $x_i = \boldsymbol{a} s_i + z$, and $\hat{A}(s_i) = A(\boldsymbol{a} s_i + z_i)$, we have the following scaling identities*

$$\boldsymbol{a}\|\hat{A}\|_{L^2_t(\partial B_i)} = \|A\|_{L^2_t(\partial D_i)}, \ \ \|\mathrm{Div}\, \hat{A}\|_{L^2_0(\partial B_i)} = \|\mathrm{Div}\, A\|_{L^2_0(\partial D_i)}, \tag{2.2.32}$$

and

$$\|[\nu \times \nabla S^0_{ii,D}]^{-1}\|_{\mathcal{L}(L^{2,0}_t(\partial D_i), L^2_0(\partial D_i))} = \|[\nu \times \nabla S^0_{ii,B}]^{-1}\|_{\mathcal{L}(L^{2,0}_t(\partial B_i), L^2_0(\partial B_i))}, \tag{2.2.33}$$

$$\|[\nu \cdot \mathrm{curl}\, S^0_{ii,D}]^{-1}\|_{\mathcal{L}(L^2_0(\partial D_i), L^{2,0}_t(\partial D_i))} = \|[\nu \cdot \mathrm{curl}\, S^0_{ii,B}{}^0]^{-1}\|_{\mathcal{L}(L^2_0(\partial B_i), L^{2,0}_t(\partial B_i))}. \tag{2.2.34}$$

Proof. We derive the identities in (2.2.32) as follows

$$\|A\|_{L^2_t(\partial D_i)} = \left(\int_{\partial D} |A(x)|^2 ds(x)\right)^{\frac{1}{2}} = \left(\int_{\partial B} \left|\hat{A}(s)\right|^2 \boldsymbol{a}^2 ds(s)\right)^{\frac{1}{2}} = \left(\boldsymbol{a}^2 \int_{\partial B} \left|\hat{A}(s)\right|^2 \, ds(s)\right)^{\frac{1}{2}}$$
$$= \boldsymbol{a}\|\hat{A}\|_{L^2_t(\partial B_i)}.$$

For the second identity, one has for $\phi \in H^1(\partial D_i)$, [3]

$$\int_{\partial D_i} \mathrm{Div}_x\, A(x)\phi(x) \ ds(x) = \int_{\partial D_i} A(x) \cdot \nabla_x \phi(x) \ ds(x) = \int_{\partial D_i} \widehat{A}(s) \cdot \frac{1}{\boldsymbol{a}} \nabla_s \widehat{\phi}(s) \boldsymbol{a}^2 ds(x),$$
$$= \int_{\partial D_i} \mathrm{Div}_s\, \widehat{A}(s) \frac{1}{\boldsymbol{a}} \widehat{\phi}(s) \ ds(x) = \int_{\partial D_i} \frac{1}{\boldsymbol{a}} \mathrm{Div}_s\, \widehat{A}(s)\phi(x) \ ds(x),$$

then $\mathrm{Div}_x\, A(x) = \frac{1}{\boldsymbol{a}} \mathrm{Div}_s\, \widehat{A}(s)$. It remains to take the norm to conclude.

Concerning (2.2.33) we have,

$$[\nu \times \nabla S^0_{ii,D}](u)(x) = \nu \times \nabla \frac{1}{4\pi} \int_{\partial D_i} \frac{1}{|x-y|} u(y) \ ds(y) = \nu \times \frac{1}{4\pi} \int_{\partial D_i} \frac{1}{|x-y|^3} (x-y) u(y) \ ds(y),$$

hence, for $x_i = \boldsymbol{a} s_i + z_i$ and $y_i = \boldsymbol{a} t_i + z_i$, and $\hat{u}(s_i) = u(\boldsymbol{a} t_i + z_i)$

$$\nu_{x_i} \times \nabla S^0_{ii,D}](u)(x) = \nu_{s_i} \times \frac{1}{4\pi} \int_{\partial B_i} \frac{1}{|\boldsymbol{a} s_i + z_i - \boldsymbol{a} t_i - z_i|^3} (\boldsymbol{a} s_i + z_i - \boldsymbol{a} t_i - z_i) u(\boldsymbol{a} t_i + z_i) \ \boldsymbol{a}^2 \ ds(t),$$
$$= \nu_{s_i} \times \frac{1}{4\pi} \int_{\partial B_i} \frac{1}{\boldsymbol{a}^3 |s_i - t_i|^3} \ \boldsymbol{a} \ (s_i - t_i) u(\boldsymbol{a} t_i + z_i) \ \boldsymbol{a}^2 \ ds(t),$$
$$= \nu_{s_i} \times \frac{1}{4\pi} \int_{\partial B_i} \frac{1}{|s_i - t_i|^3} \ (s_i - t_i) \hat{u}(t_i) \ ds(t) = [\nu_{x_i} \times \nabla S^0_{ii,B}](\hat{u})(s_i).$$

From the equality $[\nu_{x_i} \times \nabla S^0_{ii,D}]([\nu_{x_i} \times \nabla S^0_{ii,D}]^{-1}(u)) = u$, we get replacing in the previous identity u by $[\nu_{x_i} \times \nabla S^0_{ii,D}]^{-1}(u)$, that $[\nu_{x_i} \times \nabla S^0_{ii,B}]([\nu_{x_i} \times \widehat{\nabla S^0_{ii,D}}]^{-1}(u)) = \widehat{u}$. Finally inverting the left-hand side operator, we have the scales

$$[\nu_{x_i} \times \widehat{\nabla S^0_{ii,D}}]^{-1}(u) = \nu_{x_i} \times \nabla S^0_{ii,B}]^{-1} \widehat{u}. \tag{2.2.35}$$

[3]The gradient stands for the surface gradient.

As

$$\|[\nu \times \nabla S^0_{ii,D}]^{-1}\|_{\mathcal{L}(L^{2,0}_t(\partial D_i),L^2_0(\partial D_i))} = \sup_{\substack{u\in L^2_0(\partial D_i)\\ u\neq 0}} \frac{\|[\nu \times \nabla S^0_{ii,D}]^{-1}(u)\|_{L^2(\partial D_i)}}{\|u\|_{L^2(\partial D_i)}} \tag{2.2.36}$$

(2.2.32) implies

$$\|[\nu \times \nabla S^0_{ii,D}]^{-1}\|_{\mathcal{L}(L^{2,0}_t(\partial D_i),L^2_0(\partial D_i))} = \sup_{\substack{\widehat{u}\in L^2_0(\partial B_i)\\ \widehat{u}\neq 0}} \frac{\boldsymbol{a}\|[\nu \times \widehat{\nabla S^0_{ii,D}}]^{-1}(u)\|_{L^2(\partial B_i)}}{\boldsymbol{a}\|\widehat{u}\|_{L^2(\partial B_i)}}, \tag{2.2.37}$$

and using (2.2.35)

$$\|[\nu \times \nabla S^0_{ii,D}]^{-1}\|_{\mathcal{L}(L^{2,0}_t(\partial D_i),L^2_0(\partial D_i))} = \sup_{\substack{\widehat{u}\in L^2_0(\partial B_i)\\ \widehat{u}\neq 0}} \frac{\|[\nu \times \nabla S^0_{ii,B}]^{-1}(\widehat{u})\|_{L^2(\partial B_i)}}{\|\widehat{u}\|_{L^2(\partial B_i)}}, \tag{2.2.38}$$

the conclusion follows immediately. Similarly, we can derive (2.2.34). □

Proof. (Of Proposition 6) It suffices to seek for the solution of the following equation

$$\nu \times \nabla S^0_{ii,D}(v) + \nu \times S^0_{ii,D}(w) = V. \tag{2.2.39}$$

Taking the surface divergence we have, $\nu \cdot \operatorname{curl} S^0_{ii,D}(w) = \operatorname{Div} V$ and then using (2.2.25), $w = [\nu \cdot \operatorname{curl} S^0_{ii,D}]^{-1} \operatorname{Div} V$. Using (2.2.33) we get the estimate

$$\|w\|_{L^{2,0}_t(\partial D_i)} \leq \|[\nu \cdot \operatorname{curl} S^0_{\partial B_i}]^{-1}\|_{\mathcal{L}(L^2_0(\partial B_i),L^{2,0}_t(\partial B_i))}\|(\operatorname{Div} V)\|_{L^2_0(\partial D_i)}. \tag{2.2.40}$$

Put $\mathfrak{V} = \nu \times S^0_{ii,D}(w)$,

$$\begin{aligned}\|\mathfrak{V}\|_{L^2(\partial D_i)} &= \left(\int_{\partial D_i} [\nu \times S^0_{ii,D}(w)]^2 \, ds\right)^{\frac{1}{2}} = \left(\int_{\partial B_i}\left(\nu \times \int_{\partial B_i} \frac{1}{\boldsymbol{a}|s-t|}\widehat{w}(t)\boldsymbol{a}^2 \, ds(t)\right)^2 \boldsymbol{a}^2 \, ds(s)\right)^{\frac{1}{2}}, \\ &= \boldsymbol{a}^2\left(\int_{\partial B_i}\left(\nu \times \int_{\partial B_i} \frac{1}{|s-t|}\widehat{w}(t) \;\; ds(t)\right)^2 ds(s)\right)^{\frac{1}{2}} = \boldsymbol{a}^2\|(\nu \times S^0_{ii,B}(\widehat{w}))\|_{L^2(\partial B_i)}.\end{aligned}$$

Using (2.2.40) for the last coming inequality, we have

$$\begin{aligned}\|\mathfrak{V}\|_{L^2(\partial D_i)} &= \boldsymbol{a}^2\|(\nu \times S^0_{ii,B}(\widehat{w}))\|_{L^2_t(\partial B_i)}, \\ &\leq \boldsymbol{a}^2\left(\|(\nu \times S^0_{ii,B}(\widehat{w}))\|_{L^2_t(\partial B_i)} + \|\operatorname{Div}(\nu \times S^0_{ii,B}(\widehat{w}))\|_{L^2_0(\partial B_i)}\right), \\ &\leq \|(\nu \times S^0_{ii,B}(\widehat{w}))\|_{L^{2,\mathrm{Div}}_t(\partial B_i)}, \\ &\leq \boldsymbol{a}^2\|\nu \times S^0_{\partial B_i}\|_{\mathcal{L}(L^{2,0}_t(\partial B_i),L^{2,\mathrm{Div}}_t(\partial B_i))}\|\widehat{w}\|_{L^{2,0}_t(\partial B_i)}, \\ &\leq C_1^{B_i}\boldsymbol{a}^2\|\widehat{V}\|_{L^{2,\mathrm{Div}}_t\backslash(\partial B_i)} = C_1^{B_i}\boldsymbol{a}\|V\|_{L^{2,\mathrm{Div}}_t(\partial D_i)},\end{aligned} \tag{2.2.41}$$

here $C_1^{B_i} := \|\nu \times S^0_{ii,B}\|_{\mathcal{L}(L^2_0(\partial B_i),L_t^{2,\mathrm{Div}}(\partial B_i))} \|[\nu \cdot \operatorname{curl} S^0_{ii,B}]^{-1}\|_{\mathcal{L}(L_t^{2,0}(\partial B_i),L_t^{2,0}(\partial B_i))}$, which gives (2.2.28). The inequality (2.2.29) is an immediate consequence. Now, (2.2.39) becomes $\nu \times \nabla S^0_{ii,D}(v) = V - \mathfrak{V}$, with $\mathrm{Div}(V - \mathfrak{V}) = 0$, then we get successively

$$\begin{aligned}\|v\|_{L^2_0(\partial D_i)} &\leq \|[\nu \times \nabla S^0_{ii,D}]^{-1}\|_{\mathcal{L}(L_t^{2,0}(\partial D_i),L^2_0(\partial D_i))} \|V - \mathfrak{V}\|_{L_t^{2,0}(\partial D_i)},\\ &\leq \|[\nu \times \nabla S^0_{ii,B}]^{-1}\|_{\mathcal{L}(L_t^{2,0}(\partial B_i),L^2_0(\partial B_i))} \|V - \mathfrak{V}\|_{L_t^{2,0}(\partial D_i)},\\ &\leq C_3^{B_i} \|V\|_{L_t^{2,Div}(\partial D_i)},\end{aligned} \tag{2.2.42}$$

where $C_3^{B_i} := 2\|[\nu \times \nabla S^0_{ii,B}]^{-1}\|_{\mathcal{L}(L_t^{2,0}(\partial B_i),L^2_0(\partial B_i))}$. We put $\mathfrak{v} = S^0_{ii,D}(v)$ hence $\nu \times \nabla \mathfrak{v} = \nu \times \nabla S^0_{ii,D}(v)$ with $\|\nu \times \nabla \mathfrak{v}\|_{L_t^{2,0}(\partial D_i)} = \|\nu \times \nabla S^0_{ii,D}(v)\|_{L_t^{2,0}(\partial D_i)} = \|V - \mathfrak{V}\|_{L_t^{2,0}(\partial D_i)}$. Then with (2.2.41) in mind, we derive the estimate (2.2.30)

$$\begin{aligned}\|\nu \times \nabla \mathfrak{v}\|_{L_t^{2,0}(\partial D_i)} = \|V - \mathfrak{V}\|_{L_t^{2,0}(\partial D_i)} &\leq \|V\|_{L^2_t(\partial D_i)} + \|\mathfrak{V}\|_{L^2_t(\partial D_i)},\\ &\leq \|V\|_{L_t^{2,\mathrm{Div}}(\partial D_i)} + C_1^{B_i}\boldsymbol{a}\|V\|_{L_t^{2,\mathrm{Div}}(\partial D_i)},\\ &\leq (C_3^{B_i} := (1 + C_1^{B_i}))\|V\|_{L_t^{2,\mathrm{Div}}(\partial D_i)}.\end{aligned}$$

Concerning (2.2.31) we have $\|\mathfrak{v}\|_{L^2(\partial D_i)} = \|S^0_{ii,D}(v)\|_{L^2(\partial D_i)}$, hence

$$\begin{aligned}\|\mathfrak{v}\|_{L^2(\partial D_i)} &= \left(\int_{\partial B_i}\left(\int_{\partial B_i} \frac{1}{\boldsymbol{a}|s-t|}\widehat{v}(t)\boldsymbol{a}^2\, ds(t)\right)^2 \boldsymbol{a}^2\, ds(s)\right)^{\frac{1}{2}},\\ &= \boldsymbol{a}^2\|S^0_{ii,B}(v)\|_{L^2(\partial B_i)} \leq \boldsymbol{a}^2\left(\|S^0_{ii,B}(\widehat{v})\|_{L^2(\partial B_i)} + \|\nabla_t S^0_{ii,B}(\widehat{v})\|_{L^2(\partial B_i)}\right),\\ &\leq \boldsymbol{a}^2\|S^0_{ii,B}\|_{\mathcal{L}(L^2(\partial B_i),H^1(\partial B_i))}\|\widehat{v}\|_{L^2(\partial B_i)} = \boldsymbol{a}\|S^0_{ii,B}\|_{\mathcal{L}(L^2(\partial B_i),H^1(\partial B_i))}\|v\|_{L^2(\partial D_i)},\end{aligned}$$

and with (2.2.42) we get,

$$\|\mathfrak{v}\|_{L^2(\partial D_i)} \leq \boldsymbol{a}\big(C_4^{B_i} := \|S^0_{ii,B}\|_{\mathcal{L}(L^2(\partial B_i),H^1(\partial B_i))}C_3^{B_i}\big)\|V\|_{L_t^{2,Div}(\partial D_i)},$$

to conclude we put for $l = 1, 2, 3, 4$, $C_l = \max i \in \{1, ..., m\}C_l^{B_i}$. □

We have the following theorem

Theorem 8. *There exist constants $C_{B,2}$, $C_{B,1}$ and C_e which depend only on B_i's and independent of their number such that if*

$$C_{B,1}|k|^2\boldsymbol{a} < 1, \quad C_{B,2}\left(\frac{\ln m^{\frac{1}{3}}}{\boldsymbol{\delta}^3} + \frac{2km^{\frac{1}{3}}}{\boldsymbol{\delta}^2} + \frac{m^{\frac{2}{3}}}{2\boldsymbol{\delta}}k^2\right)\boldsymbol{a}^3 < 1 \tag{2.2.43}$$

then

$$\|A\|_{L_t^{2,\mathrm{Div}}} \leq C_e\boldsymbol{a} \tag{2.2.44}$$

Further, in view of Proposition 6 each $a_i \in L_t^{2,\mathrm{Div}}(\partial D_i)$ can be decomposed as the sum of

$$A_i^{[1]} \in L_t^{p,Div}(\partial D_i) \setminus L_t^{p,0}(\partial D_i) \quad \textit{and} \quad A_i^{[2]} = \nu \times \nabla u_i \in \ \in L_t^{p,0}(\partial D_i),\ u \in H^1(\partial D_i) \setminus C; \tag{2.2.45}$$

which satisfies

$$\|A^{[1]}\|_{L^2(\partial D_i)} \leq C_1 \boldsymbol{a} \|A\|_{L_t^{2,\mathrm{Div}}(\partial D_i)}, \tag{2.2.46}$$

$$\|A^{[1]}\|_{L_t^{2,\mathrm{Div}}(\partial D_i)} \leq C_2 \|A\|_{L_t^{2,\mathrm{Div}}(\partial D_i)}, \tag{2.2.47}$$

$$\|A^{[2]}\|_{L_t^{2,0}(\partial D_i)} \leq C_3 \|A\|_{L_t^{2,\mathrm{Div}}(\partial D_i)}, \tag{2.2.48}$$

$$\|u_i\|_{L^2(\partial D_i)} \leq C_4 \boldsymbol{a} \|A\|_{L_t^{2,\mathrm{Div}}(\partial D_i)}, \tag{2.2.49}$$

where $(C_i)_{i=1,2,3,4.}$ *are constants which depends only on the shape of the* B_i*'s (i.e their Lipschitz character) and not on their number* m*.*

From (2.2.20), we have successively

$$(I/2 + \mathcal{M}_D)(I + (I/2 + \mathcal{M}_D)^{-1}\mathcal{M}_N)A = -\nu \times E^I,$$
$$(I + (I/2 + \mathcal{M}_D)^{-1}\mathcal{M}_N)A = -(I/2 + \mathcal{M}_D)^{-1}\nu \times E^I,$$

and if

$$\|(I/2 + \mathcal{M}_D)^{-1}\| \|\mathcal{M}_N\| < 1, \tag{2.2.50}$$

we get

$$\|A\| \leq \frac{\|(I/2 + \mathcal{M}_D)^{-1}\|}{1 - \|(I/2 + \mathcal{M}_D)^{-1}\| \|\mathcal{M}_N\|} \|\nu \times E^I\|,$$

where

$$\|A\| := \max_{m \in \{1,\ldots,M\}} \|A_m\|_{L_t^{2,\mathrm{Div}}(\partial D_m)},$$
$$\|(I/2 + \mathcal{M}_D)^{-1}\| := \max_{i \in \{1,\ldots,m\}} \|(I/2 + M^k_{ii,D})^{-1}\|_{\mathcal{L}(L_t^{2,\mathrm{Div}}(\partial D_m), L_t^{2,\mathrm{Div}}(\partial D_m))},$$
$$\|\mathcal{M}_N\| := \max_{i \in \{1,\ldots,m\}} \sum_{\substack{j=1 \\ j \neq m}}^{M} \|M^k_{ij,D}\|_{\mathcal{L}(L_t^{2,\mathrm{Div}}(\partial D_j), L_t^{2,\mathrm{Div}}(\partial D_i))}.$$

Proof. We prove that under the condition (2.2.43), we have (2.2.44). The properties (2.2.45)-(2.2.49) are immediate conclusions of Proposition 6 and do not rely on (2.2.43).

We set $diam(B) := \max_{i \in \{1,..,m\}} diam(B_i)$ and we suppose that $diam(B) \leq 1$.
For every $i \in \{1, ..., m\}$ and $x_i = \boldsymbol{a} s_i + z_i \in \partial D_i$, $s_i \in \partial B_i$ we write $\hat{A}(s_i) := A(\boldsymbol{a} s_i + z_i)$, and $\breve{A}(x_i) = A(\frac{x_i - z_i}{\boldsymbol{a}})$, and set

$$M^0_{ii,D} := \nu \times \int_{\partial D_i} \nabla \Phi_0(x_i, y) \times A(y)\, ds(y).$$

We have, knowing that $[I/2 + M^0_{ii,D}]$ is invertible (see Theorem 5.1 property (xix) in [57]), the identity

$$[I/2 + M^k_{ii,D}] = [I/2 + M^0_{ii,D}]\big(I + [I/2 + M^0_{ii,D}]^{-1}[M^k_{ii,D} - M^0_{ii,D}]\big), \tag{2.2.51}$$

then, by inverting the operators in each side, under the condition that

$$\|[M^k_{ii,_D} - M^0_{ii,_D}][I/2 + M^0_{ii,_D}]^{-1}\| \leq \|[I/2 + M^0_{ii,_D}]^{-1}\|\|[M^k_{ii,_D} - M^0_{ii,_D}]\| < 1, \tag{2.2.52}$$

we get $[I/2 + M^k_{ii,_D}]^{-1} = \Big(I + [I/2 + M^0_{ii,_D}]^{-1}[M^k_{ii,_D} - M^0_{ii,_D}]\Big)^{-1} [I/2 + M^0_{ii,_D}]^{-1}$. As

$$\Big(I + [I/2 + M^0_{ii,_D}]^{-1}[M^k_{ii,_D} - M^0_{ii,_D}]\Big)^{-1} = \sum_{n\geq 0} \Big(-[I/2 + M^0_{ii,_D}]^{-1}[M^k_{ii,_D} - M^0_{ii,_D}]\Big)^n$$

we have finally

$$\|[I/2 + M^k_{ii,_D}]^{-1}\| \leq \frac{\|[I/2 + M^0_{ii,_D}]^{-1}\|}{1 - \|[I/2 + M^0_{ii,_D}]^{-1}\|\|[M^k_{ii,_D} - M^0_{ii,_D}]\|}. \tag{2.2.53}$$

From here $\mathcal{L}(E) := \mathcal{L}(E, E)$ denotes the space of continuous linear operators which are defined from E to E. Hence the proof of (2.2.44), based on the condition (2.2.43), is reduced to the following two estimates:

$$\begin{aligned} \|[I/2 + M^0_{ii,_D}]^{-1}\|_{\mathcal{L}\left(L^{2,\mathrm{Div}}_t(\partial D_i)\right)} \leq & \|[I/2 + M^0_{ii,_B}]^{-1}\|_{\mathcal{L}\left(L^2_t(\partial B_i)\right)} \\ & + \|[I/2 - (K^0_{ii,_B})^*]^{-1}\|_{\mathcal{L}\left(L^2_0(\partial B_i)\right)}, \end{aligned} \tag{2.2.54}$$

and

$$\|\mathcal{M}_N\|_{\mathcal{L}\left(L^{2,\mathrm{Div}}_t(\partial B_i)\right)} \leq \frac{2^6 C_{_B}}{\boldsymbol{\delta}} \left(\frac{\ln{(n+1)}}{\boldsymbol{\delta}^2} + \frac{2kn}{\boldsymbol{\delta}} + \frac{n}{2}(n+1)k^2\right) \boldsymbol{a}^3. \tag{2.2.55}$$

where $n = O(m^{\frac{1}{3}})$, and $C_{_B}$ is a constant which depends exclusively on B_i's. In some places of the next computations, we use the notation

$$C_0 := 2^6.$$

To justify (2.2.54) and (2.2.55), we need the following lemma.

Lemma 9. *We have the following scaling estimations*

$$\|[I/2 + M^0_{ii,D}]^{-1}\|_{\mathcal{L}(L^2_t(\partial D_i))} = \|[I/2 + M^0_{ii,B}]^{-1}\|_{\mathcal{L}(L^2_t(\partial B_i))}, \tag{2.2.56}$$

$$\|[I/2 - K^0_{ii,_D}]^{-1}\|_{\mathcal{L}(L^2_0(\partial D_i))} = \|[I/2 - K^0_{ii,_B}]^{-1}\|_{\mathcal{L}(L^2_0(\partial B_i))}, \tag{2.2.57}$$

and

$$\|[M^k_{ii,_D} - M^0_{ii,_D}]\|_{\mathcal{L}(L^{2,\mathrm{Div}}_t(\partial D_i), L^{2,\mathrm{Div}}_t(\partial D_i))} \leq 2C_{_B}|\partial B||k|^2\boldsymbol{a}\|A\|_{L^2(\partial D_i)}. \tag{2.2.58}$$

Proof. For (2.2.56) and (2.2.57), we, first have

$$\begin{aligned} M^0_{ii,_D}(A)(x_i) &= \nu \times \int_{\partial D_i} (4\pi)^2 \Phi^3_0(x_i, y)(x_i - y) \times A(y)\ ds(y), \\ &= \nu \times \int_{\partial \mathcal{Q}_i} \frac{(4\pi)^2}{\boldsymbol{a}^3} \Phi^3_0(s_i, t)\boldsymbol{a}(s_i - t) \times A(\boldsymbol{a}t + z_i)\boldsymbol{a}^2 ds(t), \\ &= \nu \times \int_{\partial \mathcal{Q}_i} (4\pi)^2 \Phi^3_0(s_i, t)(s_i - t) \times A(\boldsymbol{a}t + z_i)\ ds(t) = \{M^0_{ii,_B}\hat{A}\}\breve{}, \end{aligned}$$

which leads to $[I/2+\widehat{M^0_{ii,D}}]^{-1}(A) = [I/2+M^0_{ii,B}]^{-1}\widehat{A}$. With this in mind, considering (2.2.32) we get

$$\begin{aligned}
\|[I/2+M^0_{ii,D}]^{-1}\|_{\mathcal{L}(L^2_t(\partial D_i))} &= \sup_{(b\neq 0)\in L^2_t(\partial D_i)} \frac{\|[I/2+M^0_{ii,D}]^{-1}(A)\|_{L^2_t(\partial D_i)}}{\|A\|_{L^2_t(\partial D_i)}},\\
&= \sup_{(b\neq 0)\in L^2_t(\partial D_i)} \frac{\boldsymbol{a}\|[I/2+\widehat{M^0_{ii,D}}]^{-1}(A)\|_{L^2_t(\partial B_i)}}{\boldsymbol{a}\|\widehat{A}\|_{L^2_t(\partial B_i)}},\\
&= \sup_{(\widehat{A}\neq 0)\in L^2_t(\partial B_i)} \frac{\|[I/2+M^0_{ii,D}]^{-1}(\widehat{A})\,\|_{L^2_t(\partial B_i)}}{\|\widehat{A}\|_{L^2_t(\partial B_i)}}.
\end{aligned}$$

We obtain (2.2.57) in the same way. For (2.2.58), by Mean-value-theorem, we have

$$(\Phi_k(x,y)-\Phi_0(x,y)) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} - \frac{1}{4\pi}\frac{e^{0|x-y|}}{|x-y|} = \int_0^1 ik\frac{1}{4\pi}e^{ikl|x-y|}dl. \tag{2.2.59}$$

Taking the gradient gives

$$\nabla_x(\Phi_k(x,y)-\Phi_0(x,y)) = \frac{(ik)^2}{4\pi}\int_0^1 \frac{le^{ikl|x-y|}}{|x-y|}(x-y)dl, \tag{2.2.60}$$

thus, being $\Im k = 0$, $|\nabla_x(\Phi_k(x,y)-\Phi_0(x,y))\times A(y)| \leq \frac{|k|^2}{4\pi}|A(y)| \leq \frac{|k|^2}{4\pi}|A(y)|$, and

$$|(\Phi_k(x,y)-\Phi_0(x,y))b| \leq \frac{|k|}{4\pi}|A(y)|, \tag{2.2.61}$$

then it comes

$$\begin{aligned}
\left|[M^k_{ii,D}-M^0_{ii,D}](A)\right| = \left|\int_{\partial D_i}\nabla(\Phi_k-\Phi_0)(x_i,y)\times A(y)\,ds(y)\right| &\leq \int_{\partial D_i}\frac{|k|^2}{4\pi}|A(y)|\,ds(y),\\
&\leq \frac{|k|^2|\partial B_i|^{\frac{1}{2}}\boldsymbol{a}}{4\pi}\left(\int_{\partial D_i}|A|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Taking the norm in both sides,

$$\|[M^k_{ii,D}-M^0_{ii,D}](A)\|^2_{L^2(\partial D_i)} \leq \left(\frac{|k|^2(|\partial B_i||\partial B_i|)^{\frac{1}{2}}\boldsymbol{a}^2}{4\pi}\|A\|_{L^2(\partial D_i)}\right)^2. \tag{2.2.62}$$

We have the identities, $\mathrm{Div}[M^k_{ii,D}-M^0_{ii,D}](A)(x) = -[k^2\nu\cdot S^k_{ii,D}](A) - [(K^k_{ii,D}-K^0_{ii,D})^*](\mathrm{Div}\,A)$. To estimate $[(K^k_{ii,D}-K^0_{ii,D})^*](\mathrm{Div}\,A)$, we can reproduce the same steps as we did to obtain (2.2.62). We obtain

$$\|[(K^k_{ii,D}-K^0_{ii,D})^*]\|^2_{L^2_0(\partial D_i)} \leq \left(\frac{|k|^2(|\partial B_j||\partial B_i|)^{\frac{1}{2}}\boldsymbol{a}^2}{4\pi}\|\mathrm{Div}\,A\|_{L^2_0(\partial D_i)}\right). \tag{2.2.63}$$

Using (2.2.61), we deduce that

$$\begin{aligned}\left|[k^2\nu\cdot S^k_{ii,D}](A)\right| &= \left|[k^2\nu\cdot(S^k_{ii,D}-S^0_{ii,D})](A)+k^2\nu\cdot S^0_{ii,D}](A)\right|,\\ &\leq \frac{|k|^3|\partial B_i|\boldsymbol{a}}{4\pi}\|A\|_{L^2(\partial D_i)}+\left|[k^2\boldsymbol{a}\nu\cdot S^0_{ii,B}](\widehat{A})\right|,\end{aligned}$$

and taking the norm, we get [4]

$$\begin{aligned}\|[k^2\nu\cdot S^k_{ii,D}]b\|_{L^2(\partial D_i)} &\leq \frac{|k|^3(|\partial B_j||\partial B_i|)^{\frac{1}{2}}\boldsymbol{a}^2}{4\pi}\|A\|_{L^2(\partial D_i)}+|k|^2\boldsymbol{a}^2(|\partial B_j|)^{\frac{1}{2}}\|S^0_{ii,B}\|\|\widehat{A}\|_{L^2(\partial B_i)},\\ &\leq \frac{|k|^3(|\partial B_j||\partial B_i|)^{\frac{1}{2}}\boldsymbol{a}^2}{4\pi}\|A\|_{L^2(\partial D_i)}+|k|^2\boldsymbol{a}(|\partial B_j|)^{\frac{1}{2}}\|S^0_{ii,B}\|\|A\|_{L^2(\partial D_i)},\\ &\leq C_B|\partial B||k|^2\boldsymbol{a}\|A\|_{L^2(\partial D_i)},\end{aligned} \tag{2.2.64}$$

where $|\partial B| = \max_i|\partial B|$ and C_B is the maximum of the constants that appear in the inequalities (2.2.62) and (2.2.63). Hence

$$\|[M^k_{ii,D}-M^0_{ii,D}](A)\|_{L^{2,\mathrm{Div}}_t(\partial D_i)} \leq 2C_B|\partial B||k|^2\boldsymbol{a}\|A\|_{L^2(\partial D_i)}. \tag{2.2.65}$$

□

To prove (2.2.54), let us recall that we have

$$\|[I/2+M^0_{ii,D}]^{-1}\| = \sup_{b\neq 0}\left(\frac{\|[I/2+M^0_{ii,D}]^{-1}A\|^2_{L^2(\partial D_i)}+\|\mathrm{Div}[I/2+M^0_{ii,D}]^{-1}A\|^2_{L^2(\partial D_i)}}{\|A\|^2_{L^2(\partial D_i)}+\|\mathrm{Div}\,A\|^2_{L^2(\partial D_i)}}\right)^{\frac{1}{2}} \tag{2.2.66}$$

Considering the fact that[5] $\mathrm{Div}[I/2+M^0_{ii,D}]^{-1} = [I/2-(K^0_{ii,D})^*]^{-1}\,\mathrm{Div}$, then (2.2.66) gives

$$\begin{aligned}\|[I/2+M^0_{ii,D}]^{-1}\| &= \sup_{b\neq 0}\left(\frac{\|[I/2+M^0_{ii,D}]^{-1}A\|^2_{L^2(\partial D_i)}+\|[I/2-(K^0_{ii,D})^*]^{-1}\,\mathrm{Div}\,A\|^2_{L^2(\partial D_i)}}{\|A\|^2_{L^2(\partial D_i)}+\|\mathrm{Div}\,A\|^2_{L^2(\partial D_i)}}\right)^{\frac{1}{2}},\\ &\leq \sup_{b\neq 0}\left(\frac{\|[I/2+M^0_{ii,D}]^{-1}A\|^2_{L^2(\partial D_i)}}{\|A\|^2_{L^2(\partial D_i)}+\|\mathrm{Div}\,A\|^2_{L^2(\partial D_i)}}\right)^{\frac{1}{2}}\\ &\quad+\sup_{b\neq 0}\left(\frac{\|[I/2-(K^0_{ii,D})^*]^{-1}\,\mathrm{Div}\,A\|^2_{L^2(\partial D_i)}}{\|A\|^2_{L^2(\partial D_i)}+\|\mathrm{Div}\,A\|^2_{L^2(\partial D_i)}}\right)^{\frac{1}{2}},\end{aligned}$$

where the last inequality is due to the fact that $\alpha^2+\beta^2 \leq \alpha^2+\beta^2+2\alpha\beta$ for $\alpha,\beta \geq 0$. Finally, as $\alpha^2 \leq \beta^2+\alpha^2$, we get [6]

$$\|[I/2+M^0_{ii,D}]^{-1}\| \leq \sup_{b\neq 0}\left(\frac{\|[I/2+M^0_{ii,D}]^{-1}A\|^2}{\|A\|^2}\right)^{\frac{1}{2}}+\sup_{b\neq 0}\left(\frac{\|[I/2-(K^0_{ii,D})^*]^{-1}\,\mathrm{Div}\,A\|^2}{\|\mathrm{Div}\,A\|^2}\right)^{\frac{1}{2}}.$$

To prove (2.2.55), we will need the following lemma

[4] Notice that $\boldsymbol{a}\|\widehat{A}\|_{L^2(\partial B_i)} = \|A\|_{L^2(\partial D_i)}$.

[5] It suffices to write, for $b,c \in L^{2,\mathrm{Div}}$ such that $b = [I/2+M^0_{ii,D}]^{-1}c$, $\mathrm{Div}\,c = \mathrm{Div}[I/2+M^0_{ii,D}]b = [I/2-(K^0_{ii,D})^*]\,\mathrm{Div}\,A$ and inverting $[I/2-(K^0_{ii,D})^*]$ to get $\mathrm{Div}\,A = [I/2-(K^0_{ii,D})^*]^{-1}\,\mathrm{Div}\,c$.

[6] With L^2-norm.

Lemma 10. *For every $i, j \in \{1, ..., m\}$, under the condition that $\boldsymbol{a} \leq \boldsymbol{\delta} < 1$ we have*

$$\|M^k_{ij,D}\|_{\mathcal{L}(L^{2,Div}_t(\partial B), L^{2,Div}_t(\partial B))} \leq \frac{4(|\partial B_i||\partial B_j|)^{\frac{1}{2}}}{\pi \boldsymbol{\delta}_{ij}} (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^2 \boldsymbol{a}^3. \tag{2.2.67}$$

Proof. For $i \neq j$, $x_i \in \partial D_i$ we have, recalling (2.2.5), that

$$\begin{aligned}
\int_{\partial D_j} \Phi_k(x_i, y) A(y)\ ds(y) &= \int_{\partial D_j} \big(\nabla_y (y - z_j)\big)\big(\Phi_k(x_i, y) A(y)\big)\ ds(y), \\
&= \int_{\partial D_j} (y - z_j) \operatorname{Div} \left(\Phi_k(x_i, y) A(y)\right)\ ds(y), \\
&= -\int_{\partial D_j} (y - z_j) \Phi_k(x_i, y) \operatorname{Div} A(y)\ ds(y) \\
&\quad - \int_{\partial D_j} (y - z_j) \nabla_y \Phi_k(x_i, y) \cdot A(y)\ ds(y).
\end{aligned} \tag{2.2.68}$$

Being $-M^k_{ij,D}(A) = -\nu \times \nabla_x \times \int_{\partial D_j} \Phi_k(x_i, y) \times A(y)\ ds(y)$, it comes from (2.2.68)

$$\begin{aligned}
-M^k_{ij,D}(A) =& \nu \times \nabla_x \times \int_{\partial D_j} (y - z_j) \Phi_k(x_i, y) \operatorname{Div} A(y)\ ds(y) \\
&+ \nu \times \nabla_x \times \int_{\partial D_j} (y - z_j) \nabla_y \Phi_k(x_i, y) \cdot A(y))\ ds(y), \\
=& \nu \times \int_{\partial D_j} (y - z_j) \times \nabla_x \Phi_k(x_i, y) \operatorname{Div} A(y)\ ds(y) \\
&+ \nu \times \int_{\partial D_j} (y - z_j) \times \nabla_x \nabla_y \Phi_k(x_i, y) \cdot A(y))\ ds(y).
\end{aligned} \tag{2.2.69}$$

As [7]

$$\begin{aligned}
-\nabla_x \nabla_y \Phi_k(x, y) = (4\pi)^4 \Phi_k(x, y) \Phi_0^2 \left\{(\Phi_0 - ik)^2 + (\Phi_0 - ik)\Phi_0 + \Phi_0^2\right\}(x, y) \left((x - y)(x - y)^T\right) \\
+ (4\pi)^2 \Phi_k \Phi_0 (\Phi_0 - ik)(x, y)\ I,
\end{aligned}$$

where $(x - y)^T$ stands for the transpose vector of $(x - y)$, we get[8]

$$|\nabla_x \nabla_y \Phi_k(x, y)| \leq \frac{3}{4\pi} \frac{1}{\boldsymbol{\delta}_{ij}} (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^2, \tag{2.2.70}$$

and

$$|\nabla_y \Phi_k(x, y)| \leq \frac{1}{4\pi} \frac{1}{\boldsymbol{\delta}_{ij}} \left(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\right). \tag{2.2.71}$$

[7] Notice that $\left((x - y)(x - y)^T\right) b = (b \cdot (x - y))\,(x - y)$ and $(x - y) \times (x - y) = 0$.

[8] As $e^{-\Im k\ \boldsymbol{\delta}_{ij}} \leq 1$.

Hence, in view of (2.2.69), using Holder's inequality,

$$\begin{aligned}
\left| M^k_{ij,D}(A)(x_i) \right| \leq & |\nabla_x \Phi_k(x_i,y)| \|(y-z_j)\|_{L^2(\partial D_i)} \|\text{Div}\, A\|_{L^2(\partial D_i)} \\
& + |\nabla_y \nabla_x \Phi_k(x_i,y)| \|(y-z_j)\|_{L^2(\partial D_i)} \|A(y)\|_{L^2(\partial D_i)}, \\
\leq & \frac{2\ |\partial B_j|^{\frac{1}{2}}}{\pi \boldsymbol{\delta}_{ij}} \big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^2 \boldsymbol{a}^2 \|A(y)\|_{L^{2,\text{Div}}_{\text{t}}(\partial D_i)},
\end{aligned}$$

from which follows that

$$\|M^k_{ij,D}(A)\|_{L^2(\partial D_i)} \leq \frac{2(|\partial B_i||\partial B_j|)^{\frac{1}{2}}}{\pi \boldsymbol{\delta}_{ij}} \big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^2 \boldsymbol{a}^3 \|A\|_{L^{2,\text{Div}}_{\text{t}}(\partial D_i)}. \tag{2.2.72}$$

Taking the surface divergence of $M^k_{ij,D}(A)$ gives

$$\text{Div}\, M^k_{ij,D}(A)(x_i) = -\int_{\partial D_j} \frac{\partial \Phi_k(x_i,y)}{\partial \nu_x} \text{Div}\, A(y)\ ds(y) - k^2 \nu \cdot \int_{\partial D_j} \Phi_k(x_i,y) A(y)\ ds(y) \tag{2.2.73}$$

and using (2.2.68)

$$\begin{aligned}
\left| \text{Div}\, M^k_{ij,D}(A)(x_i) \right| \leq & \left| \int_{\partial D_j} \frac{\partial \Phi_k(x_i,y)}{\partial \nu_x} \text{Div}\, A(y)\ ds(y) \right| + \left| \int_{\partial D_j} (y-z_j) \nabla \Phi_k(x_i,y) \cdot A(y)\ ds(y) \right| \\
& + \left| \int_{\partial D_j} (y-z_j) \Phi_k(x_i,y)\, \text{Div}\, A(y)\ ds(y) \right|.
\end{aligned}$$

Using the Mean-value-theorem, we get[9] successively

$$\begin{aligned}
\left| \int_{\partial D_j} \frac{\partial \Phi_k(x_i,y)}{\partial \nu_x} \text{Div}\, A(y)\ ds(y) \right| &= \left| \int_{\partial D_j} \frac{\partial \left(\Phi_k(x_i,y) - \Phi_k(x_i,z_j) \right)}{\partial \nu_x} \text{Div}\, A(y)\ ds(y) \right|, \\
&\leq \frac{|\partial B_j|^{\frac{1}{2}}}{\pi \boldsymbol{\delta}_{ij}} \left(\frac{1}{\boldsymbol{\delta}_{ij}} + |k| \right)^2 \boldsymbol{a}^2 \|\text{Div}\, A\|_{L^2(\partial D_j)},
\end{aligned}$$

and,

$$\begin{aligned}
\left| \int_{\partial D_j} (y-z_j) \nabla \Phi_k(x_i,y) \cdot A(y)\ ds(y) \right| &\leq \boldsymbol{a}^2 |\partial B_j|^{\frac{1}{2}} \frac{1}{4\pi \boldsymbol{\delta}_{ij}} \big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big) \|A\|_{L^2(\partial D_j)}, \\
\left| \int_{\partial D_j} (y-z_j) \Phi_k(x_i,y)\, \text{Div}\, A(y)\ ds(y) \right| &\leq \boldsymbol{a}^2 |\partial B_j|^{\frac{1}{2}} \frac{1}{4\pi \boldsymbol{\delta}_{ij}} \|\text{Div}\, A\|_{L^2(\partial D_j)}.
\end{aligned}$$

[9] Notice that

$$\int_{\partial D_i} \nabla \Phi_k(x_i,z_j)\, \text{Div}\, A\ ds(y) = \nabla \Phi_k(x_i,z_j) \int_{\partial D_i} \text{Div}\, A\ ds(y) = 0.$$

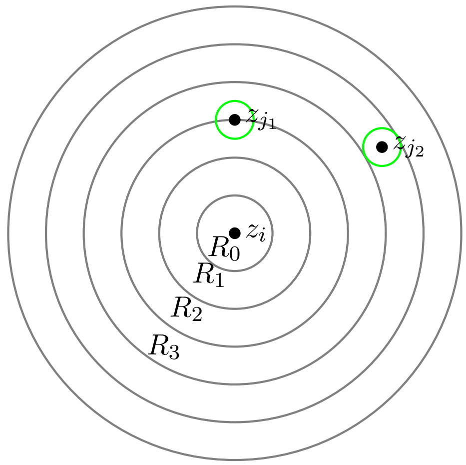

Figure 2.1: Possible configuration for the scatterers.

The sum of the three last inequalities, gives us the estimates

$$\left|\mathrm{Div}\, M^k_{ij,D}(A)(x_i)\right| \leq \frac{3|\partial B_j|^{\frac{1}{2}}}{\pi \boldsymbol{\delta}_{ij}} \left(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\right)^2 \boldsymbol{a}^2 \left(\|A\|_{L^{2,\mathrm{Div}}_{\mathrm{t}}(\partial D_i)} + \|\mathrm{Div}\, A\|_{L^2(\partial D_j)}\right), \tag{2.2.74}$$

and then

$$\|\mathrm{Div}\, M^k_{ij,D}(A)\|_{L^2(\partial D_i)} \leq \frac{3(|\partial B_i||\partial B_j|)^{\frac{1}{2}}}{\pi \boldsymbol{\delta}_{ij}} \left(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\right)^2 \boldsymbol{a}^3 \|A\|_{L^{2,\mathrm{Div}}_{\mathrm{t}}(\partial D_i)}. \tag{2.2.75}$$

We conclude by combining (2.2.72) and (2.2.75). □

Based on Lemma 10, let us show the proof of (2.2.55). Draw l spheres $(\mathcal{S}_{l\boldsymbol{\delta}}(z_i))_{\{l=1,2,\dots,n\}}$ centered at z_i with radius $l\boldsymbol{\delta}$, where n will be determined later, let $R_l = \mathcal{S}_{l+1} - \mathcal{S}_l$, and $R_0 = \mathcal{S}_1$ the volume of each R_l is given by

$$Vol(R_l) = \frac{4\pi\left((l+1)\boldsymbol{\delta}\right)^3}{3} - \frac{4\pi(l\boldsymbol{\delta})^3}{3} = \frac{4\pi(3l^2+3l+1)\boldsymbol{\delta}^3}{3}. \tag{2.2.76}$$

For $j \neq i$, we consider the spheres $\mathcal{S}_{\frac{1}{2}\boldsymbol{\delta}}(z_j)$. We claim that for $j_2 \neq j_1$, $int(\mathcal{S}_{\frac{1}{2}\boldsymbol{\delta}}(z_{j_1})) \cap int(\mathcal{S}_{\frac{1}{2}\boldsymbol{\delta}}(z_{j_2})) = \emptyset$, where $int(\mathcal{S}_{\frac{1}{2}\boldsymbol{\delta}})$ stands for the interior of $\mathcal{S}_{\frac{1}{2}\boldsymbol{\delta}}$. Indeed, if t was in the intersection, we would have $d_{j_1,j_2} = \min_{\{x\in D_{j_1}, y \in D_{j_2}\}} d(x,y) \leq d(z_{j_1}, z_{j_2}) \leq d(z_{j_1}, t) + d(t, z_{j_2}) < \frac{\boldsymbol{\delta}}{2} + \frac{\boldsymbol{\delta}}{2} = \boldsymbol{\delta}$, which contradicts the fact that $\boldsymbol{\delta}$ is the minimum distance. Hence, each domain D_{j_1} located in z_{j_1} occupy the volume of $\mathcal{S}_{\frac{1}{2}\boldsymbol{\delta}}(z_{j_2})$ which is not shared with another D_{j_i}. Then the maximum number of D_j's that could occupy each R_l, for $l = 1, \dots, m$, corresponds to the maximum number of spheres $\mathcal{S}_{\frac{1}{2}\boldsymbol{\delta}}(z_j)$ that could fit in the closure of R_l. Considering the case where z_j is on ∂R_l, only the half of the ball is in R_l, see the figure.

If m_l corresponds to the maximum amount of locations z_j that are in the closure of R_l then $m_l \leq Vol(R_l)/(\frac{Vol(\mathcal{S}_{\frac{1}{2}\boldsymbol{\delta}})}{2}) = 2^4(3l^2+3l+1) \leq 2^6 l^2$, whenever $l \geq 4$, then, being $\sum_{l=1}^n m_l = m \leq \sum_{l=1}^n 2^4(3l^2+3l+1) = 2^4 n(n^2+3n+3)$, we get $n = O(m^{\frac{1}{3}})$. Now, considering lemma 10, we have

$$\|M^k_{ij,D}\|_{\mathcal{L}(L^{2,Div}_t(\partial B),L^{2,Div}_t(\partial B))} \leq \frac{4(|\partial B_i||\partial B_j|)^{\frac{1}{2}}}{\pi\boldsymbol{\delta}_{ij}}\left(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|\right)^2\boldsymbol{a}^3 \leq \frac{C_{i,j}}{\boldsymbol{\delta}_{ij}}\left(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|\right)^2\boldsymbol{a}^3,$$

hence, for $C_{\scriptscriptstyle B} = \max_{i,j\in\{1,\dots,m\}} C_{i,j}$, we have

$$\sum_{(j\neq i)\geq 1}^{m} \|M^k_{ij,D}\|_{\mathcal{L}(L^{2,Div}_t(\partial B),L^{2,Div}_t(\partial B))} \leq C_{\scriptscriptstyle B}\sum_{l=1}^{n}\sum_{z_j\in R_l}\frac{1}{\boldsymbol{\delta}_{ij}}\left(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|\right)^2\boldsymbol{a}^3. \tag{2.2.77}$$

Since for every $z_j \in \bar{R}_l$, $l\boldsymbol{\delta} \leq \boldsymbol{\delta}_{ij}$ we get

$$\begin{aligned}
\sum_{(j\neq i)\geq 1}^{m} \|M^k_{ij,D}\|_{\mathcal{L}(L^{2,Div}_t(\partial B),L^{2,Div}_t(\partial B))} &\leq C_{\scriptscriptstyle B}\sum_{l=1}^{n}\sum_{z_j\in R_l}\frac{1}{l\boldsymbol{\delta}}\left(\frac{1}{l\boldsymbol{\delta}}+k\right)^2\boldsymbol{a}^3,\\
&\leq C_{\scriptscriptstyle B}\sum_{l=1}^{n}2^4(3l^2+3l+1)\frac{1}{l\boldsymbol{\delta}}\left(\frac{1}{l\boldsymbol{\delta}}+k\right)^2\boldsymbol{a}^3,\\
&\leq 2^6 C_{\scriptscriptstyle B}\sum_{l=1}^{n} l^2\frac{1}{l\boldsymbol{\delta}}\left(\frac{1}{l\boldsymbol{\delta}}+k\right)^2\boldsymbol{a}^3,\\
&\leq \frac{2^6 C_{\scriptscriptstyle B}}{\boldsymbol{\delta}}\left(\frac{\ln{(n+1)}}{\boldsymbol{\delta}^2}+\frac{2kn}{\boldsymbol{\delta}}+\frac{n}{2}(n+1)k^2\right)\boldsymbol{a}^3.
\end{aligned} \tag{2.2.78}$$

□

Considering (2.2.56) and (2.2.58), the condition (2.2.52) is acquired if

$$\frac{|\partial B_i|e^{C_B}}{4\pi\, diam(B)^2}(k\boldsymbol{a})^2\|[I/2+M^0_{ii,B}]^{-1}\|_{\mathcal{L}(L^2_t(\partial B_i),L^2_t(\partial B_i))} < 1, \tag{2.2.79}$$

If we set $C_{B_i} = \frac{|\partial B_i|}{4\pi\, diam(B)^2}\|[I/2+M^0_{ii,B}]^{-1}\|_{\mathcal{L}(L^2_t(\partial B_i),L^2_t(\partial B_i))}$ we get from (2.2.53)

$$\|[I/2+M^k_{ii,D}]^{-1}\| \leq C_{i,\boldsymbol{a}} := \frac{\|[I/2+M^0_{ii,B}]^{-1}\|_{\mathcal{L}(L^2_t(\partial B_i),L^2_t(\partial B_i))}}{1-C_{B_i}\left(e^{k\boldsymbol{a}}-1\right)k\boldsymbol{a}}, \tag{2.2.80}$$

so we find $\|(I/2+\mathcal{M}_D)^{-1}\| \leq C_{\boldsymbol{a}} := \max_{i\in\{1,\dots,m\}} C_{i,\boldsymbol{a}}$, under the condition (2.2.79).

2.3 Fields approximation and the linear algebraic systems

Based on the representation (2.2.12), the expression of the far field pattern is given by

$$E^\infty(\tau) = \frac{ik}{4\pi}\tau\times\int_{\partial D}A(y)e^{-ik\tau.y}ds(y),$$

where $\tau = (x/|x|) \in \mathbb{S}^2$. We put

$$\mathcal{R}_i := \int_{\partial D_i} A_i^{[1]} ds, \quad \mathcal{Q}_i := \int_{\partial D_i} \nu u_i ds, \tag{2.3.1}$$

where $A_i^{[1]}$ and u_i are defined in Theorem 8, with the notation (2.1.14) repeating the same calculations as in (2.2.78), we derive the estimate

$$\begin{aligned}
&\sum_{\substack{j\geq 1\\ j\neq i}}^{m} \frac{1}{\boldsymbol{\delta}_{ij}}\Big((\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 + (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^2 + (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|) \Big) \\
&\qquad = O\Big(\frac{1}{\boldsymbol{\delta}^4} + \frac{(|k|+1)\ \ln(m^{\frac{1}{3}})}{\boldsymbol{\delta}^3} + \frac{(|k|+1)^2\ m^{\frac{1}{3}}}{\boldsymbol{\delta}^2} + \frac{(|k|+1)^3\ m^{\frac{2}{3}}}{\boldsymbol{\delta}}\Big) \quad =: \epsilon_{k,\boldsymbol{\delta},m}.
\end{aligned} \tag{2.3.2}$$

Proposition 11. *For $\Im k = 0$, the far field pattern can be approximated by*

$$E^{\infty}(\tau) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ik\tau . z_i} \tau \times \{\mathcal{R}_i - ik\tau \times \mathcal{Q}_i\} + O\left((|k|^3 + |k|^2)\ m\boldsymbol{a}^4 \right). \tag{2.3.3}$$

For $\Im k \geq 0$, and for every $x \in \mathbb{R}^3 \setminus \cup_{i=1}^m \overline{D_i}$, such that $d_x := d(x, \cup_{i=1}^m \overline{D_i}) \geq \boldsymbol{\delta}$, the scattered electric field has the following expansion,

$$E^{sc}(x) = \sum_{i=1}^{m} \Big(\nabla_x \Phi_k(x, z_i) \times \mathcal{R}_i + \nabla_y \times \nabla_x \times (\Phi_k(x, z_i)\mathcal{Q}_i) \Big) + O\big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4} + \epsilon_{k,\boldsymbol{\delta},m}\ \boldsymbol{a}^4\big). \tag{2.3.4}$$

The elements $(\mathcal{R}_i)_{i=1}^m$ and $(\mathcal{Q}_i)_{i=1}^m$ are solutions of the following linear algebraic system

$$\begin{aligned}
\mathcal{R}_i + \left[\mathcal{P}_{\partial D_i}\right] \sum_{(j\neq i)\geq 1}^{m} \left(\Pi_k(z_i, z_j)\mathcal{R}_j - k^2 \nabla \Phi_k(z_i, z_j) \times \mathcal{Q}_j \right) &= -\left[\mathcal{P}_{\partial D_i}\right] \operatorname{curl} E^{in}(z_i) \\
&+ O\left(\frac{\boldsymbol{a}^7}{\boldsymbol{\delta}^4} + |k|\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^7 + |k|^2 \boldsymbol{a}^4 \right).
\end{aligned} \tag{2.3.5}$$

$$\begin{aligned}
\mathcal{Q}_i - \left[\mathcal{T}_{\partial D_i}\right] \sum_{(j\neq i)\geq 1}^{m} \left(-\nabla_x \Phi_k(z_i, z_j) \times \mathcal{R}_j \ + \ \Pi_k(z_i, z_j)\mathcal{Q}_j \right) &= -\left[\mathcal{T}_{\partial D_i}\right] E^{in}(z_i) \\
&+ O\left(\frac{\boldsymbol{a}^7}{\boldsymbol{\delta}^4} + \epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^7 + (1+|k|)\boldsymbol{a}^4 \right).
\end{aligned} \tag{2.3.6}$$

2.3.1 Justification of (3.3.68) and (2.3.4)

Lemma 12. *For $\Im k = 0$, the far field pattern can be approximated by*

$$E^{\infty}(\tau) = \frac{ik}{4\pi} \sum_{i=1}^{m} e^{-ik\tau . z_i} \tau \times \left\{ \int_{\partial D_i} A_i\ ds - \int_{\partial D_i} (ik\tau.(y - z_i))\ A_i(y)\ ds(y) \right\} \tag{2.3.7}$$

with an error estimate given by $O\left(e^{|k|\boldsymbol{a}}\ m\ (|k|^3)\boldsymbol{a}^4\right)$*. Precisely, in view of the decomposition* (2.2.45) *of Theorem 8 and* (2.2.79) *the far field admits the following expansion*

$$E^{\infty}(\tau) = \frac{ik}{4\pi}\sum_{i=1}^{m} e^{-ik\tau.z_i}\tau \times \{\mathcal{R}_i - ik\tau \times \mathcal{Q}_i\} + O\left(\left(|k|^3 + |k|^2\right)\ m\boldsymbol{a}^4\right). \tag{2.3.8}$$

Proof. To prove (2.3.7), we write

$$\begin{aligned} E^{\infty}(\tau) =& \frac{ik}{4\pi}\sum_{i=1}^{m} e^{-ik\tau.z_i}\tau \times \int_{\partial D_i} A_m(y)\ ds(y) \\ &+ \frac{ik}{4\pi}\sum_{i=1}^{m} \tau \times \int_{\partial D_i}\left(e^{-ik\tau.y} - e^{-ik\tau.z_i}\right) A_m(y)\ ds(y) \end{aligned} \tag{2.3.9}$$

for every $i \in \{1, ..., m\}$ and evaluate the term

$$Q_{(\tau,z_m)} := \tau \times \int_{\partial D_i}\left(e^{-ik\tau.y} - e^{-ik\tau.z_m}\right) A_m(y)\ ds(y).$$

Developing the exponential in Taylor series, we obtain

$$\begin{aligned} Q_{(\tau,z_i)} =& e^{-ik\tau.z_i}\tau \times \int_{\partial D_i}\left(e^{-ik\tau.(y-z_i)} - 1\right) A_m(y)\ ds(y), \\ =& e^{-ik\tau.z_m}\tau \times \int_{\partial D_i}\sum_{n\geq 1}\frac{(-ik\tau.(y-z_m))^n}{n!} A_m(y)\ ds(y), \\ =& e^{-ik\tau.z_m}\tau \times \Big(\int_{\partial D_i}\sum_{n\geq 2}\frac{(-ik\tau.(y-z_m))^n}{n!} A_m(y)\ ds(y) + \int_{\partial D_i}(-ik\tau.(y-z_m))\, A_m(y)\ ds(y)\Big). \end{aligned}$$

As $|y - z_i| \leq \boldsymbol{a}$, the first term gives us,[10]

$$\begin{aligned} \left|e^{-ik\tau.z_i}\tau \times \int_{\partial D_i}\sum_{n\geq 2}\frac{(-ik\tau.(y-z_i))^n}{n!} A_i(y)\ ds(y)\right| &\leq |\partial B_i|^{\frac{1}{2}}\sum_{n\geq 2}\frac{(|k|\boldsymbol{a})^n}{n!}\boldsymbol{a}\|A\|_{L^2(\partial D_i)}, \\ &\leq |\partial B_i|^{\frac{1}{2}}\sum_{n\geq 2}\frac{(|k|\boldsymbol{a})^{n-2}}{n!}|k|^2\boldsymbol{a}^4 \leq |\partial B|^{\frac{1}{2}} e^{|k|\boldsymbol{a}}|k|^2\boldsymbol{a}^4. \end{aligned}$$

Taking the sum over i, we get

$$E^{\infty}(\tau) = \frac{ik}{4\pi}\sum_{i=1}^{m} e^{-ik\tau.z_i}\tau \times \left(\mathcal{R}_i - \int_{\partial D_i}(ik\tau.(y-z_i))\, A_i(y)\ ds(y)\right) + O(e^{|k|\boldsymbol{a}}|k|^3 m\boldsymbol{a}^4). \tag{2.3.10}$$

Now, considering the decomposition (2.2.45) of theorem 8, we have

$$\begin{aligned} \int_{\partial D_i}(ik\tau.(y-z_i))\, A_i(y)\ ds(y) &= \int_{\partial D_i} ik\tau.(y-z_i)(A_i^{[1]} + A_i^{[2]})(y)\ ds(y), \\ &= \int_{\partial D_i}(ik\tau.(y-z_i))\, A_i^{[2]}(y)\ ds(y) + O(|k|\boldsymbol{a}^4), \end{aligned}$$

[10] We have $\sum_{n\geq 2}\frac{(|k|\boldsymbol{a})^{n-2}}{n!} \leq \sum_{n\geq 2}\frac{(|k|\boldsymbol{a})^{n-2}}{(n-2)!} = e^{|k|\boldsymbol{a}}$.

where $O(|k|\boldsymbol{a}^4)$ comes from

$$\left|\int_{\partial D_i} ik\tau.(y-z_i)A_i^{[1]}(y)\; ds(y)\right| \leq |k|\boldsymbol{a}^2|\partial B_i|\|A^{[1]}\|_{L^2(\partial D_i)} \leq |k|\boldsymbol{a}^2|\partial B_i|C_1\boldsymbol{a}\|A\|_{L_{\rm t}^{2,\mathrm{Div}}(\partial D_i)},$$
$$\leq |k||\partial B_i|C_1 C_e\boldsymbol{a}^4.$$

Then $\int_{\partial D_i} ik\tau.(y-z_i)\; A_i\; ds(y) = \int_{\partial D_i} ik\tau.(y-z_i)\;\nu\times\nabla u_i\; ds(y) + O(|k|\boldsymbol{a}^4)$. Multiplying by $e^{-ik\tau.z_i}$ and taking the sum over i, we obtain

$$\sum_{i=1}^m\; e^{-ik\tau.z_i}\int_{\partial D_i} ik\tau.(y-z_i)A_i\; ds(y) = \sum_{i=1}^m\; e^{-ik\tau.z_i}\left(\int_{\partial D_i} ik\tau.(y-z_i)\nu\times\nabla u_i\; ds(y) + O(|k|\boldsymbol{a}^4)\right),$$

With this last approximation, (2.3.10) gives

$$E^\infty(\tau) = \frac{ik}{4\pi}\sum_{i=1}^m\; e^{-ik\tau.z_i}\tau\times\left(\mathcal{R}_i - \int_{\partial D_i} ik\tau.(y-z_i)\;\nu\times\nabla u_i(y)\; ds(y)\right) + O\left((e^{|k|\boldsymbol{a}}|k|^3+|k|^2)\; m\boldsymbol{a}^4\right).$$

Finally, integrating by part the second term of the second member, we obtain

$$\int_{\partial D_i} ik\tau.(y-z_i)\;\nu\times\nabla u_i\; ds(y) = -ik\int_{\partial D_i}(\nu\times\nabla_y\tau.(y-z_i))\,u_i\; ds(y),$$
$$= +ik\int_{\partial D_i}(\tau\times\nu)\,u_i(y)\; ds(y) = ik\tau\times\mathcal{Q}_i.$$

□

Lemma 13. *The electric field has the following asymptotic expansion*

$$\begin{aligned}E^{sc}(x) =& \sum_{i=1}^m\Big(\nabla_x\Phi_k(x,z_i)\times\mathcal{R}_i + \nabla_y\times\nabla_x\times(\Phi_k(x,z_i)\mathcal{Q}_i)\Big)\\ &+ O\left(\frac{1}{\boldsymbol{\delta}}\left(\frac{1}{\boldsymbol{\delta}^3}+\frac{3|k|+1}{\boldsymbol{\delta}^2}+\frac{5|k|^2}{\boldsymbol{\delta}_{i_0,i}}+|k|^2+|k|^3\right)\;\boldsymbol{a}^4\right)\\ &+ O\left(\sum_{(i\neq i_0)\geq 1}^m\frac{1}{\boldsymbol{\delta}_{i_0,i}}\left(\frac{1}{\boldsymbol{\delta}_{i_0,i}^3}+\frac{3|k|+1}{\boldsymbol{\delta}_{i_0,i}^2}+\frac{5|k|^2}{\boldsymbol{\delta}_{i_0,i}}+|k|^2+|k|^3\right)\;\boldsymbol{a}^4\right).\end{aligned} \tag{2.3.11}$$

Proof. For $x\in\mathbb{R}^3\setminus\cup_{i=1}^m\overline{D_i}$, using Taylor formula with integral reminder, we get from the representation (2.2.12)

$$\begin{aligned}E^{\rm sc}(x) =& \sum_{i=1}^m\int_{\partial D_i}\big(\nabla_x\Phi_k(x,z_i) + \big(\nabla_y\nabla_x\Phi_k(x,z_i)\cdot(y-z_i)\big)\big)\times A_i(y)\; ds(y)\\ &+\sum_{i=1}^m\int_{\partial D_i}\int_0^1 D^3\Phi_k(x,ty+(1-t)z_i)\circ(y-z_i)(y-z_i)\times A_i(y)\; ds(y),\end{aligned}$$

which is[11]

$$E^{\mathrm{sc}}(x) = \sum_{i=1}^{m}\Big(\nabla_x \Phi_k(x,z_i)\times \mathcal{R}_i + \int_{\partial D_i}\Big(-\nabla_x\big(\nabla_x\Phi_k(x,z_i)\cdot(y-z_i)\big)\Big)\times A_i(y)\ ds(y)\Big) + \\ \sum_{i=1}^{m}\int_{\partial D_i}\int_0^1 D^3\Phi_k(x,ty+(1-t)z_i)\circ(y-z_i)(y-z_i)\times A_i(y)\ ds(y), \tag{2.3.12}$$

As it was done in (2.3.40), with $d_{x,i} := d(x,\partial D_i)$, we have

$$\Big|\int_{\partial D_i}\int_0^1 D^3\Phi_k(x,ty+(1-t)z_i)\circ(y-z_i)(y-z_i)\times A_i(y)\ ds(y)\Big| \\ \leq \frac{e^{-\Im k d_{x,i}}}{d_{x,i}}\Big(\frac{1}{d_{x,i}}+|k|\Big)^3 \boldsymbol{a}^2\int_{\partial D_i}|a_i|ds.$$

For a fixed $x\in\mathbb{R}^3\setminus\Omega$, set $d_x := \min_{i\in\{1,\dots,m\}} d_{x,i}$, hence there exists some i_0 such that $d_x = d(x,\partial D_{i_0})$, further

$$\boldsymbol{\delta}_{i_0,i} = d(\partial D_{i_0},\partial D_i) \leq d_{x,i} + d_{x,i_0},$$

from which follows

$$\frac{1}{d_{x,i}} \leq \frac{2}{\boldsymbol{\delta}_{i_0,i}}. \tag{2.3.13}$$

Summing over i, the reminder, remain smaller then

$$O\Big(\sum_{(i\neq i_0)\geq 1}^{m}\frac{e^{-\Im k\boldsymbol{\delta}_{i_0,i}/2}}{\boldsymbol{\delta}_{i_0,i}}\big(\frac{1}{\boldsymbol{\delta}_{i_0,i}}+|k|\big)^3\ \boldsymbol{a}^4 + \frac{e^{-\Im k\boldsymbol{\delta}}}{\boldsymbol{\delta}}\big(\frac{1}{\boldsymbol{\delta}}+|k|\big)^3\ \boldsymbol{a}^4\Big). \tag{2.3.14}$$

The second term under the sum of (2.3.12) is precisely

$$\nabla_y\times\nabla_x\times(\Phi_k(x,z_i)\mathcal{Q}_i) + O\Big(\frac{e^{-\Im k\boldsymbol{\delta}_{i_0,i}/2}}{\boldsymbol{\delta}_{i_0,i}}\big(\frac{1}{\boldsymbol{\delta}_{i_0,i}}+|k|\big)^2\ \boldsymbol{a}^4\Big). \tag{2.3.15}$$

Indeed,

$$\int_{\partial D_i}\nabla_x\big(\nabla_x\Phi_k(x,z_i)\cdot(y-z_i)\big)\times A_i(y)\ ds(y) = \int_{\partial D_i}\nabla_x\times\ \big[\big(\nabla_x\Phi_k(x,z_i)\cdot(y-z_i)\big)A_i(y)\big]\ ds(y).$$

As we did for the far field approximation, we get in view of decomposition (2.2.45)

$$\int_{\partial D_i}(\nabla_x\Phi_k(x,z_i).(y-z_i))\,A_i(y)\ ds(y) = \int_{\partial D_i}\nabla_x\Phi_k(x,z_i).(y-z_i)\Big(A_i^{[1]}+A_i^{[2]}\Big)(y)\ ds(y), \tag{2.3.16}$$

and being

$$\left|\int_{\partial D_i}\nabla_x\left(\nabla_x\Phi_k(x,z_i).(y-z_i)\right)\times A_i^{[1]}(y)\ ds(y)\right| \leq C\frac{e^{-\Im k d_{x,i}}}{d_{x,i}}\left(\frac{1}{d_{x,i}}+|k|\right)^2\ \boldsymbol{a}\int_{\partial D_i}|A_i^{[1]}|ds,$$

[11] Recall that $\nabla_y\nabla_x\Phi_k(x,y) = -\nabla_x\nabla_x\Phi_k(x,y)$.

we get, due to (2.2.46)

$$\int_{\partial D_i} \nabla\left(\nabla \Phi_k(x,z_i).(y-z_i)\right) \times A_i^{[1]}(y)\, ds(y) = 0\Big(\frac{e^{-\Im k d_{x,i}}}{d_{x,i}} \left(\frac{1}{d_{x,i}} + |k|\right)^2 \boldsymbol{a}^4\Big). \tag{2.3.17}$$

Further, integrating by part, in the second step of the following identities

$$\begin{aligned}
\int_{\partial D_i} \Big(\nabla_x \Phi_k(x,z_i).(y-z_i)\Big) A_i^{[2]}(y)\, ds(y) & \int_{\partial D_i} \left(\nabla_x \Phi_k(x,z_i).(y-z_i)\right) \nu \times \nabla u_i(y)\, ds(y), \\
&= -\int_{\partial D_i} \nu \times \nabla_y \left(\nabla_x \Phi_k(x,z_i).(y-z_i)\right) u_i(y)\, ds(y), \\
&= \int_{\partial D_i} \nabla_x \Phi_k(x,z_i) \times \nu u_i(y)\, ds(y) = \nabla_x \Phi_k(x,z_i) \times \mathcal{Q}_i,
\end{aligned}$$

and differentiating, we get

$$-\nabla_x \times \int_{\partial D_i} \left(\nabla_x \Phi_k(x,z_i).(y-z_i)\right) A_i^{[2]}(y)\, ds(y) = \nabla_y \nabla_x (\Phi_k(x,z_i)\mathcal{Q}_i). \tag{2.3.18}$$

Hence, considering (2.3.17), (2.3.15) follows from (2.3.16). Replacing (2.3.15) and (2.3.14) in (2.3.12) gives

$$\begin{aligned}
E^{\mathrm{sc}}(x) = \sum_{i=1}^{m} \Big(& \nabla_x \Phi_k(x,z_i) \times \mathcal{R}_i + \nabla_y \times \nabla_x \times (\Phi_k(x,z_i)\mathcal{Q}_i) + O\Big(\frac{e^{-\Im k d_{x,i}}}{d_{x,i}} (\frac{1}{d_{x,i}} + |k|)^2\, m\boldsymbol{a}^4\Big)\Big) \\
& + O\Big(\sum_{i=1}^{m} \frac{e^{-\Im k \boldsymbol{\delta}_{i_0,i}/2}}{\boldsymbol{\delta}_{i_0,i}} (\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|)^3\, \boldsymbol{a}^4\Big).
\end{aligned}$$

Repeating the calculations done to get (2.3.14), we obtain

$$\begin{aligned}
E^{\mathrm{sc}}(x) = & \sum_{i=1}^{m} \Big(\nabla_x \Phi_k(x,z_i) \times \mathcal{R}_i + \nabla_y \times \nabla_x \times (\Phi_k(x,z_i)\mathcal{Q}_i)\Big) \\
& + O\Big(\frac{e^{-\Im k \boldsymbol{\delta}}}{\boldsymbol{\delta}} \Big[\Big(\frac{1}{\boldsymbol{\delta}} + |k|\Big)^2 + \Big(\frac{1}{\boldsymbol{\delta}} + |k|\Big)^3\Big]\, \boldsymbol{a}^4\Big) \\
& + O\Big(\sum_{(i \neq i_0) \geq 1}^{m} \frac{e^{-\Im k \boldsymbol{\delta}_{i_0,i}/2}}{\boldsymbol{\delta}_{i_0,i}} \Big[\Big(\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|\Big)^2 + \Big(\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|\Big)^3\Big]\, \boldsymbol{a}^4\Big).
\end{aligned}$$

The approximation (2.3.4) follows using (2.3.2). □

2.3.2 Justification of (2.3.5) and (2.3.6)

We provide the justification of (2.3.6) and then the one of (2.3.5).

2.3.2.1 Justification of (2.3.6)

Let ψ be any smooth enough vectorial function. Multiplying by (2.2.17) and integrating over ∂D_i, we get

$$\int_{\partial D_i} \psi \cdot [I/2 + M^k_{ii,D}]A \, ds + \sum_{(j\neq i)\geq 1}^{m} \int_{\partial D_i} \psi \cdot [M^k_{ij,D}](A_j) \, ds = -\int_{\partial D_i} \psi \cdot \nu_i \times E^{\text{in}} \, ds. \tag{2.3.19}$$

Recalling the scaling (2.2.56) and the estimate (2.2.62)[12], we have

$$\begin{aligned}\left|\int_{\partial D_i} \psi \cdot [I/2 + M^k_{ii,D}]A^{[1]}_i \, ds\right| &\leq \|\psi\| \left(\|[I/2 + M^0_{ii,D}]A^{[1]}_i\| + \|[M^k_{ii,D} - M^0_{ii,D}]A^{[1]}_i\|\right), \\ &\leq \|\psi\|_{L^2(\partial D_i)}\Big(C_{[M^0_{ii,B}]} + \frac{|k|^2(|\partial B|)\boldsymbol{a}^2}{4\pi}\Big)\|A^{[1]}_i\|_{L^2(\partial D_i)}.\end{aligned}$$

In view of the decomposition (2.2.45), we obtain $\left|\int_{\partial D_i} \psi \cdot [I/2 + (M^k_{ii,D})]A^{[1]}_i \, ds\right| = O(\|\psi\|_{L^2(\partial D_i)}\boldsymbol{a}^2)$, and then (2.3.19) gives

$$\begin{aligned}O(\|\psi\|_{L^2(\partial D_i)}\boldsymbol{a}^2) &+ \int_{\partial D_i} \psi \cdot [I/2 + M^0_{ii,D} + (M^k_{ii,D} - M^0_{ii,D})](A^{[2]}_i) \, ds \\ &+ \sum_{(j\neq i)\geq 1}^{m} \int_{\partial D_i} \psi \cdot [M^k_{ij,D}](A^{[1]}_j + A^{[2]}_j) \, ds) \, ds = -\int_{\partial D_i} \psi \cdot \nu_i \times E^{\text{in}} \, ds.\end{aligned}$$

Using (2.2.62) and the estimates (2.2.48) of the decomposition (2.2.45), in the left hand side, we get

$$\begin{aligned}O(\|\psi\|_{L^2}\boldsymbol{a}^2) &+ \int_{\partial D_i} \psi \cdot [I/2 + M^0_{ii,D}](A^{[2]}_i) \, ds + O(|k\boldsymbol{a}|^2\|\psi\|_{L^2}\boldsymbol{a}) \\ &+ \sum_{(j\neq i)\geq 1}^{m} \int_{\partial D_i} \psi \cdot [M_{ij}](A^{[1]}_j + A^{[2]}_j) \, ds = \int_{\partial D_i} -\psi \cdot \nu_i \times E^{\text{in}} \, ds.\end{aligned} \tag{2.3.20}$$

Now, we show how we choose appropriate candidates ψ to derive the estimates (2.3.5) and (2.3.6).

Lemma 14. *There are functions $(\psi_l)_{l=1,2,3.}$ such that $\nu \times \psi_l \in L^{2,\text{Div}}_t(\partial D_i)$ and satisfiesing, for constants $C_{(M^0_{ii,B})}, C_{(M^0_{ii,B}, K^{0*}_{ii,B})}$ which depends only on $|\partial B|$,*

$$\|\nu \times \psi_l\|_{L^2(\partial Di)} \leq C_{(M^0_{ii,B})}|\partial B|\boldsymbol{a}^2, \quad \|\nu \times \psi_l\|_{L^{2,\text{Div}}_t(\partial Di)} \leq C_{(M^0_{ii,B}, K^{0*}_{ii,B})}|\partial B|\boldsymbol{a}, \tag{2.3.21}$$

and for which the approximation

$$\int_{\partial D_i} \psi_l \cdot [I/2 + M^0_{ii,D}](A^{[2]}_i) \, ds = O(\boldsymbol{a}^4 + |k|^2\boldsymbol{a}^5) + \int_{\partial D_i} \nu^l_i u_i \, ds, \tag{2.3.22}$$

hold, here $\nu \times \nabla u_i = A^{[2]}_i$.

[12] With $L^2(\partial D_i)$ norms.

Proof. Let $(b_l)_{l=1,2,3}$ be the solution of the following equation

$$[-I/2 + M^0_{ii,D}](b_l) = -\nu \times V_l, \tag{2.3.23}$$

where, $(e_l)_{l=1,2,3.}$ being the canonical base of $\mathbb{R}^3$ $V_1 = (0, 0, (x - z_i) \cdot e_2)$, $V_2 = ((x - z_i) \cdot e_3, 0, 0)$ and $V_3 = (0, (x - z_i) \cdot e_1, 0)$. It is evident that $\operatorname{curl} V_l = e_l$, further $b_l \in L^{2,Div}_{\rm t}(\partial D_i)$, and[13]

$$\nu \times (\nu \times b_l) = (\nu \cdot b_l)\nu - (\nu \cdot \nu) b_l = -b_l. \tag{2.3.24}$$

We put $\psi_l := -\nu \times b_l$, hence $\nu \times \psi_l = b_l$, and $\operatorname{Div}(\nu \times \psi_l) = \operatorname{Div} A_l$. Solving (2.3.23) amounts to solve the following problem (it suffices to take the surface divergence in the identity (2.3.24))

$$[I/2 + (K^0_{ii,D})^*](\nu \cdot \operatorname{curl} \psi_l) = -\nu^l_i. \tag{2.3.25}$$

Further, as it was done in (2.2.56) and (2.2.54), the following estimates hold

$$\|\nu \times \psi_l\|_{L^2(\partial Di)} \leq \|[-I/2 + M^0_{ii,B}]^{-1}\|_{\mathcal{L}(L^2_{\rm t}(\partial B_i))} \|\nu \times V_l\|_{L^2(\partial Di)} \leq C_{(M^0_{ii,B})} |\partial B| \boldsymbol{a}^2,$$

and

$$\|\nu \times \psi_l\|_{L^{2,\mathrm{Div}}_{\rm t}(\partial Di)} \leq \|[-I/2 + M^0_{ii,B}]^{-1}\|_{\mathcal{L}(L^{2,\mathrm{Div}}_{\rm t}(\partial B_i))} \|V_l\|_{L^{2,Div}_{\rm t}(\partial Di)} \leq C_{M^0_{ii,B}, K^{0*}_{ii,D}} |\partial B| \boldsymbol{a}.$$

We have the following relations (see Lemma 5.11 [57])

$$[I/2 + M^0_{ii,D}](A^{[2]}_i) = [I/2 + M^0_{ii,D}](\nu \times \nabla u_i) = \nu \times \nabla [I/2 + K^0_{ii,D}](u_i), \tag{2.3.26}$$

and, for every scalar function w, [14]

$$\int_{\partial D_i} w\nu \cdot \operatorname{curl} \psi \; ds = -\int_{\partial D_i} \psi \cdot (\nu \times \nabla w) \; ds. \tag{2.3.27}$$

Hence, the term under the integral of the left hand side of (2.3.20), using (2.3.26) and (2.3.27), gives

$$\begin{aligned} \int_{\partial D_i} \psi \cdot [I/2 + M^0_{ii,D}](A^{[2]}_i) \; ds &= \int_{\partial D_i} \psi \cdot \nu \times \nabla [I/2 + K^0_{ii,D}](u_i) \; ds, \\ &= -\int_{\partial D_i} (\nu \cdot \operatorname{curl} \psi)[I/2 + K^0_{ii,D}](u_i) \; ds, \\ &= -\int_{\partial D_i} [I/2 + (K^0_{ii,D})^*](\nu \cdot \operatorname{curl} \psi) \; u_i ds. \end{aligned} \tag{2.3.28}$$

Using (2.3.28) with ψ_l as in (2.3.23), we get

$$O((\boldsymbol{a}^2 + |k\boldsymbol{a}|^2 \boldsymbol{a})\boldsymbol{a}^2) + \int_{\partial D_i} [I/2 + (K^0_{ii,D})^*](\nu \cdot \operatorname{curl} \psi_l) u_i \; ds = O(\boldsymbol{a}^4) + O(|k|^2 \boldsymbol{a}\boldsymbol{a}^4) - \int_{\partial D_i} \nu^l_i u_i \; ds,$$

to conclude that

$$\int_{\partial D_i} \psi_l \cdot [I/2 + M^0_{ii,D}](A^{[2]}_i) \; ds = O(\boldsymbol{a}^4 + |k|^2 \boldsymbol{a}^5) + \int_{\partial D_i} \nu^l_i u_i \; ds. \tag{2.3.29}$$

□

[13] Having $\nu \cdot b_l = 0$ and $\nu \cdot \nu = 1$.

[14] A direct application of (2.2.3) with $a = w\psi$.

We recall the notations, for $l = 1, 2, 3$,

$$[\mathcal{T}_{\partial D_i}]^l := \int_{\partial D_i} \nu_i^l [I/2 + K^0_{ii,D}]^{-1}(x - z_i)\, ds.$$

Lemma 15. *The second term of the left hand side of* (2.3.20) *admits the following approximations*

$$\int_{\partial D_i} \psi_l \cdot [M^k_{ij,D}] A_j^{[1]}\, ds = [\mathcal{T}_{\partial D_i}]^l \cdot \nabla_x \Phi_k(z_i, z_j) \times \mathcal{R}_j + O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}\big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^2 \boldsymbol{a}^7\Big) \tag{2.3.30}$$

and

$$\begin{aligned}\int_{\partial D_i} \psi_l \cdot [M^k_{ij,D}](A_j^{[2]})\, ds =& [\mathcal{T}_{\partial D_i}]^l \cdot \left(-k^2 \Phi_k(z_i, z_j) I + \nabla_x \nabla_y \Phi_k(z_i, z_j)\right) \mathcal{Q}_j \\ &+ O\bigg(\frac{1}{\boldsymbol{\delta}_{ij}}\Big(\big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^3 + \big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)\Big)\boldsymbol{a}^7\bigg).\end{aligned} \tag{2.3.31}$$

In addition, we have the approximation

$$\int_{\partial D_i} \psi_l \cdot \nu_i \times E^{in}\, ds = \int_{\partial D_i} \nu_i^l [I/2 + K^0_{ii,D}]^{-1}(x - z_i)\, ds \cdot E^{in}(z_i) + O(k\boldsymbol{a}^4). \tag{2.3.32}$$

Proof. Adding and subtracting $\nabla_x \Phi_k(z_i, y)$, we write

$$\begin{aligned}&\int_{\partial D_i} \psi_l \cdot [M^k_{ij,D}] A_j^{[1]} ds \\ &= \int_{\partial D_i} \psi_l \cdot \Big(\nu_{x_i} \times \int_{\partial D_j} (\nabla_x \Phi_k(x_i, y) - \nabla_x \Phi_k(z_i, y)) \times A_j^{[1]}(y)\, ds(y)\Big) ds(x) \\ &+ \int_{\partial D_i} \psi_l \cdot \Big(\nu_{x_i} \times \int_{\partial D_j} (\nabla_x \Phi_k(z_i, y)) \times A_j^{[1]}(y)\, ds(y)\Big)\, ds(x).\end{aligned} \tag{2.3.33}$$

For the first integral of the right hand side, we get, using Holder's inequality then the Mean-value-theorem, with $L^2(\partial D_i)$ norm,

$$\begin{aligned}&\left|\int_{\partial D_i} \psi_l \cdot \Big(\nu_{x_i} \times \int_{\partial D_j} (\nabla_x \Phi_k(x_i, y) - \nabla_x \Phi_k(z_i, y)) \times A_j^{[1]}(y)\, ds(y)\Big)\, ds(x)\right| \\ &\qquad \leq \|\psi_l\|\, \Big\|\int_{\partial D_j} (\nabla_x \Phi_k(x_i, y) - \nabla_x \Phi_k(z_i, y)) \times A_j^{[1]}(y)\, ds(y)\Big\|.\end{aligned}$$

As

$$\begin{aligned}&\left|\Big(\int_{\partial D_j} (\nabla_x \Phi_k(x_i, y) - \nabla_x \Phi_k(z_i, y)) \times A_j^{[1]}(y)\Big)\, ds(y)\right| \\ &\leq \sup_{x \in \partial D_i} |(x - z_i)|\, \|A_j^{[1]}\| \sup_{x \in \partial D_i, y \in \partial D_j} |\nabla_x \nabla_x \Phi_k(x_i, y)|\, \boldsymbol{a} |\partial B_i|^{\frac{1}{2}},\end{aligned}$$

with (2.2.70), we get

$$\left|\Big(\int_{\partial D_j} (\nabla_x \Phi_k(x_i, y) - \nabla_x \Phi_k(z_i, y)) \times A_j^{[1]}(y)\Big)\, ds(y) \quad ds(x)\right| \leq \frac{e^{-\Im k\ \boldsymbol{\delta}_{ij}}}{4\pi} \frac{1}{\boldsymbol{\delta}_{ij}} \big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^2 \boldsymbol{a}^4. \tag{2.3.34}$$

Finally

$$\left|\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\int_{\partial D_j}(\nabla_x\Phi_k(x_i,y)-\nabla_x\Phi_k(z_i,y))\times A_j^{[1]}(y)\,ds(y))\,ds(x)\right|\leq\frac{e^{-\Im k\ \boldsymbol{\delta}_{ij}}}{4\pi\boldsymbol{\delta}_{ij}}\Big(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|\Big)^2\boldsymbol{a}^7,$$

which is

$$\begin{aligned}\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\int_{\partial D_j}(\nabla_x\Phi_k(x_i,y)-\nabla_x\Phi_k(z_i,y))\times A_j^{[1]}(y)\,ds(y))\,ds(x)&\\=O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}\Big(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|\Big)^2\boldsymbol{a}^7\Big).&\end{aligned}\tag{2.3.35}$$

For the second integral of the right hand side of (2.3.33), we get

$$\begin{aligned}\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\int_{\partial D_j}\nabla_x\Phi_k(z_i,y)\times A_j^{[1]}(y)\,ds(y))\,ds(x)&\\=\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\int_{\partial D_j}\nabla_x\left(\Phi_k(z_i,y)-\Phi_k(z_i,z_j)\right)\times A_j^{[1]}(y)\,ds(y))\,ds(x)&\\+\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\nabla_x\Phi_k(z_i,z_j)\times\int_{\partial D_j}A_j^{[1]}(y)\,ds(y))\,ds(x).&\end{aligned}$$

Using again the Mean-value-theorem as in (2.3.35) for the first integral of the second member, we get

$$\begin{aligned}&\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\int_{\partial D_j}\nabla_x\Phi_k(z_i,y)\times A_j^{[1]}(y)\,ds(y))\,ds(x)\\&=O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}\Big(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|\Big)^2\boldsymbol{a}^7\Big)+\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\nabla_x\Phi_k(z_i,z_j)\times\int_{\partial D_j}A_j^{[1]}\,ds(y))\,ds(x).\end{aligned}\tag{2.3.36}$$

In addition, considering the fact that, for any vectors a,b,c of $\mathbb{R}^3$ we have $a\cdot(b\times c)=-c\cdot(b\times a)$, we write [15]

$$\begin{aligned}\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\nabla_x\Phi_k(z_i,z_j)\times\int_{\partial D_j}A_j^{[1]}\,ds)\,ds(x)&\\=\int_{\partial D_i}\psi_l\cdot\nu_{x_i}\times\nabla\left((x-z_i)\cdot\nabla_x\Phi_k(z_i,z_j)\times\mathcal{R}_j\right)\,ds(x),&\\=-\int_{\partial D_i}\nu_{x_i}\times\psi_l\cdot\nabla\left((x-z_i)\cdot\nabla_x\Phi_k(z_i,z_j)\times\mathcal{R}_j\right)\,ds(x).&\end{aligned}$$

Integrating by parts, and considering (2.3.25), we have

$$\begin{aligned}\int_{\partial D_i}\psi_l\cdot(\nu_{x_i}\times\nabla_x\Phi_k(z_i,z_j)\times\mathcal{R}_j)\,ds(x)&=\int_{\partial D_i}\mathrm{Div}(\nu\times\psi_l)\left((x-z_i)\cdot\nabla_x\Phi_k(z_i,z_j)\times\mathcal{R}_j\right)\,ds(x),\\&=-\int_{\partial D_i}\nu\cdot\mathrm{curl}\,\psi_l\left((x-z_i)\cdot\nabla_x\Phi_k(z_i,z_j)\times\mathcal{R}_j\right)\,ds(x),\\&=\Big(\int_{\partial D_i}I/2+(K^0_{ii,D})^*]^{-1}(\nu_i^l)(x-z_i)\,ds\Big)\cdot\nabla_x\Phi_k(z_i,z_j)\times\mathcal{R}_j.\end{aligned}$$

[15] Recall the definition of $\mathcal{R}_j=\int_{\partial D_i}(A_j^{[1]}+A_j^{[2]})\,ds=\int_{\partial D_i}A_j^{[1]}\,ds$.

Replacing in (2.3.36), summing over j gives the first approximation. For (2.3.31), being $A_j^{[2]} = \nu \times \nabla u_j$ we have (see Lemma 5.11 [57])

$$[M^k_{ij,D}]\nu \times \nabla u_j = \nu \times \nabla[K^k_{i,j}]u_j - k^2\ \nu \times [S^k_{ij,D}](\nu_y u_j), \tag{2.3.37}$$

then

$$\int_{\partial D_i} \psi_l \cdot [M^k_{ij,D}](A_j^{[2]})\ ds = \psi_l \cdot (\nu \times \nabla[K^k_{i,j}]u_j - k^2\ \nu \times [S^k_{ij,D}](\nu_y u_j)). \tag{2.3.38}$$

The first term of the right hand side gives [16]

$$\begin{aligned}\int_{\partial D_i} &\psi_l \cdot \nu \times \nabla[K^k_{i,j}]u_j\ ds \\ &= -\int_{\partial D_i} \nu \cdot \operatorname{curl}\psi(x) \int_{\partial D_j} \nu_y \cdot \nabla_y \Phi_k(x_i, y) u_j(y)\ ds(y)\ ds(x), \\ &= -\int_{\partial D_i} \nu \cdot \operatorname{curl}\psi(x) \int_{\partial D_j} \nu_y \cdot \nabla_y \left(\Phi_k(x_i, y) - \Phi_k(z_i, y)\right) u_j(y)\ ds(y)\ ds(x).\end{aligned} \tag{2.3.39}$$

By Taylor formula at the first order, [17]

$$\nabla_y \left(\Phi_k(x_i, y) - \Phi_k(z_i, y)\right) = \nabla_x \nabla_y \Phi_k(z_i, y)(x - z_i) + O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^2\Big), \tag{2.3.40}$$

(2.3.39) gives

$$\begin{aligned}\int_{\partial D_i} \psi_l \cdot \nu \times \nabla[K^k_{i,j}]u_j\ ds = &-\int_{\partial D_i} \nu \cdot \operatorname{curl}\psi(x) \Big(\int_{\partial D_j} \nabla_x \nabla_y \Phi_k(z_i, y)(x - z_i) \cdot \nu_y u_j(y)\ ds(y)\Big)\ ds(x) \\ &+ \int_{\partial D_i} \nu \cdot \operatorname{curl}\psi(x) \int_{\partial D_j} \nu_y \cdot O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^2\Big) u_j(y)\ ds(y)\ ds(x).\end{aligned}$$

Adding and subtracting $\nabla_x \nabla_y \Phi_k(z_i, z_j)$ under the integral, with the approximation (2.3.40), yields

$$\begin{aligned}\int_{\partial D_i} \psi_l \cdot \nu \times \nabla[K^k_{i,j}]u_j\ ds = &-\int_{\partial D_i} \nu \cdot \operatorname{curl}\psi(x) \Big(\int_{\partial D_j} \nabla_x \nabla_y \Phi_k(z_i, z_j)(x - z_i) \cdot \nu_y u_j(y)\ ds(x)\Big)\ ds(y) \\ &+ \int_{\partial D_i} \nu \cdot \operatorname{curl}\psi(x) \int_{\partial D_j} 2 \times \nu_y \cdot O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^2\Big) u_j(y)\ ds(y)\ ds(x).\end{aligned}$$

In view of (2.2.49), we have

$$\begin{aligned}\Big|\int_{\partial D_i} &\nu \cdot \operatorname{curl}\psi(x) \int_{\partial D_j} \nu_y \cdot O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^2\Big) u_j(y)\ ds(y)\ ds(x)\Big| \\ &\leq \|\nu \cdot \operatorname{curl}\psi\| \Big\|\int_{\partial D_j} \nu_j \cdot O\big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^2\big) u_j\ ds\Big\|, \\ &\leq O\big(\frac{1}{\boldsymbol{\delta}_{ij}}\big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^3 \boldsymbol{a}^3\big) O(\boldsymbol{a}) \|u_j\| O(\boldsymbol{a}), \\ &= O\big(\frac{1}{\boldsymbol{\delta}_{ij}}\big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^3 \boldsymbol{a}^7\big).\end{aligned}$$

[16] Being $\int_{\partial D_i} (\nu \cdot \operatorname{curl}\psi) C = 0$, for any constant vector C.

[17] Actually $\left|\int_0^1 D^3\Phi_k(tx + (1-t)z_i, y) dt \circ (x - z_i)\ (x - z_i)\right| \leq \frac{C}{\boldsymbol{\delta}_{ij}}\big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^3 \boldsymbol{a}^2$.

It follows, with $(\nabla_x \nabla_y \Phi_k(z_i, z_j))^T$ standing for the transpose,

$$\int_{\partial D_i} \psi_l \cdot \nu \times \nabla[K^k_{i,j}]u_j \; ds = -\int_{\partial D_i} \nu \cdot \operatorname{curl} \psi(x)(x - z_i) \cdot \Big((\nabla_x \nabla_y \Phi_k(z_i, z_j))^T \int_{\partial D_j} \nu_j u_j \; ds \Big) \; ds(x) + O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^7\Big),$$

hence, being $(\nabla_x \nabla_y \Phi_k(z_i, z_j))^T = \nabla_x \nabla_y \Phi_k(z_i, z_j)$, we get in view of (2.3.25) that

$$\begin{aligned} &\int_{\partial D_i} \psi_l \cdot \nu \times \nabla[K^k_{i,j}]u_j \; ds \\ &\quad = O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^7\Big) \\ &\quad\quad - \int_{\partial D_i} [I/2 + (K^0_{ii,D})^*]^{-1}(-\nu^l_i)(x - z_i) \; ds \cdot (\nabla_x \nabla_y \Phi_k(z_i, z_j) \mathcal{Q}_j) \, . \end{aligned} \tag{2.3.41}$$

Now, consider the second term of (2.3.38), [18]

$$\begin{aligned} &-k^2 \int_{\partial D_i} \psi_l \cdot (\nu \times [S^k_{ij,D}](\nu_y u_j(y))) \; ds \\ &= k^2 \int_{\partial D_i} (\nu_{x_i} \times \psi_l(x)) \cdot \int_{\partial D_j} \Phi_k(x_i, y)(\nu_y u_j(y)) \; ds(y) \; ds(x), \end{aligned} \tag{2.3.42}$$

we have

$$\begin{aligned} \int_{\partial D_j} \Phi_k(x_i, y)(\nu_y u_j(y)) \; ds(y) = &\int_{\partial D_j} \Phi_k(z_i, y) \; (\nu_y u_j(y)) \; ds(y) \\ &+ \int_{\partial D_j} \int_0^1 (\nabla_x \Phi_k(tx + (1-t)z_i, y)dt) \cdot (x - z_i)) \, (\nu_y u_j(y)) \; ds(y), \end{aligned}$$

and repeating the same approximation in y, with the following estimate

$$\begin{aligned} \Big| \int_{\partial D_j} \int_0^1 (\nabla_x \Phi_k(tx + (1-t)z_i, y)dt) \cdot (x - z_i)) \, (\nu_y u_j(y)) \; ds(y) \Big| &\leq C \frac{\boldsymbol{a}^2}{\boldsymbol{\delta}_{ij}} (\frac{1}{\boldsymbol{\delta}_{ij} + |k|}) \|u_j\|^2_L(\partial D_i), \\ &\leq C \frac{1}{\boldsymbol{\delta}_{ij}} (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|) \boldsymbol{a}^4, \end{aligned}$$

we get

$$\int_{\partial D_j} \Phi_k(x_i, y)(\nu_y u_j(y)) \; ds(y) = \Phi_k(z_i, z_j) \int_{\partial D_j} \nu_j u_j \; ds + 2 \; O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\boldsymbol{a}^4\Big). \tag{2.3.43}$$

Replacing in (2.3.42), gives

$$\begin{aligned} -k^2 \int_{\partial D_i} \psi_l \cdot (\nu \times [S^k_{ij,D}](\nu_y u_j)) \; ds = &k^2 \int_{\partial D_i} (\nu_{x_i} \times \psi_l(x)) \cdot \Big(\Phi_k(z_i, z_j) \mathcal{Q}_j \\ &+ O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\boldsymbol{a}^4\Big) \Big) \; ds(x) \end{aligned}$$

[18] With the following product rule $u \cdot (v \times w) = -w \cdot (v \times u)$.

Considering the estimate (2.3.21), we have

$$-k^2 \int_{\partial D_i} \psi_l \cdot (\nu \times [S^k_{ij,D}](\nu_y u_j)) ds = k^2 \int_{\partial D_i} (\nu_{x_i} \times \psi_l(x)) \cdot \Phi_k(z_i, z_j) \mathcal{Q}_j ds(x) + O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\boldsymbol{a}^7\Big),$$

and then, with (2.2.2) for the second inequality, we derive

$$\begin{aligned}
-k^2 \int_{\partial D_i} & \psi_l \cdot (\nu \times [S^k_{ij,D}](\nu_y u_j)) \ ds \\
&= k^2 \int_{\partial D_i} (\nu_{x_i} \times \psi_l(x)) \cdot \Phi_k(z_i, z_j) \nabla \left((x - z_i) \cdot \mathcal{Q}_j\right) \ ds(x) + O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\boldsymbol{a}^7\Big), \\
&= -k^2 \int_{\partial D_i} \mathrm{Div}(\nu_{x_i} \times \psi_l(x)) \Phi_k(z_i, z_j) \left((x - z_i) \cdot \mathcal{Q}_j\right) \ ds(x) + O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\boldsymbol{a}^7\Big).
\end{aligned}$$

As consequence, being $\mathrm{Div}(\nu \times \psi) = -\nu \cdot \mathrm{curl}\, \psi$, we obtain

$$\begin{aligned}
-k^2 \int_{\partial D_i} & \psi_l \cdot (\nu \times [S^k_{ij,D}](\nu_y u_j)) \ ds \\
&= k^2 \int_{\partial D_i} [I/2 + (K^0_{ii,D})^*]^{-1}(-\nu^l_i)(x) \ \big((x - z_i) \cdot \Phi_k(z_i, z_j) \mathcal{Q}_j\big) \ ds(x) \\
&\quad + O\Big(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\boldsymbol{a}^7\Big).
\end{aligned} \tag{2.3.44}$$

It remain to put together (2.3.41), (2.3.44) and to sum over j to get the conclusion. Concerning (2.3.32), doing as in (2.3.42)

$$\begin{aligned}
\int_{\partial D_i} & \psi_l(x) \cdot \nu_{x_i} \times E^{\mathrm{in}}(x) \ ds(x) \\
&= -\int_{\partial D_i} \nu_{x_i} \times \psi_l \cdot E^{\mathrm{in}}(z_i) \ ds(x) - \int_{\partial D_i} \nu_{x_i} \times \psi_l \cdot (E^{\mathrm{in}}(x_i) - E^{\mathrm{in}}(z_i)) \ ds(x), \\
&= -\int_{\partial D_i} \nu_{x_i} \times \psi_l \cdot E^{\mathrm{in}}(z_i) \ ds(x) + O(\|\nu_{x_i} \times \psi_l\|_{L^2(\partial D_i)} \|(E^{\mathrm{in}}(x_i) - E^{\mathrm{in}}(z_i))\|_{L^2(\partial D_i)}).
\end{aligned}$$

With the Mean value Theorem, we get

$$\|(E^{\mathrm{in}}(x_i) - E^{\mathrm{in}}(z_i))\|_{L^2(\partial D_i)} = \|\int_0^1 \nabla E^{\mathrm{in}}(t x_i + (1-t) z_i) dt \cdot (x - z_i)\|_{L^2(\partial D_i)} = O(k \boldsymbol{a}^2),$$

thus, considering (2.3.21), and (2.2.2) for the last identity, we end up with

$$\begin{aligned}
\int_{\partial D_i} \psi_l(x) \nu_{x_i} \times E^{\mathrm{in}}(x) \ ds(x) &= -\int_{\partial D_i} \nu_{x_i} \times \psi_l(x) \cdot \ \nabla \left((x - z_i) \cdot E^{\mathrm{in}}(z_i)\right) \ ds(x) + O(k \boldsymbol{a}^4), \\
&= \int_{\partial D_i} -\nu_{x_i} \cdot \mathrm{curl}\, \psi_l(x) \ \left((x - z_i) \cdot E^{\mathrm{in}}(z_i)\right) \ ds(x) + O(|k| \boldsymbol{a}^4).
\end{aligned}$$

Using (2.3.25) gives the conclusion. □

Finally, the approximation for the $\mathcal{Q}_i$'s, with $\left[\mathcal{T}_{\partial D_i}\right]$ as defined in (2.2.8),

$$\begin{aligned}\mathcal{Q}_i = \left[\mathcal{T}_{\partial D_i}\right] \sum_{(j\neq i)\geq 1}^{m} \left(-\nabla_x \Phi_k(z_i,z_j)\times \mathcal{R}_j + \Pi_k(z_i,z_j)\mathcal{Q}_j\right) - \left[\mathcal{T}_{\partial D_i}\right] E^{\mathrm{in}}(z_i) \\ + O\Big(\sum_{(j\neq i)\geq 1}^{m} \frac{1}{\boldsymbol{\delta}_{ij}} \left((\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 + (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^2 + (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\right) \boldsymbol{a}^7\Big) \\ + O((1+|k|+|k|^2\boldsymbol{a})\ \boldsymbol{a}^4),\end{aligned}$$

holds. It suffices, for $l = 1,2,3$, to replace the approximations of Lemma 14 and Lemma 15 in (2.3.20) to conclude. Developing the approximation error of the above equation, as pointed in (2.3.2), gives (2.3.6).

2.3.2.2 Justification of (2.3.5)

Let ϕ be any smooth enough scalar function. Multiply each side of (2.2.17) by $\nabla\phi$ and integrate over ∂D_i to get, using the relation (2.2.1),

$$\begin{aligned}\int_{\partial D_i} \phi\, \mathrm{Div}\ [I/2 + M^k_{ii,D}]A\ ds + \sum_{(j\neq i)\geq 1}^{m} \int_{\partial D_i} \phi\ \mathrm{Div}[M^k_{ij,D}](a)\ ds \\ = -\int_{\partial D_i} \phi\ \mathrm{Div}(\nu_{x_i}\times E^{\mathrm{in}})\ ds.\end{aligned} \tag{2.3.45}$$

As $curl^2 = \mathrm{curl}\,\mathrm{curl} = -\Delta + \nabla\,\mathrm{div}$, we have

$$\begin{aligned}\int_{\partial D_i} \phi\Big([I/2 - (K^k_{\partial D_i})^*]\,\mathrm{Div}\,A - k^2\nu_{x_i}\cdot[S^k_{ii,D}]A\Big)\ ds \\ - \sum_{(j\neq i)\geq 1}^{m} \int_{\partial D_i} \phi\Big([(K^k_{i,j})^*]\,\mathrm{Div}\,A + k^2\nu_{x_i}\cdot[S^k_{ij,D}]A\Big)\ ds = -\int_{\partial D_i} \phi\ \nu_i\cdot\mathrm{curl}\,E^{\mathrm{in}}\ ds.\end{aligned} \tag{2.3.46}$$

Let now ϕ be the solution to the following integral equation

$$[-I/2 + K^0_{ii,D}](\phi)(x) = (x - z_i), \tag{2.3.47}$$

then, as result of (2.2.57), ϕ satisfies the following estimate

$$\|\phi\|_{L^2(\partial D_i)} \leq C_{K^0_{ii,B}}\,\boldsymbol{a}^2. \tag{2.3.48}$$

The tensor $\left[\mathcal{P}_{\partial D_i}\right]$ is defined in (2.2.7). The justification of (2.3.5) is a direct consequence of the following expansions.

Lemma 16. *With the previous notation we have the following three approximations*

$$\int_{\partial D_i} \phi\Big([I/2 - (K^k_{\partial D_i})^*]\,\mathrm{Div}\,A - k^2\nu_{x_i}\cdot[S^k_{ii,D}]A\Big)\ ds = \mathcal{R}_i + O((\boldsymbol{a}+1)|k|^2\boldsymbol{a}^4). \tag{2.3.49}$$

$$\begin{aligned}\int_{\partial D_i}\phi\Big([(K^k_{i,j})^*]\,\mathrm{Div}\,A+k^2\nu_{x_i}\cdot[S^k_{ij,D}]A\Big)\,ds\\ =&\big[\mathcal{P}_{\partial D_i}\big]\Big(\Pi_k(z_i,z_j)\mathcal{R}_j-k^2\nabla_x\Phi_k(z_i,z_j)\times\mathcal{Q}_j\Big)\\ &+O\bigg(\Big((\frac{1}{\boldsymbol{\delta}_{ij}}+|k|)^3+(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|)+|k|^2(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|)^2\Big)\frac{\boldsymbol{a}^7}{\boldsymbol{\delta}_{ij}}\bigg),\end{aligned} \tag{2.3.50}$$

$$\int_{\partial D_i}\phi\;\nu_i\cdot\mathrm{curl}\,E^{in}\;ds=\big[\mathcal{P}_{\partial D_i}\big]_{\partial D_i}\mathrm{curl}\,E^{in}(z_i)+O(|k|^2\boldsymbol{a}^4). \tag{2.3.51}$$

Proof. We have for (2.3.49)

$$\begin{aligned}\int_{\partial D_i}\phi\bigg([I/2-(K^k_{ii,D})^*]\,\mathrm{Div}\,A+&k^2\nu_{x_i}\cdot[S^k_{ii,D}]A\bigg)\,ds\\ =\int_{\partial D_i}\phi\bigg(&[I/2-(K^0_{ii,D})^*]\,\mathrm{Div}\,A+[(K^0_{ii,D})^*-(K^k_{ii,D})^*]\,\mathrm{Div}\,A\\ &+k^2\nu_{x_i}\cdot[S^k_{ii,D}]A\bigg)\,ds,\end{aligned} \tag{2.3.52}$$

Using (2.2.64) and (2.2.63), the right-hand side gives

$$\begin{aligned}&\int_{\partial D_i}\phi[I/2-(K^0_{ii,D})^*]\,\mathrm{Div}\,A\;ds+\;O(|k|^2\boldsymbol{a}^2\|\mathrm{Div}\,A\|_{L^2_0(\partial D_i)}\|\phi\|_{L^2(\partial D_i)})\\ &+O(|k|^2\boldsymbol{a}\|A\|_{L^2(\partial D_i)}\|\phi\|_{L^2(\partial D_i)}).\end{aligned}$$

With (2.3.48) and (2.2.44), it becomes

$$\int_{\partial D_i}\phi[I/2-(K^0_{ii,D})^*]\,\mathrm{Div}\,A\;ds+O((\boldsymbol{a}+1)|k|^2\boldsymbol{a}^4). \tag{2.3.53}$$

Hence as $\int_{\partial D_i}\phi[I/2-(K^0_{ii,D})^*]\,\mathrm{Div}\,A\;ds=\int_{\partial D_i}[I/2-K^0_{ii,D}]\phi\,\mathrm{Div}\,A\;ds$, with the definition (2.3.47) the right-hand side of (2.3.52) ends up to be

$$-\int_{\partial D_i}(x-z_i)\,\mathrm{Div}\,A(x)\;ds(x)+O((\boldsymbol{a}+1)|k|^2\boldsymbol{a}^4)=\mathcal{R}_i+O((\boldsymbol{a}+1)|k|^2\boldsymbol{a}^4). \tag{2.3.54}$$

Concerning (2.3.50), we obtain, after developing the first term of the first member in Taylor series,

$$\begin{aligned}&\int_{\partial D_i}\phi[(K^k_{i,j})^*](\mathrm{Div}\,A)ds\\ &\quad=\int_{\partial D_i}\phi(x)\nu\cdot\int_{\partial D_j}(\nabla_y\Phi_k(x,z_j)+\nabla_y\nabla_x\Phi_k(x,z_j)\cdot(y-z_j))\,\mathrm{Div}\,A(y)\;ds(y)\;ds(x)\\ &\quad+\int_{\partial D_i}\phi(x)\nu\cdot\int_{\partial D_j}\bigg(\int_0^1D^3_{y,y,x}\Phi_k(x,ty+(1-t)z_j)\circ(y-z_j)\cdot(y-z_j)\bigg)\,\mathrm{Div}\,A(y)ds(y)ds(x),\end{aligned}$$

which gives as we did it in (2.3.40)[19]

$$\int_{\partial D_i} \phi\ [(K^k_{i,j})^*](\mathrm{Div}\, A) = \int_{\partial D_i} \phi(x)\ \nu \cdot \int_{\partial D_j} (\nabla_y \nabla_x \Phi_k(x, z_j) \cdot (y - z_j))\, \mathrm{Div}\, A(y)\ ds(y)\ ds(x) + O(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^2 \boldsymbol{a} \|\mathrm{Div}\, A\| \boldsymbol{a} \|\phi\|).$$

Repeating the same computations for $x \in \partial D_i$, we get, in consideration of (2.3.48)

$$\int_{\partial D_i} \phi\ [(K^k_{i,j})^*](\mathrm{Div}\, A) = \int_{\partial D_i} \phi(x)\ \nu \cdot \int_{\partial D_j} (\nabla_y \nabla_x \Phi_k(z_i, z_j) \cdot (y - z_j))\, \mathrm{Div}\, A(y)\ ds(y)\ ds(x) + 2\ O(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^7).$$

With the notation (2.2.7), we get [20]

$$\int_{\partial D_i} \phi\ [(K^k_{i,j})^*](\mathrm{Div}\, A)\ ds = -\big[\mathcal{P}_{\partial D_i}\big]\ (\nabla_y \nabla_x \Phi_k(z_i, z_j) \mathcal{R}_j) + O(\frac{1}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^3 \boldsymbol{a}^7). \tag{2.3.55}$$

Concerning the second term of the first member of (2.3.50), we have in view of (2.2.45) theorem 8

$$\int_{\partial D_i} \phi\ k^2 \nu_i \cdot [S^k_{ij,D}](A_j)\ ds = \int_{\partial D_i} \phi\ k^2 \nu_i \cdot [S^k_{ij,D}]\Big(A^{[1]}_j + A^{[2]}_j\Big)\ ds, \tag{2.3.56}$$

and then

$$\int_{\partial D_i} \phi\ k^2 \nu_i \cdot [S^k_{ij,D}](A^{[1]}_j)\ ds = \int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \int_{\partial D_j} (\Phi_k(x, y) - \Phi(z_i, z_j))\, A^{[1]}_j(y)\ ds(y)\ ds(x) + \int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \int_{\partial D_j} \Phi_k(z_i, z_j) A^{[1]}_j(y)\ ds(y)\ ds(x),$$

Using Mean-value-theorem, we get for the right-hand side of the above equation

$$\int_{\partial D_i} \phi\ k^2 \nu_i \cdot \Big(\int_{\partial D_j} O\big((\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\frac{1}{\boldsymbol{\delta}_{ij}} \boldsymbol{a}\big) A^{[1]}_j(y)\ ds(y)\Big)\ ds + k^2 \Phi_k(z_i, z_j) \int_{\partial D_i} \phi\ \nu_i \cdot \Big(\int_{\partial D_j} A^{[1]}_j(y)\ ds(y)\Big)\ ds,$$

then considering the estimates (2.2.46) and (2.3.48), with Holder's inequality give

$$\int_{\partial D_i} \phi\ k^2 \nu_i \cdot [S^k_{ij,D}](A^{[1]}_j)\ ds = \big[\mathcal{P}_{\partial D_i}\big] k^2 \Phi_k(z_i, z_j) \mathcal{R}_j + O\big((\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)\frac{1}{\boldsymbol{\delta}_{ij}} \boldsymbol{a}^7\big). \tag{2.3.57}$$

The second term of the second member of (2.3.56) gives, again considering (2.2.45),

$$\int_{\partial D_i} \phi\ k^2 \nu_i \cdot [S^k_{ij,D}] A^{[2]}_j\ ds = \int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \int_{\partial D_j} \Phi_k(x, y) \nu \times \nabla u_j\ ds(y)\ ds(x),$$

[19] Note that $\int_{\partial D_j} \nabla_x \Phi_k(x, z_j)\, \mathrm{Div}\, A = 0$.

[20] Recall that $\int_{\partial D_j} (y - z_j)\, \mathrm{Div}\, A(y)\ ds(y) = -\int_{\partial D_j} A(y)\ ds(y) = -\mathcal{R}_j$.

which, by integrating by parts, gives

$$\int_{\partial D_i} \phi\ k^2 \nu \cdot [S^k_{ij,D}] A^{[2]}_j\ ds = \int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \Big(- \int_{\partial D_j} u_j(y) \nu_{y_i} \times \nabla_y \Phi_k(x,y)\ ds(y) \Big)\ ds(x).$$

Now, doing a first order approximation, we have

$$\begin{aligned}\int_{\partial D_i} & \phi\ k^2 \nu \cdot [S^k_{ij,D}] A^{[2]}_j\ ds \\ &= \int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \Big(- \int_{\partial D_j} u_j \nu_j\ ds \Big) \times \nabla_y \Phi_k(x, z_j)\ ds(x) \\ &\quad + \int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \Big(- \int_{\partial D_j} u_j(y) \nu_{y_i} \times \Big(\nabla_y \Phi_k(x,y) - \nabla_y \Phi_k(x, z_j) \Big)\ ds(y) \Big)\ ds(x),\end{aligned}$$

and similarly to (2.3.35) the right-hand side is equal to

$$\begin{aligned}&\int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \Big(- \int_{\partial D_j} u_j \nu_j\ ds \Big) \times \nabla_y \Phi_k(x, z_j)\ ds(x) \\ &+ \int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \Bigg(- \int_{\partial D_j} u_j \nu_j \times O\Big(\frac{1}{\boldsymbol{\delta}_{ij}} (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^2 \boldsymbol{a} \Big)\ ds \Bigg)\ ds(x),\end{aligned}$$

which, in view of the estimates (2.2.49) and (2.3.48), becomes

$$\int_{\partial D_i} \phi(x)\ k^2 \nu_{x_i} \cdot \Big(- \int_{\partial D_j} u_j \nu_j\ ds \Big) \times \nabla_y \Phi_k(x, z_j)\ ds(x) + O\Big(\frac{|k|^2}{\boldsymbol{\delta}_{ij}} (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^2 \boldsymbol{a}^7 \Big). \tag{2.3.58}$$

Repeating the same calculation, for $x \in \partial D_i$, gives

$$\begin{aligned}\int_{\partial D_i} \phi\ k^2 \nu \cdot [S^k_{ij,D}] A^{[2]}_j\ ds = &\int_{\partial D_i} \phi\ k^2 \nu_i \cdot \Big(- \int_{\partial D_j} u_j \nu_j\ ds \Big) \times \nabla_y \Phi_k(z_i, z_j)\ ds \\ &+ O\Big(\frac{|k|^2}{\boldsymbol{\delta}_{ij}} (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^2 \boldsymbol{a}^7 \Big).\end{aligned}$$

Hence, being $V \times U = -U \times V$ for any vectors U, V, and $\nabla_y \Phi_k(x,y) = -\nabla_x \Phi_k(x,y)$ we get with the notations (2.2.7)

$$\int_{\partial D_i} \phi\ k^2 \nu \cdot [S^k_{ij,D}] A^{[2]}_j\ ds = -\big[\mathcal{P}_{\partial D_i}\big] k^2 \nabla_x \Phi_k(z_i, z_j) \times \mathcal{Q}_j + O\left(|k|^2 \frac{1}{\boldsymbol{\delta}_{ij}} (\frac{1}{\boldsymbol{\delta}_{ij}} + |k|)^2 \boldsymbol{a}^7 \right). \tag{2.3.59}$$

The last approximation of Lemma 16 being obvious, we end the proof of (2.3.5) by taking the sum over j of the two first approximations and replacing in (2.3.46). $\square$

2.4 Invertibility of the linear system

For μ^+ and μ^- defined as in (2.2.10) and $\mathcal{E} = (\mathcal{E})_{i=1}^{2m}$ defined as

$$\mathcal{E}_i = \begin{cases} E^{\text{in}}(z_i), & i \in \{1, ..., m\}, \\ \operatorname{curl} E^{\text{in}}(z_{i-m}), & i \in \{m+1, ..., 2m\}, \end{cases} \tag{2.4.1}$$

we have the following proposition.

Proposition 17. *Under the condition*

$$C_{Li} := 1 - C_{Ls} \frac{\mu^+ \boldsymbol{a}^3}{\boldsymbol{\delta}^3} > 0, \tag{2.4.2}$$

for some constant C_{Ls}, [21] *the following linear system is invertible*

$$\begin{aligned} \widehat{\mathcal{Q}}_i - \left[\mathcal{T}_{\partial D_i}\right] \sum_{(j\neq i)\geq 1}^{m} \Big(- \nabla_x \Phi_k(z_i, z_j) \times \widehat{\mathcal{R}}_j \ + \Pi_k(z_i, z_j) \widehat{\mathcal{Q}}_j \Big) &= -\left[\mathcal{T}_{\partial D_i}\right] E^{in}(z_i), \\ \widehat{\mathcal{R}}_i + \left[\mathcal{P}_{\partial D_i}\right] \sum_{(j\neq i)\geq 1}^{m} \Big(\Pi_k(z_i, z_j) \widehat{\mathcal{R}}_j - k^2 \nabla \Phi_k(z_i, z_j) \times \widehat{\mathcal{Q}}_j \Big) &= -\left[\mathcal{P}_{\partial D_i}\right] \operatorname{curl} E^{in}(z_i), \end{aligned} \tag{2.4.3}$$

and the solution satisfies the following estimate

$$\left(\sum_{i=1}^{m} (|\widehat{\mathcal{R}}_i|^2 + |\widehat{\mathcal{Q}}_i|^2) \right)^{\frac{1}{2}} := \Big(\big\langle \widehat{\mathcal{Q}}, \widehat{\mathcal{Q}} \big\rangle_{\mathbb{C}^{3\times m}} + \big\langle \widehat{\mathcal{R}}, \widehat{\mathcal{R}} \big\rangle_{\mathbb{C}^{3\times m}} \Big)^{\frac{1}{2}} \leq \frac{1}{C_{Li} \mu^-} \boldsymbol{a}^3 \big\langle \widehat{\mathcal{E}}, \widehat{\mathcal{E}} \big\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}}. \tag{2.4.4}$$

Further, if the condition (2.2.43) *is satisfied, then the system could be inverted using Neumann series with the following estimate*

$$|\mathcal{R}_i| \leq \frac{1}{C_{L_i^2} \mu^-} \boldsymbol{a}^3 |\mathcal{E}_i|, \ \ |\mathcal{Q}_i| \leq \frac{1}{C_{L_i^2} \mu^-} \boldsymbol{a}^3 |\mathcal{E}_{i+m}|. \tag{2.4.5}$$

To prove this result, we need to introduce some notations. Let $(\widehat{\mathcal{C}}_i)_{i\in\{1,\dots,2m\}}$ be defined as

$$\widehat{\mathcal{C}}_i = \begin{cases} \left[\mathcal{T}_{\partial D_i}\right]^{-1} \widehat{\mathcal{Q}}_i, \ i \in \{1, \dots, m\}, \\ - \left[\mathcal{P}_{\partial D_{i-m}}\right]^{-1} \widehat{\mathcal{R}}_{i-m}, \ i \in \{m+1, \dots, 2m\}, \end{cases}$$

and let $\mathcal{Q}$ be the following diagonal Bloc matrix

$$Q_i := \begin{cases} \left[\mathcal{T}_{\partial D_i}\right], & \text{for } i \in \{1, \dots, m\}, \\ - \left[\mathcal{P}_{\partial D_{i-m}}\right] & \text{for } i \in \{m+1, \dots, 2m\}, \end{cases} \tag{2.4.6}$$

with $\mathcal{Q}_{1,1} = Diag(Q_i)_{i=1}^m$, and $\mathcal{Q}_{2,2} = Diag(Q_{i+m})_{i=1}^m$.

Consider

$$\widehat{\theta}_{i,j} \widehat{\mathcal{C}}_j := \nabla_x \Phi_k(z_i, z_j) \times \widehat{\mathcal{C}}_j = \begin{bmatrix} 0 & -\partial_3 \Phi_k(z_i, z_j) & \partial_2 \Phi_k(z_i, z_j) \\ \partial_3 \Phi_k(z_i, z_j) & 0 & -\partial_1 \Phi_k(z_i, z_j) \\ -\partial_2 \Phi_k(z_i, z_j) & \partial_1 \Phi_k(z_i, z_j) & 0 \end{bmatrix} \widehat{\mathcal{C}}_j, \tag{2.4.7}$$

and define the following bloc matrix

$$\Sigma^k := \begin{bmatrix} \Sigma^k_{1,1} & 0_{\mathbb{C}^m \times \mathbb{C}^m} \\ 0_{\mathbb{C}^m \times \mathbb{C}^m} & \Sigma^k_{2,2} \end{bmatrix} = (\sigma^k_{i,j})_{i,j=1}^{2m}, \quad \Theta^k := \begin{bmatrix} 0_{\mathbb{C}^m \times \mathbb{C}^m} & \Theta^k_{1,2} \\ \Theta^k_{2,1} & 0_{\mathbb{C}^m \times \mathbb{C}^m} \end{bmatrix} = (\theta^k_{i,j})_{i,j=1}^{2m},$$

[21] The constant C_{Ls} is provided in (2.4.36).

where

$$\sigma^k{}_{i,j} := \begin{cases} -\Pi_k(z_i, z_j), & i \neq j, i, j \in \{1, ..., m\}, \\ -\Pi_k(z_{i-m}, z_{j-m}), & i \neq j,\ i, j \in \{m+1, ..., 2m\}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.4.8}$$

and

$$\theta^k_{i,j} = \begin{cases} \widehat{\theta}_{i,j-m}, & i \in \{1, ..., m\},\ j \in \{1+m, ..., 2m\}, \\ k^2 \widehat{\theta}_{i-m,j} & j \in \{1, ..., m\},\ i \in \{1+m, ..., 2m\}, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.4.9}$$

With these notations, solving the system (2.4.3) is equivalent to solve the equation

$$\widehat{\mathcal{C}} + \Sigma^k \mathcal{Q}\widehat{\mathcal{C}} + \Theta^k \mathcal{Q}\widehat{\mathcal{C}} = \mathcal{E}. \tag{2.4.10}$$

If we multiply both sides of the last system by $\mathcal{Q}\widehat{\mathcal{C}}$ we get

$$\big\langle \widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}} + \big\langle \Sigma^k \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}} + \big\langle \Theta^k \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}} = \big\langle \mathcal{E}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}} \tag{2.4.11}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^{3\times 2m}}$ stands for the usual scalar product in $\mathbb{C}^{3\times 2m}$.

Adding and subtracting $\big\langle \Sigma^0 \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}}$ gives

$$\begin{aligned} \big\langle \widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}} + \Big\langle \Big(\Sigma^k - \Sigma^0\Big) \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \Big\rangle_{\mathbb{C}^{3\times 2m}} &+ \big\langle \Sigma^0 \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}} \\ &+ \big\langle \Theta^k \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}} = \big\langle \mathcal{E}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}}, \end{aligned} \tag{2.4.12}$$

Let $\chi_{\partial\widehat{\Omega}}$ denotes the characteristic function on $\partial\widehat{\Omega} = \cup_{i=1}^m \partial B_{\boldsymbol{\delta}/4}(z_i)$ where $B_r(z) := B(z, r)$ denotes a ball of center z and radius r, and $\langle \cdot, \cdot \rangle_{L^2(\partial\widehat{\Omega})}$ denotes the usual scalar product of $L^2(\partial\Omega)$.

Lemma 18. *For* $\mathcal{U} := \sum_{i=1}^m \nu_{xi} \cdot \overline{Q_i \widehat{\mathcal{C}}_i} \chi_{\partial B^{z_i}_{\boldsymbol{\delta}/4}}$, *and* $\mathcal{V} := \sum_{i=m+1}^{2m} \nu_{x_{i-m}} \cdot \overline{Q_i \widehat{\mathcal{C}}_i} \chi_{\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4}}$ *with* ν_{xi} *being the outward unit normal vector to* $\partial B^{z_i}_{\boldsymbol{\delta}/4}$, *we have*

$$\Sigma^0_{1,1} \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1 \cdot \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1 = \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \Big(\langle S^0_{\partial\widehat{\Omega}}\, \mathcal{U}, \mathcal{U} \rangle_{L^2(\partial\widehat{\Omega})} - \sum_{i=1}^m \langle S^0_{\partial B^{z_i}_{\boldsymbol{\delta}/4}}\, \mathcal{U}, \mathcal{U} \rangle_{L^2(\partial B^{z_i}_{\boldsymbol{\delta}/4})} \Big), \tag{2.4.13}$$

$$\Sigma^0_{2,2} \mathcal{Q}_{2,2} \widehat{\mathcal{C}}_2 \cdot \mathcal{Q}_{2,2} \widehat{\mathcal{C}}_2 = \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \Big(\langle S^0_{\partial\widehat{\Omega}} \mathcal{V}, \mathcal{V} \rangle_{L^2(\partial\widehat{\Omega})} - \sum_{i=m+1}^{2m} \langle S^0_{\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4}}\, \mathcal{V}, \mathcal{V} \rangle_{L^2(\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4})} \Big), \tag{2.4.14}$$

$$|\Big\langle \Big(\Sigma^k - \Sigma^0\Big) \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \Big\rangle_{\mathbb{C}^{3\times 2m}}| \leq \frac{63^{\frac{1}{2}} |k|^2 m^{\frac{2}{3}}}{4\pi \boldsymbol{\delta}} \sum_{i=1}^{2m} |Q_i \widehat{\mathcal{C}}_i|^2 = \frac{63^{\frac{1}{2}} |k|^2 D(\Omega)^{\frac{2}{3}}}{4\pi \boldsymbol{\delta}^3} \big\langle \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}}, \tag{2.4.15}$$

and

$$|\big\langle \Theta^k \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}}| \leq \frac{1}{8\pi} \frac{(1+|k|^2) C_{k,D(\Omega)}}{\boldsymbol{\delta}^3} \big\langle \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \big\rangle_{\mathbb{C}^{3\times 2m}}. \tag{2.4.16}$$

where $C_{k,D(\Omega)} := \max(C_0 D(\Omega)^{\frac{1}{3}}, |k| C_0 \big(D(\Omega)^{\frac{2}{3}} + D(\Omega)^{\frac{1}{3}}\big))$.

Proof. To prove (2.4.13), using Mean-value-theorem for harmonic function, for $j \neq i$, we have

$$\nabla_x \nabla_y \Phi_0(z_i, z_j) = \frac{3 \times 4^3}{4\pi \boldsymbol{\delta}^3} \int_{B(z_j, \frac{\boldsymbol{\delta}}{4})} \nabla_x \nabla_y \Phi_0(z_i, y) dy = \frac{48}{\pi \boldsymbol{\delta}^3} \nabla_x \int_{B(z_j, \frac{\boldsymbol{\delta}}{4})} \nabla_y \Phi_0(z_i, y)\ dy \tag{2.4.17}$$

then using Gauss divergence theorem,

$$\nabla_x \nabla_y \Phi_0(z_i, z_j) = \frac{48}{\pi \boldsymbol{\delta}^3} \nabla_x \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(z_i, y) \nu_y\ ds(y). \tag{2.4.18}$$

Repeating the same for z_i we get

$$\nabla_x \nabla_y \Phi_0(z_i, z_j) = \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(x, y) \nu_x \nu_y^T\ ds(y)\ ds(x), \tag{2.4.19}$$

and from $\left\langle \Sigma^0_{1,1} \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1, \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1 \right\rangle_{\mathbb{C}^{3\times m}} = \sum_{i=1}^m \sum_{1\leq j \neq i}^m (\nabla_x \nabla_y \Phi_0(z_i, z_j) Q_j \widehat{\mathcal{C}}_j) \cdot \overline{Q_i \widehat{\mathcal{C}}_i}$, using (2.4.19), we get

$$\begin{aligned} &\left\langle \Sigma^0_{1,1} \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1, \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1 \right\rangle \\ &\quad = \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \sum_{i=1}^m \Big(\sum_{1\leq j\neq i}^m \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(x, y) \nu_x \nu_y^T\ ds(y)\ ds(x)\ Q_j \widehat{\mathcal{C}}_j \Big) \cdot \overline{Q_i \widehat{\mathcal{C}}_i}, \\ &\quad = \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \sum_{i=1}^m \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Big(\sum_{1\leq j\neq i}^m \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(x, y) \nu_y \cdot Q_j \widehat{\mathcal{C}}_j\ ds(y) \Big) \nu_x \cdot \overline{Q_i \widehat{\mathcal{C}}_i}\ ds(x). \end{aligned} \tag{2.4.20}$$

Adding and subtracting $\sum_{i=1}^m \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(x, y) \nu_y \cdot Q_j \widehat{\mathcal{C}}_i\ ds(y) \nu_x \cdot \overline{Q_i \widehat{\mathcal{C}}_i}\ ds(x)$, gives

$$\begin{aligned} \left\langle \Sigma^0_{1,1} \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1, \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1 \right\rangle = & \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \sum_{i=1}^m \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Big(\sum_{j=1}^m \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(x, y) \nu_y \cdot Q_j \widehat{\mathcal{C}}_j\ ds(y) \Big) \nu_x \cdot \overline{Q_i \widehat{\mathcal{C}}_i} ds(x), \\ & - \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \sum_{i=1}^m \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(x, y) \nu_y \cdot Q_j \widehat{\mathcal{C}}_i\ ds(y) \nu_x \cdot \overline{Q_i \widehat{\mathcal{C}}_i} ds(x), \end{aligned} \tag{2.4.21}$$

and then (2.4.20) becomes,

$$\begin{aligned} \left\langle \Sigma^0_{1,1} \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1, \mathcal{Q}_{1,1} \widehat{\mathcal{C}}_1 \right\rangle_{\mathbb{C}^{3\times m}} = & \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \int_{\partial \widehat{\Omega} := \cup_{i=1}^{2m} \partial B^{z_i}_{\boldsymbol{\delta}/4}} \int_{\cup_{j=1}^{2m} \partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(x, y) \mathcal{U}(y)\ ds(y)\ \overline{\mathcal{U}}(x)\ ds(x) \\ & - \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \sum_{i=1}^{2m} \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \int_{\partial B^{z_i}_{\boldsymbol{\delta}/4}} \Phi_0(x, y) \nu_y \cdot Q_j \widehat{\mathcal{C}}_i\ ds(y) \nu_x \cdot \overline{Q_i \widehat{\mathcal{C}}_i}\ ds(x). \end{aligned}$$

The same arguments remain valid for (2.4.14).
Concerning (2.4.15), we have

$$\left| (-k^2 \Phi_k(z_i, z_j) I + \nabla_x \nabla_y \Phi_k(z_i, z_j) - \nabla_x \nabla_y \Phi_0(z_i, z_j)) \right| \leq \frac{6|k|^2}{4\pi \boldsymbol{\delta}_{ij}}. \tag{2.4.22}$$

Indeed, recalling (2.2.59) and (2.2.60)

$$\nabla_y(\Phi_k(x,y)-\Phi_0(x,y)) = \frac{-(ik)^2}{4\pi}\int_0^1 \frac{le^{ikl|x-y|}}{|x-y|}(x-y)dl \tag{2.4.23}$$

then we have also

$$\begin{aligned}\nabla_x\nabla_y(\Phi_k(x,y)-\Phi_0(x,y)) = &\frac{-(ik)^3}{4\pi}\int_0^1 l^2 e^{ikl|x-y|}dl\frac{(x-y)(x-y)^T}{|x-y|^2}\\ &\frac{-(ik)^2}{4\pi}\int_0^1 le^{ikl|x-y|}dl\left(\frac{I}{|x-y|}+\frac{(x-y)(x-y)^T}{|x-y|^3}\right).\end{aligned}$$

Integrating by part the first term of the right-hand side, gives

$$\begin{aligned}\nabla_x\nabla_y(\Phi_k(x,y)-\Phi_0(x,y)) =&\Big(k^2\Phi_k(x,y)-\frac{k^2}{4\pi}\int_0^1 2l\frac{e^{ikl|x-y|}}{|x-y|}dl\Big)\frac{(x-y)(x-y)^T}{|x-y|^2}\\ &+\frac{-(ik)^2}{4\pi}\int_0^1 le^{ikl|x-y|}dl\Big(\frac{I}{|x-y|}+\frac{(x-y)(x-y)^T}{|x-y|^3}\Big).\end{aligned}$$

Hence, we get

$$\begin{aligned}|\Pi_k(z_i,z_j)-\Pi_0(z_i,z_j)| &= \left|k^2\Phi_k(z_i,z_j)I-\nabla_x\nabla_y\Phi_k(z_i,z_j)+\nabla_x\nabla_y\Phi_0(z_i,z_j)\right|,\\ &\leq \left|k^2\Phi_k(z_i,z_j)\right| + |\nabla_x\nabla_y\Phi_k(z_i,z_j)-\nabla_x\nabla_y\Phi_0(z_i,z_j)| \leq \frac{6|k|^2}{4\pi\boldsymbol{\delta}_{ij}}.\end{aligned}$$

Now, as

$$\left\langle\left(\Sigma^k_{1,1}-\Sigma^0_{1,1}\right)\mathcal{Q}_{1,1}\widehat{\mathcal{C}}_1,\mathcal{Q}_{1,1}\widehat{\mathcal{C}}_1\right\rangle_{\mathbb{C}^{3\times m}} = \sum_{i=1}^m\Big(\sum_{1\leq j\neq i}^m(\Pi_k(z_i,z_j)-\Pi_0(z_i,z_j))Q_j\widehat{\mathcal{C}}_j\Big)\cdot\overline{Q_i\widehat{\mathcal{C}}_i} \tag{2.4.24}$$

using Holder's inequality, for the inner sum, we get

$$\left|\left\langle\left(\Sigma^k_{1,1}-\Sigma^0_{1,1}\right)\mathcal{Q}_{1,1}\widehat{\mathcal{C}}_1,\mathcal{Q}_{1,1}\widehat{\mathcal{C}}_1\right\rangle\right| \leq \sum_{i=1}^m\Big(\sum_{1\leq j\neq i}^m|\Pi_k(z_i,z_j)-\Pi_0(z_i,z_j)|^2\Big)^{\frac{1}{2}}\Big(\sum_{1\leq j\neq i}^m\left|Q_j\widehat{\mathcal{C}}_j\right|^2\Big)^{\frac{1}{2}}\left|Q_i\widehat{\mathcal{C}}_i\right|,$$

which gives in view of (2.4.22)

$$\left|\left\langle\left(\Sigma^k_{1,1}-\Sigma^0_{1,1}\right)\mathcal{Q}_{1,1}\widehat{\mathcal{C}}_1,\mathcal{Q}_{1,1}\widehat{\mathcal{C}}_1\right\rangle_{\mathbb{C}^{3\times m}}\right| \leq \sum_{i=1}^m\Big(\sum_{1\leq j\neq i}^m(\frac{3|k|^2}{2\pi\boldsymbol{\delta}_{ij}})^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^m\left|Q_j\widehat{\mathcal{C}}_j\right|^2\Big)^{\frac{1}{2}}\left|Q_i\widehat{\mathcal{C}}_i\right|.$$

Repeating Holder's inequality for the outer sum, we obtain

$$\begin{aligned}\left|\left\langle\left(\Sigma^k_{1,1}-\Sigma^0_{1,1}\right)\mathcal{Q}_{1,1}\widehat{\mathcal{C}}_1,\mathcal{Q}_{1,1}\widehat{\mathcal{C}}_1\right\rangle_{\mathbb{C}^{3\times m}}\right| &\leq\Big(\sum_{i=1}^m\sum_{1\leq j\neq i}^m(\frac{3|k|^2}{2\pi\boldsymbol{\delta}_{ij}})^2\sum_{j=1}^m\left|Q_j\widehat{\mathcal{C}}_j\right|^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^m\left|Q_i\widehat{\mathcal{C}}_i\right|^2\Big)^{\frac{1}{2}},\\ &\leq\Big(\sum_{i=1}^m\sum_{1\leq j\neq i}^m(\frac{6|k|^2}{4\pi\boldsymbol{\delta}_{ij}})^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^m\left|Q_j\widehat{\mathcal{C}}_j\right|^2\Big).\end{aligned} \tag{2.4.25}$$

The inner sum gives, as we did it in (2.2.78),

$$\sum_{1\leq j\neq i}^{m}(\frac{6|k|^2}{4\pi\boldsymbol{\delta}_{ij}})^2 \leq \sum_{l=2}^{m^{\frac{1}{3}}} 7l^2 \ \frac{3^2|k|^4}{4^2\pi^2 l^2\boldsymbol{\delta}^2} = 7m^{\frac{1}{3}}\frac{3^2|k|^4}{4^2\pi^2\boldsymbol{\delta}^2},$$

and then

$$(\sum_{i=1}^{m}\sum_{1\leq j\neq i}^{m}(\frac{3|k|^2}{2\pi\boldsymbol{\delta}_{ij}})^2)^{\frac{1}{2}} \leq m^{\frac{1}{2}}\big(7m^{\frac{1}{3}}\frac{3^2|k|^4}{4^2\pi^2\boldsymbol{\delta}^2}\big)^{\frac{1}{2}} \leq \frac{63^{\frac{1}{2}}|k|^2 m^{\frac{2}{3}}}{4\pi\boldsymbol{\delta}}.$$

Repeating the same calculation for $\left|\left\langle\left(\Sigma^k_{2,2}-\Sigma^0_{2,2}\right)\mathcal{Q}_{2,2}\widehat{\mathcal{C}}_2, \mathcal{Q}_{2,2}\widehat{\mathcal{C}}_2\right\rangle_{\mathbb{C}^{3\times m}}\right|$ leads to the conclusion. For the last assertion (2.4.16), we proceed as follows:

$$\begin{aligned}\left\langle\Theta^k\mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}}\right\rangle_{\mathbb{C}^{3\times 2m}} =& \sum_{i=1}^{m}\big(\sum_{j\neq i}^{m}\nabla\Phi_k(z_i,z_j)\times Q_{j+m}\widehat{\mathcal{C}}_{j+m}\big)\cdot\overline{Q_i\widehat{\mathcal{C}}_i} \\ &+\sum_{i=1}^{m}\big(\sum_{j\neq i}^{m}k^2\nabla\Phi_k(z_i,z_j)Q_j\widehat{\mathcal{C}}_j\big)\cdot\overline{Q}_{i+m}\overline{\widehat{\mathcal{C}}}_{i+m},\end{aligned} \tag{2.4.26}$$

The first term of the right hand side of (2.4.26) is smaller then

$$\frac{1}{4\pi}\sum_{i=1}^{m}\sum_{j\neq i}^{m}\frac{|Q_{j+m}\widehat{\mathcal{C}}_{j+m}|}{\boldsymbol{\delta}_{ij}}(\frac{1}{\boldsymbol{\delta}_{ij}}+|k|)|Q_i\widehat{\mathcal{C}}_i|, \tag{2.4.27}$$

which is,

$$\frac{1}{4\pi}\sum_{i=1}^{m}\sum_{j\neq i}^{m}\frac{|Q_{j+m}\widehat{\mathcal{C}}_{j+m}|}{\boldsymbol{\delta}_{ij}}\frac{|Q_i\widehat{\mathcal{C}}_i|}{\boldsymbol{\delta}_{ij}}+\frac{1}{4\pi}\sum_{i=1}^{m}\sum_{j\neq i}^{m}\frac{|k|^{\frac{1}{2}}|Q_{j+m}\widehat{\mathcal{C}}_{j+m}|}{\boldsymbol{\delta}_{ij}^{\frac{1}{2}}}\frac{|k|^{\frac{1}{2}}|Q_i\widehat{\mathcal{C}}_i|}{\boldsymbol{\delta}_{ij}^{\frac{1}{2}}}, \tag{2.4.28}$$

and do not exceed [22]

$$\frac{1}{8\pi}\sum_{i=1}^{m}\sum_{j\neq i}^{m}\Big(\frac{|Q_{j+m}\widehat{\mathcal{C}}_{j+m}|^2}{\boldsymbol{\delta}_{ij}^2}+\frac{|Q_i\widehat{\mathcal{C}}_i|^2}{\boldsymbol{\delta}_{ij}^2}\Big)+\frac{1}{8\pi}\sum_{i=1}^{m}\sum_{j\neq i}^{m}\Big(\frac{|k||Q_{j+m}\widehat{\mathcal{C}}_{j+m}|^2}{\boldsymbol{\delta}_{ij}}+\frac{|k||Q_i\widehat{\mathcal{C}}_i|^2}{\boldsymbol{\delta}_{ij}}\Big), \tag{2.4.29}$$

which in its turn, is not greater than[23]

$$\begin{aligned}&\frac{1}{8\pi}\sum_{j=1}^{m}|Q_{j+m}\widehat{\mathcal{C}}_{j+m}|^2\sum_{i\neq j}^{m}\frac{1}{\boldsymbol{\delta}_{ij}^2}+\frac{1}{8\pi}\sum_{i=1}^{m}|Q_i\widehat{\mathcal{C}}_i|^2\sum_{j\neq i}^{m}\frac{1}{\boldsymbol{\delta}_{ij}^2} \\ &+\frac{|k|}{8\pi}\sum_{j=1}^{m}|Q_{j+m}\widehat{\mathcal{C}}_{j+m}|^2\sum_{i\neq j}^{m}\frac{1}{\boldsymbol{\delta}_{ij}}+\frac{|k|}{8\pi}\sum_{i=1}^{m}|Q_i\widehat{\mathcal{C}}_i|^2\sum_{j\neq i}^{m}\frac{1}{\boldsymbol{\delta}_{ij}}.\end{aligned}$$

[22] Comes from $2ab \leq a^2+b^2$ for every real numbers A, b.
[23] being $\sum_{i=1}^{m}\sum_{j\neq i}^{m}a_{i,j}=\sum_{j=1}^{m}\sum_{i\neq j}^{m}a_{i,j}$, for every real numbers $a_{i,j}$.

We have, as done in (2.2.78), for $m \leq D(\Omega)/\boldsymbol{\delta}^3$,

$$\sum_{j\neq i}^{m} \frac{1}{\boldsymbol{\delta}_{ij}^2} = \sum_{l=2}^{(m/2)^{\frac{1}{3}}} \frac{C_0 l^2}{l^2 \boldsymbol{\delta}^2} = \frac{C_0 (\frac{m}{2})^{\frac{1}{3}}}{\boldsymbol{\delta}^2} \leq \frac{C_0 D(\Omega)^{\frac{1}{3}}}{\boldsymbol{\delta}^3} = \frac{C^{(1)}_{D(\Omega)}}{\boldsymbol{\delta}^3}. \tag{2.4.30}$$

We get, in a similar way,

$$\sum_{j\neq i}^{m} \frac{1}{\boldsymbol{\delta}_{ij}} = \sum_{l=2}^{(m/2)^{\frac{1}{3}}} \frac{C_0 l^2}{l \boldsymbol{\delta}^2} = \frac{C_0 (\frac{m}{2})^{\frac{1}{3}}\left((\frac{m}{2})^{\frac{1}{3}}+1\right)}{2\boldsymbol{\delta}} \leq \frac{C_0 \left(D(\Omega)^{\frac{2}{3}} + D(\Omega)^{\frac{1}{3}}\right)}{2\boldsymbol{\delta}^3} = \frac{C^{(2)}_{D(\Omega)}}{\boldsymbol{\delta}^3}. \tag{2.4.31}$$

Then, for $C_{k,D(\Omega)} := \max(C^{(1)}_{D(\Omega)}, |k| C^{(2)}_{D(\Omega)})$, we obtain

$$\left| \sum_{i=1}^{m} \sum_{j\neq i}^{m} \nabla \Phi_k(z_i, z_j) \times Q_{i+m} \widehat{\mathcal{C}}_{i+m} \cdot \overline{Q_i \widehat{\mathcal{C}}_i} \right| \leq \frac{1}{8\pi} \frac{C_{k,D(\Omega)}}{\boldsymbol{\delta}^3} \sum_{i=1}^{2m} \left| Q_i \widehat{\mathcal{C}}_i \right|. \tag{2.4.32}$$

Repeating the same argument for the second term of the right-hand side of (2.4.26) gives the conclusion. □

Proof. (of Proposition 17) If we consider (2.4.12), in view of (2.4.13) and (2.4.14), we get

$$\begin{aligned}
&\langle \widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}} + \langle \Theta^k \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}} + \left\langle \left(\Sigma^k - \Sigma^0\right) \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \right\rangle_{\mathbb{C}^{3\times 2m}} \\
&\quad + \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \Big(\langle S^0_{\partial\widehat{\Omega}}\, \mathcal{U}, \mathcal{U} \rangle_{L^2(\partial\widehat{\Omega})} - \sum_{i=1}^{m} \langle S^0_{\partial B^{z_i}_{\boldsymbol{\delta}/4}}\, \mathcal{U}, \mathcal{U} \rangle_{L^2(\partial B^{z_i}_{\boldsymbol{\delta}/4})} \Big) \\
&\quad + \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \left(\langle S^0_{\partial\widehat{\Omega}} \mathcal{V}, \mathcal{V} \rangle_{L^2(\partial\widehat{\Omega})} - \sum_{i=m+1}^{2m} \langle S^0_{\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4}}\, \mathcal{V}, \mathcal{V} \rangle_{L^2(\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4})} \right) = \langle \mathcal{E}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}}.
\end{aligned}$$

Using the fact that $\langle S^0_{\partial\widehat{\Omega}}\, \phi, \phi \rangle_{L^2(\partial\widehat{\Omega})} \geq 0$, for every $\phi \in L^2(\partial\Omega)$ we get

$$\begin{aligned}
&\langle \widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}} + \langle \Theta^k \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}} + \left\langle \left(\Sigma^k - \Sigma^0\right) \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \right\rangle_{\mathbb{C}^{3\times 2m}} \\
&\quad - \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \Big(\sum_{i=1}^{m} \langle S^0_{\partial B^{z_i}_{\boldsymbol{\delta}/4}}\, \mathcal{U}, \mathcal{U} \rangle_{L^2(\partial B^{z_i}_{\boldsymbol{\delta}/4})} + \sum_{i=m+1}^{2m} \langle S^0_{\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4}}\, \mathcal{V}, \mathcal{V} \rangle_{L^2(\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4})} \Big) \leq \langle \mathcal{E}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}}.
\end{aligned}$$

By the definition of $\mathcal{Q}$ and the inequalities (2.2.11), we have

$$\langle \mathcal{Q}\widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}} \leq \mu^+ \boldsymbol{a}^3 \langle \widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}}. \tag{2.4.33}$$

Using (2.4.15) and (2.4.16), we arrive at

$$\begin{aligned}
&\Big(1 - \Big[\frac{(1+|k|^2) C_{k,D(\Omega)}}{8\pi} + \frac{63^{\frac{1}{2}} |k|^2 D(\Omega)^{\frac{2}{3}}}{4\pi}\Big] \frac{\mu^+ \boldsymbol{a}^3}{\boldsymbol{\delta}^3} \Big) \langle \widehat{\mathcal{C}}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}} \\
&\quad - \frac{48^2}{\pi^2 \boldsymbol{\delta}^6} \Big(\sum_{i=1}^{m} \langle S^0_{\partial B^{z_i}_{\boldsymbol{\delta}/4}}\, \mathcal{U}, \mathcal{U} \rangle_{L^2(\partial B^{z_i}_{\boldsymbol{\delta}/4})} + \sum_{i=m+1}^{2m} \langle S^0_{\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4}}\, \mathcal{V}, \mathcal{V} \rangle_{L^2(\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4})} \Big) \leq \langle \mathcal{E}, \mathcal{Q}\widehat{\mathcal{C}} \rangle_{\mathbb{C}^{3\times 2m}}.
\end{aligned} \tag{2.4.34}$$

Further, the following scaling inequality holds

$$\begin{aligned}\langle S^0_{\partial B^{z_i}_{\boldsymbol{\delta}/4}}\,\mathcal{U},\mathcal{U}\rangle_{L^2(\partial B^{z_i}_{\boldsymbol{\delta}/4})} &= \int_{B^{z_i}_{\boldsymbol{\delta}/4}}\int_{B^{z_i}_{\boldsymbol{\delta}/4}} (\Phi_0(x,y)\nu_x\cdot Q_i\widehat{\mathcal{C}}_i\; ds(y))\nu_y\cdot\overline{Q_i\widehat{\mathcal{C}}_i}\; ds(x),\\ &= \int_{\partial B^{z_i}_1}\Big(\int_{\partial B^{z_i}_1}\frac{1}{(\frac{\boldsymbol{\delta}}{4})}\;\Phi_0(s,t)\;\nu_t\cdot Q_i\widehat{\mathcal{C}}_i(\frac{\boldsymbol{\delta}}{4})^2\;ds(t)\Big)\nu_s\;\cdot\overline{Q_i\widehat{\mathcal{C}}_i}(\frac{\boldsymbol{\delta}}{4})^2\;ds(s),\\ &\leq (\frac{\boldsymbol{\delta}^3}{4^3})\|S_{B^1}\|\Big|Q_i\widehat{\mathcal{C}}_i\Big|\big|\partial B^0_1\big| = (\frac{4\pi\boldsymbol{\delta}^3}{4^3})\|S_{B^1}\|\Big|Q_i\widehat{\mathcal{C}}_i\Big|^2,\end{aligned}$$

which remains valid for $\langle S^0_{\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4}}\;\mathcal{V},\mathcal{V}\rangle_{L^2(\partial B^{z_{i-m}}_{\boldsymbol{\delta}/4})}$, gives in (2.4.34)

$$\begin{aligned}\Big(1-\big[\frac{(1+|k|^2)C_{k,D(\Omega)}}{8\pi}+\frac{63^{\frac{1}{2}}|k|^2D(\Omega)^{\frac{2}{3}}}{4\pi}\big]\frac{\mu^+\boldsymbol{a}^3}{\boldsymbol{\delta}^3}\Big)\langle\widehat{\mathcal{C}},\mathcal{Q}\widehat{\mathcal{C}}\rangle_{\mathbb{C}^{3\times 2m}}&\\ -\frac{12^2\|S_{B^1}\|}{\pi\boldsymbol{\delta}^3}\sum_{i=1}^{2m}\Big|Q_i\widehat{\mathcal{C}}_i\Big|^2 &\leq \langle\mathcal{E},\mathcal{Q}\widehat{\mathcal{C}}\rangle_{\mathbb{C}^{3\times 2m}}.\end{aligned}$$

As $\langle\mathcal{E},\mathcal{Q}\widehat{\mathcal{C}}\rangle_{\mathbb{C}^{3\times 2m}}\leq\langle\widehat{\mathcal{C}},\mathcal{Q}\widehat{\mathcal{C}}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}}\langle\mathcal{E},\mathcal{Q}\mathcal{E}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}}$ we get, with (2.4.33),

$$\begin{aligned}\Big(1-\big[\frac{(1+|k|^2)C_{k,D(\Omega)}}{8\pi}+\frac{12^2\|S_{B^1}\|}{\pi}+\frac{63^{\frac{1}{2}}|k|^2D(\Omega)^{\frac{2}{3}}}{4\pi}\big]\frac{\mu^+\boldsymbol{a}^3}{\boldsymbol{\delta}^3}\Big)\langle\widehat{\mathcal{C}},\mathcal{Q}\widehat{\mathcal{C}}\rangle_{\mathbb{C}^{3\times 2m}}&\leq\\ \langle\widehat{\mathcal{C}},\mathcal{Q}\widehat{\mathcal{C}}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}}\langle\mathcal{E},\mathcal{Q}\mathcal{E}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}},&\end{aligned}$$

which is precisely

$$\left(1-C_{Ls}\frac{\mu^+\boldsymbol{a}^3}{\boldsymbol{\delta}^3}\right)\langle\widehat{\mathcal{C}},\mathcal{Q}\widehat{\mathcal{C}}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}}\leq\langle\mathcal{E},\mathcal{Q}\mathcal{E}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}} \tag{2.4.35}$$

where we set

$$C_{Ls}:=\big[\frac{(1+|k|^2)C_0\Big((1+\frac{|k|}{2})D(\Omega)^{\frac{1}{3}}+\frac{|k|}{2}D(\Omega)^{\frac{2}{3}}\Big)}{8\pi}+\frac{12^2\|S_{B^1}\|}{\pi}+\frac{63^{\frac{1}{2}}|k|^2D(\Omega)^{\frac{2}{3}}}{4\pi}\big] \tag{2.4.36}$$

recalling the constants $C_{k,D(\Omega)}$. Then a sufficient condition for the solvability of (2.4.3) is given by $C_{Ls}\frac{\mu^+\boldsymbol{a}^3}{\boldsymbol{\delta}^3}<1$. Further, if the previous condition is satisfied, then from(2.4.35), considering (2.2.11), we get

$$\left(1-C_{Ls}\frac{\mu^+\boldsymbol{a}^3}{\boldsymbol{\delta}^3}\right)\mu^-\boldsymbol{a}^3\langle\widehat{\mathcal{C}},\widehat{\mathcal{C}}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}}\leq\mu^+\boldsymbol{a}^3\langle\mathcal{E},\mathcal{E}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}}, \tag{2.4.37}$$

and the definition of $\mathcal{C}$, yields inverting (2.2.11)

$$\begin{aligned}\langle\widehat{\mathcal{C}},\widehat{\mathcal{C}}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}} &= \left(\langle\mathcal{Q}^{-1}_{1,1}\widehat{\mathcal{Q}},\mathcal{Q}^{-1}_{1,1}\widehat{\mathcal{Q}}\rangle_{\mathbb{C}^{3\times m}}+\langle\mathcal{Q}^{-1}_{2,2}\widehat{\mathcal{R}},\mathcal{Q}^{-1}_{2,2}\widehat{\mathcal{R}}\rangle_{\mathbb{C}^{3\times m}}\right)^{\frac{1}{2}},\\ &\geq\frac{\mu^+}{\boldsymbol{a}^3}\left(\langle\widehat{\mathcal{Q}},\widehat{\mathcal{Q}}\rangle_{\mathbb{C}^{3\times m}}+\langle\widehat{\mathcal{R}},\widehat{\mathcal{R}}\rangle_{\mathbb{C}^{3\times m}}\right)^{\frac{1}{2}}.\end{aligned}$$

Replacing in (2.4.37), we obtain

$$\left(1 - C_{Ls}\frac{\mu^+\boldsymbol{a}^3}{\boldsymbol{\delta}^3}\right)\mu^-\mu^+\left(\langle\widehat{\mathcal{Q}},\widehat{\mathcal{Q}}\rangle_{\mathbb{C}^{3\times m}} + \langle\widehat{\mathcal{R}},\widehat{\mathcal{R}}\rangle_{\mathbb{C}^{3\times m}}\right)^{\frac{1}{2}} \leq \mu^+\boldsymbol{a}^3\langle\mathcal{E},\mathcal{E}\rangle^{\frac{1}{2}}_{\mathbb{C}^{3\times 2m}}. \tag{2.4.38}$$

Concerning (2.4.5), it suffice to observe that

$$\left|\Sigma^k\right| \leq 2\sum_{i=1}^{m}|\Pi_k(z_i,z_j)| \leq \frac{1}{2\pi}\sum_{i=1}^{m}\Big(\frac{|k|^2}{\boldsymbol{\delta}_{ij}} + \frac{3}{\boldsymbol{\delta}_{ij}}\big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big)^2\Big),$$

and

$$\left|\Theta^k\right| \leq 2\sum_{i=1}^{m}|\nabla\Phi_k(z_i,z_j)| \leq \frac{1}{2\pi}\sum_{i=1}^{m}\frac{1}{\boldsymbol{\delta}_{ij}}\big(\frac{1}{\boldsymbol{\delta}_{ij}} + |k|\big),$$

which, summing as in (2.2.78) gives,

$$\left|\Sigma^k\mathcal{Q}\right| + \left|\Theta^k\mathcal{Q}\right| \leq \left|\Sigma^k\right| + \left|\Theta^k\right| \leq 4\mu^+\left(\frac{\ln m^{\frac{1}{3}}}{\boldsymbol{\delta}^3} + \frac{2km^{\frac{1}{3}}}{\boldsymbol{\delta}^2} + \frac{m^{\frac{2}{3}}}{2\boldsymbol{\delta}}k^2\right)\boldsymbol{a}^3$$

with this, it comes that, for

$$C_{L^2_i} = 1 - 4\mu^+\left(\frac{\ln m^{\frac{1}{3}}}{\boldsymbol{\delta}^3} + \frac{2km^{\frac{1}{3}}}{\boldsymbol{\delta}^2} + \frac{m^{\frac{2}{3}}}{2\boldsymbol{\delta}}k^2\right)\boldsymbol{a}^3$$

$$|\mathcal{C}_i| \leq \frac{1}{C_{L^2_i}}|\mathcal{E}_i|. \tag{2.4.39}$$

□

2.5 End of the proof of Theorem 4

With the notations of the previous section, the linear system ((2.3.6),(2.3.5)) becomes

$$\mathcal{C} + \Sigma^k\mathcal{Q}\mathcal{C} + \Theta^k\mathcal{Q}\mathcal{C} = \mathcal{E} + \epsilon(\boldsymbol{a},\boldsymbol{\delta},|k|,m)\boldsymbol{a}^3, \tag{2.5.1}$$

with $(\mathcal{C}_i)_{i\in\{1,\dots,2m\}}$ defined as

$$\mathcal{C}_i := \begin{cases} \left[\mathcal{T}_{\partial D_i}\right]^{-1}\mathcal{Q}_i, & i \in \{1,\dots,m\}, \\ -\left[\mathcal{P}_{\partial D_{i-m}}\right]^{-1}\mathcal{R}_{i-m}, & i \in \{m+1,\dots,2m\}, \end{cases} \tag{2.5.2}$$

and $\epsilon(\boldsymbol{a},\boldsymbol{\delta},|k|,m) = (\epsilon_i(\boldsymbol{a},\boldsymbol{\delta},|k|,m))_{i=1}^m$ with

$$\epsilon_i(\boldsymbol{a},\boldsymbol{\delta},|k|,m) := \begin{cases} \dfrac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4} + \epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4 + (1+|k|)\boldsymbol{a}, & i \in \{1,\dots,m\}, \\ \dfrac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4} + |k|\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4 + |k|^2\boldsymbol{a}, & i \in \{m+1,\dots,2m\}. \end{cases} \tag{2.5.3}$$

The difference between (2.5.1) and (2.4.10) implies

$$\mathcal{C}-\widehat{\mathcal{C}})+\Sigma^{k}\mathcal{Q}(\mathcal{C}-\widehat{\mathcal{C}})+\Theta^{k}\mathcal{Q}(\mathcal{C}-\widehat{\mathcal{C}})=\epsilon(\boldsymbol{a},\boldsymbol{\delta},|k|,m), \tag{2.5.4}$$

which gives, with the estimates (2.4.4)

$$\sum_{i=1}^{m}\Big(|\widehat{\mathcal{R}}_i-\mathcal{R}_i|^2+|\widehat{\mathcal{Q}}_i-\mathcal{Q}_i|^2\Big)\leq\frac{1}{C_{Li}\mu^-}\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}+(1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4+\max(|k|^2,1+|k|)\boldsymbol{a}\Big)^2\ 2m\ \boldsymbol{a}^6, \tag{2.5.5}$$

and, with (2.4.5)

$$\begin{cases}|\widehat{\mathcal{R}}_i-\mathcal{R}_i|\leq\dfrac{1}{C_{L_i^2}\mu^-}\Big(\dfrac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}+(1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4+\max(|k|^2,1+|k|)\boldsymbol{a}\Big)^2\ \boldsymbol{a}^3,\\ |\widehat{\mathcal{Q}}_i-\mathcal{Q}_i|\leq\dfrac{1}{C_{L_i^2}\mu^-}\Big(\dfrac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}+(1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4+\max(|k|^2,1+|k|)\boldsymbol{a}\Big)^2\ \boldsymbol{a}^3.\end{cases} \tag{2.5.6}$$

We set

$$O^{\epsilon}(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}):=O\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}+(1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4+\max(|k|^2,1+|k|)\boldsymbol{a}\Big).$$

Lemma 19. *We have the following asymptotic approximation for the far field,*

$$\begin{aligned}E^{\infty}(\tau)=&\frac{ik}{4\pi}\sum_{i=1}^{m}e^{-ik\tau.z_i}\tau\times\big(\widehat{\mathcal{R}}_i-ik\tau\times\widehat{\mathcal{Q}}_i\big)+O\Big(\ (|k|^3+|k|^2)\ m\boldsymbol{a}^4\Big)\\&+\frac{|k|}{2\pi}\ \frac{\max(1,|k|)}{C_{Li}\mu^-}\ O^{\epsilon}\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}\Big)m\boldsymbol{a}^3\end{aligned}$$

and the following one for the scattered field

$$E^{sc}(x)=\sum_{i=1}^{m}\Big(\nabla\Phi_k(x,z_i)\times\widehat{\mathcal{R}}_i+\operatorname{curl}\operatorname{curl}(\Phi_k(x,z_i)\widehat{\mathcal{Q}}_i)\Big)+\frac{(C_{L_i^2}\mu^-)^{-1}}{\mu^+}\times O^{\epsilon}(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}) \tag{2.5.7}$$

$$+O\big((C_{L_i^2}\mu^-)^{-1}\frac{\boldsymbol{a}^7}{\boldsymbol{\delta}^7}+\epsilon(\boldsymbol{\delta}^6,|k|+|k|^2+|k|^3)\boldsymbol{a}^7+\epsilon(\boldsymbol{\delta}^5,|k|^2)\boldsymbol{a}^7\big). \tag{2.5.8}$$

Proof. Recalling the approximation of the far field in Proposition 11, we have

$$\begin{aligned}E^{\infty}(\tau)=&\frac{ik}{4\pi}\sum_{i=1}^{m}e^{-ik\tau.z_i}\tau\times\big((\mathcal{R}_i-\widehat{\mathcal{R}}_i)-ik\tau\times(\mathcal{Q}_i-\widehat{\mathcal{Q}_i})\\&+\frac{ik}{4\pi}\sum_{i=1}^{m}e^{-ik\tau.z_i}\tau\times\big(\widehat{\mathcal{R}}_i-ik\tau\times\widehat{\mathcal{Q}}_i\big)+O\Big((|k|^3+|k|^2)\ m\boldsymbol{a}^4\Big).\end{aligned} \tag{2.5.9}$$

For the first term of the right hand side, we have

$$
\begin{aligned}
&|\frac{ik}{4\pi}\sum_{i=1}^{m} e^{-ik\tau.z_i}\tau\times\big((\mathcal{R}_i-\widehat{\mathcal{R}}_i)-ik\tau\times(\mathcal{Q}_i-\widehat{\mathcal{Q}_i})\big)|\\
&\qquad\qquad\leq 2\,\frac{|k|}{4\pi}\max(1,|k|)\;m^{\frac{1}{2}}\Big(\sum_{i=1}^{m}\big(\Big|\mathcal{R}_i-\widehat{\mathcal{R}}_i\Big|^2+\Big|\mathcal{Q}_i-\widehat{\mathcal{Q}_i}\Big|^2\big)\Big)^{\frac{1}{2}},\\
&\qquad\qquad\leq \frac{|k|}{2\pi}\frac{\max(1,|k|)}{C_{L_i\mu^-}}\;O^{\epsilon}\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}\Big)m\boldsymbol{a}^3.
\end{aligned}
$$

With this estimate, (2.5.9) becomes

$$
\begin{aligned}
E^{\infty}(\tau)=&\frac{ik}{4\pi}\sum_{i=1}^{m} e^{-ik\tau.z_i}\tau\times\big(\widehat{\mathcal{R}}_i-ik\tau\times\widehat{\mathcal{Q}}_i\big)+O\Big(\;(|k|^3+|k|^2)\;m\boldsymbol{a}^4\Big)\\
&+\frac{|k|}{2\pi}\frac{\max(1,|k|)}{C_{L_i\mu^-}}\;O^{\epsilon}\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}\Big)m\boldsymbol{a}^3.
\end{aligned}
\tag{2.5.10}
$$

Again, in view of Proposition 11, we have

$$
\begin{aligned}
E^{\mathrm{sc}}(x)=&\sum_{i=1}^{m}\Big(\nabla\Phi_k(x,z_i)\times\big(\mathcal{R}_i-\widehat{\mathcal{R}}_i\big)+\operatorname{curl}\operatorname{curl}(\Phi_k(x,z_i)\big(\mathcal{Q}_i-\widehat{\mathcal{Q}}_i\big))\Big)\\
&+\sum_{i=1}^{m}\Big(\nabla\Phi_k(x,z_i)\times\widehat{\mathcal{R}}_i+\operatorname{curl}\operatorname{curl}(\Phi_k(x,z_i)\widehat{\mathcal{Q}}_i)\Big)\\
&+O\big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}+\epsilon_{k,\boldsymbol{\delta},m}\;\boldsymbol{a}^4\big).
\end{aligned}
\tag{2.5.11}
$$

Let i_0 be as in (2.3.13), from the representation of the linear system we have [24]

$$
\begin{aligned}
&\sum_{(i\neq i_0)\geq 1}^{m}\Big(\nabla\Phi_k(z_{i_0},z_i)\times\big(\mathcal{R}_i-\widehat{\mathcal{R}}_i\big)+\operatorname{curl}\operatorname{curl}(\Phi_k(z_{i_0},z_i)\big(\mathcal{Q}_i-\widehat{\mathcal{Q}}_i\big))\Big)\\
&\qquad\qquad=[\mathcal{T}_{\partial D_{i_0}}]^{-1}\big(\mathcal{Q}_{i_0}-\widehat{\mathcal{Q}}_{i_0}\big)+\epsilon_{i_0}(\boldsymbol{a},\boldsymbol{\delta},|k|,m),
\end{aligned}
$$

[24] Notice that $-\mathit{\Pi}_k(x,y)=\nabla_y\times\nabla_x\times\Phi_k(x,y)\;I$.

hence, adding and subtracting the last identity to (2.5.11) gives

$$\begin{aligned}
E^{\mathrm{sc}}(x) =& \Big(\nabla\Phi_k(x,z_{i_0}) \times \big(\mathcal{R}_{i_0} - \widehat{\mathcal{R}}_{i_0}\big) + \operatorname{curl}\operatorname{curl}(\Phi_k(x,z_{i_0})\big(\mathcal{Q}_{i_0} - \widehat{\mathcal{Q}}_{i_0})\big)\Big) \\
&+ \sum_{(i\neq i_0)\geq 1}^{m} \Big[(\nabla\Phi_k(x,z_i) - \nabla\Phi_k(z_{i_0},z_i)) \times \big(\mathcal{R}_{i_0} - \widehat{\mathcal{R}}_{i_0}\big) \\
&\qquad + \operatorname{curl}\operatorname{curl}\Big(\big(\Phi_k(x,z_i) - \Phi_k(z_{i_0},z_i)\big)\big(\mathcal{Q}_i - \widehat{\mathcal{Q}}_i\big)\Big)\Big] \\
&+ [\mathcal{T}_{\partial D_{i_0}}]^{-1}\big(\mathcal{Q}_{i_0} - \widehat{\mathcal{Q}}_{i_0}\big) + \epsilon_{i_0}(\boldsymbol{a},\boldsymbol{\delta},|k|,m) \\
&+ \sum_{i=1}^{m}\Big(\nabla\Phi_k(x,z_i) \times \widehat{\mathcal{R}}_i + \operatorname{curl}\operatorname{curl}(\Phi_k(x,z_i)\widehat{\mathcal{Q}}_i)\Big) \\
&+ O\big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4} + \epsilon_{k,\boldsymbol{\delta},m}\ \boldsymbol{a}^4\big).
\end{aligned} \tag{2.5.12}$$

For $x \in \partial\Omega$, $\frac{1}{d_{x,i_0}} = \frac{1}{\boldsymbol{\delta}}$ and then the first term of the right hand side of (2.5.12) is smaller then

$$\Big(\frac{1}{\boldsymbol{\delta}}\big(\frac{1}{\boldsymbol{\delta}} + |k|\big)|\mathcal{R}_{i_0} - \widehat{\mathcal{R}}_{i_0}| + \Big(\frac{|k|^2}{\boldsymbol{\delta}} + \frac{1}{\boldsymbol{\delta}}\big(\frac{1}{\boldsymbol{\delta}} + |k|\big)^2\Big)|\mathcal{Q}_{i_0} - \widehat{\mathcal{Q}}_{i_0}|\Big) \tag{2.5.13}$$

which gives, considering (2.5.6)

$$(C_{L_i^2}\mu^-)^{-1}\Big(\frac{1}{\boldsymbol{\delta}^3} + \frac{1+2|k|}{\boldsymbol{\delta}^2} + \frac{|k|+2|k|^2}{\boldsymbol{\delta}}\Big)\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4} + (1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4 + \max(|k|^2, 1+|k|)\boldsymbol{a}\Big)^2\ \boldsymbol{a}^3,$$

which is

$$O\big((C_{L_i^2}\mu^-)^{-1}\frac{\boldsymbol{a}^7}{\boldsymbol{\delta}^7} + \epsilon(\boldsymbol{\delta}^6, |k|)\boldsymbol{a}^7 + \epsilon(\boldsymbol{\delta}^5, |k|^2)\boldsymbol{a}^7\big). \tag{2.5.14}$$

The second term of (2.5.12) is exactly

$$\begin{aligned}
\sum_{(i\neq i_0)\geq 1}^{m} \Big[&\big(\int_{[0,1]} \nabla_x\nabla_x\Phi_k(tx + (1-t)z_{i_0}, z_i)\ dt \cdot (x - z_{i_0})\big) \times \big(\mathcal{R}_{i_0} - \widehat{\mathcal{R}}_{i_0}\big) \\
&+ \big(\int_{[0,1]} \nabla_x\mathit{\Pi}_k(tx + (1-t)z_{i_0}, z_i)\ dt \cdot (x - z_{i_0})\big)\big(\mathcal{Q}_i - \widehat{\mathcal{Q}}_i\big)\Big],
\end{aligned}$$

which turns out to be not greater then

$$\sum_{\substack{(i\neq i_0)\\ i\geq 1}}^{m}\Big[\frac{1}{\boldsymbol{\delta}_{i_0,i}}\big(\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|\big)^2\boldsymbol{\delta}|\mathcal{R}_{i_0} - \widehat{\mathcal{R}}_{i_0}| + \Big(\frac{|k|^2}{\boldsymbol{\delta}_{i_0,i}}\big(\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|\big) + \frac{1}{\boldsymbol{\delta}_{i_0,i}}\big(\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|\big)^3\Big)\boldsymbol{\delta}\Big|\mathcal{Q}_i - \widehat{\mathcal{Q}}_i\Big|\Big],$$

and, due to (2.5.6), not exceed

$$\begin{aligned}
\sum_{(i\neq i_0)\geq 1}^{m}&\Big[\frac{1}{\boldsymbol{\delta}_{i_0,i}}\big(\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|\big)^2\boldsymbol{\delta} + \Big(\frac{|k|^2}{\boldsymbol{\delta}_{i_0,i}}\big(\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|\big) + \frac{1}{\boldsymbol{\delta}_{i_0,i}}\big(\frac{1}{\boldsymbol{\delta}_{i_0,i}} + |k|\big)^3\Big)\boldsymbol{\delta}\Big] \\
&\times(C_{L_i^2}\mu^-)^{-1}\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4} + (1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4 + \max(|k|^2, 1+|k|)\boldsymbol{a}\Big)^2\ 2\boldsymbol{a}^3,
\end{aligned}$$

hence summing as in (2.2.78) gives

$$O\Big(\frac{1}{\boldsymbol{\delta}^4}+\frac{(1+|k|)\ln(m^{1/3})}{\boldsymbol{\delta}^3}+\frac{(|k|+|k|^2)m^{1/3}}{\boldsymbol{\delta}^2}+\frac{(|k|^3+|k|^2)m^{2/3}}{\boldsymbol{\delta}}\Big)\boldsymbol{\delta}$$
$$\times(C_{L_i^2}\mu^-)^{-1}\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}+(1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4+\max(|k|^2,1+|k|)\boldsymbol{a}\Big)^2 2\boldsymbol{a}^3,$$

which is, for $m=O(1/\boldsymbol{\delta}^3)$, the analogue of (2.5.14) with the following additional term

$$\epsilon(\boldsymbol{\delta}^6,|k|^2+|k|^3)\boldsymbol{a}^7+\epsilon(\boldsymbol{\delta}^5,|k|^2)\boldsymbol{a}^7. \tag{2.5.15}$$

The third term of (2.5.12), due to (2.2.11) and (2.5.6), is bounded by

$$\frac{(C_{L_i^2}\mu^-)^{-1}}{\mu^+}\Big(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}+(1+|k|)\epsilon_{k,\boldsymbol{\delta},m}\boldsymbol{a}^4+\max(|k|^2,1+|k|)\boldsymbol{a}\Big)^2=\frac{(C_{L_i^2}\mu^-)^{-1}}{\mu^+}\times O^{\epsilon}(\frac{\boldsymbol{a}^4}{\boldsymbol{\delta}^4}), \tag{2.5.16}$$

compiling this last error with (2.5.14) and the additional term (2.5.15) gives us the result. □

Chapter 3

Foldy-Lax approximation of the electromagnetic fields generated by anisotropic inhomogeneities in the mesoscale regime with complements for the perfectly conducting case

3.1 General setting and main results

3.1.1 General setting

Understanding the interaction between the waves (as the light or the acoustic fluctuations or the elastic displacements) with the matter has been of fundamental importance since a long time. Since the pioneering works of Rayleigh and Kirchhoff, it was known that the wave diffracted by small scaled inhomogeneities is dominated by the first multipoles (poles or dipoles). In modern terminology, the dominating fields are given by (polarized) point sources located at the center of the particles. As far as the three types of waves, cited above, are concerned these point sources are the Green's functions of the corresponding propagator. In this direction, the next key step is achieved by Mie [55] in his full expansion of the electromagnetic field for spherically shaped particles. These formal expansions were later mathematically justified, see for instance [25] in the framework of low frequencies expansions. A further step was achieved in [10] where the full expansion at any order is derived and justified.

These works dealt with single or well separated inhomogeneities. In other words, only the interaction of the single inhomogeneity and the wave is taken into account. In the presence of multiple and close inhomogeneities, then mutual interactions between them and the waves should be taken into account. In this respect, a formal argument to handle such multiple interaction was proposed by Foldy in his seminal work [29]. To state his formulas, he looks the inhomogeneities as point-like potentials (i.e. Dirac-like potentials). Then, he states a close form of the scattered wave by simply eliminating the singularity on the locations of these potentials. This elimination of the singularity translates a physical motivation saying that 'the scattering coefficient of each point-like scatterer is proportional to the external field acting on it', which is known as the Foldy assumption. These formal representations of the scattered waves are then stated on more sound mathematical arguments by Berezin-Faddeev, see [12], in the framework of the Krein's selfadjoint extension of symmetric operators. Another related method is the so called regularization method (or the renormalization technique) which aims at computing the Green's function, and hence the Schwartz integral operator, in the presence of the collections of inhomogeneities. This idea consists in taking the Fourier transform of the formal equation,'cut' or regularize and invert the related equations, via Weinstein-Aronsza theorem, in the Fourier domain and then comeback. More details on these ideas can be found in the book [3]. The Faddeev approach was extended to singular potentials supported not only on points but also on curves and surfaces, see [47, 48] for more details. This approach gives as, via the Krein's resolvent representations, exact formulas to represent the scattered waves generated by singular potentials. However, our goal is to deal with cluster of small scaled inhomogeneities. The intuitive believe is that the dominant part of the generated waves would be reminiscent to the exact formulas described above, for Dirac-like potentials supported on the centers of the inhomogeneities, but with scattering coefficients modeled by geometric or contrasts properties of the inhomogeneities. This is called the Foldy-Lax approximation or the point-interaction approximation.

Several methods were proposed in the literature to justify such approximations, see [64, 53, 18, 20] for instance regarding acoustic and elastic waves. Descriptions and relation/differences between these works can be found in [21]. Let us emphasize here that those works dealt with exterior problems (impenetrable inclusions, holes or voids). Regarding electromagnetic waves, very few works are proposed, apart from [66] where both the results and the justifications are quite questionable. In our previous work [13], we considered the case of perfectly conductive inclusions and we gave a rigorous justification of this Foldy-Lax approximation under general conditions on the cluster of such inclusions as their minimum distance between them δ and their maximum radius a of the form $\ln(\delta^{-1})\ \frac{a}{\delta}$ is bounded by a constant depending only on the Lipschitz character of the shapes of the inclusions. The only limitation of this result, to handle the mesoscale regime (i.e. $\delta \sim a$), is the appearance of the term $\ln(\delta^{-1})$. This term appears naturally in the analysis, in that work, which is heavily based on the scales of the related layer potentials knowing that the dyadic Maxwell fundamental solution has a singularity of order 3 (while the ones of Laplace or Lamé have singularities of the order 1).

In this present work, we get rid of this logarithmic term and state the approximation in the mesoscale regime. But the most important contribution is to handle the transmission problem and the impenetrable problem in a unified way. In addition, anisotropic and eventually complex valued electromagnetic material parameters can be handled as well. Our arguments can be summarized as follows. To handle the anisotropic transmission problem, we provided a representation of the solution using the electromagnetic Lippmann-Schwinger operator and give an estimate of the total field. To overcome the logarithmic constraint, instead of using Neumann series to estimates the density of the used representation, we make use of a Rellich identity and, inspired by some arguments from [57], with appropriate changes, we prove that both the exterior and the interior traces have an equivalent norm modulo a constant which depends on the geometry of the inhomogeneities and the material parameters (via their contrasts). Regarding the perfect conductor problem, we use layer potential representation of the solution. Using an appropriate Helmholtz decomposition for the density, appearing in the layer potential representation, we transforme the boundary integral interaction operator into a volume one to get Lippmann-Schwinger like integral representation. This allows us to translate the result obtained for the transmission problem to the perfectly conductive case.

It is worth mentioning that even in the scalar case, i.e. related to the Laplace operator, with a cluster of small obstacles with Neumann boundary conditions was left open to our best knowledge. The approach we follow here definitely handles this case and provides the corresponding Foldy-Lax approximation in the same generality as we are proposing in it this work.

In the case of periodically distributed small inhomogeneities, the homogenization applies, see [11, 38], and provides the equivalent media with averaged materials. As compared to homogenization, the Foldy-Lax approximation has several advantages. The first one is that we have the dominating field (i.e. the Foldy-Lax field) for general (and not only periodic) distributions of the small inhomogeneities. This reduces the complexity of the forward problem to compute the scattered fields by inverting an algebraic system. Second, higher order approximations are possible with more effective dominating fields, i.e. with generalized Foldy-Lax fields. So far, this is not fully justified, but we believe it to be true and we will report on it in the future. Third, as we have freedom in distributing these inhomogeneities, then we can generate not only volumetric equivalent materials but also low dimensional ones as surfaces and curves. This opens the way to applications in low dimensions metamaterials as well, see [6, 5] for instance. As far as the Maxwell model is concerned, the Foldy-Lax approximation provided here shows that one can generate volumetric materials and Gradient-metasurfaces. The first situation is modeled by modifying both the background permittivity and permeability. The second is modeled by an equivalent interface with jumps of both the electric and magnetic fields across it.

The rest of the paper is described as follows. In the next subsection, we state clearly the models and the obtained results with critical discussion about them. In section 2 and section 3, we provide the full proofs of the results for the transmission and the perfectly conducting models respectively. A short Appendix is added at the end to include technical tools and in particular a useful lemma on the counting of the number of small particles distributed in any given bounded set in terms of the parameters δ and a.

3.1.2 Main results

We deal with the scattering of time-harmonic electromagnetic plane waves at a frequency ω in a medium composed of an isotropic and constant background and an anisotropic material represented by multiply connected, bounded, Lipschitz[1] domain $D^- = \cup_{m=1}^{\aleph} D_m$ where $\aleph$ is the number of

[1] This means that the boundary is locally described by the graph of a Lipschitz function. More details are given later.

connected components. The Maxwell equations read as follows

$$\begin{cases} \nabla \times \mathcal{E} - i\omega\mu\mathcal{H} = 0, \quad \text{in } \mathbb{R}^3 \setminus \partial D, \\ \nabla \times \mathcal{H} + i\omega\varepsilon\mathcal{E} = \sigma\mathcal{E}, \quad \text{in } \mathbb{R}^3 \setminus \partial D, \\ \mathcal{E} = \mathcal{E}^{\text{in}} + \mathcal{E}^{\text{sc}}, \\ \nu \times \mathcal{E}|_+ = \nu \times \mathcal{E}|_-, \ \nu \times \mathcal{H}|_+ = \nu \times \mathcal{H}|_- \text{ on } \partial D \end{cases} \tag{3.1.1}$$

with the notation $\partial D := \cup_{m=1}^{\aleph} \partial D_m$ where ε and σ are respectively the electric permittivity and the conductivity and μ corresponds to the magnetic permeability. These parameters can be real or complex tensor or scalar valued functions. Here $\mathcal{E}^{\text{in}}$ stands for the incident wave. It is solution of the first two equations above everywhere in the space. The vector field $\mathcal{E}^{\text{sc}}$ stands for the scattered vector field.

We also consider the scattering from a perfect conductor modeled by the following problem

$$\begin{cases} \nabla \times \mathcal{E} - i\omega\mu\mathcal{H} = 0, \\ \nabla \times \mathcal{H} + i\omega\varepsilon\mathcal{E} = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \mathcal{E} = \mathcal{E}^{\text{in}} + \mathcal{E}^{\text{sc}}, \\ \nu \times \mathcal{E}|_+ = 0, \ \text{on } \partial D, \end{cases} \tag{3.1.2}$$

where ν is the unit outward normal vector to the boundary of D^-. The surrounding background of D^- is homogeneous with constant parameter ε_0, μ_0 and null conductivity σ. In both the two models above, the scattered field must satisfy the radiation conditions

$$\left(\sqrt{\mu_0\varepsilon_0^{-1}}\right)\mathcal{H}^{\text{sc}}(x) \times \frac{x}{|x|} - \mathcal{E}^{\text{sc}}(x) = O\left(\frac{1}{|x|^2}\right). \tag{3.1.3}$$

Setting $k := w\sqrt{\epsilon_0\mu_0}$, $\mathcal{E} := \sqrt{\varepsilon_0{\mu_0}^{-1}}E$ and $\mathcal{H} := \sqrt{\varepsilon_0{\mu_0}^{-1}}H$ with $\mu_r := (\mu_0)^{-1}\mu$ and $\varepsilon_r := (\varepsilon_0)^{-1}(\varepsilon + i\sigma/\omega)$, we arrive at

$$\begin{cases} \operatorname{curl} E - ik\mu_r H = 0, \quad \text{in } \mathbb{R}^3 \setminus \partial D, \\ \operatorname{curl} H + ik\varepsilon_r E = 0, \quad \text{in } \mathbb{R}^3 \setminus \partial D, \\ E = E^{\text{sc}} + E^{\text{in}}, \\ \nu \times E|_- - \nu \times E|_+ = \nu \times H|_- - \nu \times H|_+ = 0, \\ H^{\text{sc}} \times \dfrac{x}{|x|} - E^{\text{sc}} = O\left(\dfrac{1}{|x|^2}\right), \quad |x| \to \infty, \end{cases} \tag{$\mathcal{P}_1$}$$

for the inhomogeneous (i.e. transmission) problem and

$$\begin{cases} \nabla \times E - ikH = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{D} \\ \nabla \times H + ikE = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ E = E^{\text{in}} + E^{\text{sc}}, \\ \nu \times E|_+ = 0, \ \text{on } \partial D, \end{cases} . \tag{$\mathcal{P}_2$}$$

for the conductive (i.e. exterior) problem. For both the two models the incident field $(E^{\text{in}}, H^{\text{in}})$ satisfies in the whole space the system

$$\begin{cases} \operatorname{curl} E^{\text{in}} - ikH^{\text{in}} = 0, \\ \operatorname{curl} H^{\text{in}} + ikE^{\text{in}} = 0. \end{cases} \tag{3.1.4}$$

Motivated by applications, typical incident electric fields are plane waves, i.e of the form $E^{\text{in}} := E^{\text{in}}(x, \theta) := P\, e^{ik\theta\cdot x}, x \in \mathbb{R}^3$, where P is the (constant) vector modeling the polarization direction and θ, with $|\theta| = 1$, is the incident direction such that $P \cdot \theta = 0$. The related magnetic incident field is then $H^{\text{in}} := H^{\text{in}}(x, \theta) := P \times \theta\, e^{ik\theta\cdot x}/ik$.

We suppose that,

$$D_m := \boldsymbol{a}\boldsymbol{\mathcal{D}}_m + \boldsymbol{z}_i, i = 1, ..., \aleph, \tag{3.1.5}$$

where each set $\boldsymbol{\mathcal{D}}_i$, contained in the ball $B_0^{1/2} := B(0, 1/2)$, and contains the origin, is assumed to be a Lipschitz bounded domain. The points $(\boldsymbol{z}_i)_{i=1}^{\aleph}$ are their given locations in $\mathbb{R}^3$ and $\boldsymbol{a} \in \mathbb{R}^+$ is a small parameter measuring the maximum relative radius.
Let $\boldsymbol{\delta}_{mj} := \min_{x\in D_m, y\in D_j} d(x, y)$, be the distance between two bodies $D_m, D_j, m \neq j$, and set

$$\boldsymbol{\delta} := \min_{m\neq j\in\{1,...,\aleph\}} \boldsymbol{\delta}_{mj}.$$

Let us recall that a bounded open connected domain B, is said to be a Lipschitz domain with character $(l_{\partial B}, L_{\partial B})$ if for each $x \in \partial D$ there exist a coordinate system $(y_i)_{i=1,2,3}$, a truncated cylinder $\mathfrak{C}$ centered at x whose axis is parallel to y_3 with length l satisfying $l_{\partial B} \leq l \leq 2l_{\partial B}$, and a Lipschitz function f that is $|f(s_1) - f(s_2)| \leq L_{\partial B}|s_1 - s_2|$ for every $s_1, s_2 \in \mathbb{R}^2$, such that $B \cap \mathfrak{C} = \{(y_i)_{i=1,2,3} : y_3 > f(y_1, y_2)\}$ and $\partial B \cap \mathfrak{C} = \{(y_i)_{i=1,2,3} : y_3 = f(y_1, y_2)\}$. In this work, we assume that the sequence of Lipschitz characters $(l_{\partial\boldsymbol{\mathcal{D}}_m}, L_{\partial\boldsymbol{\mathcal{D}}_m})_{i=1}^{\aleph}$ of the bodies $\boldsymbol{\mathcal{D}}_m, i = 1, ..., \aleph$, is bounded.

Regarding the problem $\mathcal{P}_1$, we need some assumptions on the electromagnetic material properties of the small particles. Precisely, we suppose that the contrast of magnetic permeability and electric permittivity , which are assumed to be respectively real and complex valued 3×3-tensor, are, with their derivative, essentially uniformly bounded , i.e.

$$(\|\boldsymbol{\mathcal{C}}_A\|_{\mathbb{W}^{1,\infty}(\cup_{i=1^m} D_m)})_{A=\varepsilon_r,\ \mu_r} \leq \boldsymbol{c}_\infty, \tag{3.1.6}$$

and essentially uniformly coercive that is, for almost every $x \in D$, we have

$$\begin{aligned} \Re\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}(x)U \cdot \overline{U}\Big) &\geq c_\infty^{\varepsilon-}|U|^2, \\ \boldsymbol{\mathcal{C}}_{\mu_r}(x)U \cdot U &\geq c_\infty^{\mu-}|U|^2, \end{aligned} \tag{3.1.7}$$

with positive constants $c_\infty^{\varepsilon-}$ and $c_\infty^{\mu-}$. Here we used the notation $\boldsymbol{\mathcal{C}}_A := A - \mathbb{I}$, where $\mathbb{I}$ is the identity matrix of $\mathbb{R}^3 \times \mathbb{R}^3$. I will stand for the identity operator.

Under these conditions, both the scattering problem $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ under their respective transmission/boundary and radiating conditions are well posed in appropriate spaces following the lines described in [22, 62] for instance. More details are given in the text. In addition, due to the Stratton-Chu formula when $\Im k$ is different from zero, the scattered electromagnetic fields have a fast decay at infinity as we have attenuation. But when $\Im k = 0$, i.e. in the absence of attenuation, we have the following behavior (as spherical-waves) of the scattered electric fields far away from the sources D_m's

$$E^{\rm sc}(x) = \frac{e^{ik|x|}}{|x|}\{E^{\infty}(\hat{x}) + O(|x|^{-1})\}, \quad |x| \longmapsto \infty, \tag{3.1.8}$$

and we have a similar behavior for the scattered magnetic field as well

$$H^{\rm sc}(x) = \frac{e^{ik|x|}}{|x|}\{H^{\infty}(\hat{x}) + O(|x|^{-1})\}, \quad |x| \longmapsto \infty \tag{3.1.9}$$

where $(E^{\infty}(\hat{x}), H^{\infty}(\hat{x}))$ is the electromagnetic far field pattern in the direction of propagation $\hat{x} := \frac{x}{|x|}$.

We set, for $m \in \{1, ..., \aleph\}$,

$$\boldsymbol{\mathcal{A}}_f^m := \frac{1}{|D_m|}\int_{D_m} f dv, \tag{3.1.10}$$

and

$$\boldsymbol{\mathcal{A}}_f(x) := \sum_{m=1}^{\aleph} \boldsymbol{\mathcal{A}}_f^m \chi_{D_m}(x). \tag{3.1.11}$$

Here dv is the volume measure of $\mathbb{R}^3$, and $dv(x)$ will be denoted dx while for the surface measure of $\mathbb{R}^3$ write ds and ds_x when the variable of integration is specified. Finally, we recall the Green's function for the Helmholtz operator (i.e. the fundamental solution for the Helmholtz equation)

$$\Phi_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y,$$

and the electromagnetic dyadic Green's function

$$\Pi(x,y) := k^2\Phi_k(x,y)\mathbb{I} + \nabla_x\nabla_x\Phi_k(x,y) = k^2\Phi_k(x,y)\mathbb{I} - \nabla_x\nabla_y\Phi_k(x,y), \quad x \neq y. \tag{3.1.12}$$

Our main results are stated in the following two theorems.

Theorem 20. *For the scattering by a cluster of small anisotropic particles embedded in a homogeneous background whose parameter satisfy the conditions (3.1.7), with maximal diameter $\boldsymbol{a}$ and minimal distance separating them $\boldsymbol{\delta} = \boldsymbol{c}_r\boldsymbol{a}$. The far field of the scattered wave admits, provided that $\boldsymbol{c}_r = O(|k|)$, the following expansions,*

- *If both ε_r and μ_r are symmetric, then we have the following approximation*

$$\begin{aligned} E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\left(\mathcal{R}_m^{\varepsilon_r}\times\hat{x}\right) + \frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\mathcal{Q}_m^{\mu_r}\Big) \\ + O\Big(\frac{|k|(2|k|+1)}{\boldsymbol{c}_r^3}\Big[\frac{1}{\boldsymbol{c}_r^4} + \boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})| + \boldsymbol{a}\Big]\Big), \end{aligned} \tag{3.1.13}$$

where $(\mathcal{R}_m^{\varepsilon_r}, \mathcal{Q}_m^{\mu_r})_{m=1}^{\aleph}$ *is the solution of the following invertible linear system*

$$\begin{aligned}
[\mathcal{P}_{D_m}^{\mu_r^*}]^{-1}\mathcal{Q}_m^{\mu_r} &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{Q}_j^{\mu_r} - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{R}_j^{\varepsilon_r}\Big] + H^{in}(\boldsymbol{z}_m) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{For} \quad m=1,...,\aleph. \\
[\mathcal{P}_{D_m}^{\varepsilon_r^*}]^{-1}\mathcal{R}_m^{\varepsilon_r} &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{R}_j^{\varepsilon_r} + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{Q}_j^{\mu_r}\Big] + E^{in}(\boldsymbol{z}_m).
\end{aligned} \tag{3.1.14}$$

- *If, in the contrary,* ε_r *or* μ_r *is not symmetric, then, with* $(\boldsymbol{\mathcal{A}}_{\varepsilon_r}^m)_{m=1}^{\aleph}$ *and* $(\boldsymbol{\mathcal{A}}_{\mu_r}^m)_{m=1}^{\aleph}$ *standing for their respective average in each particle* D_m*, we have*

$$\begin{aligned}
E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}&\Bigg(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\mathcal{R}_m\times\hat{x}\Big) + \frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\mathcal{Q}_m\Bigg) \\
&+ O\Big(\frac{|k|(2|k|+1)}{\boldsymbol{c}_r^3}\Big[\frac{1}{\boldsymbol{c}_r^4} + \boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})| + \boldsymbol{a}\Big]\Big),
\end{aligned} \tag{3.1.15}$$

where, in this case, $\Big(\mathcal{R}_m\Big)_{m=1}^{\aleph}$ *and* $\Big(\mathcal{Q}_m\Big)_{m=1}^{\aleph}$ *is the solution the invertible following linear system*

$$\begin{aligned}
[\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}_{\mu_r^*}^m}]^{-1}\mathcal{Q}_m &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{Q}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{R}_j\Big] + H^{in}(\boldsymbol{z}_m), \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{For} \quad m=1,...,\aleph. \\
[\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}_{\varepsilon_r^*}^m}]^{-1}\mathcal{R}_m &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{R}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{Q}_j\Big] + E^{in}(\boldsymbol{z}_m),
\end{aligned} \tag{3.1.16}$$

$$[\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}_{\mu_r^*}^m}]^{-1} := \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r^*}}^m[\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}_{\mu_r^*}^m}]^{-1}(\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}^m)^{-1},$$

and

$$[\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}_{\varepsilon_r^*}^m}]^{-1} := \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r^*}}^m[\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}_{\varepsilon_r^*}^m}]^{-1}(\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}^m)^{-1}.$$

For a given matrix B*, we have*

$$[\mathcal{P}_{D_m}^{B}] := \int_{D_m}\nabla V\ (B-\mathbb{I})dv \tag{3.1.17}$$

with V *is the solution of the following integral equation*

$$V - \operatorname{div}\int_{D_m}\frac{1}{|x-y|}(B-\mathbb{I})\nabla V(y)dy = (y-\boldsymbol{z}_m), \quad y\in D_m. \tag{3.1.18}$$

Theorem 21. *For the scattering by a cluster of small conducting particles, with maximal diameter $\boldsymbol{a}$ and minimal distance separating them $\boldsymbol{\delta} = \boldsymbol{c}_r\boldsymbol{a}$, with $\boldsymbol{c}_r = O(|k|)$, the far field of the scattered wave admits the following expansion*

$$\begin{aligned} E^\infty(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\mathcal{R}_m\times\hat{x}\Big) + \frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\mathcal{Q}_m\Big) \\ +O\Big(\frac{\boldsymbol{c}_L|k|(2|k|+1)}{\boldsymbol{c}_r^3}\Big[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Big]\Big), \end{aligned} \tag{3.1.19}$$

where $(\mathcal{R}_m, \mathcal{Q}_m)_{m=1}^{\aleph}$ is the solution the invertible linear system

$$\begin{aligned} \big[\mathcal{T}_{Dm}\big]^{-1}\mathcal{Q}_m &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{Q}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{R}_j\Big] + H^{in}(\boldsymbol{z}_m) \\ \big[\mathcal{P}_{Dm}\big]^{-1}\mathcal{R}_m &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{R}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{Q}_j\Big] + E^{in}(\boldsymbol{z}_m), \end{aligned} \quad \textit{For}\ \ m=1,...,\aleph, \tag{3.1.20}$$

with

$$\begin{aligned} \big[\mathcal{P}_{Dm}\big] &:= \int_{\partial D_m}[-I/2+(\boldsymbol{K}^0_{\partial D_m})^*]^{-1}(\nu)y^*ds_y, \\ \big[\mathcal{T}_{Dm}\big] &:= \int_{\partial D_m}[\frac{1}{2}I+(\boldsymbol{K}^0_{\partial D_m})^*]^{-1}(\nu)y^*ds_y. \end{aligned} \tag{3.1.21}$$

Before we provide the proofs of the above theorem, we would like to address some remarks.

- In the error of approximation, the constant appearing in the Landau notation are bounded by the largest ratio of the eigenvalues of both ε, μ, for the Theorem 20, and the largest Lipschitz constant for Theorem 21.
- In our opinion, and in the current form of the algebraic systems, it is hard to improve the approximation error order, except maybe for rotation invariant geometries by using the fundamental Newton's theorem for fields that are also rotation invariant.
- The tensor that appears could be explicitly calculated for simple geometries (sphere, ellipsoid) for more details see [9]. Further more, for a perfect conductor case, the tensor can be explicitly calculated, for convex geometries, using Neumann series as the spectral radius of the double layer potential have a spectral radius that is smaller than $\frac{1}{2}$.
- It is also possible to evaluate $[\mathcal{P}^{\mu_r}_{D_m}]$ and $[\mathcal{P}^{\varepsilon_r}_{D_m}]$, using boundary integral equation when both ε_r^* and μ_r^* are symmetric definite positive matrix see Section 3.2.1 here after.

3.2 Scattering by Anisotropic Inhomogenities. Problem ($\mathcal{P}_1$)

Let us introduce the Newton-like potential

$$\mathcal{S}^k_D(V) := \int_D \Phi_k(x,y)V(y)dy, \tag{3.2.1}$$

defined, for V in $\mathbb{L}^2(D)$ and maps continuously $\mathbb{L}^2(D)$ into $\mathbb{H}^2(D)$, (see Theorem 9.11 [30]) precisely

$$\left\|\mathcal{S}_D^k(V)\right\|_{\mathbb{H}^2(D)} \leq c_{2,k}\|V\|_{\mathbb{L}^2(D)}, \tag{3.2.2}$$

which is a compact integral operator from $L^2(D)$ to $H^s(D)$, $s < 1$. The constant $c_{2,k}$ remains independent of D. To show it, it suffices to write

$$\mathcal{S}_D^k(V) = \mathcal{S}_{B(v,R)}^k(V\chi_D)$$

for any sufficient large radius R to contain D and $v \in D$, and

$$\|\mathcal{S}_D^k(V)\|_{\mathbb{H}^2(D)} \leq \|\mathcal{S}_{B(v,R)}^k(V\chi_D)\|_{\mathbb{H}^2(B(v,R))} \leq \boldsymbol{c}_{2,k}\|V\chi_D\|_{\mathbb{L}^2(B(v,R))}. \tag{3.2.3}$$

Finally by $\mathbb{H}(\text{curl}, D)$, we mean the subspace of $\mathbb{L}^2(D)$-vector fields, with $\mathbb{L}^2(D)$ rotational, that is

$$\mathbb{H}(\text{curl}, D) = \Big\{u \in \mathbb{L}^2(D) \mid \operatorname{curl} u \in \mathbb{L}^2(D)\Big\}. \tag{3.2.4}$$

We also define, for a bounded tensor C,

$$\mathcal{S}_D^{k,C}(V) := \int_D \Phi_k(x,y)C(y)V(y)dy, \tag{3.2.5}$$

for V in $\mathbb{L}^2(D)$.

3.2.1 Anisotropic polarization tensor

Let us set

$$[\mathcal{P}_{D_m}^B] := \int_{D_m} \nabla V_m^{\boldsymbol{\mathcal{C}}_B}\, \boldsymbol{\mathcal{C}}_B\, dv, \tag{3.2.6}$$

where $V_m^{\boldsymbol{\mathcal{C}}_B} := V_m^{\text{sc}} + (x - \boldsymbol{z}_m)$ is the solution of the following problem

$$\begin{cases} \Delta V_m^{\boldsymbol{\mathcal{C}}_B} = 0 \text{ in } \mathbb{R}^3 \setminus D_m, \\ \operatorname{div}(A\nabla V_m^{\boldsymbol{\mathcal{C}}_B}) = 0 \text{ in } D_m, \\ V_m^{\boldsymbol{\mathcal{C}}_B} = V^{\text{sc}} + (x - z_i), \\ V_m^{\boldsymbol{\mathcal{C}}_B}|_- - V_m^{\boldsymbol{\mathcal{C}}_B}|_+ = 0 \text{ on } \partial D_m, \\ \nu \cdot B\nabla V_m^{\boldsymbol{\mathcal{C}}_B}|_- - \nu \cdot \nabla V_m^{\boldsymbol{\mathcal{C}}_B}|_+ = 0, \\ V_m^{\text{sc}} \to 0, |x| \to \infty. \end{cases} \tag{$\mathcal{P}r^{Ani}(1)$}$$

The problem ($\mathcal{P}r^{Ani}(1)$) is solved by the following Lippmann-Schwinger integral equation with $V_l := (V_m^{\boldsymbol{\mathcal{C}}_B})_l = V_m^{\boldsymbol{\mathcal{C}}_B} \cdot e_l$, for $l = 1, 2, 3$,

$$V_l - \operatorname{div} \mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_B}(\nabla V_l) = (x - \boldsymbol{z}_m) \cdot e_l. \tag{3.2.7}$$

The following proposition summarizes the needed properties of the polarization tensor introduced above.

Proposition 22. *Every solution to ($\mathcal{P}r^{Ani}(1)$) satisfies the following estimate*

$$\|V\|_{\mathbb{H}^1(D_m)} \leq \left(\frac{1}{2} - \boldsymbol{a}^2 \|\boldsymbol{\mathcal{C}}_B\|_{\mathbb{L}^\infty(D_m)} \sqrt{\frac{3}{\pi}}\right)^{-1} (\|\boldsymbol{\mathcal{C}}_B\|_{\mathbb{L}^\infty(D_m)} + 1)\boldsymbol{a}^{3/2}. \tag{3.2.8}$$

Furthermore, whenever $B \in \mathbb{W}^{1,\infty}(D_m)$ and $\boldsymbol{a}$ sufficiently small, the tensor (3.2.6) behaves like $\boldsymbol{\mathcal{C}}_B$ in terms of positive or negative definiteness and symmetry, namely

$$([\mathcal{P}^B_{D_m}]U, \overline{U}) > 0, \ \textit{whenever} \ (\boldsymbol{\mathcal{C}}_B U, \overline{U}) > 0, \tag{3.2.9}$$

and

$$\Re([\mathcal{P}^B_{D_m}]U, \overline{U}) > 0, \textit{whenever} \ \Re(\boldsymbol{\mathcal{C}}_B U, \overline{U}) > 0. \tag{3.2.10}$$

In addition, we have the following scaling property

$$[\mathcal{P}^B_{D_m}] = \boldsymbol{a}^3 [\mathcal{P}^{\widehat{B}}_{\boldsymbol{\mathcal{D}}_m}], \tag{3.2.11}$$

where for $s \in \boldsymbol{\mathcal{D}}_m$, $\widehat{B}(s) := B(\boldsymbol{a}s + z_m)$.

Before stating the proof, we need to introduce the anisotropic polarization tensor, defined in ([9], p.121-122), as

$$\left([\boldsymbol{\mathcal{P}}^B_{\partial D_m}]\right)_{ij} = \int_{\partial D_m} \nu \cdot (\boldsymbol{\mathcal{C}}_B e_j)(\Theta_m)_i|_- ds,$$

where Θ_m solves, for a fixed $m \in \{1, ..., \aleph\}$, the following transmission problem

$$\begin{cases} \Delta\Theta = 0 \text{ in } \mathbb{R}^3 \setminus D_m, \\ \operatorname{div}(B\nabla\Theta) = 0 \text{ in } D_m, \\ \Theta|_- - \Theta|_+ = (x - \boldsymbol{z}_m) \text{ on } \partial D_m, \\ \nu \cdot B\nabla\Theta|_- - \nu \cdot \nabla\Theta|_+ = \nu \cdot \nabla(x - \boldsymbol{z}_m), \\ \Theta \to 0, |x| \to \infty. \end{cases} \tag{$\mathcal{P}r^{Ani}$}$$

Obviously, we have

$$\Theta_m := \begin{cases} V^{\text{sc}}_m \text{ in } \mathbb{R}^3 \setminus D_m, \\ V^{\text{sc}}_m + (x - \boldsymbol{z}_m) \text{ in } D_m. \end{cases} \tag{3.2.12}$$

Hence, being

$$\begin{aligned} \left([\boldsymbol{\mathcal{P}}^B_{\partial D_m}]\right)_{ij} &= \int_{\partial D_m} \nu \cdot (\boldsymbol{\mathcal{C}}_B e_j)(\Theta_m)_i|_- ds = \int_{D_m} \operatorname{div}\Big((\boldsymbol{\mathcal{C}}_B e_j) \ (\Theta_m \cdot e_i)\Big) dv, \\ &= \int_{D_m} ((\nabla\Theta_m)^* e_i) \cdot (\boldsymbol{\mathcal{C}}_B e_j) dv + \int_{D_m} (\Theta_m \cdot e_i) \operatorname{div}(\boldsymbol{\mathcal{C}}_B e_j) dv. \end{aligned}$$

we get[2]

$$[\boldsymbol{\mathcal{P}}^B_{\partial D_m}] = [\mathcal{P}^B_{D_m}] + \int_{D_m} \Theta_m \otimes \operatorname{div}(\boldsymbol{\mathcal{C}}^*_B) dv, \tag{3.2.13}$$

[2] Recall that, for a given matrix A $Ae_j \cdot e_i = (A)_{ij}$

or

$$[\mathcal{P}^B_{D_m}] = [\boldsymbol{\mathcal{P}}^B_{\partial D_m}] - \int_{D_m} \Theta_m \otimes \operatorname{div}(\boldsymbol{\mathcal{C}}^*_B) dv. \tag{3.2.14}$$

where the divergence is applied to each line of the matrix taken as a vector field, and $\boldsymbol{\mathcal{C}}^*_B$ stands for the transpose of $\boldsymbol{\mathcal{C}}_B$. With the above notations, we have the following lemma.

Lemma 23. *(Theorem 3.4 of [39]) The polarization tensor $[\boldsymbol{\mathcal{P}}^B_{\partial D_m}]$ behaves like $\boldsymbol{\mathcal{C}}_B$ in term of positive or negative definiteness and symmetry.*

Proof. (of Proposition 22) The operator $[I - \operatorname{div} \mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} \nabla]$ is one-to-one on $\mathbb{H}^1(D_m)$, provided that $\boldsymbol{\mathcal{C}}_B$ is definite-positive, and it satisfies the following estimates (see [41] or the proof of (3.2.32) in the next subsection.)

$$\begin{aligned} \|[I - \operatorname{div} \mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} \nabla](V)\|_{\mathbb{H}^{1,\boldsymbol{\mathcal{C}}_B}(D_m)} &:= \int_{D_m} [I - \operatorname{div} \mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} \nabla](V) \cdot \overline{V} dx \\ &+ \int_{D_m} \nabla [I - \operatorname{div} \mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} \nabla](V) \cdot \boldsymbol{\mathcal{C}}_B \nabla \overline{V} dx \geq \frac{\|V\|_{\mathbb{H}^1(D_m)}}{2}. \end{aligned} \tag{3.2.15}$$

We have

$$\|[\operatorname{div}(\mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} - \mathcal{S}^{0,\boldsymbol{\mathcal{C}}_B}_{D_m}) \nabla V_l]\|_{\mathbb{H}^1(D_m)} \leq \frac{3}{\sqrt{\pi}} \boldsymbol{a}^2 \|V_l\|_{\mathbb{H}^1(D_m)}, \tag{3.2.16}$$

since, due to (A.0.3) and (A.0.4), we have both

$$\begin{cases} |[\operatorname{div}(\mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} - \mathcal{S}^{0,\boldsymbol{\mathcal{C}}_B}_{D_m}) \nabla](V)(x)| \leq \dfrac{1}{4\pi} \displaystyle\int_{D_m} |\nabla V_l(y)| \, dy \leq \dfrac{1}{\sqrt{3}} \dfrac{\boldsymbol{a}^{\frac{3}{2}}}{4} \|V_l\|_{\mathbb{H}^1(D_m)} \\ |\nabla [\operatorname{div}(\mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} - \mathcal{S}^{0,\boldsymbol{\mathcal{C}}_B}_{D_m}) \nabla](V)(x)| \leq \displaystyle\int_{D_m} \dfrac{1}{4\pi |x-y|} \Big[1 + \big(2 + \dfrac{1}{|x-y|}\big)|x-y|\Big] |\nabla V_l(y)| \, dy, \\ \qquad \leq \displaystyle\int_{D_m} \dfrac{1}{2\pi} \Big[1 + \dfrac{1}{|x-y|}\Big] |\nabla V_l(y)| \, dy \leq \dfrac{2}{\sqrt{\pi}} \sqrt{\boldsymbol{a}} \|V_l\|_{\mathbb{H}^1(D_m)}, \end{cases} \tag{3.2.17}$$

being, obviously,

$$\int_{D_m} \frac{1}{|x-y|^2} \, dy \leq \lim_{r \to 0} \int_{B(y,\boldsymbol{a}) \setminus B(y,r)} \frac{1}{|x-y|^2} \, dy \leq \lim_{r \to 0} \int_0^{2\pi} \int_0^{\pi} \int_r^{\boldsymbol{a}} \frac{1}{R^2} \, R^2 dR \sin(\theta) d\theta \, d\phi \leq 2\pi \boldsymbol{a}. \tag{3.2.18}$$

We write

$$(x - z_i) = [I - \operatorname{div} \mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m}](V) = [I - \operatorname{div} \mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m}](V) + [\operatorname{div}(\mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} - \mathcal{S}^{k,\boldsymbol{\mathcal{C}}_B}_{D_m}) \nabla](V), \tag{3.2.19}$$

then from (3.2.15) and (3.2.16), we get

$$\begin{aligned} \|(x - z_i)\|_{\mathbb{H}^{1,Q}(D_m)} &\geq \|[I - \operatorname{div} \mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m}](V)\|_{\mathbb{H}^{1,Q}(D_m)} - \|[\operatorname{div}(\mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m} - \mathcal{S}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m}) \nabla](V)\|_{\mathbb{H}^{1,Q}(D_m)}, \\ &\geq \frac{\|V\|_{\mathbb{H}^1(D_m)}}{2} - \frac{\sqrt{3}\boldsymbol{a}^2}{\sqrt{\pi}} \|\boldsymbol{\mathcal{C}}_B\|_{\mathbb{L}^\infty(D_m)} \|V\|_{\mathbb{H}^1(D_m)}. \end{aligned}$$

The remaining part of the proof is due to the fact that, for $\boldsymbol{a}$ sufficiently small, $[\mathcal{P}^B_{D_m}]$ inherits its property from $[\boldsymbol{\mathcal{P}}^B_{\partial D_m}]$ through Lemma 23.

□

Finally, we have the following, obvious, statement.

Proposition 24. *For $U_m^{\boldsymbol{\mathcal{C}}_B} := \boldsymbol{\mathcal{C}}_B \nabla V_m^{\boldsymbol{\mathcal{C}}_B}$, we have*

$$\int_{D_m} U_m^{\boldsymbol{\mathcal{C}}_B} dv = \int_{D_m} \boldsymbol{\mathcal{C}}_B \nabla V_m^{\boldsymbol{\mathcal{C}}_B} dv = [\mathcal{P}^B_{D_m}], \tag{3.2.20}$$

for $\boldsymbol{\mathcal{C}}_B$ symmetric. If B is a constant matrix and not necessarily symmetric, we have

$$\int_{D_m} U_m^{\boldsymbol{\mathcal{C}}_B} dv = \boldsymbol{\mathcal{C}}_B [\mathcal{P}^B_{D_m}] \boldsymbol{\mathcal{C}}_B^{-1}. \tag{3.2.21}$$

3.2.2 Lippmann-Schwinger integral formulation and apriori estimates.

In this section, we establish the wellposedness of our problem ($\mathcal{P}_1$). We start with showing the uniqueness of its solution and then prove its existence using a Lippmann-Schwinger integral representation of the electromagnetic field and at the same time we give an estimate of the total field taking into account the cluster of our small inhomogeneities.

We first recall that the contrast of magnetic permeability and electric permittivity, which are assumed to be respectively real and complex valued 3×3-tensor, essentially uniformly bounded with their derivatives, i.e.

$$(\|\boldsymbol{\mathcal{C}}_A\|_{\mathbb{W}^{1,\infty}(\cup_{i=1}{m} D_m)})_{A=\varepsilon_r,\ \mu_r} \leq \boldsymbol{c}_\infty, \tag{3.2.22}$$

and essentially uniformly coercive, that is, for almost every $x \in D$, we have

$$\begin{aligned} \Re\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}(x) U \cdot \overline{U}\Big) &\geq c_\infty^{\varepsilon-} |U|^2, \\ \boldsymbol{\mathcal{C}}_{\mu_r}(x) U \cdot U &\geq c_\infty^{\mu-} |U|^2. \end{aligned} \tag{3.2.23}$$

A sufficient condition to get the above inequality for the contrast of the relative electric permittivity is given by

$$\rho^-(\Re \boldsymbol{\mathcal{C}}_{\varepsilon_r}) - \rho^+(\Im \boldsymbol{\mathcal{C}}_{\varepsilon_r}) > 0,$$

where $\rho^+(A)$ and $\rho^-(A)$ stand respectively, for the largest and the smallest eigenvalue of A. As $\boldsymbol{\mathcal{C}}_{\mu_r}$ is a real valued 3×3-tensor, it suffices for it to be definite positive.
Indeed, we have

$$\begin{aligned} \Re\big\langle V, \overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r} V}\big\rangle = \Re\big\langle \overline{V}, \boldsymbol{\mathcal{C}}_{\varepsilon_r} V\big\rangle &= \big\langle \Im V, \Re\boldsymbol{\mathcal{C}}_{\varepsilon_r} \Im V\big\rangle + \big\langle \Re V, \Re\boldsymbol{\mathcal{C}}_{\varepsilon_r} \Re V\big\rangle \\ &\quad + \Big(\big\langle \Re V, \Im\boldsymbol{\mathcal{C}}_{\varepsilon_r} \Im V\big\rangle - \big\langle \Im V, \Im\boldsymbol{\mathcal{C}}_{\varepsilon_r} \Re V\big\rangle\Big), \\ &\geq \rho^+(\Re\boldsymbol{\mathcal{C}}_{\varepsilon_r})(\|\Im V\|^2 + \|\Re V\|^2) - 2\rho^-(\Im\boldsymbol{\mathcal{C}}_{\varepsilon_r})(\|\Im V\|\|\Re V\|), \\ &\geq \Big(\rho^-(\Re\boldsymbol{\mathcal{C}}_{\varepsilon_r}) - \rho^+(\Im\boldsymbol{\mathcal{C}}_{\varepsilon_r})\Big)\|V\|^2. \end{aligned} \tag{3.2.24}$$

Proposition 25. *The Problem ($\mathcal{P}_1$) admits a unique solution.*

Proof. We suppose that ε_r and μ_r are 3×3-tensors. Let ($E = E_1 - E_2$, $H = H_1 - H_2$) be the difference of two solutions of the problem ($\mathcal{P}_1$). Obviously both the normal trace of E and H are continuous across the boundary and satisfy the Silver–Müller radiation condition, hence, due to Green's formula, we have

$$\int_{\partial D} \nu \times E|_{\pm} \cdot \overline{H} ds = ik \int_D \mu_r H \cdot \overline{H} dv + \overline{ik} \int_D E \cdot \varepsilon_r \overline{E} dv = ik\Big(\int_D \mu_r H \cdot \overline{H} dv - \int_D E \cdot \varepsilon_r \overline{E} dv\Big), \quad (3.2.25)$$

and taking the real part gives

$$-\Re \int_{\partial D} \nu \times E \cdot \overline{H} ds = \int_D (\mu_r - \mu_r^*) \Im H \cdot \Re H dv + \int_D (\varepsilon_r - \varepsilon_r^*) \Im E \cdot \Re E dv. \quad (3.2.26)$$

Furthermore, we have, $\Im(H) \times \Re(H) = \Im(E) \times \Re(E) = 0$,[3] and, with $(\varepsilon_r)_{ij}$ standing for the component of ε_r, we have

$$(\varepsilon_r - \varepsilon_r^*) V = \overbrace{\begin{pmatrix} (\varepsilon_r)_{12} - (\varepsilon_r)_{21} \\ (\varepsilon_r)_{13} - (\varepsilon_r)_{31} \\ (\varepsilon_r)_{23} - (\varepsilon_r)_{32} \end{pmatrix}}^{U(\varepsilon_r) :=} \times V, \quad (3.2.27)$$

which guaranties that $(\mu_r - \mu_r^*) \Im H \cdot \Re H = (U(\varepsilon_r) \times \Im(H)) \cdot \Re H = 0$. The same observation concerning the second integral of the right-hand side of (3.2.26), implies that

$$\Re \int_{\partial D} \nu \times E \cdot \overline{H} ds = 0.$$

The Rellich lemma (see [22]) induces that $E \equiv H \equiv 0$. □

To prove the exitence of the solution and derive the needed estimates, we use the equivalent Lippmann-Schwinger equation. For that, we define the operator of Lippmann-Schwinger to be

$$\mathcal{LS}^{(\varepsilon_r, \mu_r)} := \begin{pmatrix} I - (k^2 + \nabla \operatorname{div}) \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}} & -ik \operatorname{curl} \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\mu_r}} \\ +ik \operatorname{curl} \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}} & I - (k^2 + \nabla \operatorname{div}) \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\mu_r}} \end{pmatrix}. \quad (3.2.28)$$

We set (E, H) to be a solution, provided that it is solvable, of the integral equation,

$$\mathcal{LS}^{(\varepsilon_r, \mu_r)} \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} E^{\text{in}} \\ H^{\text{in}} \end{pmatrix}. \quad (\mathcal{M}_{\mathcal{L}.\mathcal{S}})$$

and $(E_{\mathcal{A}}, H_{\mathcal{A}})$ the solution of the Lippmann-Schwinger equation with averaged parameter, that is

$$\mathcal{LS}^{(\boldsymbol{\mathcal{A}}_{\varepsilon_r}, \boldsymbol{\mathcal{A}}_{\mu_r})} \begin{pmatrix} E_{\mathcal{A}} \\ H_{\mathcal{A}} \end{pmatrix} = \begin{pmatrix} E^{\text{in}} \\ H^{\text{in}} \end{pmatrix}. \quad (\mathcal{A} - \mathcal{M}_{\mathcal{L}.\mathcal{S}})$$

With the previous notations, we have the following proposition.

[3]Due to the fact that $H \times H = E \times E = 0$.

Proposition 26. *A solution of the equation ($\mathcal{M}_{\mathcal{L}.\mathcal{S}}$) for $\Re(k) \geq 0$, $\Im(k) \geq 0$ solves the problem ($\mathcal{P}_1$). Further, under the conditions (3.2.23), the operator $\mathcal{LS}^{(\varepsilon_r,\mu_r)}$ is an isomorphism of $\mathcal{H}(\mathrm{curl}, \cup_{m=1}^{\aleph} D_m)$ provided that*

$$\boldsymbol{c}_r := \frac{\boldsymbol{\delta}}{\boldsymbol{a}} \geq \frac{c_0 2|k|\boldsymbol{c}_\infty^2}{\max(c_\infty^{\varepsilon^-}, c_\infty^{\mu^-})}, \tag{3.2.29}$$

and $V := E, H$, satisfy the following estimates

$$\|V\|_{\mathbb{L}^2(\cup_{m=1}^{\aleph} D_m))} \leq C\|(E^{in}, H^{in})\|_{\mathbb{H}(\mathrm{curl},\cup_{m=1}^{\aleph} D_m))}. \tag{3.2.30}$$

Besides for $\boldsymbol{\mathcal{C}}_{\mu_r}$, $\boldsymbol{\mathcal{C}}_{\varepsilon_r}$ in $\mathbb{W}^{1,\infty}(\cup_{i=1^m} D_m)$ and $(E_{\mathcal{A}}, H_{\mathcal{A}})$ solution of ($\mathcal{A} - \mathcal{M}_{\mathcal{L}.\mathcal{S}}$) then

$$\begin{aligned} \|E_{\mathcal{A}} - E\|_{\mathbb{L}^2(\cup D_m)} &\leq \boldsymbol{c}_{2,k} c_\infty \boldsymbol{a}\Big(\|H\|_{\mathbb{L}^2(\cup D_m)} + \|E\|_{\mathbb{L}^2(\cup D_m)}\Big), \\ \|H_{\mathcal{A}} - H\|_{\mathbb{L}^2(\cup D_m)} &\leq \boldsymbol{c}_{2,k} c_\infty \boldsymbol{a}\Big(\|H\|_{\mathbb{L}^2(\cup D_m)} + \|E\|_{\mathbb{L}^2(\cup D_m)}\Big), \end{aligned} \tag{3.2.31}$$

for some positive constant which depends only on k.

To prove this proposition, we need the following lemma.

Lemma 27. *The potential $\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}(V) := \mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_A}(V) := ((i\alpha)^2 + \nabla \operatorname{div})\mathcal{S}_D^{i\alpha,\boldsymbol{\mathcal{C}}_A}(V)$ solves*

$$\big(\mathrm{curl}^2 \mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_A} + \alpha^2 \mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_A}\big)(V) = -\alpha^2 \boldsymbol{\mathcal{C}}_A V$$

and we have for $\alpha \geq 0$

$$-\int_D \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \overline{\boldsymbol{\mathcal{C}}_A V} dv \geq \|\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|_{\mathbb{H}(\mathrm{curl},\mathbb{R}^3)} \geq \alpha^2 \|\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)} \geq 0. \tag{3.2.32}$$

Further, both $\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_A} := ik \operatorname{curl} \mathcal{S}_D^{i\alpha,\boldsymbol{\mathcal{C}}_A}$ and $\mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_A} - \mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_A}$ are compact operators, and it holds that

$$\int_D (\mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_A} - \mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_A})(V) \cdot \overline{\boldsymbol{\mathcal{C}}_A V} dv \leq \frac{|k|(|k|+1)}{4\pi}\Big((|k|+5)\boldsymbol{a}^{\frac{5}{2}} + 2^3 c_0 \frac{|k|+5}{(1+2\boldsymbol{c}_r)^2 \boldsymbol{c}_r}\Big)\|\boldsymbol{\mathcal{C}}_A V\|^2_{\mathbb{L}^2(D_m)}, \tag{3.2.33}$$

$$\int_D \mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_{A_1}}(V_1) \cdot \overline{\boldsymbol{\mathcal{C}}_{A_2} V_2} dv \leq \Big((\frac{1}{2} + \frac{k}{4}\boldsymbol{a})\boldsymbol{a} + c_0 \frac{(1+|k|)}{4\pi \boldsymbol{c}_r^3}\Big)\|\boldsymbol{\mathcal{C}}_{A_2} V_2\|_{\mathbb{L}^2(D)} \|\boldsymbol{\mathcal{C}}_{A_1} V_1\|_{\mathbb{L}^2(D)}, \tag{3.2.34}$$

where we recall that $\boldsymbol{c}_r = \frac{\boldsymbol{\delta}}{\boldsymbol{a}}$.

Proof. First, we write

$$\begin{aligned} \int_D (\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \overline{\boldsymbol{\mathcal{C}}_A V}) dv &= -\int_D \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \Big(\mathrm{curl}^2 \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} + \alpha^2 \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}}\Big) dx, \\ &= -\int_D \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \mathrm{curl}^2 \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} dx - \alpha^2 \|\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(D)}. \end{aligned} \tag{3.2.35}$$

Due to Green's formula inside D, we have

$$\int_{\partial D} \nu \times \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl} \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} ds = \|\operatorname{curl} \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(D)} - \int_D \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl}^2 \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} dx. \tag{3.2.36}$$

As, outside of D, $\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}$ satisfies the Maxwell equation for $k = i\alpha$, a direct application of Green's identity outside of D, for a sufficiently large $R > 0$,[4] implies that

$$\begin{aligned}\int_{\partial B_R} \nu \times \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl} \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} ds \\ - \int_{\partial D} \nu \times \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl} \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} ds = \|\operatorname{curl} \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(B_R\setminus D)} + \alpha^2 \|\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(B_R\setminus D)}\end{aligned}$$

which guaranties[5] that

$$-\Re\Big(\int_{\partial D} \nu \times \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl} \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} ds\Big) \geq \|\operatorname{curl} \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)} + \alpha^2 \|\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}.$$

Further, we have

$$\Im\Big(\int_{\partial B_R} \nu_{\partial B_R} \times \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl} \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}}\ ds\Big) - \Im\Big(\int_{\partial D} \nu \times \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl} \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}}\ ds\Big) = 0,$$

which, with the radiation condition, gives

$$-\int_{\partial D} \nu \times \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl} \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} ds \geq \|\operatorname{curl} \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)} + \alpha^2 \|\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}. \tag{3.2.37}$$

The above inequality in (3.2.36) gives

$$\|\operatorname{curl} \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)} + \alpha^2 \|\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)} \leq -\|\operatorname{curl} \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}\|^2_{\mathbb{L}^2(D)} + \int_D \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl}^2 \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} dv.$$

Then this last inequality in (3.2.35) ends the proof of (3.2.32).

Let us now prove (3.2.34) and (3.2.33). We write, for $\alpha = 1$,

$$\begin{aligned}\Big\|[\mathcal{N}_{D,V}^{i,\boldsymbol{\mathcal{C}}_A} - \mathcal{N}_{D,V}^{k,\boldsymbol{\mathcal{C}}_A}]\Big\|_{\mathbb{L}^2(D_m)} \leq & \Big(\sum_{m=1}^{\aleph} \Big\|[\mathcal{N}_{D_m,V}^{i,\boldsymbol{\mathcal{C}}_A} - \mathcal{N}_{D_m,V}^{k,\boldsymbol{\mathcal{C}}_A}]\Big\|^2_{\mathbb{L}^2(D_m)}\Big)^{\frac{1}{2}} \\ & + \Big(\sum_{m=1}^{\aleph} \Big\|[\mathcal{N}_{D\setminus D_m,V}^{i,\boldsymbol{\mathcal{C}}_A} - \mathcal{N}_{D\setminus D_m,V}^{k,\boldsymbol{\mathcal{C}}_A}]\Big\|_{\mathbb{L}^2(D_m)}\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.2.38}$$

[4] The continuity of the normal trace of $\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}$ across ∂D is due the facts that the operator $\nu \times \nabla$ is an isomorphism From $\mathbb{H}^s(\partial D) \setminus \mathbb{R}$ to $\mathbb{H}^{1-s}(\partial D) \setminus \mathbb{R}$ and $\operatorname{div} \mathcal{S}_D^{i\alpha,\boldsymbol{\mathcal{C}}_A}(\cdot)$ have a continuous Dirichlet trace.

[5] The obvious thing is that $\Re \int_{\partial B_R} \nu \times \mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A} \cdot \operatorname{curl} \overline{\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_A}} ds \underset{R\longrightarrow\infty}{\longrightarrow 0}$, due to the exponential decay of the kernel as $\alpha > 0$.

Obviously, we have, due to (A.0.2), that

$$[\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_A}_{D/D_m,V} - \mathcal{N}^{k,\boldsymbol{\mathcal{C}}_A}_{D/D_m,V}](x) = Int_1^m + Int_2^m$$

where

$$Int_1^m :=k^2 \int_{D\setminus D_m} \frac{e^{ik|x-y|}}{4\pi|x-y|}(\boldsymbol{\mathcal{C}}_A V)(y)dy - \int_{D\setminus D_m} \frac{e^{-|x-y|}}{4\pi|x-y|}(\boldsymbol{\mathcal{C}}_A V)(y)dy$$

and

$$Int_2^m := \int_{D\setminus D_i}\int_0^1 \frac{e^{\left((ik+1)t-1\right)|x-y|}\left((ik+1)t-1\right)}{4\pi(ik-1)^{-1}|x-y|}\Big[I+\Big((ik+1)t-1-\frac{1}{|x-y|}\Big)\frac{(x-y)^{\overset{2}{\otimes}}}{|x-y|}\Big]dt(\boldsymbol{\mathcal{C}}_A V)(y)dy.$$

We have, using Hölder's inequality, [6]

$$|Int_1^m| \leq \frac{(|k|^2+1)}{4\pi}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \frac{\boldsymbol{a}^{\frac{3}{2}}}{\boldsymbol{\delta}_{mj}}\|\boldsymbol{\mathcal{C}}_A V\|_{\mathbb{L}^2(D_j)},$$

and

$$\begin{aligned}|Int_2^m| &\leq \int_{D\setminus D_m} \frac{|k|(|k|+1)}{4\pi}\Big[\frac{2}{|x-y|}+|k|\Big]|\boldsymbol{\mathcal{C}}_A V|(y)dy,\\ &\leq \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{|k|(|k|+1)}{4\pi}\Big[\frac{2}{\boldsymbol{\delta}_{jm}}+|k|\Big]\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_A V\|_{\mathbb{L}^2(D_j)},\end{aligned}$$

which helps conclude, using Hölder's inequality for the second step once more and (A.0.6) of Lemma 43, that

$$\begin{aligned}&\sum_{m=1}^{\aleph}\Big\|\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_A}_{D/D_m,V} - \mathcal{N}^{k,\boldsymbol{\mathcal{C}}_A}_{D/D_m,V}\Big\|^2_{\mathbb{L}^2(D_m)}\\ &\leq \sum_{m=1}^{\aleph}\Big(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{|k|(|k|+1)}{4\pi}\Big[\frac{3}{\boldsymbol{\delta}_{jm}}+|k|\Big]\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_A V\|_{\mathbb{L}^2(D_j)}\Big)^2\boldsymbol{a}^3,\\ &\leq c_0\Big(\frac{(|k|+1)}{4\pi|k|^{-1}}\Big)^2\aleph\boldsymbol{a}^6\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\frac{9}{\boldsymbol{\delta}^2_{jm}}+\frac{2|k|}{\boldsymbol{\delta}_{jm}}+|k|^2\Big]\ \|\boldsymbol{\mathcal{C}}_A V\|^2_{\mathbb{L}^2(D_j)},\\ &\leq c_0\Big(\frac{(|k|+1)}{4\pi|k|^{-1}}\Big)^2\aleph\boldsymbol{a}^6\Big[\frac{9\aleph^{\frac{1}{3}}}{\boldsymbol{\delta}^2}+\frac{2\aleph^{\frac{2}{3}}|k|}{\boldsymbol{\delta}}+\aleph|k|^2\Big]\ \|\boldsymbol{\mathcal{C}}_A V\|^2_{\mathbb{L}^2(D)}.\end{aligned} \tag{3.2.39}$$

[6] We have considered $|k| > 1$.

Hence

$$\sum_{m=1}^{\aleph}\Big\|\mathcal{N}^{i,\mathcal{C}_A}_{D/D_m,V}-\mathcal{N}^{k,\mathcal{C}_A}_{D/D_m,V}\Big\|^2_{\mathbb{L}^2(D_m)}$$
$$\begin{aligned}&\leq c_0\Big(\frac{(|k|^2+|k|)}{4\pi}\Big)^2\Big[\frac{9\boldsymbol{a}^6}{(\boldsymbol{a}/2+\boldsymbol{\delta})^4\boldsymbol{\delta}^2}+\frac{2\boldsymbol{a}^6|k|}{(\boldsymbol{a}/2+\boldsymbol{\delta})^5\boldsymbol{\delta}}+\frac{\boldsymbol{a}^6|k|^2}{(\boldsymbol{a}/2+\boldsymbol{\delta})^6}\Big]\ \|\boldsymbol{\mathcal{C}}_AV\|^2_{\mathbb{L}^2(D)},\\ &\leq c_0\Big(\frac{(|k|^2+|k|)}{4\pi}\Big)^2\Big[\frac{9(2^4)}{(1+2\boldsymbol{c}_r)^4\boldsymbol{c}_r^2}+\frac{2^6|k|}{(1+2\boldsymbol{c}_r)^5\boldsymbol{c}_r}+\frac{2^6|k|^2}{(1+2\boldsymbol{c}_r)^6}\Big]\ \|\boldsymbol{\mathcal{C}}_AV\|^2_{\mathbb{L}^2(D)},\\ &\leq 2^3c_0\Big(\frac{(|k|^2+|k|)}{4\pi}\Big)^2\Big[\frac{|k|^2+4|k|+18}{(1+2\boldsymbol{c}_r)^4\boldsymbol{c}_r^2}\Big]\ \|\boldsymbol{\mathcal{C}}_AV\|^2_{\mathbb{L}^2(D)}.\end{aligned}\tag{3.2.40}$$

We deal now with the first term of the right-hand side of (3.2.38). We write

$$\begin{aligned}\Big|[\mathcal{N}^{i,\mathcal{C}_A}_{D_m,V}-\mathcal{N}^{k,\mathcal{C}_A}_{D_m,V}](x)\Big|&\leq\int_{D_m}\frac{|k|(|k|+1)}{4\pi}\Big[\Big(\frac{3}{|x-y|}+|k|\Big)\Big]|\boldsymbol{\mathcal{C}}_AV|(y)dy,\\ &\leq\frac{|k|(|k|+1)}{4\pi}\Big[\Big(\int_{D_m}\frac{3}{|x-y|^2}dy\Big)^{\frac{1}{2}}+|k|\boldsymbol{a}^{\frac{3}{2}}\Big]\|\boldsymbol{\mathcal{C}}_AV\|_{\mathbb{L}^2(D_m)}\end{aligned}$$

and, as done in (3.2.18), we obtain

$$\begin{aligned}\sum_{m=1}^{\aleph}\Big\|[\mathcal{N}^{i,\mathcal{C}_A}_{D_m,V}-\mathcal{N}^{k,\mathcal{C}_A}_{D_m,V}](x)\Big\|^2_{\mathbb{L}^2(D_m)}&\leq\sum_{m\geq1}^{\aleph}\boldsymbol{a}^3\Big(\frac{|k|(|k|+1)}{4\pi}\Big)^2\Big[\sqrt{6\pi\boldsymbol{a}}+|k|\boldsymbol{a}^{\frac{3}{2}}\Big]^2\|\boldsymbol{\mathcal{C}}_AV\|^2_{\mathbb{L}^2(D_m)},\\ &\leq\Big(\frac{|k|(|k|+1)}{4\pi}\Big)^2\Big[\sqrt{6\pi\boldsymbol{a}}+|k|\boldsymbol{a}^{\frac{3}{2}}\Big]^2\boldsymbol{a}^3\|\boldsymbol{\mathcal{C}}_AV\|^2_{\mathbb{L}^2(\cup_{m\geq1}D_m)}.\end{aligned}\tag{3.2.41}$$

Gathering (3.2.40) and (3.2.41), we have

$$\Big\|[\mathcal{N}^{i,\mathcal{C}_A}_{D,V}-\mathcal{N}^{k,\mathcal{C}_A}_{D,V}]\Big\|_{\mathbb{L}^2(D)}\leq\frac{|k|(|k|+1)}{4\pi}\Big((|k|+5)\boldsymbol{a}^{\frac{5}{2}}+2^3c_0\frac{|k|+5}{(1+2\boldsymbol{c}_r)^2\boldsymbol{c}_r}\Big)\|\boldsymbol{\mathcal{C}}_AV\|_{\mathbb{L}^2(D_m)}.\tag{3.2.42}$$

To derive (3.2.34), we write

$$,\sum_{m=1}^{\aleph}\int_{D_m}\mathcal{M}^{k,\mathcal{C}_{A_1}}_{D,V_1}\cdot\overline{\boldsymbol{\mathcal{C}}_{A_2}V_2}dv=\sum_{m=1}^{\aleph}\int_{D_m}\mathcal{M}^{k,\mathcal{C}_{A_1}}_{D_m,V_1}\cdot\overline{\boldsymbol{\mathcal{C}}_{A_2}V_2}dv+\sum_{m=1}^{\aleph}\int_{D_m}\sum_{\substack{j\geq1\\ j\neq m}}^{\aleph}\mathcal{M}^{k,\mathcal{C}_{A_1}}_{D_j,V_1}\cdot\overline{\boldsymbol{\mathcal{C}}_{A_1}V_2}dv.\tag{3.2.43}$$

The first term of the second member can be estimated using young's inequality

$$\Big|\int_{\mathbb{R}^3}u(x)(v*w)(x)dx\Big|\leq\|u\|_{\mathbb{L}^2(\mathbb{R}^3)}\|w\|_{\mathbb{L}^2(\mathbb{R}^3)}\|v\|_{\mathbb{L}^1(\mathbb{R}^3)},$$

with $u(x) = |\boldsymbol{\mathcal{C}}_{A_2} V_2|(x)\chi_{D_m}(x), v(x) = |\nabla\Phi_k|(x)\chi_{B(0,\boldsymbol{a})}(x), w(x) = |\boldsymbol{\mathcal{C}}_{A_1} V_1|(x)\chi_{D_m}(x)$. We get[7]

$$\begin{aligned}\sum_{m=1}^{\aleph}\int_{R^3}&|\boldsymbol{\mathcal{C}}_{A_2} V_2|(x)\chi_{D_m}(x)\int_{R^3}|\nabla\phi_k(x-y)|\chi_{B(0,\boldsymbol{a})}(x-y)\ |\boldsymbol{\mathcal{C}}_{A_1} V_1|(y)\chi_{D_m}(y)dy\ dx\\ &\leq \sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{A_2} V_2\chi_{D_m}\|_{\mathbb{L}^2(\mathbb{R}^3)}\|\boldsymbol{\mathcal{C}}_{A_1} V_1\chi_{D_m}\|_{\mathbb{L}^2(\mathbb{R}^3)}\left\|\nabla\Phi_k\chi_{B(0,\boldsymbol{a})}\right\|_{\mathbb{L}^1(\mathbb{R}^3)},\\ &\leq \|\boldsymbol{\mathcal{C}}_{A_2} V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{A_1} V_1\|_{\mathbb{L}^2(D)}\int_{B(0,\boldsymbol{a})}\frac{1}{4\pi}\Big(\frac{1}{|x-y|^2}+\frac{|k|}{|x-y|}\Big)dx,\\ &\leq (\frac{1}{2}+\frac{k}{4}\boldsymbol{a})\boldsymbol{a}\|\boldsymbol{\mathcal{C}}_{A_2} V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{A_1} V_1\|_{\mathbb{L}^2(D)}.\end{aligned} \tag{3.2.44}$$

The later sum in the right hand side of (3.2.43) is smaller than

$$S_1 := \sum_{m=1}^{\aleph}\int_{D_m}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\int_{D_j}\frac{1}{\boldsymbol{\delta}_{mj}}(\frac{1}{\boldsymbol{\delta}_{mj}}+|k|)\ |\boldsymbol{\mathcal{C}}_{A_1} V_1|\ |\boldsymbol{\mathcal{C}}_{A_2} V_2|dv, \tag{3.2.45}$$

and similar calculation, as done above and using (A.0.6) of Lemma 43, gives successively

$$\begin{aligned}S_1 &\leq \sum_{m=1}^{\aleph}\boldsymbol{a}^3\|\boldsymbol{\mathcal{C}}_{A_2} V_2\|_{\mathbb{L}^2(D_m)}\Big(c_0(\frac{\aleph^{\frac{1}{3}}}{\boldsymbol{\delta}^2}+\frac{|k|\aleph^{\frac{2}{3}}}{\boldsymbol{\delta}})\Big)^{\frac{1}{2}}\Big(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}(\frac{1}{\boldsymbol{\delta}_{ij}^2}+\frac{|k|}{\boldsymbol{\delta}_{mj}})\|\boldsymbol{\mathcal{C}}_{A_1} V_1\|^2_{\mathbb{L}^2(D_j)}\Big)^{\frac{1}{2}},\\ &\leq \Big(c_0(\frac{\aleph^{\frac{1}{3}}}{\boldsymbol{\delta}^2}+\frac{|k|\aleph^{\frac{2}{3}}}{\boldsymbol{\delta}})\Big)^{\frac{1}{2}}\boldsymbol{a}^3\left(\|\boldsymbol{\mathcal{C}}_{A_2} V_2\|^2_{\mathbb{L}^2(D)}\sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}(\frac{1}{\boldsymbol{\delta}_{ij}^2}+\frac{|k|}{\boldsymbol{\delta}_{mj}})\|\boldsymbol{\mathcal{C}}_{A_1} V_1\|^2_{\mathbb{L}^2(D_j)}\right)^{\frac{1}{2}},\\ &\leq c_0(\frac{\aleph^{\frac{1}{3}}}{\boldsymbol{\delta}^2}+\frac{|k|\aleph^{\frac{2}{3}}}{\boldsymbol{\delta}})\boldsymbol{a}^3\|\boldsymbol{\mathcal{C}}_{A_2} V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{A_1} V_1\|_{\mathbb{L}^2(D)},\end{aligned}$$

which implies that

$$\left|\sum_{m=1}^{\aleph}\int_{D_m}\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_{A_1}}(V_1)\cdot\boldsymbol{\mathcal{C}}_{A_2} V_2 dv\right| \leq c_0\frac{(1+|k|)}{4\pi\boldsymbol{c}_r^3}\|\boldsymbol{\mathcal{C}}_{A_2} V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{A_1} V_1\|_{\mathbb{L}^2(D)}. \tag{3.2.46}$$

□

Proof. (Proposition 26) Consider the equation ($\mathcal{M}_{\mathcal{L}.\mathcal{S}}$), which is, with the notation of Lemma 27 and $\langle\cdot,\cdot\rangle$ standing for the scalar product in $\mathbb{L}^2(D)$,

$$\begin{aligned}\big\langle E,\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle - \big\langle\mathcal{N}_{D,E}^{i\alpha,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle - \big\langle\mathcal{N}_{D,E}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}-\mathcal{N}_{D,E}^{i\alpha,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle - \big\langle ik\mathcal{M}_{D,H}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle &= \big\langle E^{\mathrm{in}}\cdot\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle,\\ \big\langle H,\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle - \big\langle\mathcal{N}_{D,H}^{i\alpha,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle - \big\langle\mathcal{N}_{D,H}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}-\mathcal{N}_{D,H}^{i\alpha,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle + \big\langle ik\mathcal{M}_{D,E}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle &= \big\langle H^{\mathrm{in}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle,\end{aligned}$$

[7] Notice that $D_m \subset B(\boldsymbol{z}_m,\frac{\boldsymbol{a}}{2}) \subset B(y,\boldsymbol{a})$ whenever $y\in D_m$, and $\chi_{B(0,\boldsymbol{a})}(x-y) = \chi_{B(y,\boldsymbol{a})}(x)$.

and with (3.2.32), taking the real part of the above equation, we get

$$\begin{aligned}\Re\big\langle E,\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle-\Re\big\langle\mathcal{N}_{D,E}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}-\mathcal{N}_{D,E}^{i\alpha,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle-\Re\big\langle ik\mathcal{M}_{D,H}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle\leq&\Re\big\langle E^{\text{in}}\cdot\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle,\\ \Re\big\langle H,\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle-\Re\big\langle\mathcal{N}_{D,H}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}-\mathcal{N}_{D,H}^{i\alpha,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle+\Re\big\langle ik\mathcal{M}_{D,E}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle\leq&\Re\big\langle H^{\text{in}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle.\end{aligned}\tag{3.2.47}$$

With the estimations (3.2.33), (3.2.34) of Lemma 27 and the assumption (3.2.23) we get[8]

$$\begin{aligned}&c_\infty^{\varepsilon^-}\|E\|^2-c_\infty^2\frac{|k|(|k|+1)}{4\pi}\Big(\boldsymbol{a}^{\frac{3}{2}}+2^3c_0\frac{|k|+5}{(1+2\boldsymbol{c}_r)^2\boldsymbol{c}_r}\Big)\|E\|^2\\ &\qquad\qquad\qquad-\boldsymbol{c}_\infty^2|k|\Big(\boldsymbol{a}+c_0\frac{(1+|k|)}{4\pi\boldsymbol{c}_r^3}\Big)\|H\|\|E\|\leq\big\langle E^{\text{in}}\cdot\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\big\rangle,\\ &c_\infty^{\mu^-}\|H\|^2-c_\infty^2\frac{|k|(|k|+1)}{4\pi}\Big(\boldsymbol{a}^{\frac{3}{2}}+2^3c_0\frac{|k|+5}{(1+2\boldsymbol{c}_r)^2\boldsymbol{c}_r}\Big)\|H\|^2\\ &\qquad\qquad\qquad-\boldsymbol{c}_\infty^2|k|\Big(\boldsymbol{a}+c_0\frac{(1+|k|)}{4\pi\boldsymbol{c}_r^3}\Big)\|H\|\|E\|\leq\big\langle H^{\text{in}},\boldsymbol{\mathcal{C}}_{\mu_r}H\big\rangle,\end{aligned}$$

which, under the conditions

$$\boldsymbol{c}_r\geq\frac{c_0 2|k|\boldsymbol{c}_\infty^2}{\max(c_\infty^{\varepsilon^-},c_\infty^{\mu^-})},\quad\frac{|k|(|k|+1)}{4\pi}\boldsymbol{a}<1,\ \text{and}\ \boldsymbol{a}\leq\frac{1}{16},\tag{3.2.48}$$

gives

$$\|E\|^2-\frac{1}{4\pi}\|H\|\|E\|\leq\frac{\boldsymbol{c}_\infty}{c_\infty^{\varepsilon^-}}\|E^{\text{in}}\|\|E\|,$$

and

$$\|H\|^2-\frac{1}{4\pi}\|H\|\|E\|\leq\frac{\boldsymbol{c}_\infty}{c_\infty^{\mu^-}}\|H^{\text{in}}\|\|H\|.$$

More precisely,

$$\|E\|\leq\frac{5\boldsymbol{c}_\infty}{4c_\infty^{\varepsilon^-}}\Big(\|E^{\text{in}}\|+\frac{1}{4\pi}\|H^{\text{in}}\|\Big),\tag{3.2.49}$$

and

$$\|H\|\leq\frac{5\boldsymbol{c}_\infty}{4c_\infty^{\mu^-}}\Big(\|H^{\text{in}}\|+\frac{1}{4\pi}\|E^{\text{in}}\|\Big).\tag{3.2.50}$$

Concerning the estimate (3.2.31), identifying both left hand sides of the eqs. ($\mathcal{M}_{\mathcal{L}.\mathcal{S}}$) and ($\mathcal{A}-\mathcal{M}_{\mathcal{L}.\mathcal{S}}$), we have

$$\begin{aligned}(H_{\mathcal{A}}-H)-(k^2+\nabla\operatorname{div})\mathcal{S}_D^{k,\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}}(H_{\mathcal{A}}-H)+ik\operatorname{curl}\mathcal{S}_D^{k,\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}}(E_{\mathcal{A}}-E)\\ =-(k^2+\nabla\operatorname{div})\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\mu_r}-\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}}(H)+ik\operatorname{curl}\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}-\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}}(E),\end{aligned}$$

[8] Assuming that $|k|\boldsymbol{a}\leq 1$.

then, due to estimates (3.2.30) for the solution of Lippmann-Schwinger integral equation, we obtain

$$\begin{aligned}\|H_{\mathcal{A}} - H\|_{\mathbb{L}^2(D)} &\leq \left\| -(k^2 + \nabla \operatorname{div}) \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\mu_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}}(H) + ik \operatorname{curl} \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}}(E) \right\|_{\mathbb{L}^2(D)}, \\ &\leq \boldsymbol{c}_{2,k} \Big(\left\| (\boldsymbol{\mathcal{C}}_{\mu_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}) H \right\|_{\mathbb{L}^2(D)} + \left\| (\boldsymbol{\mathcal{C}}_{\varepsilon_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}) E \right\|_{\mathbb{L}^2(D)} \Big), \\ &\leq \boldsymbol{c}_{2,k} c_\infty \boldsymbol{a} \Big(\|H\|_{\mathbb{L}^2(D)} + \|E\|_{\mathbb{L}^2(D)} \Big).\end{aligned}$$

□

Remark 28. *We have two observations:*

- *We can consider either both real tensor or complex valued ones for the relative electric permittivity and magnetic permeability inducing similar assumption concerning the corresponding contrast in* (3.2.23)*.*
- *We could improve the condition on the ratio $\frac{\boldsymbol{a}}{\boldsymbol{\delta}}$ by taking a larger α. But this would increase the constant that appears in the estimation* (3.2.30) *which will result in the worsening of the error of approximation in the latter calculation.*

3.2.3 Field approximation and the related linear system

We set

$$\mathcal{Q}_m^B := \int_{D_m} \boldsymbol{\mathcal{C}}_B H \, dv, \quad \mathcal{R}_m^B := \int_{D_m} \boldsymbol{\mathcal{C}}_B E \, dv \tag{3.2.51}$$

and write, with $(E_{\mathcal{A}},\ H_{\mathcal{A}})$ solution of $(\mathcal{A} - \mathcal{M}_{\mathcal{L}.\mathcal{S}})$,

$$\mathcal{Q}_m := \int_{D_m} H_{\mathcal{A}} \, dv, \quad \mathcal{R}_m := \int_{D_m} E_{\mathcal{A}} \, dv. \tag{3.2.52}$$

For $U_m^{\mu_r^*} = \boldsymbol{\mathcal{C}}_{\mu_r}^* \nabla V_m^{\mu_r^*}$ and $U_m^{\varepsilon_r^*} = \boldsymbol{\mathcal{C}}_{\varepsilon_r}^* \nabla V_m^{\varepsilon_r^*}$, where $V_m^{\mu_r^*}$, $V_m^{\varepsilon_r^*}$ are the respective solutions of $({\mathcal{P}r}^{Ani}(1))$ with $B = \mu_r^*$ and $B = \varepsilon_r^*$, we recall that

$$U_m^{\mu_r^*} - \boldsymbol{\mathcal{C}}_{\mu_r}^* \nabla \operatorname{div} \mathcal{S}_{D_m}^0 (U_m^{\mu_r^*}) = \boldsymbol{\mathcal{C}}_{\mu_r}^* \tag{3.2.53}$$

and

$$U_m^{\varepsilon_r^*} - \boldsymbol{\mathcal{C}}_{\mu_r}^* \nabla \operatorname{div} \mathcal{S}_{D_m}^0 (U_m^{\varepsilon_r^*}) = \boldsymbol{\mathcal{C}}_{\varepsilon_r}^*. \tag{3.2.54}$$

Due to Equation (3.2.8), we get

$$\|U_m^{\mu_r^*}\|_{\mathbb{L}^2(D_m)} \leq \|\boldsymbol{\mathcal{C}}_{\mu_r}^* \nabla V\|_{\mathbb{L}^2(D_m)} \leq c_\infty \boldsymbol{a}^{\frac{3}{2}}, \tag{3.2.55}$$

and

$$\|U_m^{\varepsilon_r^*}\|_{\mathbb{L}^2(D_m)} \leq \|\boldsymbol{\mathcal{C}}_{\varepsilon_r}^* \nabla V\|_{\mathbb{L}^2(D_m)} \leq c_\infty \boldsymbol{a}^{\frac{3}{2}}, \tag{3.2.56}$$

where

$$c_\infty := \max_{m\leq\aleph} \sup_{x\in D_m,\, B=\varepsilon_r,\mu_r.} \frac{(\|B\|_{\mathbb{L}^\infty(D_m)}+2)^2}{\left(\frac{1}{2}-\boldsymbol{a}^2\|B\|_{\mathbb{L}^\infty(D_m)}\sqrt{\frac{3}{\pi}}\right)}$$

is a positive constant.

The following assumption could be seen as a consequence of the scaling (3.2.11) and Lemma 23, for $A=\varepsilon_r,\mu_r$,

$$\mu_B^-\boldsymbol{a}^3|V|^2 \leq [\mathcal{P}_{D_m}^B]V\cdot V \leq \mu_B^+\boldsymbol{a}^3|V|^2 \qquad \text{whenever } B\in\mathbb{W}^\infty(D_m,\mathbb{R}^3\times\mathbb{R}^3), \tag{3.2.57}$$

and

$$\mu_B^-\boldsymbol{a}^3|V|^2 \leq \Re\Big([\mathcal{P}_{D_m}^B]V\cdot\overline{V}\Big) \leq \mu_B^+\boldsymbol{a}^3|V|^2 \qquad \text{whenever } B\in\mathbb{W}^\infty(D_m,\mathbb{C}^3\times\mathbb{C}^3), \tag{3.2.58}$$

where $\mu_A^+ := \max_m \mu_{A,m}^+$, $\mu_A^- := \min_m \mu_{A,m}^-$ and $(\mu_{A,m}^\pm)_{m=1}^\aleph$ are constants that satisfy the previous inequalities for each D_m.

We set for U, $V\in\mathbb{L}^2(D_m)$

$$Er_m(U,V) := O\Bigg(\sum_{\substack{(j\geq 1)\\ j\neq m}}^{\aleph}\Big[\frac{\|V\|_{\mathbb{L}^2(D_j)}}{\boldsymbol{\delta}_{mj}^4}+\Big(\frac{3|k|}{\boldsymbol{\delta}_{mj}^3}+\frac{3|k|^2+1}{\boldsymbol{\delta}_{mj}^2}+\frac{|k|(|k|^2+1)}{\boldsymbol{\delta}_{mj}}\Big)\Big(\|U\|_{\mathbb{L}^2(D_j)}+\|V\|_{\mathbb{L}^2(D_j)}\Big)\Big]\boldsymbol{a}^{\frac{11}{2}}\Bigg). \tag{3.2.59}$$

With (A.0.6) of Lemma 43, we obtain

$$\sum_{m=1}^{\aleph} Er_m(U,V)^2 = O\Bigg(\|U\|_{\mathbb{L}^2(D)}^2\frac{\boldsymbol{a}^{11}}{\boldsymbol{\delta}^8}+\Big(\|U\|_{\mathbb{L}^2(D)}^2+\|V\|_{\mathbb{L}^2(\cup_m D_m)}^2\Big)\frac{(|k|+2)^3\boldsymbol{a}^{11}|\ln(\boldsymbol{\delta})|}{\boldsymbol{\delta}^6}\Bigg). \tag{3.2.60}$$

The far field of the scattered wave is given by, see ([22], (6.26)-(6.27) p. 199),

$$E^\infty(\hat{x}) = \frac{k^2}{4\pi}\hat{x}\times\Big(\int_{\cup_{m=1}^\aleph D_m} e^{-ik\hat{x}\cdot y}\boldsymbol{\mathcal{C}}_{\varepsilon_r}E(y)dy\times\hat{x}\Big)+\frac{ik}{4\pi}\hat{x}\times\int_{\cup_{m=1}^\aleph D_m} e^{-ik\hat{x}\cdot y}\boldsymbol{\mathcal{C}}_{\mu_r}H(y)dy. \tag{3.2.61}$$

Furthermore, for the scattering of plane wave, (3.2.30) gives, with (3.2.49) and (3.2.50),

$$\|E\|_{\mathbb{L}^2(D)} \leq \frac{5\boldsymbol{c}_\infty}{4c_\infty^{\varepsilon^-}}\Big(|P|+\frac{|k|}{4\pi}|\theta\times P|\Big)\Big(\frac{1}{\boldsymbol{c}_r}\Big)^{\frac{3}{2}} \tag{3.2.62}$$

and

$$\|H\|_{\mathbb{L}^2(D)} \leq \frac{5\boldsymbol{c}_\infty}{4c_\infty^{\varepsilon^-}}\Big(\frac{1}{4\pi}|P|+|k||\theta\times P|\Big)\Big(\frac{1}{\boldsymbol{c}_r}\Big)^{\frac{3}{2}}. \tag{3.2.63}$$

With the above estimates, we have the following result.

Proposition 29. *For $\hat{x}\in\mathbb{S}^1$, we have the following approximations for the far field of the scattered waves*

1. *if $\boldsymbol{\mathcal{C}}_{\mu_r}$, $\boldsymbol{\mathcal{C}}_{\varepsilon_r}$ are symmetric.*

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph} \left(\frac{k^2}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \left(\mathcal{R}_m^{\varepsilon_r} \times \hat{x} \right) + \frac{ik}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \mathcal{Q}_m^{\mu_r} \right) + \left(\frac{|k|^3}{\boldsymbol{c}_r^3} \boldsymbol{a} \right), \tag{3.2.64}$$

2. *if $\boldsymbol{\mathcal{C}}_{\mu_r}$, $\boldsymbol{\mathcal{C}}_{\varepsilon_r}$ are not symmetric*

$$\begin{aligned} E^{\infty}(\hat{x}) = & \sum_{m=1}^{\aleph} \left(\frac{k^2}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \left(\boldsymbol{\mathcal{C}}_{\varepsilon_r} \mathcal{R}_m \times \hat{x} \right) + \frac{ik}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \boldsymbol{\mathcal{C}}_{\mu_r} \mathcal{Q}_m \right) \\ & + O\left(\frac{|k|^3 + |k|^2}{\boldsymbol{c}_r^3} \boldsymbol{a} + \frac{|k| \boldsymbol{c}_\infty (\boldsymbol{c}_\infty \boldsymbol{c}_{2,k} + 1)}{\boldsymbol{c}_r^3} \boldsymbol{a} \right), \end{aligned} \tag{3.2.65}$$

uniformly for $\hat{x} \in \mathbb{S}^2$, where $\left(\mathcal{R}_m^{\varepsilon_r} \right)_{m=1}^{\aleph}$ and $\left(\mathcal{Q}_m^{\mu_r} \right)_{m=1}^{\aleph}$ are solutions of the following system

$$\begin{aligned} \mathcal{Q}_m^{\mu_r} = & [\mathcal{P}_{D_m}^{\mu_r}] \left(\sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \left[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \mathcal{Q}_j^{\mu_r} - ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \mathcal{R}_j^{\varepsilon_r} \right] + H^{in}(\boldsymbol{z}_m) \right) \\ & + Er_m(H, E) + O\left(k^2 \|\boldsymbol{\mathcal{C}}_{\mu_r} H\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{C}}_{\varepsilon_r} E\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}} + |k|^2 \boldsymbol{a}^4 \right), \\ \mathcal{R}_m^{\varepsilon_r} = & [\mathcal{P}_{D_m}^{\varepsilon_r}] \left(\sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \left[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \mathcal{R}_j^{\varepsilon_r} + ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \mathcal{Q}_j^{\mu_r} \right] + \cdot E^{in}(\boldsymbol{z}_m) \right) \\ & + Er_m(E, H) + O\left(k^2 \|\boldsymbol{\mathcal{C}}_{\varepsilon_r} E\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{C}}_{\mu_r} H\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}} + |k| \boldsymbol{a}^4 \right), \end{aligned} \tag{3.2.66}$$

and $\left(\mathcal{R}_m \right)_{m=1}^{\aleph}$ and $\left(\mathcal{Q}_m \right)_{m=1}^{\aleph}$ are solutions the following system

$$\begin{aligned} \mathcal{Q}_m = & [\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}_{\mu_r^*}^m}] (\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r^*}}^m)^{-1} \left(\sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \left[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}^j \mathcal{Q}_j - ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}^j \mathcal{R}_j \right] + H^{in}(\boldsymbol{z}_m) \right) \\ & + Er_m(H_{\mathcal{A}}) + O(k^2 \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}}), \\ \mathcal{R}_m = & [\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}_{\varepsilon_r^*}^m}] (\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r^*}}^m)^{-1} \left(\sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \left[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}^j \mathcal{R}_j + ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}^j \mathcal{Q}_j \right] + E^{in}(\boldsymbol{z}_m) \right) \\ & + Er_m(E_{\mathcal{A}}) + O(k^2 \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}}). \end{aligned} \tag{3.2.67}$$

Proof. For $x \in D_m$, we have from ($\mathcal{M}_{\mathcal{L}.\mathcal{S}}$)

$$\begin{aligned} & H - (k^2 + \nabla \operatorname{div}) \mathcal{S}_{D_m}^{k, \boldsymbol{\mathcal{C}}_{\mu_r}}(H) \\ & \quad + ik \operatorname{curl} \mathcal{S}_{D_m}^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E) = \sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \left[(k^2 + \nabla \operatorname{div}) \mathcal{S}_{D_j}^{k, \boldsymbol{\mathcal{C}}_{\mu_r}}(H) - ik \operatorname{curl} \mathcal{S}_{D_j}^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E)(x) \right] + E^{\text{in}}(x), \end{aligned} \tag{3.2.68}$$

and

$$E - (k^2 + \nabla \operatorname{div}) \mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}}_{D_m}(E) \\ - ik \operatorname{curl} \mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m}(H) = \sum_{\substack{j \geq 1 \\ j \neq m}}^{\aleph} \Big[(k^2 + \nabla \operatorname{div}) \mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}}_{D_j}(E) + ik \operatorname{curl} \mathcal{S}^{k, (\boldsymbol{\mathcal{C}}_{\mu_r})}_{D_j}(H)(x) \Big] + H^{\text{in}}(x), \tag{3.2.69}$$

- (*Derivation of* (3.2.66)) Multiplying the first member of (3.2.68), by $U_m^{\mu_r^*}$ and integrating over D_m, we derive

$$\begin{aligned}
&\int_{D_m} U_m^{\mu_r^*}(x) \cdot \Big(H - (k^2 + \nabla \operatorname{div}) \mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m}(H) + ik \operatorname{curl} \mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{A}}_{D_m}(E) \Big)(x) dx \\
&\quad = \int_{D_m} U_m^{\mu_r^*} \cdot \big[I - \nabla \operatorname{div} \mathcal{S}^{0, \boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m} \big](H)(x) dx - k^2 \int_{D_m} U_m^{\mu_r^*}(x) \cdot \mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m}(H)(x) dx \\
&\quad - \int_{D_m} U_m^{\mu_r^*}(x) \cdot \Big(\nabla \operatorname{div} \Big[\mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{A}}_{D_m} - \mathcal{S}^{0, \boldsymbol{\mathcal{C}}_{A}}_{D_m} \Big](H) - ik \operatorname{curl} \mathcal{S}^{k, Q^{\boldsymbol{\mathcal{C}}}(\varepsilon_r)}_{D_m}(E) \Big)(x) dx, \\
&\quad = \int_{D_m} U_m^{\mu_r^*} \cdot \big[I - \nabla \operatorname{div} \mathcal{S}^{0, \boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m} \big](H)(x) dx \\
&\quad + O\Big(k^2 \boldsymbol{a}^{7/2} \| \boldsymbol{\mathcal{C}}_{\mu_r} H \|_{\mathbb{L}^2(D_m)} + \boldsymbol{a}^{5/2} \| \boldsymbol{\mathcal{C}}_{\varepsilon_r} E \|_{\mathbb{L}^2(D_m)} \Big).
\end{aligned} \tag{3.2.70}$$

Indeed, due to the third identity (A.0.4), for $\alpha = 0$, it is obvious that

$$\begin{aligned}
&\Big| \nabla \operatorname{div} \big[\mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{A}}_{D_m} - \mathcal{S}^{0, \boldsymbol{\mathcal{C}}_{A}}_{D_m} \big](H)(x) \Big| \\
&\qquad \leq \Big| \int_{D_m} \Big(\int_0^1 \frac{e^{\big((ik)t\big)|(x-y)|} k^2 t}{4\pi |x-y|} \Big[\mathbb{I} + \Big(ikt - \frac{1}{|x-y|} \Big) \frac{(x-y)^{\overset{2}{\otimes}}}{|x-y|} \Big] dt \Big) (\boldsymbol{\mathcal{C}}_{\mu_r} H)(y) dy \Big|, \\
&\qquad \leq \int_{D_m} \Big[\frac{2k^2}{4\pi |x-y|} + \frac{k^3}{4\pi} \Big) \Big] \, \big| (\boldsymbol{\mathcal{C}}_{\mu_r} H)(y) \big| dy, \\
&\qquad \leq \Big[\frac{k^2}{2\pi} \Big(\lim_{r \to 0} \int_{B(y, \boldsymbol{a}) \setminus B(\boldsymbol{z}_m, r)} \frac{1}{|x-y|^2} dy \Big)^{\frac{1}{2}} + \frac{k^3}{4\pi} \boldsymbol{a}^{\frac{3}{2}} \Big] \big\| \boldsymbol{\mathcal{C}}_{\mu_r} H \big\|_{\mathbb{L}^2(D_m)}, \\
&\qquad \leq \Big[\frac{k^2}{2\pi} \pi \boldsymbol{a}^{\frac{1}{2}} + \frac{k^3}{4\pi} \boldsymbol{a}^{\frac{3}{2}} \Big] \big\| \boldsymbol{\mathcal{C}}_{\mu_r} H \big\|_{\mathbb{L}^2(D_m)},
\end{aligned}$$

hence,

$$\Big\| \nabla \operatorname{div} \big[\mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m} - \mathcal{S}^{0, \boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m} \big](H) \Big\|_{\mathbb{L}^2(D_m)} \leq \frac{\pi}{3} \Big[\frac{k^2}{\sqrt{2\pi}} + \frac{k^3}{4\pi} \boldsymbol{a} \Big] \boldsymbol{a}^2 \big\| \boldsymbol{\mathcal{C}}_{\mu_r} H \big\|_{\mathbb{L}^2(D_m)}.$$

We also have, for similar reasons,

$$\Big\| k^2 \big[\mathcal{S}^{k, \boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m} \big](H) \Big\|_{\mathbb{L}^2(D_m)} \leq \frac{\pi}{3} \Big[\frac{k^2}{2\pi} \sqrt{2\pi} \boldsymbol{a}^{\frac{1}{2}} \Big] \boldsymbol{a}^{\frac{3}{2}} \big\| \boldsymbol{\mathcal{C}}_{\mu_r} H \big\|_{\mathbb{L}^2(D_m)}.$$

Regarding the third term of (3.2.70), we have

$$\begin{aligned}
&\Big|\int_{D_m} U_m^{\mu_r^*}(x)\cdot\Big(ik\,\mathrm{curl}\mathcal{S}_{D_m}^{k,Q^{\boldsymbol{\mathcal{C}}}(\varepsilon_r)}(H)\Big)(x)dx\Big|\\
&\qquad\le \|U_m^{\mu_r^*}\|_{\mathbb{L}^2(D_m)}\big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(D_m)}\int_{B(0,\boldsymbol{a})}|\nabla\Phi_k(x)|dx,\\
&\qquad\le\frac{\boldsymbol{c}_{\mu_r}\boldsymbol{a}^{\frac{3}{2}}\big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(D_m)}}{4\pi}\int_{B(0,\boldsymbol{a})}\Big(\frac{1}{|x-y|^2}+\frac{|k|}{|x-y|}\Big)dx,\\
&\qquad\le\boldsymbol{c}_{\mu_r}\big(\frac{1}{2}+\frac{k}{4}\boldsymbol{a}\big)\boldsymbol{a}^{\frac{5}{2}}\big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(D_m)}.
\end{aligned}$$

Hence, we get from (3.2.70)

$$\begin{aligned}
&\int_{D_m}U_m^{\mu_r^*}\cdot\Big[H-(k^2+\nabla\,\mathrm{div})\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}(H)-ik\,\mathrm{curl}\,\mathcal{S}_{D_m}^{k,Q^{\boldsymbol{\mathcal{C}}}(\varepsilon_r)}(E)\Big]dv\\
&=\int_{D_m}\Big[I-\boldsymbol{\mathcal{C}}_{\mu_r}^*\nabla\,\mathrm{div}\,\mathcal{S}_{D_m}^0\Big](U_m^{\mu_r^*})\cdot H\;dv+O(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}),\\
&=\int_{D_m}\boldsymbol{\mathcal{C}}_{\mu_r}^*H\;dv+O(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}).
\end{aligned}\tag{3.2.71}$$

Multiplying the second member of (3.2.68) by $U_m^{\mu_r^*}$ ind integrating over D_m, gives

$$\begin{aligned}
&\sum_{\substack{j\ge1\\ j\neq m}}^{\aleph}\int_{D_m}U_m^{\mu_r^*}(x)\cdot\Big[\int_{D_j}\Pi_k(x,y)(\boldsymbol{\mathcal{C}}_{\mu_r}H)(y)dy+ik\int_{D_j}\nabla\Phi_k(x,y)\times(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E)(y)dy\Big]dx\\
&\qquad\qquad+\int_{D_m}U_m^{\mu_r^*}\cdot H^{\mathrm{in}}dv,
\end{aligned}$$

and a first order approximation, considering (A.0.1), gives

$$\begin{aligned}
&\int_{D_m}U_m^{\mu_r^*}dv\cdot\sum_{\substack{j\ge1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\int_{D_j}(\boldsymbol{\mathcal{C}}_{\mu_r}H)(y)dy+ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\int_{D_j}(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E)(y)dy\Big]\\
&+\int_{D_m}U_m^{\mu_r^*}dv\;H^{\mathrm{in}}(\boldsymbol{z}_j)+O(|k|^2\boldsymbol{a}^4)+2Er_m(H,E),
\end{aligned}\tag{3.2.72}$$

which gives, when the contrasts are symmetric, joined with (3.2.71), the first approximation in (3.2.66). Similar calculation gives the second one starting from (3.2.69), and multiplying by $U_m^{\varepsilon_r^*}$ as defined in (3.2.54).

- (*Derivation of* (3.2.67)) When, in the other hand, both $\boldsymbol{\mathcal{C}}_{\varepsilon_r},\boldsymbol{\mathcal{C}}_{\mu_r}$ are in $\mathbb{W}^{1,\infty}(\cup_{m=1}^{\aleph}D_m)$, and not necessarily symmetric, we get, with a first order approximation, for the combined eqs. (3.2.71) and (3.2.72) of the corresponding integral formulation of the Problem ($\mathcal{A}-\mathcal{M}_{\mathcal{L}.\mathcal{S}}$)

$$\int_{D_m} \boldsymbol{\mathcal{A}}^*_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}}\, dv + O(k^2 \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}})$$
$$= \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}_{\mu_r^*}} dv \cdot \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \int_{D_j} (\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}})(y) dy + ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \int_{D_j} (\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}})(y) dy\Big]$$
$$+ \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}_{\mu_r^*}} \cdot H^{\mathrm{in}} dv + Er_m^{(E_{\mathcal{A}}, H_{\mathcal{A}})}(H_{\mathcal{A}}),$$

which we rewrite as

$$\boldsymbol{\mathcal{A}}^*_{\boldsymbol{\mathcal{C}}_{\mu_r}} \mathcal{R}_m + O(k^2 \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}})$$
$$= \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}_{\mu_r^*}} dv \cdot \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} \mathcal{R}_m + ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} \mathcal{Q}_m\Big]$$
$$+ \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}_{\mu_r^*}} \cdot H^{\mathrm{in}} dv + Er_m^{(E_{\mathcal{A}}, H_{\mathcal{A}})}(H_{\mathcal{A}}),$$

or more precisely

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r^*}} \mathcal{R}_m + O(k^2 \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}})$$
$$= \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r^*}} [\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^*}}] \boldsymbol{\mathcal{A}}^{-1}_{\boldsymbol{\mathcal{C}}_{\mu_r^*}} \cdot \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} \mathcal{R}_m + ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} \mathcal{Q}_m\Big]$$
$$+ \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}_{\mu_r^*}} dv \cdot H^{\mathrm{in}}(\boldsymbol{z}_m) + Er_m^{(E_{\mathcal{A}}, H_{\mathcal{A}})}(H_{\mathcal{A}}) + O(|k|^2 \boldsymbol{a}^4).$$

as for the symmetric case, the slights changes, for the second equation, are retrieved in the error of approximation.

- (*Derivation of* (3.2.64)) Concerning the far field approximation (3.2.64), a first order approximation gives

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph} \Big(\frac{k^2}{4\pi} \hat{x} \times \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E(y) \times \hat{x} dy + \frac{ik}{4\pi} \hat{x} \times \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \boldsymbol{\mathcal{C}}_{\mu_r} H(y) dy\Big)$$
$$+ \sum_{m=1}^{\aleph} \Bigg(\frac{k^2}{4\pi} \hat{x} \times \Big(\int_{D_m} (e^{-ik\hat{x}\cdot y} - e^{-ik\hat{x}\cdot\boldsymbol{z}_m}) \boldsymbol{\mathcal{C}}_{\varepsilon_r} E(y) \times \hat{x} dy \qquad (3.2.73)$$
$$+ \frac{ik}{4\pi} \int_{D_m} (e^{-ik\hat{x}\cdot y} - e^{-ik\hat{x}\cdot\boldsymbol{z}_m}) \boldsymbol{\mathcal{C}}_{\mu_r} H(y) dy\Big)\Bigg).$$

As

$$\Big|(e^{-ik\hat{x}\cdot y} - e^{-ik\hat{x}\cdot\boldsymbol{z}_m})\Big| \leq \Big| ik\hat{x} \int_0^1 e^{-ik\hat{x}\cdot(ty+(1-t)z_m)} dt \cdot (y - \boldsymbol{z}_m)\Big| \leq |k| \boldsymbol{a}, \qquad (3.2.74)$$

we get

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{C}}_{\varepsilon_r}E(y)\times\hat{x}dy + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{C}}_{\mu_r}H(y)dy\Big) + O\left(\sum_{m=1}^{\aleph}\frac{|k|^3+|k|^2}{4\pi}\boldsymbol{a}\Big(\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)} + \boldsymbol{a}^{\frac{3}{2}}\big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(D_m)}\Big)\right) \tag{3.2.75}$$

then follows

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{C}}_{\varepsilon_r}E(y)\times\hat{x}dy + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{C}}_{\mu_r}H(y)dy\Big) + O\left(\frac{|k|^3+|k|^2}{4\pi}(\aleph\boldsymbol{a}^3)^{\frac{1}{2}}\Big(\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(\cup D_m)} + \big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(\cup D_m)}\Big)^{\frac{1}{2}}\boldsymbol{a}\right). \tag{3.2.76}$$

The conclusion comes using the estimations (3.2.62) and (3.2.63).
The approximation (3.2.65) could be achieved by adding-subtracting to (3.2.76) above the following expression

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}(y)\times\hat{x}dy + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}(y)dy\Big).$$

Precisely

$$\begin{aligned} E^{\infty}(\hat{x}) &= \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}(y)\times\hat{x}dy + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}(y)dy\Big) \\ &+ \sum_{m=1}^{\aleph}\left(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\Big)(y)\times\hat{x}dy\right. \\ &\left. + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\mu_r}H - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}\Big)(y)dy\right) + O\left(\frac{|k|^3+|k|^2}{\boldsymbol{c}_r^3}\boldsymbol{a}\right). \end{aligned} \tag{3.2.77}$$

Obviously

$$\begin{aligned} \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\Big)(y)dy &= \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big[\boldsymbol{\mathcal{C}}_{\varepsilon_r}\Big(E - E_{\mathcal{A}}\Big)\Big](y)dy \\ &+ \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big[\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\Big)E_{\mathcal{A}}\Big](y)dy, \end{aligned}$$

achieving that

$$\sum_{m=1}^{\aleph}\left|\int_{D_m} e^{-ik\hat{x}\cdot \boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E-\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\Big)(y)dy\right| \leq \sum_{m=1}^{\aleph}\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}\|_{\mathbb{L}^\infty(D_m)}\|E-E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} + \sum_{m=1}^{\aleph}\boldsymbol{a}^{\frac{3}{2}}\boldsymbol{a}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}\|_{\mathbb{W}^\infty(D_m)}\|E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)}.$$

Using Hölder's inequality, with both (3.2.31) and similar (3.2.62) and (3.2.63) for $E_{\mathcal{A}}$, we get

$$\sum_{m=1}^{\aleph}\left|\int_{D_m} e^{-ik\hat{x}\cdot \boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E-\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\Big)(y)dy\right| = O\left(\frac{\boldsymbol{c}_\infty^2\boldsymbol{c}_{2,k}+\boldsymbol{c}_\infty}{\boldsymbol{c}_r^3}\boldsymbol{a}\right). \tag{3.2.78}$$

With similar calculation we get

$$\sum_{m=1}^{\aleph}\left|\int_{D_m} e^{-ik\hat{x}\cdot \boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\mu_r}H-\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}\Big)(y)dy\right| = O\left(\frac{\boldsymbol{c}_\infty^2\boldsymbol{c}_{2,k}+\boldsymbol{c}_\infty}{\boldsymbol{c}_r^3}\boldsymbol{a}\right). \tag{3.2.79}$$

Injecting both of the above estimations in (3.2.77) gives the results.

□

Proposition 30. *Both the linear systems*

$$\begin{aligned}
[\mathcal{P}_{D_m}^{\mu_r^*}]^{-1}\widehat{\mathcal{Q}}_m^{\mu_r} &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\widehat{\mathcal{Q}}_j^{\mu_r} - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\widehat{\mathcal{R}}_j^{\varepsilon_r}\Big] + \widehat{H}^{in}(\boldsymbol{z}_m),\\
[\mathcal{P}_{D_m}^{\varepsilon_r^*}]^{-1}\widehat{\mathcal{R}}_m^{\varepsilon_r} &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\widehat{\mathcal{R}}_j^{\varepsilon_r} + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\widehat{\mathcal{Q}}_j^{\mu_r}\Big] + \widehat{E}^{in}(\boldsymbol{z}_m),
\end{aligned} \tag{3.2.80}$$

and

$$\begin{aligned}
\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r^*}}[\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^*}}]^{-1}\mathcal{Q}_m &= \left(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\mu_r}}\widehat{\mathcal{Q}}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\widehat{\mathcal{R}}_j\Big] + \widehat{H}^{in}(\boldsymbol{z}_m)\right),\\
\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r^*}}[\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^*}}]^{-1}\mathcal{R}_m &= \left(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\widehat{\mathcal{R}}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\mu_r}}\widehat{\mathcal{Q}}_j\Big] + \widehat{E}^{in}(\boldsymbol{z}_m)\right),
\end{aligned} \tag{3.2.81}$$

are invertible, provided

$$\frac{\boldsymbol{\delta}}{\boldsymbol{a}} = \boldsymbol{c}_r \geq 3|k|\mu^+_{(\varepsilon_r,\mu_r)}. \tag{3.2.82}$$

For $|k| > 1$ we have the following estimates

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}|^2\Big)^{\frac{1}{2}} \leq \frac{9\mu^+_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3}{8}\Bigg(\Big(\sum_{m=1}^{\aleph}|H^{in}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}} + \frac{1}{3}\Big(\sum_{m=1}^{\aleph}|E^{in}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}\Bigg), \tag{3.2.83}$$

and

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}_m^{\varepsilon_r}|^2\Big)^{\frac{1}{2}} \leq \frac{9\mu^+_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3}{8}\Bigg(\frac{1}{3}\Big(\sum_{m=1}^{\aleph}|H^{in}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}} + \Big(\sum_{m=1}^{\aleph}|E^{in}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}\Bigg). \tag{3.2.84}$$

Furthermore $(\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\widehat{\mathcal{R}}_m, \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}}\widehat{\mathcal{Q}}_m)_{m=1}^{\aleph}$ satisfy similar estimates.

Proof. We deal with the case of real valued coefficients (i.e assumption (3.2.57) is fulfilled). When either one of them or both are complex valued, we procceed in the same way taking the real parts in the the coming derivations.

- (*First linear system*) We start writing the system (3.2.80) as follows

$$\begin{aligned}
\boldsymbol{\mathcal{Q}}_m - \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{P}_{D_m}^{\varepsilon_r^*}]\boldsymbol{\mathcal{R}}_j\Big] &= H^{\text{in}}(\boldsymbol{z}_m),\\
\boldsymbol{\mathcal{R}}_m - \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_m}^{\varepsilon_r}]\boldsymbol{\mathcal{R}}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j\Big] &= E^{\text{in}}(\boldsymbol{z}_m),
\end{aligned} \tag{3.2.85}$$

  where $\boldsymbol{\mathcal{Q}}_j := [\mathcal{P}_{D_j}^{\mu_r^*}]^{-1}\widehat{\mathcal{Q}}_j^{\mu_r}$ and $\boldsymbol{\mathcal{R}}_j := [\mathcal{P}_{D_j}^{\mu_r^*}]^{-1}\mathcal{R}_j^{\mu_r}$ for $j \in \{1, ..., \aleph\}$. Observing that, due to mean value property for harmonic function,

$$\begin{aligned}
|B(0,\boldsymbol{\delta}/4)|^2 &\sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_0(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big]\\
&= \sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)}\int_{B(\boldsymbol{z}_j,\boldsymbol{\delta}/p)}\Big[\Pi_0(x,y)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big]dydx,\\
&= \sum_{m=1}^{\aleph}\Bigg(\int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)}\int_{B(\boldsymbol{z},\rho)}\Big[\Pi_0(x,y)\sum_{j\geq 1}^{\aleph}\chi_{D_j}(x)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big]dydx\\
&\quad - \int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)}\int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)}\Big[\Pi_0(x,y)\chi_{D_j}(x)[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\cdot[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big]dydx\Bigg),
\end{aligned} \tag{3.2.86}$$

  for some $\boldsymbol{z}$ and ρ such that $\boldsymbol{z} \in \cup_{m=1}^{\aleph}B(\boldsymbol{z}_m,\boldsymbol{\delta}/4) \subset B(\boldsymbol{z},\rho)$. Also for $Q(x) = \sum_{j\geq 1}^{\aleph}\chi_{D_j}(x)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j$

we get

$$-|B(0,\boldsymbol{\delta}/4)|^2 \sum_{m=1}^{\aleph} \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big[\Pi_0(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big]$$

$$= -\int_{B(\boldsymbol{z},\rho)} \mathcal{N}^{0,I}_{B(\boldsymbol{z},\rho)}(Q)(x)\cdot Q(x)dx$$

$$+ \sum_{m=1}^{\aleph} \int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)} \int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)} \Big[\Pi_0(x,y)[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big] dydx,$$

in such a way that

$$\begin{aligned} &-\sum_{\substack{m,j=1\\ j\neq m}}^{\aleph} \Big[\Pi_0(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big] \\ &\geq \frac{\sum_{m=1}^{\aleph}}{|B(0,\boldsymbol{\delta}/4)|^2} \int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)} \int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)} \Big[\Pi_0(x,y)[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big] dydx. \end{aligned} \tag{3.2.87}$$

With this in mind we have, from the first equation of linear system,

$$\sum_{m=1}^{\aleph} \boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m - \sum_{\substack{m,j=1\\ j\neq m}}^{\aleph} \Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m$$

$$+ \sum_{\substack{m,j=1\\ j\neq m}}^{\aleph} ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{P}_{D_m}^{\varepsilon_r^*}]\boldsymbol{\mathcal{R}}_j \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m = \sum_{m=1}^{\aleph} H^{\mathrm{in}}(\boldsymbol{z}_m)\cdot[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m.$$

Using (3.2.87) we get

$$\begin{aligned} &\sum_{m=1}^{\aleph} \boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m - \sum_{\substack{m,j=1\\ j\neq m}}^{\aleph} (\Pi_k - \Pi_0)(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \\ &+ \sum_{\substack{m,j=1\\ j\neq m}}^{\aleph} ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{P}_{D_j}^{\varepsilon_r^*}]\boldsymbol{\mathcal{R}}_j \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \\ &+ \frac{\sum_{m=1}^{\aleph}}{|\frac{\boldsymbol{\delta}}{4}|^2} \int_{B(\boldsymbol{z}_m,\frac{\boldsymbol{\delta}}{4})} \int_{B(\boldsymbol{z}_m,\frac{\boldsymbol{\delta}}{4})} \Big[\Pi_0(x,y)[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big] dydx \leq \sum_{m=1}^{\aleph} H^{\mathrm{in}}(\boldsymbol{z}_m)\cdot[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m. \end{aligned} \tag{3.2.88}$$

As we have, see ([30], Theorem 9.9),

$$\left\|\int_{\mathbb{R}^3} \Pi_0(\cdot,y)\chi_{B(\boldsymbol{z}_m,1)}(y)dy\right\|_{\mathbb{L}^2(\mathbb{R}^3)} = \left\|\chi_{B(\boldsymbol{z}_m,1)}\right\|_{\mathbb{L}^2(\mathbb{R}^3)} \tag{3.2.89}$$

we get, with $S_\aleph(\Pi_0)$ standing for the third term of the left-hand side of (3.2.88),

$$\begin{aligned}
S_\aleph(\Pi_0) &\leq \frac{\sum_{m=1}^{\aleph}}{|B(0,1)|^2} \int_{B(\boldsymbol{z}_m,1)} \int_{B(\boldsymbol{z}_m,1)} \Big[\Pi_0(x,y)[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m\Big] dydx, \\
&\leq \frac{\sum_{m=1}^{\aleph}}{|B(0,1)|^2} \Big\| \int_{B(\boldsymbol{z}_m,1)} \Pi_0(x,y)[\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m dy \Big\|_{\mathbb{L}^2(B(\boldsymbol{z}_m,1))} \Big\| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big\|_{\mathbb{L}^2(B(\boldsymbol{z}_m,1))} \\
&\leq \sum_{m=1}^{\aleph} \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big|^2,
\end{aligned}$$

which replaced in (3.2.88), together with (A.0.4), for $\alpha = 0$, gives

$$\begin{aligned}
&\sum_{m=1}^{\aleph} \boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m - \sum_{\substack{m,j=1 \\ j\neq m}}^{\aleph} \left(|k|^2 \Big(\frac{2}{\boldsymbol{\delta}_{mj}} + |k| \Big) + \frac{|k|^2}{\boldsymbol{\delta}_{mj}} \right) \Big| [\mathcal{P}_{D_j}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_j \Big| \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big| \\
&- \sum_{\substack{m,j=1 \\ j\neq m}}^{\aleph} \left(\frac{|k|}{\boldsymbol{\delta}_{mj}} \Big(\frac{1}{\boldsymbol{\delta}_{mj}} + |k| \Big) \right) \Big| [\mathcal{P}_{D_j}^{\varepsilon_r^*}]\boldsymbol{\mathcal{R}}_j \Big| \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big| \\
&- \sum_{m=1}^{\aleph} \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big|^2 \leq \sum_{m=1}^{\aleph} \big| H^{\mathrm{in}}(\boldsymbol{z}_m) \big| \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big|.
\end{aligned} \tag{3.2.90}$$

From the above inequality, with calculations similar to (3.2.39), we get

$$\begin{aligned}
&\sum_{m=1}^{\aleph} \boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m - \Big(\frac{3|k|^2}{(\boldsymbol{a}/2+\boldsymbol{\delta})^{\frac{1}{2}}\boldsymbol{\delta}} + \frac{|k|(|k|^2+1)}{(\boldsymbol{a}/2+\boldsymbol{\delta})^{\frac{3}{2}}} \Big) \frac{1}{(\boldsymbol{a}/2+\boldsymbol{\delta})^{\frac{3}{2}}} \sum_{m=1}^{\aleph} \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big|^2 \\
&- \left(\frac{|k|}{\boldsymbol{\delta}} \Big(\frac{1}{(\boldsymbol{a}/2+\boldsymbol{\delta})\boldsymbol{\delta}} + \frac{|k|}{(\boldsymbol{a}/2+\boldsymbol{\delta})^2} \Big) \right) \Big(\sum_{m=1}^{\aleph} \Big| [\mathcal{P}_{D_m}^{\varepsilon_r^*}]\boldsymbol{\mathcal{R}}_m \Big|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big|^2 \Big)^{\frac{1}{2}} \\
&- \sum_{m=1}^{\aleph} \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big|^2 \leq \sum_{m=1}^{\aleph} \big| H^{\mathrm{in}}(\boldsymbol{z}_m) \big| \Big| [\mathcal{P}_{D_m}^{\mu_r^*}]\boldsymbol{\mathcal{Q}}_m \Big|,
\end{aligned}$$

then, with the original notations, we get

$$\begin{aligned}
&\sum_{m=1}^{\aleph} [\mathcal{P}_{D_m}^{\mu_r^*}]^{-1} \widehat{\mathcal{Q}}_m^{\mu_r} \cdot \widehat{\mathcal{Q}}_m^{\mu_r} - \Big(\frac{3\sqrt{2}|k|^2}{(1+2\boldsymbol{c}_r)^{\frac{1}{2}}\boldsymbol{c}_r} + \frac{\sqrt{2}^3|k|(|k|^2+1)}{(1+2\boldsymbol{c}_r)^{\frac{3}{2}}} \Big) \frac{\sqrt{2}^3}{(1+2\boldsymbol{c}_r)^{\frac{3}{2}}} \frac{\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}_m^{\mu_r}|^2}{\boldsymbol{a}^3} \\
&- \left(\frac{|k|}{\boldsymbol{c}_r} \Big(\frac{2}{(1+2\boldsymbol{c}_r)} + \frac{2|k|}{(1+\boldsymbol{c}_r)^2} \Big) \right) \frac{\Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}_m^{\varepsilon_r}|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}_m^{\mu_r}|^2 \Big)^{\frac{1}{2}}}{\boldsymbol{a}^3} \\
&- \sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}_m^{\mu_r}|^2 \leq \sum_{m=1}^{\aleph} |H^{\mathrm{in}}(\boldsymbol{z}_m)| |\mathcal{Q}_m^{\mu_r}|.
\end{aligned} \tag{3.2.91}$$

Due to (3.2.57), considering that, both

$$\frac{1}{\boldsymbol{a}^3 \mu^{+}_{(\varepsilon_r,\mu_r)}} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \leq [\mathcal{P}^{\mu^*_r}_{D_m}]^{-1} \widehat{\mathcal{Q}}^{\mu_r}_m \cdot \widehat{\mathcal{Q}}^{\mu_r}_m \leq \frac{1}{\boldsymbol{a}^3 \mu^{-}_{(\varepsilon_r,\mu_r)}} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \tag{3.2.92}$$

and

$$\frac{1}{\boldsymbol{a}^3 \mu^{+}_{(\varepsilon_r,\mu_r)}} \left|\widehat{\mathcal{R}}^{\varepsilon_r}_m\right|^2 \leq [\mathcal{P}^{\varepsilon^*_r}_{D_m}]^{-1} \widehat{\mathcal{R}}^{\varepsilon_r}_m \cdot \widehat{\mathcal{R}}^{\varepsilon_r}_m \leq \frac{1}{\boldsymbol{a}^3 \mu^{-}_{(\varepsilon_r,\mu_r)}} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \tag{3.2.93}$$

are satisfied, both (3.2.91) and the obtained result from repeating the same calculation for the second equation, gives, with $\boldsymbol{c}_r \geq 3|k|\mu^{+}_{(\varepsilon_r,\mu_r)}$ and $|k| > 1$,

$$(\frac{7}{9} - \boldsymbol{a}^3) \sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 - \frac{2}{9} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}^{\varepsilon_r}_m|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \Big)^{\frac{1}{2}} \leq \mu^{+}_{(\varepsilon_r,\mu_r)} \boldsymbol{a}^3 \sum_{m=1}^{\aleph} |H^{\text{in}}(\boldsymbol{z}_m)| |\widehat{\mathcal{Q}}^{\mu_r}_m|,$$

and

$$(\frac{7}{9} - \boldsymbol{a}^3) \sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}^{\mu_r}_m|^2 - \frac{2}{9} \Big(\sum_{m=1}^{\aleph} |\mathcal{Q}^{\varepsilon_r}_m|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}^{\mu_r}_m|^2 \Big)^{\frac{1}{2}} \leq \mu^{+}_{(\varepsilon_r,\mu_r)} \boldsymbol{a}^3 \sum_{m=1}^{\aleph} |E^{\text{in}}(\boldsymbol{z}_m)| |\widehat{\mathcal{Q}}^{\mu_r}_m|.$$

which, for $\boldsymbol{a} \leq \frac{1}{3}$, gives

$$\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 - \frac{1}{3} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}^{\varepsilon_r}_m|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \Big)^{\frac{1}{2}} \leq \mu^{+}_{(\varepsilon_r,\mu_r)} \frac{\boldsymbol{a}^3}{3} \Big(\sum_{m=1}^{\aleph} |H^{\text{in}}(\boldsymbol{z}_m)|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \Big)^{\frac{1}{2}},$$

and

$$\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}^{\mu_r}_m|^2 - \frac{1}{3} \Big(\sum_{m=1}^{\aleph} |\mathcal{Q}^{\varepsilon_r}_m|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}^{\mu_r}_m|^2 \Big)^{\frac{1}{2}} \leq \mu^{+}_{(\varepsilon_r,\mu_r)} \frac{\boldsymbol{a}^3}{3} \Big(\sum_{m=1}^{\aleph} |E^{\text{in}}(\boldsymbol{z}_m)|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \Big)^{\frac{1}{2}}.$$

Then follows the conclusion, namely

$$\Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \Big)^{\frac{1}{2}} \leq \mu^{+}_{(\varepsilon_r,\mu_r)} \boldsymbol{a}^3 \Big(\sum_{m=1}^{\aleph} |H^{\text{in}}(\boldsymbol{z}_m)|^2 \Big)^{\frac{1}{2}} + \frac{1}{3} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}^{\varepsilon_r}_m|^2 \Big)^{\frac{1}{2}},$$

and

$$\Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}^{\mu_r}_m|^2 \Big)^{\frac{1}{2}} \leq \mu^{+}_{(\varepsilon_r,\mu_r)} \boldsymbol{a}^3 \Big(\sum_{m=1}^{\aleph} |E^{\text{in}}(\boldsymbol{z}_m)|^2 \Big)^{\frac{1}{2}} + \frac{1}{3} \Big(\sum_{m=1}^{\aleph} |\mathcal{Q}^{\varepsilon_r}_m|^2 \Big)^{\frac{1}{2}}.$$

- (*Non symmetric case*) We could deduce the estimate from the above calculation writing down the linear system as follows, first we set, for $m \in \{1, ..., \aleph\}$

$$[\mathcal{T}^{\boldsymbol{\mathcal{A}}^m_{\mu^*_r}}_{D_m}]^{-1} := \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu^*_r}} [\mathcal{P}^{\boldsymbol{\mathcal{A}}^m_{\mu^*_r}}_{D_m}]^{-1} (\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}})^{-1},$$
$$[\mathcal{T}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon^*_r}}_{D_m}]^{-1} := \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon^*_r}} [\mathcal{P}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon^*_r}}_{D_m}]^{-1} (\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}})^{-1},$$

and, this time

$$\boldsymbol{\mathcal{R}}_m := [\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^*}}]^{-1} \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} \widehat{\mathcal{R}}_m, \tag{3.2.94}$$

$$\boldsymbol{\mathcal{Q}}_m := [\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^*}}]^{-1} \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}} \widehat{\mathcal{Q}}_m \tag{3.2.95}$$

the system (3.2.81) can, hence, be written as

$$\begin{aligned} \boldsymbol{\mathcal{Q}}_m - \sum_{\substack{j \geq 1 \\ j \neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) [\mathcal{T}_{D_j}^{\boldsymbol{\mathcal{A}}_{\mu_r^*}}] \boldsymbol{\mathcal{Q}}_j - ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times [\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}_{\varepsilon_r^*}}] \boldsymbol{\mathcal{R}}_j \Big] &= H^{\text{in}}(\boldsymbol{z}_m), \\ \boldsymbol{\mathcal{R}}_m - \sum_{\substack{j \geq 1 \\ j \neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) [\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}_{\varepsilon_r^*}}] \boldsymbol{\mathcal{R}}_j + ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times [\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}_{\mu_r^*}}] \boldsymbol{\mathcal{Q}}_j \Big] &= E^{\text{in}}(\boldsymbol{z}_m), \end{aligned} \tag{3.2.96}$$

hence, following the same line as in the proof for the first linear system, we get

$$\begin{aligned} \Big(\sum_{m=1}^{\aleph} |\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} \widehat{\mathcal{Q}}_m|^2 \Big)^{(1/2)} &\leq \frac{9 \mu^+_{(\varepsilon_r, \mu_r)} \boldsymbol{a}^3}{8} \Bigg(\Big(\sum_{m=1}^{\aleph} |H^{\text{in}}(\boldsymbol{z}_m)|^2 \Big)^{\frac{1}{2}} + \frac{1}{3} \Big(\sum_{m=1}^{\aleph} |E^{\text{in}}(\boldsymbol{z}_m)|^2 \Big)^{\frac{1}{2}} \Bigg), \\ \Big(\sum_{m=1}^{\aleph} |\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} \widehat{\mathcal{R}}_m|^2 \Big)^{(1/2)} &\leq \frac{9 \mu^+_{(\varepsilon_r, \mu_r)} \boldsymbol{a}^3}{8} \Bigg(\frac{1}{3} \Big(\sum_{m=1}^{\aleph} |H^{\text{in}}(\boldsymbol{z}_m)|^2 \Big)^{\frac{1}{2}} + \Big(\sum_{m=1}^{\aleph} |E^{\text{in}}(\boldsymbol{z}_m)|^2 \Big)^{\frac{1}{2}} \Bigg), \end{aligned} \tag{3.2.97}$$

with unchanged condition on

$$\boldsymbol{c}_r = 3|k| \mu^+_{(\varepsilon_r, \mu_r)}, |k| > 1. \tag{3.2.98}$$

For the last statement, it suffices to observe, that $\boldsymbol{\mathcal{C}}_A$ and its transpose $\boldsymbol{\mathcal{C}}_A^*$ share the same eigenvalues, and hence leave those of $[\mathcal{P}_{D_m}^{A^*}]^{-1}$ identical to those of $[\mathcal{T}_{D_m}^{A^*}]^{-1}$.

□

Proof. (Of Theorem 20)

- We start with (3.1.13), noticing that, for $(\widehat{\mathcal{Q}}_m^{\mu_r}, \widehat{\mathcal{R}}_m^{\mu_r})_{m=1}^{\aleph}$ solution of (3.2.80) and $(\mathcal{Q}_m^{\mu_r}, \mathcal{R}_m^{\mu_r})_{m=1}^{\aleph}$ solution of (3.2.66) then $(\widehat{\mathcal{Q}}_m^{\mu_r} - \mathcal{Q}_m^{\mu_r}, \widehat{\mathcal{R}}_m^{\mu_r} - \mathcal{R}_m^{\mu_r})_{m=1}^{\aleph}$ solves (3.2.80) with

$$\widehat{H}^{\text{in}}(\boldsymbol{z}_m) = [\mathcal{P}_{D_m}^{\mu_r^*}]^{-1} \left(Er_m(H, E) + O\Big(k^2 \|\boldsymbol{\mathcal{C}}_{\mu_r} H\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{C}}_{\varepsilon_r} E\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}} \Big) \right) \tag{3.2.99}$$

and

$$\widehat{E}^{\text{in}}(\boldsymbol{z}_m) = [\mathcal{P}_{D_m}^{\varepsilon_r^*}]^{-1} \left(Er_m(E, H) + O\Big(k^2 \|\boldsymbol{\mathcal{C}}_{\varepsilon_r} E\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{C}}_{\mu_r} H\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}} \Big) \right). \tag{3.2.100}$$

Thus, with the estimates (3.2.83), (3.2.84) and (3.2.92), we have, with the aid of (3.2.60) for the third inequality

$$
\begin{aligned}
&\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}-\mathcal{Q}_m^{\mu_r}|^2\Big)^{\frac{1}{2}} \\
&\leq \frac{9\mu^+_{(\varepsilon_r,\mu_r)}}{8\mu^-_{(\varepsilon_r,\mu_r)}}\Bigg[\Bigg\{\sum_{m=1}^{\aleph}\Big(Er_m(H,E)+O\big(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}\big)\Big)^2\Bigg\}^{\frac{1}{2}} \\
&\qquad +\frac{1}{3}\Bigg\{\sum_{m=1}^{\aleph}\Big(Er_m(E,H)+O\big(k^2\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}\big)\Big)^2\Bigg\}^{\frac{1}{2}}\Bigg], \\
&\leq \frac{9\mu^+_{(\varepsilon_r,\mu_r)}}{8\mu^-_{(\varepsilon_r,\mu_r)}}\Bigg[\Big(\sum_{m=1}^{\aleph}Er_m(H,E)^2\Big)^{\frac{1}{2}}+\Bigg(\sum_{m=1}^{\aleph}O\Big(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}\Big)^2\Bigg)^{\frac{1}{2}} \\
&\qquad +\Big(\sum_{m=1}^{\aleph}Er_m(E,H)^2\Big)^{\frac{1}{2}}+\Bigg(\sum_{m=1}^{\aleph}O\Big(k^2\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}\Big)^2\Bigg)^{\frac{1}{2}}\Bigg], \\
&\leq \frac{9\mu^+_{(\varepsilon_r,\mu_r)}}{8\mu^-_{(\varepsilon_r,\mu_r)}}\Bigg[O\Bigg(\|H\|^2_{\mathbb{L}^2(D)}\frac{\boldsymbol{a}^{11}}{\boldsymbol{\delta}^8}+\Big(\|E\|^2_{\mathbb{L}^2(D)}+\|H\|^2_{\mathbb{L}^2(D)}\Big)\frac{(|k|+2)^3\boldsymbol{a}^{11}|\ln(\boldsymbol{\delta})|}{\boldsymbol{\delta}^6}\Bigg)^{\frac{1}{2}} \\
&\qquad +\Bigg(O\Bigg(k^2\Big(\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|^2_{\mathbb{L}^2(D_m)}\boldsymbol{a}^7\Big)^{\frac{1}{2}}+\Big(\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|^2_{\mathbb{L}^2(D_m)}\boldsymbol{a}^5\Big)^{\frac{1}{2}}\Bigg) \\
&\qquad +O\Bigg(\|E\|^2_{\mathbb{L}^2(D)}\frac{\boldsymbol{a}^{11}}{\boldsymbol{\delta}^8}+\Big(\|E\|^2_{\mathbb{L}^2(D)}+\|H\|^2_{\mathbb{L}^2(D)}\Big)\frac{(|k|+2)^3\boldsymbol{a}^{11}|\ln(\boldsymbol{\delta})|}{\boldsymbol{\delta}^6}\Bigg)^{\frac{1}{2}} \\
&\qquad +\Bigg(O\Big(k^2\Big(\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|^2_{\mathbb{L}^2(D_m)}\boldsymbol{a}^7\Big)^{\frac{1}{2}}+\Big(\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|^2_{\mathbb{L}^2(D_m)}\boldsymbol{a}^5\Big)^{\frac{1}{2}}\Big)^2\Bigg)^{\frac{1}{2}}\Bigg].
\end{aligned}
$$

With the estimates (3.2.62) and (3.2.63), for the scattering of plane waves, we obtain

$$
\begin{aligned}
\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}-\mathcal{Q}_m^{\mu_r}|^2\Big)^{\frac{1}{2}} \leq \frac{4\mu^+_{(\varepsilon_r,\mu_r)}}{8\mu^-_{(\varepsilon_r,\mu_r)}}\times\frac{\boldsymbol{c}_\infty(|k|+1)}{\min(c_\infty^{\varepsilon^-},c_\infty^{\mu^-})\boldsymbol{c}_r^{\frac{3}{2}}}\Bigg[\;O\Bigg(\frac{\boldsymbol{a}^{11}}{\boldsymbol{\delta}^8}+\frac{(|k|+2)^3\boldsymbol{a}^{11}|\ln(\boldsymbol{\delta})|}{\boldsymbol{\delta}^6}\Bigg)^{\frac{1}{2}} \\
+O\big(k^2c_\infty\boldsymbol{a}^{7/2}+c_\infty\boldsymbol{a}^{5/2}\big)\Bigg],
\end{aligned}
\tag{3.2.101}
$$

hence,

$$
\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}-\mathcal{Q}_m^{\mu_r}|^2\Big)^{\frac{1}{2}}=O\Big(\frac{\boldsymbol{c}_{(\varepsilon_r,\mu_r)}(|k|+1)}{\boldsymbol{c}_r^{3/2}}\Big[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Big]\boldsymbol{a}^{3/2}\Big), \tag{3.2.102}
$$

and with similar calculations follows

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}_m^{\mu_r}-\mathcal{R}_m^{\mu_r}|^2\Big)^{\frac{1}{2}}=O\Big(\frac{\boldsymbol{c}_{(\varepsilon_r,\mu_r)}(|k|+1)}{\boldsymbol{c}_r^{3/2}}\Big[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Big]\boldsymbol{a}^{3/2}\Big), \tag{3.2.103}$$

with

$$\boldsymbol{c}_{(\varepsilon_r,\mu_r)}:=\frac{4\mu^+_{(\varepsilon_r,\mu_r)}}{8\mu^-_{(\varepsilon_r,\mu_r)}}\times\frac{\boldsymbol{c}_\infty(|k|+1)}{\min(c_\infty^{\varepsilon^-},c_\infty^{\mu^-})}.$$

Recalling the approximation (3.2.64), we have

$$\begin{aligned}
E^\infty(\hat{x})&=\sum_{m=1}^{\aleph}\bigg(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\mathcal{R}_m^{\varepsilon_r}\times\hat{x}\Big)+\frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\mathcal{Q}_m^{\mu_r}\bigg)+O(\boldsymbol{a}),\\
&=\sum_{m=1}^{\aleph}\bigg(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\widehat{\mathcal{R}}_m^{\varepsilon_r}\times\hat{x}\Big)+\frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\widehat{\mathcal{Q}}_m^{\mu_r}\bigg)+O(\boldsymbol{a})\\
&\quad+\Big(\sum_{m=1}^{\aleph}\frac{|k|^2}{16\pi^2}\Big|e^{-ik\hat{x}\cdot z_m}\Big|\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}-\mathcal{Q}^{\mu_r}|^2\Big)^{\frac{1}{2}}\\
&\quad+\Big(\sum_{m=1}^{\aleph}\frac{|k|^2}{16\pi^2}\Big|e^{-ik\hat{x}\cdot z_m}\Big|\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}_m^{\varepsilon_r}-\mathcal{R}^{\varepsilon_r}|^2\Big)^{\frac{1}{2}},
\end{aligned} \tag{3.2.104}$$

which with the estimations (3.2.102) and (3.2.103) reduces to

$$\begin{aligned}
E^\infty(\hat{x})&=\sum_{m=1}^{\aleph}\bigg(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\widehat{\mathcal{R}}_m^{\varepsilon_r}\times\hat{x}\Big)+\frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\widehat{\mathcal{Q}}_m^{\mu_r}\bigg)+O(\boldsymbol{a})\\
&\quad+\frac{|k|}{2\pi}\frac{1}{\boldsymbol{\delta}^{3/2}}O\Big(\frac{\boldsymbol{c}_{(\varepsilon_r,\mu_r)}(|k|+1)}{\boldsymbol{c}_r^{3/2}}\Big[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Big]\boldsymbol{a}^{3/2}\Big),
\end{aligned} \tag{3.2.105}$$

or

$$\begin{aligned}
E^\infty(\hat{x})&=\sum_{m=1}^{\aleph}\bigg(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\widehat{\mathcal{R}}_m^{\varepsilon_r}\times\hat{x}\Big)+\frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\widehat{\mathcal{Q}}_m^{\mu_r}\bigg)+O(\boldsymbol{a})\\
&\quad+\frac{|k|}{2\pi}O\Big(\frac{\boldsymbol{c}_{(\varepsilon_r,\mu_r)}(|k|+1)}{\boldsymbol{c}_r^{3}}\Big[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Big]\Big).
\end{aligned} \tag{3.2.106}$$

- The approximation (3.1.15) can be justified in a similar way replacing, respectively, $\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\mathcal{R}_m$ and $\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}}\mathcal{Q}_m$ by $\mathcal{R}_m$ and $\mathcal{Q}_m$.

□

3.3 Scattering by perfect conductors, i.e. ($\mathcal{P}_2$)

3.3.1 Preliminaries and notations

3.3.1.1 Boundary traces, Surface gradient and related operator

Let ϕ and U be smooth, respectively, scalar and vector valued functions. The tangential gradient of ϕ, is given by $\nabla_{\mathrm{t}}\phi := -\nu \times \big(\nu \times \nabla\phi\big)$, where ν is the outward unit normal to D. Then the (weak) surface divergence of A, of tangential fields, that is $\nu \cdot A = 0$, is defined by the duality identity

$$\int_{\partial D} \phi \ \mathrm{Div}\, A \ ds = -\int_{\partial D} \nabla_{\mathrm{t}}\phi \cdot A \ ds. \tag{3.3.1}$$

or by the relation, for $A = \nu \times U$,

$$\mathrm{Div}\, A = -\nu \cdot \mathrm{curl}\, U. \tag{3.3.2}$$

Obviously, from (3.3.1), if $\mathrm{Div}\, A$ exists, taking $\phi_m(x) = 1$, gives $\int_{\partial D} \mathrm{Div}\, A(x) \ ds_x = 0$, see [57] and Chapter 2 in [22] for more details.

The spaces $\mathbb{L}_t^{2,Div}(\partial D)$ denote the space of all tangential fields of $\mathbb{L}^2(\partial D)$ that have an $\mathbb{L}_0^2(\partial D)$ weak surface divergence, precisely

$$\mathbb{L}_t^{2,Div}(\partial D) = \Big\{ A \in \mathbb{L}^2(\partial D);\ A \cdot \nu = 0, \text{ such that } \mathrm{Div}\, A \in \mathbb{L}_0^2(\partial D) \Big\},$$

where

$$\mathbb{L}_0^2(\partial D) := \Big\{ A \in \mathbb{L}^2(\partial D) \text{ such that } \int_{\partial D} A ds = 0 \Big\}.$$

We recall, the usual Single and Double layer-potentials, either scalar or vector field, defined, respectively, as follows

$$\boldsymbol{S}_{\partial D}^k(\ \cdot\ ) := \int_{\partial D} \Phi_k(x,y)(\ \cdot\ (y)) \ ds_y, \tag{3.3.3}$$

$$\boldsymbol{K}_{\partial D}^k(\ \cdot\ ) := \int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu_y}(\ \cdot\ (y)) \ ds_y. \tag{3.3.4}$$

These fields are $C^\infty(\mathbb{R}^3 \setminus \partial D)$. In addition, they admit the following Dirichlet trace, for almost every $x \in \partial D$

$$\boldsymbol{S}_{\partial D}^k(A)(x)|_\pm := [S_{\partial D}^k](A)(x) := \lim_{\substack{s \to x \\ s \in \Gamma_\pm(x)}} \boldsymbol{S}_{\partial D}^k(A)(s) = \int_{\partial D} \Phi_k(x,y) A(y) \ ds_y, \tag{3.3.5}$$

$$\boldsymbol{K}_{\partial D}^k(A)(x)|_\pm := [\pm I/2 + K_{\partial D}^k](A)(x) := \lim_{\substack{s \to x \\ s \in \Gamma_\pm(x)}} \boldsymbol{K}_{\partial D}^k(A)(s) = \pm\frac{1}{2}A(x) + p.v. \int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu_y} A(y) \ ds_y, \tag{3.3.6}$$

were $\left\{\Gamma_{\pm}(x),\ x \in \cup_{m=1}^{\aleph} \partial D_m\right\}$ is a family of doubly truncated cones with a vertex at x, which lies in both sides of the boundary, such that $\Gamma_{\pm}(x) \cap D^{\mp} = \emptyset$ and the integral (3.3.6) is taken in the principal value of Cauchy sense. For every $s \in [0, 1]$, all the operators

$$\begin{aligned} [S^0_{\partial D}] &: H^{-s}(\partial D) \longrightarrow H^{1-s}(\partial D), \\ [\pm I/2 + K^0_{\partial D}] &: H^{s}(\partial D) \to H^{s}(\partial D), \\ [\pm I/2 + (K^0_{\partial D})^*] &: H^{-s}_0(\partial D) \to H^{-s}_0(\partial D), \end{aligned}$$

are isomorphisms. More details can be found in [69] and [56].

3.3.1.2 Virtual-mass and Polarization tensor

The following two quantities will play an important role in the sequel

$$\left[\mathcal{P}_{Dm}\right] := \int_{\partial D_m} [-I/2 + (\boldsymbol{K}^0_{\partial D_m})^*]^{-1}(\nu) y^* ds_y, \quad \left[\mathcal{T}_{Dm}\right] := \int_{\partial D_m} [\frac{1}{2} I + (\boldsymbol{K}^0_{\partial D_m})^*]^{-1}(\nu) y^* ds_y. \tag{3.3.7}$$

Both $-\left[\mathcal{P}_{Dm}\right]$ and $\left[\mathcal{T}_{Dm}\right]$ are positive-definite symmetric matrices, (see Lemma 5 and Lemma 6 in [16] or Theorem 4.11 in [9]) and satisfy the following scales

$$\left[\mathcal{P}_{Dm}\right] = \boldsymbol{a}^3 \left[\mathcal{P}_{\boldsymbol{\mathcal{D}}_i}\right], \text{ and } \left[\mathcal{T}_{Dm}\right] = \boldsymbol{a}^3 \left[\mathcal{T}_{\boldsymbol{\mathcal{D}}_m}\right]. \tag{3.3.8}$$

For $i \in \{1, ..., \aleph\}$ let $(\mu_i^{\mathcal{T}})^+$, $(\mu_i^{\mathcal{P}})^+$ be the respective maximal eigenvalues of $\left[\mathcal{T}_{\boldsymbol{\mathcal{D}}_m}\right]$, $-\left[\mathcal{P}_{\boldsymbol{\mathcal{D}}_m}\right]$, and let $(\mu_i^{\mathcal{T}})^-$, $(\mu_i^{\mathcal{P}})^-$ be their minimal ones. We define

$$\mu^+ := \max_{i \in \{1, ..., \aleph\}} ((\mu_i^{\mathcal{T}})^+, (\mu_i^{\mathcal{P}})^+), \quad \mu^- := \min_{i \in \{1, ..., \aleph\}} ((\mu_i^{\mathcal{T}})^-, (\mu_i^{\mathcal{P}})^-). \tag{3.3.9}$$

Hence for every vector $\mathcal{C}$, we get

$$\begin{aligned} \mu^- |\mathcal{C}|^2 \boldsymbol{a}^3 &\leq [\mathcal{T}_{\partial D_m}]\, \mathcal{C} \cdot \mathcal{C} \leq \boldsymbol{a}^3 \mu^+ |\mathcal{C}|^2, \\ \mu^- |\mathcal{C}|^2 \boldsymbol{a}^3 &\leq -[\mathcal{P}_{\partial D_m}]\, \mathcal{C} \cdot \mathcal{C} \leq \boldsymbol{a}^3 \mu^+ |\mathcal{C}|^2. \end{aligned} \tag{3.3.10}$$

3.3.2 Existence and uniqueness of the solution

3.3.2.1 Maxwell layer potentials and boundary traces

The well-posedness of the problem ($\mathcal{P}_2$) is addressed in [22] and [61], for instance, and the unique solution can be expressed in terms of layer potentials. Especially, when $k \in \mathbb{C} \setminus \mathbb{R}^*$, with $\Im k > 0$, or with additional hypothesis on $k \in \mathbb{R}^+$, we can use the following representation,

$$E(x) := E^{\text{in}}(x) + \operatorname{curl} \boldsymbol{S}^k_{\partial D}(A)(x), \;\; x \in D^+ = \mathbb{R}^3 \setminus \cup_{m=1}^{\aleph} \overline{D}_i \tag{3.3.11}$$

where A is the unknown vector density in $\mathbb{L}^{2,Div}_t(\partial D)$, to be found to solve the problem. In this case, the magnetic field is given by

$$H(x) := H^{\text{in}}(x) + \frac{1}{ik} \operatorname{curl} \operatorname{curl} \boldsymbol{S}^k_{\partial D}(A)(x), \tag{3.3.12}$$

with, for almost every $x \in \cup_{m=1}^{\aleph} \partial D_m$, the following traces are valid, see [57]), are valid

$$
\begin{aligned}
[I/2 + \boldsymbol{M}^k_{\partial D}](A)(x) :=& \nu \times \lim_{\substack{s \to x \\ s \in \Gamma_{\pm}(x)}} \operatorname{curl} \boldsymbol{S}^k_{\partial D}(A)(s) = \pm \frac{1}{2} A + \nu \times \operatorname{curl} \int_{\partial D} \Phi_k(x,y) A(y)\, ds_y, \\
[\boldsymbol{N}^k_{\partial D}](A[2])(x) :=& \nu \times \lim_{\substack{s \to x \\ s \in \Gamma_{\pm}(x)}} \operatorname{curl}^2 \boldsymbol{S}^k_{\partial D}(A)(s) \\
=& k^2 \nu \times \int_{\partial D} \Phi_k(x,y)\, A(y)\, ds_y + \nabla \int_{\partial D} \Phi_k(x,y) \operatorname{Div} A(y)\, ds_y.
\end{aligned}
\tag{3.3.13}
$$

For the perfect conductor boundary condition, such a representation gives birth to the following singular integral equation

$$
[I/2 + \boldsymbol{M}^k_{\partial D}](A) = \nu \times E^{\text{in}},
$$

which can be written as

$$
[I/2 + \boldsymbol{M}^k_{m_{D_m}} + \sum_{\substack{j \geq 1 \\ j \neq m}}^{\aleph} \boldsymbol{M}^k_{m_{D_j}}](A) = \nu \times E^{\text{in}}, \tag{3.3.14}
$$

with, for $x \in \partial D_m$ and $j \in \{1, ..., \aleph\}$,

$$
\boldsymbol{M}^k_{m_{D_j}} A(x) := \nu \times \int_{\partial D_j} \nabla_x \Phi_k(x,y) \times A(y)\, ds_y. \tag{3.3.15}
$$

For the surface divergence, we have

$$
\operatorname{Div}[I/2 + \boldsymbol{M}^k_{\partial D}](A)(x) = -[k^2 \nu \cdot S^k_{\partial D}](A) - [I/2 - (K^k_{\partial D})^*](\operatorname{Div} A). \tag{3.3.16}
$$

In addition, for every $u \in H^1(\partial D_m)$ (cf. Lemma 5.11 in [57])

$$
[I/2 + \boldsymbol{M}^k_{\partial D}](\nu \times \nabla u) = \nu \times \nabla [I/2 + K^k_{\partial D}](u) - k^2\, \nu \times [S^k_{\partial D}](\nu u). \tag{3.3.17}
$$

The fact that the operators appearing in (3.3.14) are isomorphisms on $\mathbb{L}^{2,Div}_t(\partial D)$ is addressed in the next section together with an estimate of the density.

3.3.2.2 A priori estimates of the densities

The following result together with the fact that $[\pm I/2 + \boldsymbol{M}^k_{\partial D}]$ is of closed range, makes it describe an isomorphism of $\mathbb{L}^{2,Div}_t(\partial D)$. The proof goes in two steps. In the first step, we prove the estimate when the wave number k has a positive imaginary part, $\Im k > 0$, (ex. $k = i$). As $[\boldsymbol{M}^k_{\partial D} - \boldsymbol{M}^i_{\partial D}]$ is compact, we deduce, in a second step, a similar estimates for $k \in \mathbb{R}^+$, under an appropriate condition on $\boldsymbol{c}_r$ (i.e. the dilution parameter).

Theorem 31. *Let $k \in \mathbb{C}$ such that $\Im(k) > 0$. For any $A \in \mathbb{L}^{2,Div}_t(\partial D)$, there exists a positive constant $\boldsymbol{c}_{L,k}$, which depends only on k and the Lipschitz character of the D's, such that*

$$
\|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D)} = \sum_{m=1}^{\aleph} \|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)} \leq \boldsymbol{c}_{L,k} \|[\pm I/2 + \boldsymbol{M}^k_{\partial D}] A\|^2_{\mathbb{L}^{2,Div}_t(\partial D)}. \tag{3.3.18}
$$

In order to derive this estimate, we need to use the following key Helmholtz decomposition of the densities

Lemma 32. *Each element A_m of $\mathbb{L}_t^{2,Div}(\partial D_m)$ can be decomposed as $A_m = A_m^1 + A_m^2$ where*

$$A_m^1 = \nu \times \nabla v_m^A \in \mathbb{L}_t^{2,0}(\partial D_m),\ v_m \in \mathbb{H}^1(\partial D_m) \setminus \mathbb{C} \ \text{ and } A_m^2 \ \in \mathbb{L}_t^{2,Div}(\partial D_m) \setminus \mathbb{L}_t^{2,0}(\partial D_m), \text{ and} \tag{3.3.19}$$

with the estimates

$$\|A_m^2\|_{\mathbb{L}^2(\partial D_m)} \leq \boldsymbol{c}_1 \boldsymbol{a} \|A_m\|_{\mathbb{L}_t^{2,Div}(\partial D_m)}, \quad \|A_m^2\|_{\mathbb{L}_t^{2,Div}(\partial D_m)} \leq \boldsymbol{c}_2 \|A\|_{\mathbb{L}_t^{2,Div}(\partial D_m)},$$
$$\|A_m^1\|_{\mathbb{L}_t^{2,0}(\partial D_m)} \leq \boldsymbol{c}_3 \|A_m\|_{\mathbb{L}_t^{2,Div}(\partial D_m)}, \quad \|v_m^A\|_{\mathbb{L}^2(\partial D_m)} \leq \boldsymbol{c}_4 \boldsymbol{a} \|A\|_{\mathbb{L}_t^{2,Div}(\partial D_m)}.$$

where $(\boldsymbol{c}_i)_{i=1,2,3,4.}$ are constants which depend only on the Lipschitz character of the $D_m's$.

To prove Lemma 32, it suffices to seek for the solution of the following equation,

$$\begin{cases} [\nu \times \nabla \boldsymbol{S}_{\partial D_m}^0](w) + [\nu \times \boldsymbol{S}_{\partial D_m}^0](W) = A_m, \\ \qquad\qquad [\nu \cdot \operatorname{curl} \boldsymbol{S}_{\partial D_m}^0](W) = \operatorname{Div} A_m, \end{cases} \tag{3.3.20}$$

and get the desired estimations, using the scaling properties of the implied operators, with $A_m^2 = [\nu \times \boldsymbol{S}_{\partial D_i}^0](W)$ and $A_m^1 = [\nu \times \nabla \boldsymbol{S}_{\partial D_i}^0](w)$. The details can be found in [14].

In what follows, $\boldsymbol{c}_L$ and c_k will designate generic constants independent of $\boldsymbol{a}$ and depend, respectively, on the Lipschitz character of D and k.

Suppose that E and $H = \operatorname{curl} E/ik$ are two vector fields that satisfy the Maxwell system, with a complex wave number k, $\Im(k) > 0$ and such that $\nu \times E$ is in $\mathbb{L}^{2,\mathrm{Div}}(\partial D := \cup_{m=1}^{\aleph} \partial D_m)$. Hence, in view of the decomposition of Lemma 38, there exist $\mathfrak{e}_i$ and $\mathfrak{E}_i$ such that, in each ∂D_m, $\nu_i \times E = \nu \times \nabla \mathfrak{e}_i + \nu \times \mathfrak{E}_i$ with the following estimate,

$$\|\nu \times \mathfrak{E}_i\|_{\mathbb{L}^2(\partial D_m)} \leq \boldsymbol{c}_1 \boldsymbol{a} \|E\|_{\mathbb{L}_t^{2,Div}(\partial D_m)}, \quad \|\mathfrak{e}_i\|_{\mathbb{L}^2(\partial D_m)} \leq \boldsymbol{c}_4 \boldsymbol{a} \|E\|_{\mathbb{L}_t^{2,Div}(\partial D_m)}.$$

recalling that $\|E\|_{\mathbb{L}_t^{2,Div}(\partial D_m)} = \|\nu \times E\|_{\mathbb{L}^2(\partial D_m)} + \|\nu \cdot \operatorname{curl} E\|_{\mathbb{L}^2(\partial D_m)}$.

Let us set $\mathfrak{e} := \sum_{m=1}^{\aleph} \chi_{\partial D_m} \mathfrak{e}_i$, $\ \mathfrak{E} := \sum_{m=1}^{\aleph} \chi_{\partial D_m} \mathfrak{E}_i$. With this notations, for every $x \in \partial D = \cup_{m=1}^{\aleph} \partial D_m$, we get

$$\nu \times E = \nu \times \nabla \mathfrak{e} + \nu \times \mathfrak{E}, \tag{3.3.21}$$

and the following estimates hold

$$\|\nu \times \mathfrak{E}\|_{\mathbb{L}^2(\cup_{m=1}^{\aleph} \partial D_m)} \leq \boldsymbol{c}_1 \boldsymbol{a} \|E\|_{\mathbb{L}_t^{2,Div}(\partial \cup_{m=1}^{\aleph} D_m)}, \quad \|\mathfrak{e}_i\|_{\mathbb{L}^2(\cup_{m=1}^{\aleph} \partial D_m)} \leq \quad \boldsymbol{c}_4 \boldsymbol{a} \|E\|_{\mathbb{L}_t^{2,Div}(\partial \cup_{m=1}^{\aleph} D_m)}. \tag{3.3.22}$$

Indeed, with Lemma 38,

$$\|\nu \times \mathfrak{E}\|^2_{\mathbb{L}^2(\cup_{m=1}^{\aleph} \partial D_m)} = \sum_{m=1}^{\aleph} \int_{\partial D_m} (\nu_i \times \mathfrak{E})^2 ds \leq \sum_{m=1}^{\aleph} \boldsymbol{c}_1^2 \boldsymbol{a}^2 \left(\int_{\partial D_m} (\nu \times E_i)^2 + (\operatorname{div} \nu \times E_i)^2 ds \right),$$
$$\leq \boldsymbol{c}_1^2 \boldsymbol{a}^2 \sum_{m=1}^{\aleph} \int_{\partial D_m} (\nu \times E_i)^2 ds = \boldsymbol{c}_1^2 \boldsymbol{a}^2 \|E\|^2_{\mathbb{L}_t^{2,Div}(\partial D)}.$$

The second estimation can be done in the same way.

Lemma 33. *For any solution E to the Maxwell system, continuous up to the boundary, we have*

$$\Big|\frac{1}{ik}\int_{\partial D}\nu\cdot E\times \operatorname{curl}\overline{E}\ ds\Big| \leq \boldsymbol{c}_L\ \boldsymbol{a}\ \|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E\|_{\mathbb{L}_t^{2,Div}(\partial D)}, \tag{3.3.23}$$

with $\boldsymbol{c}_L$ independent of $\boldsymbol{a}$ and k, from which follows

$$\|E\|^2_{\mathbb{H}(\mathrm{curl},D)} \leq \boldsymbol{c}_L c_k \boldsymbol{a}\|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E\|_{\mathbb{L}_t^{2,Div}(\partial D)} \tag{3.3.24}$$

where $c_k := 1/\min\big(\Im k/|k|^2, \Im(k)\big)$.

Proof. Considering the decomposition (3.3.21), and using (3.3.1) for the second identity, we have

$$\begin{aligned}\int_{\partial D} E\cdot\nu\times\overline{\operatorname{curl}E}\ ds &= \int_{\partial D}(\nabla\mathfrak{e}+\mathfrak{E})\cdot\nu\times\overline{\operatorname{curl}E}\ ds = -\int_{\partial D}\mathfrak{e}\ \operatorname{Div}(\nu\times\overline{\operatorname{curl}E})\ ds + \int_{\partial D}\mathfrak{E}\cdot\nu\times\overline{\operatorname{curl}E}\ ds,\\ &= k^2\int_{\partial D}\mathfrak{e}\ \nu\cdot E\ ds + \int_{\partial D}\mathfrak{E}\cdot\nu\times\overline{\operatorname{curl}E}\ ds.\end{aligned}$$

Taking the absolute value, with Hölder's inequality and involving the estimates (3.3.22), we get successively

$$\begin{aligned}\Big|\frac{1}{ik}\int_{\partial D}E\cdot\nu\times\operatorname{curl}\overline{E}\ ds\Big| &\leq \|\mathfrak{e}\|_{\mathbb{L}^2(\partial D)}\|\frac{k^2}{ik}\nu\cdot E\|_{\mathbb{L}^2(\partial D)} + \|\mathfrak{E}\|_{\mathbb{L}^2(\partial D)}\|\frac{1}{ik}\nu\times\operatorname{curl}E\|_{\mathbb{L}^2(\partial D)},\\ &\leq \boldsymbol{c}_L\ \boldsymbol{a}\ \|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E\|_{\mathbb{L}_t^{2,Div}(\partial D)}.\end{aligned}$$

For (3.3.24), consider the following Green's identity

$$\frac{1}{ik}\int_D|\operatorname{curl}E|^2dx + ik\int_D|E|^2dx = \frac{1}{ik}\int_{\partial D}\nu\times E\cdot\operatorname{curl}\overline{E}\ ds. \tag{3.3.25}$$

The real part gives,

$$\frac{\Im k}{|k|^2}\int_D|\operatorname{curl}E|^2dx + \Im k\int_D|E|^2dx = -\Re\Big(\frac{1}{ik}\int_{\partial D}\nu\times E\cdot\operatorname{curl}\overline{E}\ ds\Big),$$

then, with $c_k := 1/\min\big(\Im k/|k|^2, \Im(k)\big)$ involving the estimates (3.3.23), we obtain

$$\int_D|\operatorname{curl}E|^2dx + \int_D|E|^2dx \leq c_k\boldsymbol{c}_L\ \boldsymbol{a}\ \|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E\|_{\mathbb{L}_t^{2,Div}(\partial D)}.$$

□

Lemma 34. *We have, the following estimates for the exterior (E^+) and interior (E^-) traces of the radiating electric field E*

$$\int_{\partial D}\big|\nu\times(\nu\times E^\pm)\big|^2 \leq \boldsymbol{c}_L c_k\Big(\|H^\pm\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E^\pm\|_{\mathbb{L}_t^{2,Div}(\partial D)} + \|\nu\cdot E^\pm\|^2_{\mathbb{L}^2(\partial D)}\Big), \tag{3.3.26}$$

and

$$\int_{\partial D}\big|\nu\cdot E^\pm\big|^2 \leq \boldsymbol{c}_L c_k\Big(\|H^\pm\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E^\pm\|_{\mathbb{L}_t^{2,Div}(\partial D)} + \|\nu\times(\nu\times E^\pm)\|^2_{\mathbb{L}^2(\partial D)}\Big), \tag{3.3.27}$$

and due to the Maxwell equation symmetry, same estimates remain true for the magnetic field H.

Proof. Let E and X be two vector fields, with X real valued, and consider the following identity

$$\int_{D=\cup_{m=1}^{\aleph} D_m} \operatorname{div}\left(\frac{1}{2}|E|^2 X - Re(\overline{E}\cdot X)\ E\right)\ dx = \int_{\partial D=\cup_{m=1}^{\aleph}\partial D_m}\left(\frac{1}{2}|E|^2 X - Re(\overline{E}\cdot X)\ E\right)\cdot \nu ds. \tag{3.3.28}$$

Following [58], let $\widehat{X}_m \in \mathbb{C}^\infty(\overline{\boldsymbol{\mathcal{D}}_m})$ be compactly supported in $\mathbb{R}^3$, such that $\widehat{X}_m \cdot \nu_{\partial\boldsymbol{\mathcal{D}}_m} \geq \boldsymbol{c}_{L_m} > 0$ with support as small as wanted. Such a choice is possible due to Lemma 1.5.1.9 in [31]. Recall that $\boldsymbol{\mathcal{D}}_m \subset B(0,\frac{1}{2})$ and $D_m = \boldsymbol{a}\boldsymbol{\mathcal{D}}_m + \boldsymbol{z}_m$.

Set $\boldsymbol{c}_L^- := \min_{m\in\{1,\dots,\aleph\}} \boldsymbol{c}_{L_m}$, and define, for every $x\in\mathbb{R}^3$,

$$X(x) := \sum_{m=1}^{\aleph}\Big(\widehat{X}_m\big(\frac{(x-z_m)}{\epsilon}\big)\chi_{B(\boldsymbol{z}_m,\frac{3\boldsymbol{a}}{2})}\Big).$$

Then X satisfies, for any positive scalar function V,

$$\begin{aligned}\int_{\partial D} VX\cdot\nu ds = \sum_{m=1}^{\aleph}\int_{\partial\boldsymbol{\mathcal{D}}_m} V(\epsilon s_m + z_m)\widehat{X}(s_m)\cdot\nu_{\partial\boldsymbol{\mathcal{D}}_m}\boldsymbol{a}^2 ds_{s_m} &\geq \sum_{m=1}^{\aleph}\boldsymbol{c}_{L_m}\int_{\partial\boldsymbol{\mathcal{D}}_m} V(\epsilon s_m+z_m)\boldsymbol{a}^2 ds\\ &\geq \boldsymbol{c}_L^-\int_{\partial D} V ds,\end{aligned} \tag{3.3.29}$$

and for $\boldsymbol{c}_L^+ := \max_m \sup_{\boldsymbol{\mathcal{D}}_m}\Big(|\operatorname{div}\widehat{X}|, |\nabla\ \widehat{X}|, |\widehat{X}|\Big)$, we obtain

$$\begin{cases}\sup\limits_{x\in D_m}|\nabla X(x)| \leq \dfrac{1}{\boldsymbol{a}}\sup\limits_{s\in\boldsymbol{\mathcal{D}}_m}\left|\nabla_s\widehat{X}(s)\right| \leq \dfrac{\boldsymbol{c}_L^+}{\boldsymbol{a}},\\ \sup\limits_{x\in D_m}|\operatorname{div} X(x)| \leq \dfrac{1}{\boldsymbol{a}}\sup\limits_{s\in\boldsymbol{\mathcal{D}}_m}\left|\operatorname{div}_s\widehat{X}(s)\right| \leq \dfrac{\boldsymbol{c}_L^+}{\boldsymbol{a}}.\end{cases} \tag{3.3.30}$$

Being $|E|^2 = E\cdot\overline{E}$, we get both

$$\operatorname{div}(|E|^2 X/2) = \big(|E|^2\operatorname{div} X + (\nabla E)\overline{E}\cdot X + (\nabla\overline{E})\ E\cdot X\big)/2,$$

and

$$\operatorname{div}((\overline{E}\cdot X)\ E) = (\overline{E}\cdot X)\ \operatorname{div} E - (\nabla\overline{E})X\cdot E - (\nabla X)\overline{E}\cdot E.$$

Taking their real parts then their difference[9], gives

$$\operatorname{div}\big(\frac{1}{2}|E|^2X - \Re\big((\overline{E}\cdot X)E\big)\big) = \Re\big(\frac{1}{2}|E|^2\ \operatorname{div} X - \overline{E}\cdot X\ \operatorname{div} E - \overline{E}\cdot\nabla X\ E + \overline{E}\times X\cdot\operatorname{curl} E\big).$$

[9]Noticing that

$$\Re\left((\nabla\overline{E})X\cdot E\right) = \Re\left((\nabla E)X\cdot\overline{E}\right),$$

and

$$\Re\left((\nabla\overline{E})X\cdot E - (\nabla\overline{E})E\cdot X\right) = \Re\left(\operatorname{curl}\overline{E}\cdot(E\times X)\right) = \Re\left(\operatorname{curl} E\cdot(\overline{E}\times X)\right).$$

As E is divergence free, replacing in (3.3.28), we have

$$\int_{\partial D}\Big(\frac{|E|^2 X}{2} - \Re(\overline{E}\cdot X)E\Big)\cdot \nu ds = \Re\int_D \Big(\frac{|E|^2 \operatorname{div} X}{2} - \overline{E}\cdot(\nabla X)E + \overline{E}\times X\cdot \operatorname{curl} E\Big)dv. \quad (3.3.31)$$

Since $E = (\nu\cdot E)\nu - \nu\times(\nu\times E)$, we obtain

$$\Re\int_{\partial D}(\overline{E}\cdot X)\ E\cdot\nu\ ds = -\Re\int_{\partial D}\Big(\nu\times(\nu\times\overline{E})\cdot X\ \nu\cdot E - |\nu\cdot E|^2\nu\cdot X\Big)ds$$

and then the left hand side of (3.3.31) becomes

$$\int_{\partial D}\Big(\frac{1}{2}|E|^2 - |\nu\cdot E|^2\Big)X\cdot\nu\ ds + \Re\int_{\partial D}\nu\times(\nu\times E)\cdot X\ \overline{\nu}\cdot E\ ds,$$

hence it follows, with (3.3.30),

$$\Big|\int_{\partial D}\Big(\frac{|E|^2}{2} - |\nu\cdot E|^2\Big)\nu\cdot X ds\Big| \leq \boldsymbol{c}_L^+\Big(\frac{1}{\boldsymbol{a}}\int_D\frac{3|E|^2}{2}dv + \int_D|\overline{E}|\,|\operatorname{curl} E|dv\Big) + \Big|\int_{\partial D}\nu\times(\nu\times E)\cdot X\ \overline{\nu}\cdot E\ ds\Big|,$$

As $|E|^2 = |\nu\times(\nu\times E)|^2 + |(\nu\cdot E)\nu|^2$, using Hölder's inequality, we have

$$\Big|\int_{\partial D}\frac{1}{2}\left(|\nu\times(\nu\times E)|^2 - |\nu\cdot E|^2\right)X\cdot\nu\ ds\Big| \leq \boldsymbol{c}_L^+\Big(\frac{1}{\boldsymbol{a}}\|E\|_{\mathbb{H}(\mathrm{curl},D)^2} + \|\nu\times E\|_{\mathbb{L}^2(\partial D)}\|\nu\cdot E\|_{\mathbb{L}^2(\partial D)}\Big), \quad (3.3.32)$$

which, in view of (3.3.29), gives both

$$\int_{\partial D}|\nu\times(\nu\times E)|^2 ds \leq \frac{\boldsymbol{c}_L^+}{\boldsymbol{c}_L^-}\Big(\frac{\|E\|_{\mathbb{H}(\mathrm{curl},D)^2}}{\boldsymbol{a}} + \|\nu\times E\|_{\mathbb{L}^2(\partial D)}\|\nu\cdot E\|_{\mathbb{L}^2(\partial D)}\Big) + \|\nu\cdot E\|^2_{\mathbb{L}^2(\partial D)},$$

and

$$\int_{\partial D}|\nu\cdot E|^2 ds \leq \frac{\boldsymbol{c}_L^+}{\boldsymbol{c}_L^-}\Big(\frac{\|E\|_{\mathbb{H}(\mathrm{curl},D)^2}}{\boldsymbol{a}} + \|\nu\times E\|_{\mathbb{L}^2(\partial D)}\|\nu\cdot E\|_{\mathbb{L}^2(\partial D)}\Big) + \|\nu\times(\nu\times E)\|^2_{\mathbb{L}^2(\partial D)}.$$

Using (3.3.24), we get

$$\int_{\partial D}|\nu\times(\nu\times E)|^2 ds \leq \boldsymbol{c}_{L,k}\Big(\|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E\|_{\mathbb{L}_t^{2,Div}(\partial D)} + \|\nu\times E\|_{\mathbb{L}^2(\partial D)}\|\nu\cdot E\|_{\mathbb{L}^2(\partial D)} + \|\nu\cdot E\|^2_{\mathbb{L}^2(\partial D)}\Big),$$

and

$$\int_{\partial D}|\nu\cdot E|^2 ds \leq \boldsymbol{c}_{L,k}\Big(\|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E\|_{\mathbb{L}_t^{2,Div}(\partial D)} + \|\nu\times E\|_{\mathbb{L}^2(\partial D)}\|\nu\cdot E\|_{\mathbb{L}^2(\partial D)} + \|\nu\times(\nu\times E)\|^2_{\mathbb{L}^2(\partial D)}\Big).$$

Then [10]

$$\begin{cases}\displaystyle\int_{\partial D}|\nu\times(\nu\times E)|^2 ds \leq \boldsymbol{c}_{L,k}\Big((1+\frac{1}{|k^2|})\|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E\|_{\mathbb{L}_t^{2,Div}(\partial D)} + \|\nu\cdot E\|^2_{\mathbb{L}^2(\partial D)}\Big),\\ \displaystyle\int_{\partial D}|\nu\cdot E|^2 ds \leq \boldsymbol{c}_{L,k}\Big((1+\frac{1}{|k^2|})\|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E\|_{\mathbb{L}_t^{2,Div}(\partial D)} + \|\nu\times(\nu\times E)\|^2_{\mathbb{L}^2(\partial D)}\Big).\end{cases} \quad (3.3.33)$$

[10] Being $\|\nu\times E\|_{\mathbb{L}^2(\partial D)} \leq \|E\|_{\mathbb{L}_t^{2,Div}(\partial D)}$, $\|\nu\cdot E\|_{\mathbb{L}^2(\partial D)} \leq \frac{1}{|k|^2}\|H\|_{\mathbb{L}_t^{2,Div}(\partial D)}$,

Concerning the exterior field, we consider a ball $B_r(0)$, with a sufficiently large radius r, which contains $D = \cup_{m=1}^{\aleph} D_m$ and the support of X. Let $D_r := B_r \setminus D$, applying Green's formula (3.3.25) on D_r gives

$$\int_{D_r} \big(\frac{1}{ik}|\mathrm{curl}\, E|^2 + ik|E|^2\big)dx = \int_{\partial B(0,r)} \nu \times \overline{E} \cdot \frac{1}{ik}\,\mathrm{curl}\, E\; ds - \int_{\partial D} \nu \times \overline{E} \cdot \frac{1}{ik}\,\mathrm{curl}\, E\; ds. \tag{3.3.34}$$

As $\Im k > 0$, due to the asymptotic behavior (3.1.8), we have

$$2\Re\Big(\int_{\partial B_r} E \cdot \; \frac{1}{ik}\,\mathrm{curl}\, E \times \nu \; ds\Big) \longrightarrow 0,$$

then, the real part of (3.3.34) and letting r to ∞ gives

$$-\int_{\mathbb{R}^3 \setminus D} \big(\frac{\Im(k)}{|k|^2}|\mathrm{curl}\, E|^2 + \Im k|E|^2\big)dv \geq -\Re\big(\int_{\partial D} \nu \times \overline{E} \cdot \frac{1}{ik}\,\mathrm{curl}\, E\; ds\big),$$

hence

$$\int_{\mathbb{R}^3 \setminus \cup_{m=1}^{\aleph} \overline{D_m}} \big(|\mathrm{curl}\, E|^2 + |E|^2\big)dv \leq c_k \Re\big(\int_{\partial D} \nu \times \overline{E} \cdot \frac{1}{ik}\,\mathrm{curl}\, E\; ds\big), \tag{3.3.35}$$

which is the analogue formula for (3.3.24). The rest of the proof is identical. □

As a direct consequence of Lemma 34, we have the following assertion

Lemma 35. *For every solution (E, H) to the Maxwell system, continuous up to the boundary, with wave number k, we have both*

$$\|E^{\pm}\|_{\mathbb{L}_t^{2,Div}(\partial D)} \leq \boldsymbol{c}_{L,k}^2 \|H^{\pm}\|_{\mathbb{L}_t^{2,Div}(\partial D)} \tag{3.3.36}$$

and

$$\|H^{\pm}\|_{\mathbb{L}_t^{2,Div}(\partial D)} \leq \boldsymbol{c}_{L,k}^2 \|E^{\pm}\|_{\mathbb{L}_t^{2,Div}(\partial D)}. \tag{3.3.37}$$

In addition, taking $E = \mathrm{curl}\, \boldsymbol{S}_{\partial D}^k(A)$ for some $A \in \mathbb{L}_t^{2,Div}(\partial D)$ with the trace limites as in (3.3.13) we have

$$\|[\pm I/2 + M_{\partial D}^k]A\|_{\mathbb{L}_t^{2,Div}(\partial D)} \leq \boldsymbol{c}_{L,k} \|[N_{\partial D}^k]A\|_{\mathbb{L}_t^{2,Div}(\partial D)}, \tag{3.3.38}$$

and

$$\|[N_{\partial D}^k]A\|_{\mathbb{L}_t^{2,Div}(\partial D)} \leq \boldsymbol{c}_{L,k} \|[\pm I/2 + M_{\partial D}^k]A\|_{\mathbb{L}_t^{2,Div}(\partial D)}. \tag{3.3.39}$$

Proof. (Lemma 35) For the first part, being $E = \nu \times (\nu \times E) + \nu \cdot E\; \nu$, we have

$$\|E\|_{\mathbb{L}^2(\partial D)}^2 \leq \|\nu \times (\nu \times E)\|_{\mathbb{L}^2(\partial D)}^2 + \|\nu \cdot E\|_{\mathbb{L}^2(\partial D)}^2.$$

Hence, with the first inequality of Lemma 34, we get

$$\|E^{\pm}\|_{\mathbb{L}^2(\partial D)}^2 \leq \boldsymbol{c}_{L,k}\Big(\frac{|k|^2+1}{|k|^2}\|H^{\pm}\|_{\mathbb{L}_t^{2,Div}(\partial D)}\|E^{\pm}\|_{\mathbb{L}_t^{2,Div}(\partial D)} + \|\nu \cdot E^{\pm}\|_{\mathbb{L}^2(\partial D)}^2\Big) + \|\nu \cdot E^{\pm}\|_{\mathbb{L}^2(\partial D)}^2 \tag{3.3.40}$$

and with the same argument, for the magnetic field using the second inequality of Lemma 34, we get

$$\|H^\pm\|^2_{\mathbb{L}^2(\partial D)} \leq \boldsymbol{c}_{L,k}\Big(\frac{|k|^2+1}{|k|^2}\|H^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)}\|E^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)} + \|\nu\times H^\pm\|^2_{\mathbb{L}^2(\partial D)}\Big) + \|\nu\times H^\pm\|^2_{\mathbb{L}^2(\partial D)}. \tag{3.3.41}$$

Adding the last two inequalities and observing that

$$\|E^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)} \leq \|E^\pm\|^2_{\mathbb{L}^2(\partial D)} + \|H^\pm\|^2_{\mathbb{L}^2(\partial D)} \tag{3.3.42}$$

and that

$$\|\nu\cdot E^\pm\|^2_{\mathbb{L}^2(\partial D)} + \|\nu\times H^\pm\|^2_{\mathbb{L}^2(\partial D)} \leq (1+1/|k|)\|H^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)}(\|E\|_{\mathbb{L}^2(\partial D)} + \|H\|_{\mathbb{L}^2(\partial D)}), \tag{3.3.43}$$

gives

$$\begin{aligned}
&\|H^\pm\|^2_{\mathbb{L}^2(\partial D)} + \|E^\pm\|^2_{\mathbb{L}^2(\partial D)} \\
&\qquad \leq \boldsymbol{c}_{L,k}\Bigg(\Big(\|H^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)}\|E^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)} + \|H^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)}(\|E\|_{\mathbb{L}^2(\partial D)} + \|H\|_{\mathbb{L}^2(\partial D)})\Big) \\
&\qquad\qquad + \|\nu\times H^\pm\|^2_{\mathbb{L}^2(\partial D)} + \|\nu\cdot E^\pm\|^2_{\mathbb{L}^2(\partial D)}\Bigg), \\
&\qquad \leq \ \boldsymbol{c}_{L,k} \quad \|H^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)}\Big(\|H^\pm\|^2_{\mathbb{L}^2(\partial D)} + \|E^\pm\|^2_{\mathbb{L}^2(\partial D)}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\Big(\|H^\pm\|^2_{\mathbb{L}^2(\partial D)} + \|E^\pm\|^2_{\mathbb{L}^2(\partial D)}\Big)^{\frac{1}{2}} \leq \boldsymbol{c}_{L,k}\|H^\mp\|_{\mathbb{L}^{2,Div}_t(\partial D)} \leq \boldsymbol{c}_{L,k}\Big(\|H^\mp\|^2_{\mathbb{L}^2(\partial D)} + \|E^\mp\|^2_{\mathbb{L}^2(\partial D)}\Big)^{\frac{1}{2}}.$$

Interchanging E and H in (3.3.40) and (3.3.42) then repeating the same calculation gives

$$\Big(\|H^\pm\|^2_{\mathbb{L}^2(\partial D)} + \|E^\pm\|^2_{\mathbb{L}^2(\partial D)}\Big)^{\frac{1}{2}} \leq \boldsymbol{c}_{L,k}\|E^\pm\|_{\mathbb{L}^{2,Div}_t(\partial D)} \leq \boldsymbol{c}_{L,k}\Big(\|H^\pm\|^2_{\mathbb{L}^2(\partial D)} + \|E^\pm\|^2_{\mathbb{L}^2(\partial D)}\Big)^{\frac{1}{2}}.$$

The two last inequalities give us the desired result. □

The estimates in Theorem 31 follow using the triangular inequality and (3.3.38). Indeed,

$$\begin{aligned}
\|A\|_{\mathbb{L}^{2,Div}_t}(\partial D) &= \Big\|(I/2 + M^k_{\partial D})A - (-I/2 + M^k_{\partial D})A\Big\|_{\mathbb{L}^{2,Div}_t(\partial D)}, \\
&\leq \Big\|(I/2 + M^k_{\partial D})A\Big\|_{\mathbb{L}^{2,Div}_t(\partial D)} + \Big\|(-I/2 + M^k_{\partial D})A\Big\|_{\mathbb{L}^{2,Div}_t(\partial D)}, \\
&\leq (1 + \boldsymbol{c}^2_{L,k})\Big\|(\pm I/2 + M^k_{\partial D})A\Big\|_{\mathbb{L}^{2,Div}_t(\partial D)}.
\end{aligned} \tag{3.3.44}$$

Remark 36. *Similar calculations lead to the following estimates*

$$\|[\pm I/2 + M^0_{\partial D}]A\|_{\mathbb{L}^2(D)} \leq \boldsymbol{c}_L \|[\mp I/2 + M^0_{\partial D}]A\|_{\mathbb{L}^2(D)} + \|[S^0_{\partial D}]A\|_{\mathbb{L}^2(D)} \tag{3.3.45}$$

with instead of (3.3.25)*, we use the following inequality*

$$\int_{\partial D} \nu \times [S^0_{\partial D}]A \cdot \operatorname{curl}[S^0_{\partial} D]A|_{\pm} ds \geq \left\|\operatorname{curl} S^0_{\partial D}(A)\right\|^2 - \left\|\nabla S^0_{\partial D}(\operatorname{Div} A)\right\| \left\|S^0_{\partial D}(\operatorname{Div} A)\right\|$$

including the following one

$$\int_{\partial D} \nu \times \operatorname{curl}[S^0_{\partial} D]A|_{\pm} \cdot \nabla[S^0_{\partial D}] \operatorname{Div} A ds \geq \left\|\nabla S^0_{\partial D}(\operatorname{Div} A)\right\|,$$

where the norms are either of $\mathbb{L}^2(D)$ *or* $\mathbb{L}^2(\mathbb{R}^3 \setminus D)$ *with the use of a similar decomposition (i.e Lemma 32).*

Theorem 37. *There exist a positive constant* $\boldsymbol{c}_{L,k}$ *which depends only on the Lipschitz character of* $\boldsymbol{\mathcal{D}}_m$*'s and* k *such that,*

- *If* $\Im k = 0$ *and* $\boldsymbol{c}_r = \frac{\boldsymbol{\delta}}{\boldsymbol{a}} \geq c_0 2\boldsymbol{c}_L(|k|+4)\boldsymbol{c}_{L,1}$*, then*

$$\sum_{m=1}^{\aleph} \|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)} \leq 3\boldsymbol{c}_{L,1} \aleph \boldsymbol{a}^2, \tag{3.3.46}$$

  for $\boldsymbol{c}_{L,1}$ *stands for the constant that appears in* (31) *of Theorem 31 for* $k = i$*.*

- *If* $\Im(k) > 0$ *and* $\boldsymbol{c}_r = 1$

$$\sum_{m=1}^{\aleph} \|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)} \leq \boldsymbol{c}_{L,k} \aleph \boldsymbol{a}^2. \tag{3.3.47}$$

The following lemma will help us provide a short cut to derive the desired estimates and approximations, using the results of section 3.2.3.

Lemma 38. *For each* $m \in \{1, ..., \aleph\}$ *and* $x \in D_m$*, we set,*

$$\mathcal{H}^{1,A}_m(x) := \boldsymbol{S}^0_{\partial D_m}(w)(x) \quad \text{and} \quad \mathcal{H}^{2,A}_m(x) := \boldsymbol{S}^0_{\partial D_m}(W)(x). \tag{3.3.48}$$

where w *and* W *are solutions of* (3.3.20)*, then*

$$A = \nu \times \nabla \mathcal{H}^{1,A}_m(x) + \nu \times \mathcal{H}^{2,A}_m(x), \tag{3.3.49}$$

and we have the following estimates holds

$$\|\nabla \mathcal{H}^{1,A}_m\|_{\mathbb{L}^2(D_m)} \leq \boldsymbol{c}_L \boldsymbol{a}^{\frac{1}{2}} \|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}, \tag{3.3.50}$$

and

$$\|\operatorname{curl} \mathcal{H}^{2,A}_m\|_{\mathbb{L}^2(D_m)} \leq \boldsymbol{c}_L \boldsymbol{a}^{\frac{1}{2}} \|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)} \tag{3.3.51}$$

are satisfied with some positive constant $\boldsymbol{c}_L$ *depending only on the Lipschitz character of* D_m*'s.*

Proof. With the aid of the Green formula, being $\operatorname{curl}^2 \mathcal{H}^{2,A} = 0$ in D_m,

$$\begin{aligned}
\|\operatorname{curl} \mathcal{H}^{2,A}_m\|^2_{\mathbb{L}^2(D_m)} = & \int_{\partial D_m} \nu \times \mathcal{H}^{2,A}_m \cdot \operatorname{curl} \mathcal{H}^{2,A}_m|_- ds, \\
\leq & \big\|\nu \times \mathcal{H}^{2,A}_m\big\|_{\mathbb{L}^2(\partial D_m)} \big\|\nu \times \operatorname{curl} \mathcal{H}^{2,A}_m|_-\big\|_{\mathbb{L}^2(\partial D_m)}, \\
\leq & \boldsymbol{c}_L \boldsymbol{a} \|A_m\|_{\mathbb{L}^{2,Div}_t}(\partial D_m) \boldsymbol{c}_L \big\|\nu \cdot \operatorname{curl} \mathcal{H}^{2,A}_m|_-\big\|_{\mathbb{L}^2(\partial D_m)} \leq \boldsymbol{c}^+_L \boldsymbol{a} \|A_m\|^2_{\mathbb{L}^{2,Div}_t}(\partial D_m).
\end{aligned}$$

Again, using Green formula, we have

$$\begin{aligned}
\int_{D_m} \big|\nabla \mathcal{H}^{1,A}_m\big|^2 dv = & \int_{\partial D_m} \mathcal{H}^{1,A}_m \nu \cdot \nabla \mathcal{H}^{1,A}_m|_- ds \\
\leq & \big\|\mathcal{H}^{1,A}_m\big\|_{\mathbb{L}^2(\partial D_m)} \big\|\nu \cdot \nabla \mathcal{H}^{1,A}_m|_-\big\|_{\mathbb{L}^2(\partial D_m)}, \\
\leq & \boldsymbol{c}_L \boldsymbol{a} \|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)} \boldsymbol{c}_L \big\|\nu \times \nabla \mathcal{H}^{1,A}_m|_-\big\|_{\mathbb{L}^2(\partial D_m)} \leq \boldsymbol{c}^+_L \boldsymbol{a} \|A_m\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)}.
\end{aligned}$$

□

Lemma 39. *For every $k \in \mathbb{R}^+$, the operator $\boldsymbol{M}^k_{\partial D} - \boldsymbol{M}^i_{\partial D}$ is compact and we have, under the condition on c_r,*

$$\|\Big(\boldsymbol{M}^k_{\partial D} - \boldsymbol{M}^i_{\partial D}\Big) A\|_{\mathbb{L}^{2,Div}_t(\partial D)} \leq \Big(\frac{1}{16\pi^2 \boldsymbol{c}^2_{L,1}} + \boldsymbol{c}_{L,k} \boldsymbol{a}^2\Big)^{\frac{1}{2}} \|A\|_{\mathbb{L}^{2,Div}_t(\partial D)}. \tag{3.3.52}$$

Proof. We have, due to the representation (3.3.14)

$$\begin{aligned}
\|\Big(\boldsymbol{M}^k_{\partial D} - \boldsymbol{M}^i_{\partial D}\Big) A\|^2_{\mathbb{L}^{2,Div}_t(\partial D)} \leq & \Bigg[\Big(\sum_{m=1}^{\aleph} \|\Big(\boldsymbol{M}^k_{m_{D_m}} - \boldsymbol{M}^i_{m_{D_m}}\Big) A_m\|^2\Big)^{\frac{1}{2}} \\
& + \Big(\sum_{m=1}^{\aleph} \|\sum_{\substack{j \geq 1 \\ j \neq m}}^{\aleph} (\boldsymbol{M}^k_{m_{D_j}} - \boldsymbol{M}^i_{m_{D_j}}) A_j\|^2\Big)^{\frac{1}{2}}\Bigg]^2.
\end{aligned} \tag{3.3.53}$$

Using (A.0.3), we obtain

$$|[\boldsymbol{M}^k_{m_{D_m}} - \boldsymbol{M}^i_{m_{D_m}}](A)| \leq \boldsymbol{c}_{L,k} \boldsymbol{a} \|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}. \tag{3.3.54}$$

Due to (3.3.16), (A.0.2) for $\alpha = 1$ and (A.0.3), we get

$$|[\operatorname{Div}(\boldsymbol{M}^k_{m_{D_m}} - \boldsymbol{M}^i_{m_{D_m}})](A)| \leq \boldsymbol{c}_L \Big((c_k + \boldsymbol{c}_L)\Big) \boldsymbol{a} \|A\|_{\mathbb{L}^{2,Div}_t(\partial D_m)} + (1 + k^2) |[\nu \cdot \boldsymbol{S}^0_{\partial D_i}](A)|.$$

With

$$C^1_{L,k} := (1 + k^2) \|[S^0_{m_{\boldsymbol{\mathcal{D}}_m}}]\|_{\mathcal{L}(\mathbb{L}^2(\partial \boldsymbol{\mathcal{D}}_m))},$$

we get

$$\|[\boldsymbol{M}^k_{m_{D_m}} - \boldsymbol{M}^i_{m_{D_m}}](A)\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)} \leq \boldsymbol{c}_{L,k} \boldsymbol{a}^4 \|A_m\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)} + C^1_{L,k} \boldsymbol{a}^2 \|A_m\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)}. \tag{3.3.55}$$

Summing over m

$$\sum_{m=1}^{\aleph} \|[\boldsymbol{M}^k_{m_{D_m}} - \boldsymbol{M}^i_{m_{D_m}}](A)\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)} \leq \boldsymbol{c}_{L,k}\boldsymbol{a}^2 \sum_{m=1}^{\aleph} \|A_m\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)}. \tag{3.3.56}$$

For $x \in \partial D_m$, we have, with the notation of Lemma 27

$$\boldsymbol{M}^k_{m_{D_j}}(A)(x) = -\nu \times \mathcal{N}^{k,\mathbb{I}}_{D/D_m}(\nabla \mathcal{H}^{1,A}_j + \mathcal{H}^{2,A}_j) + \nu \times \mathcal{M}^{k,\mathbb{I}}_{D/D_m}(\operatorname{curl} \mathcal{H}^{2,A}_j), \tag{3.3.57}$$

and

$$\operatorname{div} \boldsymbol{M}^k_{m_{D_j}}(A)(x) = -\nu \cdot \mathcal{N}^{k,\mathbb{I}}_{D/D_m}(\operatorname{curl} \mathcal{H}^{2,A}_j) + k^2 \nu \cdot \mathcal{M}^{k,\mathbb{I}}_{D/D_m}(\nabla \mathcal{H}^{1,A}_j + \mathcal{H}^{2,A}_j). \tag{3.3.58}$$

Indeed,

$$\begin{aligned}
\boldsymbol{M}^k_{m_{D_j}}(A)(x) =& \nu \times \operatorname{curl} \int_{\partial D_j} \Phi_k(x,y)(\nu \times \nabla \mathcal{H}^{1,A}_j + \nu \times \mathcal{H}^{2,A}_j)(y) ds_y, \\
=& \nu \times \operatorname{curl} \int_{D_j} \operatorname{curl}_y \Big(\Phi_k(x,y)(\nabla \mathcal{H}^{1,A}_m + \mathcal{H}^{2,A}_m)(y) \Big) ds_y, \\
=& -\nu \times \Big(\operatorname{curl}^2 \int_{D_j} \Phi_k(x,y)(\nabla \mathcal{H}^{1,A}_j + \mathcal{H}^{2,A}_j)(y) ds_y - \operatorname{curl} \int_{D_j} \Phi_k(x,y) \operatorname{curl} \mathcal{H}^{2,A}_j(y) ds_y \Big).
\end{aligned}$$

taking the surface divergence clear the second equation. Now, from (3.3.57) similarly to what was done to get (3.2.40), (3.2.46) we get both

$$\begin{aligned}
\sum_{m=1}^{\aleph} \|\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big(\boldsymbol{M}^k_{m_{D_j}} - \boldsymbol{M}^i_{m_{D_j}}\Big)(A)\|^2_{\mathbb{L}^2(\partial D_m)} \leq& c_0 \frac{(1+|k|)^2}{16\pi^2 \boldsymbol{c}_r^6} \sum_{j=1}^{m} \frac{\|\operatorname{curl} \mathcal{H}^{2,A}_j(y)\|^2_{\mathbb{L}^2(D_j)}}{\boldsymbol{a}} \\
&+ 2^3 c_0 \Big(\frac{|k|(|k|+1)}{4\pi}\Big)^2 \Big[\frac{(|k|+4)^2}{(1+2\boldsymbol{c}_r)^4 \boldsymbol{c}_r^2}\Big] \sum_{j=1}^{m} \frac{\|\nabla \mathcal{H}^{1,A}_j + \mathcal{H}^{2,A}_j\|^2_{\mathbb{L}^2(D_j)}}{\boldsymbol{a}}, \\
\sum_{m=1}^{\aleph} \|\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \operatorname{Div}\Big(\boldsymbol{M}^k_{m_{D_j}} - \boldsymbol{M}^i_{m_{D_j}}\Big)(A)\|_{\mathbb{L}^2(\partial D_m)} \leq& 2^3 c_0 \frac{|k|^2(1+|k|)^2}{\boldsymbol{c}_r^6} \sum_{j=1}^{\aleph} \frac{\|\nabla \mathcal{H}^{1,A}_j + \mathcal{H}^{2,A}_j\|^2_{\mathbb{L}^2(D_j)}}{\boldsymbol{a}} \\
&+ 2^3 c_0 \Big(\frac{|k|(|k|+1)}{4\pi}\Big)^2 \Big[\frac{(|k|+4)^2}{(1+2\boldsymbol{c}_r)^4 \boldsymbol{c}_r^2}\Big] \sum_{j=1}^{m} \frac{\|\operatorname{curl} \mathcal{H}^{2,A}_j(y)\|^2_{\mathbb{L}^2(D_j)}}{\boldsymbol{a}},
\end{aligned}$$

From Lemma 38 we get

$$\begin{aligned}
&\sum_{m=1}^{\aleph} \|\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big(\boldsymbol{M}^k_{m_{D_j}} - \boldsymbol{M}^i_{m_{D_j}}\Big)(A)\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)} \\
&\qquad \leq 2^3 c_0 \boldsymbol{c}_L \Big(\frac{|k|(|k|+1)}{4\pi}\Big)^2 \Big[\frac{1}{\boldsymbol{c}_r^6} + \frac{(|k|+4)^2}{(1+2\boldsymbol{c}_r)^4 \boldsymbol{c}_r^2}\Big] \sum_{j=1}^{m} \|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)},
\end{aligned} \tag{3.3.59}$$

which give, with $\boldsymbol{c}_r$ as stated in Theorem 37,

$$\sum_{m=1}^{\aleph} \| \sum_{\substack{j \geq 1 \\ j \neq m}}^{\aleph} \left(\boldsymbol{M}^k_{m_{D_j}} - \boldsymbol{M}^i_{m_{D_j}} \right)(A) \|^2_{\mathbb{L}^{2,Div}_t(\partial D_m)} \leq \frac{1}{(4\pi)^2 \boldsymbol{c}^2_{L,1}} \|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D)}. \tag{3.3.60}$$

$\square$

Proof. (Theorem 37). Now, we end up the proof of Proposition 37 as follows

$$\begin{aligned} \left\| [I/2 + \boldsymbol{M}^k_{\partial D}](A) \right\| \geq & \left\| [I/2 + \boldsymbol{M}^i_{\partial D}](A) \right\| - \left\| [\boldsymbol{M}^i_{\partial D} - \boldsymbol{M}^k_{\partial D}](A) \right\|, \\ \geq & \left\| [I/2 + \boldsymbol{M}^i_{\partial D}](A) \right\| - \left(\frac{1}{16\pi^2 \boldsymbol{c}^2_{L,1}} + \boldsymbol{c}_{L,k} \boldsymbol{a}^2 \right)^{\frac{1}{2}} \|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D)} \\ \geq & \left(\frac{1}{\boldsymbol{c}_{L,1}} - \left(\frac{1}{16\pi^2 \boldsymbol{c}^2_{L,1}} + \boldsymbol{c}_{L,k} \boldsymbol{a}^2 \right)^{\frac{1}{2}} \right) \|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D)}. \end{aligned}$$

Then for $\boldsymbol{c}_{L,k} \boldsymbol{a} \leq \frac{1}{3\boldsymbol{c}_{L,1}}$

$$\left\| \nu \times E^{\text{in}} \right\|^2_{\mathbb{L}^{2,Div}_t(\partial D)} = \left\| [I/2 + \boldsymbol{M}^k_{\partial D}](A) \right\| \geq \frac{1}{3\boldsymbol{c}_{L,1}} \|A\|^2_{\mathbb{L}^{2,Div}_t(\partial D)}.$$

$\square$

3.3.3 Fields approximation and the linear algebraic systems

Based on the representation (3.3.11), the expression of the far field pattern is given by

$$E^\infty(\hat{x}) = \frac{ik}{4\pi} \hat{x} \times \int_{\partial D} A(y) e^{-ik\hat{x}.y} ds_y, \tag{3.3.61}$$

where $\hat{x} = (x/|x|) \in \mathbb{S}^2$ and A is the solution of the (3.3.14). As it was done in [14], let us consider $(\psi^m_l)^3_{l=1}$ as the solution of

$$[-I/2 + \boldsymbol{M}^0_{i_{D_i}}](\nu \times \psi^l_m) = -\nu \times V_l, \tag{3.3.62}$$

with

$$V^*_1 = (0, 0, (x - \boldsymbol{z}_i) \cdot e_2), \;\; V^*_2 = ((x - \boldsymbol{z}_i) \cdot e_3, 0, 0), \;\;\; \text{and } V^*_3 = (0, (x - \boldsymbol{z}_i) \cdot e_1, 0).$$

Taking the surface divergence of (3.3.62) gives

$$[I/2 + (\boldsymbol{K}^0_{\partial D_m})^*](\nu \cdot \operatorname{curl} \psi^l_m) = -\nu^l_i. \tag{3.3.63}$$

Notice that

$$\begin{aligned} \int_{\partial D_m} \nu \times \psi^l_m ds &= - \int_{\partial D_m} (y - \boldsymbol{z}_m) \nu \cdot \operatorname{curl} \psi^l_m(y) ds_y \\ &= - \int_{\partial D_m} [I/2 + (\boldsymbol{K}^0_{\partial D_m})]^{-1} (y - \boldsymbol{z}_m) \; [I/2 + (\boldsymbol{K}^0_{\partial D_m})^*] \nu \cdot \operatorname{curl} \psi^l_m(y) ds_y \\ &= \int_{\partial D_m} [I/2 + (\boldsymbol{K}^0_{\partial D_m})]^{-1} (y - \boldsymbol{z}_m) \; \nu^l_i ds_y. \end{aligned} \tag{3.3.64}$$

As $\operatorname{curl} V_l = e_l$, it holds that

$$\int_{\partial D_m} \nu^l u ds = \int_{\partial D_m} \nu^l \cdot \operatorname{curl} V \; u ds = -\int_{\partial D_m} \nu \times V_l \cdot \nabla u ds = \int_{\partial D_m} V_l \cdot \nu \times \nabla u ds \tag{3.3.65}$$

for any scalar function $u \in \mathbb{H}^1(\partial D_m)$. Finally, let ϕ_m be the solution to the following integral equation

$$[-\frac{1}{2}I + \boldsymbol{K}^0_{\partial D_m}](\phi_m)(x) = (x - \boldsymbol{z}_i). \tag{3.3.66}$$

Lemma 40. *Due to the scale invariance of both the double-layer and the Maxwell-dipole operators, the following estimates hold*

$$\|\nu \times \psi^l_m\|_{\mathbb{L}^2(\partial D_i)} \leq \boldsymbol{c}_L \boldsymbol{a}^2, \;\; \|\nu \times \psi^l_m\|_{\mathbb{L}^{2,Div}_t(\partial Di)} \leq \boldsymbol{c}_L \boldsymbol{a}, \quad \textit{and} \;\; \|\phi_m\|_{\mathbb{L}^2(\partial D_m)} \leq \boldsymbol{c}_L \boldsymbol{a}^2. \tag{3.3.67}$$

Proposition 41. *For $\Im k = 0$, the far field pattern can be approximated by*

$$E^\infty(\hat{x}) = \frac{ik}{4\pi} \sum_{m=1}^{\aleph} e^{-ik\hat{x}.\boldsymbol{z}_m} \hat{x} \times \left\{ \mathcal{Q}^1_m - ik\hat{x} \times \mathcal{Q}^2_m \right\} + O\Big(\frac{|k|^3}{\boldsymbol{c}_r} \; \boldsymbol{a} \Big). \tag{3.3.68}$$

The elements $(\mathcal{Q}^1_m)_{i=1}^{\aleph}$ and $(\mathcal{Q}^2_m)_{i=1}^{\aleph}$ are solutions of the following linear algebraic system

$$\begin{aligned} \mathcal{Q}^2_m = &-\big[\mathcal{P}_{Dm}\big] \sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \Big(\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \mathcal{Q}^2_j - k^2 \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \mathcal{Q}^1_j \Big) - \big[\mathcal{P}_{Dm}\big] \operatorname{curl} E^{in}(\boldsymbol{z}_i), \\ &+ Er_m\big(\operatorname{curl} \mathcal{H}^{2,A}, \nabla \mathcal{H}^{1,A} + \mathcal{H}^{2,A}\big) + O\Big((\boldsymbol{a}+1)|k|^2 \boldsymbol{a}^3 \|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)} + |k|\boldsymbol{a} \Big), \end{aligned} \tag{3.3.69}$$

$$\begin{aligned} \mathcal{Q}^1_m = &\big[\mathcal{T}_{Dm}\big] \sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \Big(-\nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \mathcal{Q}^2_j + \Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \; \mathcal{Q}^1_j \Big) - \big[\mathcal{T}_{Dm}\big] E^{in}(\boldsymbol{z}_i) \\ &+ Er_m\big((\nabla \mathcal{H}^{1,A} + \mathcal{H}^{2,A}), \operatorname{curl} \mathcal{H}^{2,A}\big) + O\Big((\boldsymbol{a}+1)|k|^2 \boldsymbol{a}^3 \|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)} + \boldsymbol{a} \Big). \end{aligned} \tag{3.3.70}$$

Proof.
- (*The far field Approximation*) Starting from the expression (3.3.61) with representation of the density as in (3.3.49) we get with a simple integration by part

$$\begin{aligned} E^\infty(\widehat{x}) = \frac{ik}{4\pi} \sum_{m=1}^{\aleph} \Bigg(&\hat{x} \times \int_{D_m} e^{-ik\hat{x}.y} \operatorname{curl} \mathcal{H}^{2,A}_m(y) ds_y \\ &+ ik\hat{x} \times \int_{D_m} e^{-ik\hat{x}.y} \Big(\nabla \mathcal{H}^{1,A}_m(y) + \mathcal{H}^{2,A}_m(y) \Big) \times \hat{x} ds_y \Bigg). \end{aligned} \tag{3.3.71}$$

  This representation (3.3.71) is the analogous of (3.2.61), stated for the transmission problem, in such a way that, the far field approximation is done in similar way, with the appropriate changes that involve (38) (i.e the appearance of the constant $\boldsymbol{c}_L$).

- (*Derivation of the linear system*)

  For (3.3.69), multiplying (3.3.14) by $\nabla\phi_m$ and integrating over D_m, we obtain

$$\int_{\partial D_m} \nabla\phi_m \cdot [I/2 + \boldsymbol{M}^k_{m_{D_m}}](A_m)\ ds \\ = -\sum_{\substack{j\geq 1 \\ j\neq m}} \int_{\partial D_m} \nabla\phi_m \cdot [\boldsymbol{M}^k_{m_{D_j}}](A_j) ds + \int_{\partial D_m} \nabla\phi_m \cdot \nu \times E^{\mathrm{in}} ds. \tag{3.3.72}$$

Integrating by part and considering (3.3.58), we get

$$\int_{\partial D_m} \nabla\phi_m \cdot [I/2 + \boldsymbol{M}^k_{m_{D_m}}](A_m)\ ds \\ = \sum_{\substack{j\geq 1 \\ j\neq m}} \int_{\partial D_m} \Big(-\phi_m \nu \cdot \mathcal{N}^{k,\mathbb{I}}_{D_j}(\mathrm{curl}\, \mathcal{H}^{2,A}_j) + k^2 \nu \cdot \mathcal{M}^{k,\mathbb{I}}_{D_j}(\nabla \mathcal{H}^{1,A}_j + \mathcal{H}^{2,A}_j)\Big) ds \\ + \int_{\partial D_m} \nabla\phi_m \cdot \nu \times E^{\mathrm{in}} ds. \tag{3.3.73}$$

We have

$$\int_{\partial D_m} \nabla\phi_m \cdot [I/2 + \boldsymbol{M}^k_{m_{D_m}}](A_m)\ ds = \mathcal{Q}^2_m + O\Big(|k|^2 \boldsymbol{a}^3 \|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}\Big). \tag{3.3.74}$$

Indeed, using the relation (3.3.1), we have

$$\int_{\partial D_m} \phi_m \,\mathrm{Div}[I/2 + \boldsymbol{M}^k_{m_{D_m}}]A\ ds \\ = \int_{\partial D_m} \phi_m \Big([I/2 - (\boldsymbol{K}^k_{\partial D_i})^*]\,\mathrm{Div}\, A - k^2 \nu_{x_i} \cdot [\boldsymbol{S}^k_{\partial D_i}]A \Big)\ ds. \\ = \int_{\partial D_m} \phi_m \Big([I/2 - (\boldsymbol{K}^0_{\partial D_m})^* + (\boldsymbol{K}^0_{\partial D_m} - \boldsymbol{K}^k_{\partial D_i})^*]\,\mathrm{Div}\, A + k^2 \nu \cdot [\boldsymbol{S}^k_{\partial D_i}]A \Big)\ ds,$$

Hence, with the definition (3.3.66), we get

$$\int_{\partial D_m} \phi_m [I/2 - (\boldsymbol{K}^0_{\partial D_m})^*]\,\mathrm{Div}\, A\ ds = \int_{\partial D_m} [I/2 - \boldsymbol{K}^0_{\partial D_m}]\phi_m \,\mathrm{Div}\, A\ ds, \\ = -\int_{\partial D_m} (x - \boldsymbol{z}_i)\,\mathrm{Div}\, A(x)\ ds_x, \\ = \int_{\partial D_m} \mathrm{curl}\, \mathcal{H}^{2,A}\ dv$$

and the left-hand side of (3.3.73) ends up to be

$$\mathcal{Q}^2_m + O((\boldsymbol{a}+1)|k|^2 \boldsymbol{a}^3 \|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}). \tag{3.3.75}$$

As already done in (3.2.72), with a first order approximation, the second member of (3.3.73) can be approximated by

$$-\Big[\int_{\partial D_m}\phi_m\otimes\nu ds\Big]\ \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\left(\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\int_{D_j}\operatorname{curl}\mathcal{H}_j^{2,A}dv-k^2\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\int_{D_j}(\nabla\mathcal{H}_j^{1,A}+\mathcal{H}_j^{2,A})dv\right)$$
$$+Er_m\big(\operatorname{curl}\mathcal{H}^{2,A},\nabla\mathcal{H}^{1,A}+\mathcal{H}^{2,A}\big)\tag{3.3.76}$$

where

$$\nabla\mathcal{H}^{1,A}+\mathcal{H}^{2,A}=\sum_{m=1}^{\aleph}(\nabla\mathcal{H}_j^{1,A}+\mathcal{H}_j^{2,A})\chi_{D_m}(x)$$

and

$$\operatorname{curl}\mathcal{H}^{2,A}=\sum_{m=1}^{\aleph}\operatorname{curl}\mathcal{H}_j^{2,A}\chi_{D_m}(x).$$

Replacing both (3.3.74) and (3.3.76) in (3.3.73) gives the result;

$$\mathcal{Q}_m^2=-\big[\mathcal{P}_{Dm}\big]\left[\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\left(\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{Q}_j^2-k^2\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{Q}_j^1\right)+\operatorname{curl}E^{\text{in}}(\boldsymbol{z}_m)\right]$$
$$+Er_m\big(\operatorname{curl}\mathcal{H}^{2,A},\nabla\mathcal{H}^{1,A}+\mathcal{H}^{2,A}\big)+O\Big((\boldsymbol{a}+1)|k|^2\boldsymbol{a}^3\|A_m\|_{\mathbb{L}_t^{2,Div}(\partial D_m)}+|k|\boldsymbol{a}\Big).\tag{3.3.77}$$

Concerning (3.3.70) we have, for $[\Psi_m]=(\psi_m^l)_{(l=1,2,3)}$ as described in (3.3.62),

$$\int_{\partial D_m}[\Psi]_m\cdot[I/2+\boldsymbol{M}^k_{m_{D_m}}](A_m)\ ds$$
$$=-\sum_{\substack{j\geq 1\\ j\neq m}}\int_{\partial D_m}[\Psi]_m\cdot[\boldsymbol{M}^k_{m_{D_j}}](A_j)ds+\int_{\partial D_m}[\Psi]_m\cdot\nu\times E^{\text{in}}ds.\tag{3.3.78}$$

The left-hand-side is equal to

$$\mathcal{Q}_m^1+O(\boldsymbol{a}^3\|A\|_{\mathbb{L}_t^{2,Div}(\partial D_m)}).\tag{3.3.79}$$

Indeed,

$$\int_{\partial D_m}\psi_m^l\cdot[I/2+\boldsymbol{M}^k_{m_{D_m}}]A_m\ ds=O(\boldsymbol{a}^3\|A_m\|_{\mathbb{L}_t^{2,Div}(\partial D_m)})$$
$$+\int_{\partial D_m}\psi_m^l\cdot[I/2+\boldsymbol{M}^0_{i_{D_i}}](A_m^1)\ ds\tag{3.3.80}$$

and, in view of the decomposition Lemma 38, we have, due to the scale invariance of $[I/2 + \boldsymbol{M}^0_{m_{D_m}}]$ (see [14]), the estimates (3.3.67), those of Lemma 32, and similar estimate for (3.3.55) with $\alpha = 0$, that

$$\begin{aligned}\left|\int_{\partial D_m} \psi^l_m \cdot [I/2 + \boldsymbol{M}^k_{m_{D_m}}]A^2_m \, ds\right| \leq & \|\psi^l_m\| \left(\|[I/2 + \boldsymbol{M}^0_{m_{D_m}}]A^2_m\| + \|[\boldsymbol{M}^k_{m_{D_m}} - \boldsymbol{M}^0_{m_{D_m}}]A^2_m\|\right), \\ \leq & \|\psi^l_m\|_{\mathbb{L}^2(\partial D_m)}\Big(\boldsymbol{c}_L + \frac{|k|^2(|\partial\boldsymbol{\mathcal{D}}|)\boldsymbol{a}^2}{4\pi}\Big)\|A^2_m\|_{\mathbb{L}^2(\partial D_m)}, \\ \leq & \Big(\boldsymbol{c}_l + \frac{|k|^2(|\partial\boldsymbol{\mathcal{D}}|)\boldsymbol{a}^2}{4\pi}\Big)\boldsymbol{a}^3\|A_m\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}.\end{aligned}$$

Further with (3.3.63), (3.3.17) and the representation lemma 32 we have, for the second term of the left-hand member of (3.3.80)

$$\begin{aligned}\int_{\partial D_m} \psi^l_m \cdot \nu \times \nabla[I/2 + \boldsymbol{K}^0_{\partial D_m}](u^A_m) ds = & \int_{\partial D_m} -[I/2 + (\boldsymbol{K}^0_{\partial D_m})^*](\nu \cdot \operatorname{curl} \psi^l_m) \; u^A_m ds. \\ = & \int_{\partial D_m} \nu_l u^A_m ds.\end{aligned}$$

Due to (3.3.65) and Lemma 32, we have

$$\int_{\partial D_m} \nu_l u^A_m ds = \int_{\partial D_m} V_l \cdot \nu \times \left(\nabla u^A_m + \mathcal{H}^{1,A}\right) ds + O(\boldsymbol{a}^3\|A\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}), \tag{3.3.81}$$

hence follows, integrating by part for the second step,

$$\begin{aligned}\int_{\partial D_m} \nu_l u^A_m ds = \int_{\partial D_m} V_l \cdot \nu \times \nabla u^A_m ds = & \int_{\partial D_m} V_l \cdot \nu \times \left(\nabla u^A_m + \mathcal{H}^{1,A}\right) ds + O(\boldsymbol{a}^3\|A\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}), \\ = & \int_{D_m} \nabla(\mathcal{H}^{2,A} + \mathcal{H}^{1,A}) \cdot e_l \; dv - \int_{D_m} V_l \cdot \operatorname{curl} \mathcal{H}^{1,A} dv \\ & + O(\boldsymbol{a}^3\|A\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}).\end{aligned} \tag{3.3.82}$$

Then (3.3.51) gives

$$\int_{\partial D_m} [\Psi]_m \nu \times \nabla[I/2 + \boldsymbol{K}^0_{\partial D_m}](u^A_i) ds = \int_{\partial D_m} \nu u^A_m ds = \mathcal{Q}^1_m + O(\boldsymbol{a}^3\|A\|_{\mathbb{L}^{2,Div}_t(\partial D_m)}). \tag{3.3.83}$$

For the first term of the right-hand-side of (3.3.78), we have, with (3.3.57) and a first order approximation,

$$\begin{aligned}
&\sum_{\substack{j\geq 1\\ j\neq m}} \int_{\partial D_m} \psi_m^l \cdot [\boldsymbol{M}^k_{mD_j}] A_j ds \\
&\quad = -\sum_{\substack{j\geq 1\\ j\neq m}} \int_{\partial D_m} \psi_m^l \cdot \Big(\nu \times \mathcal{N}^{k,\mathbb{I}}_{D_j}(\nabla \mathcal{H}^{1,A}_j + \mathcal{H}^{2,A}_j) - \nu \times \mathcal{M}^{k,\mathbb{I}}_{D_j}(\operatorname{curl} \mathcal{H}^{2,A}_j)\Big) \\
&\quad = -\sum_{\substack{j\geq 1\\ j\neq m}} \int_{\partial D_m} \psi_m^l \cdot \Bigg(\nu \times \Big(\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \int_{D_j} (\nabla \mathcal{H}^{1,A}_j + \mathcal{H}^{2,A}_j) dv\Big) \\
&\qquad\qquad - \nu \times \Big(\nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \int_{D_j} \operatorname{curl} \mathcal{H}^{2,A}_j dv\Big)\Bigg) \\
&\qquad + Er_m\big((\nabla \mathcal{H}^{1,A} + \mathcal{H}^{2,A}), \operatorname{curl} \mathcal{H}^{2,A}\big).
\end{aligned} \tag{3.3.84}$$

We rewrite (3.3.84) as follows

$$\begin{aligned}
\sum_{\substack{j\geq 1\\ j\neq m}} \int_{\partial D_m} \psi_m^l \cdot [\boldsymbol{M}^k_{mD_j}] A_j ds = \sum_{\substack{j\geq 1\\ j\neq m}} \int_{\partial D_m} \nu \times \psi_m^l \cdot \Bigg(\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \mathcal{Q}^1_j - \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \mathcal{Q}^2_j\Bigg) \\
+ Er_m\big((\nabla \mathcal{H}^{1,A} + \mathcal{H}^{2,A}), \operatorname{curl} \mathcal{H}^{2,A}\big),
\end{aligned} \tag{3.3.85}$$

or, more precisely with (3.3.64) in mind and $l = 1, 2, 3$ gives, according to (3.3.7),

$$\begin{aligned}
\sum_{\substack{j\geq 1\\ j\neq m}} \int_{\partial D_m} [\Psi]_m \cdot [\boldsymbol{M}^k_{mD_j}] A_j ds = \big[\mathcal{T}_{Dm}\big] \sum_{\substack{j\geq 1\\ j\neq m}} \Bigg(\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \mathcal{Q}^1_j - k^2 \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \mathcal{Q}^2_j\Bigg) \\
+ Er_m\big((\nabla \mathcal{H}^{1,A} + \mathcal{H}^{2,A}), \operatorname{curl} \mathcal{H}^{2,A}\big).
\end{aligned} \tag{3.3.86}$$

Assembling both (3.3.86) and (3.3.79) in (3.3.78) gives us (3.3.70).

□

As in the proof of Proposition 30, with μ^+ and μ^- as defined in (3.3.9), we have the following proposition.

Proposition 42. *Under the condition that*

$$\frac{\boldsymbol{\delta}}{\boldsymbol{a}} = \boldsymbol{c}_r \geq 3|k|\mu^+. \tag{3.3.87}$$

the following linear system is invertible

$$\begin{aligned}\widehat{\mathcal{Q}^2_m} &= -\big[\mathcal{P}_{Dm}\big] \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big(\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j)\widehat{\mathcal{Q}^2_j} - k^2\nabla\Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \widehat{\mathcal{Q}^1_j}\Big) - \big[\mathcal{P}_{Dm}\big]\,\mathrm{curl}\, E^{in}(\boldsymbol{z}_i), \\ \widehat{\mathcal{Q}^1_m} &= \big[\mathcal{T}_{Dm}\big] \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big(-\nabla\Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \widehat{\mathcal{Q}^2_j} + \Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j)\ \widehat{\mathcal{Q}^1_j}\Big) - \big[\mathcal{T}_{Dm}\big]E^{in}(\boldsymbol{z}_i).\end{aligned} \tag{3.3.88}$$

and the solution satisfies the following estimates

$$\left(\sum_{m=1}^{\aleph} \big(|\widehat{\mathcal{Q}^1_m}|^2\big)\right)^{\frac{1}{2}} \leq \frac{9\mu^+ \boldsymbol{a}^3}{8} \left(\sum_{m=1}^{\aleph} \big(|E(\boldsymbol{z}_i)|^2 + |\mathrm{curl}\, E(\boldsymbol{z}_i)|^2\big)\right)^{\frac{1}{2}}, \tag{3.3.89}$$

and

$$\left(\sum_{m=1}^{\aleph} \big(|\widehat{\mathcal{Q}^2_m}|^2\right)^{\frac{1}{2}} \leq \frac{9\mu^+ \boldsymbol{a}^3}{8} \left(\sum_{m=1}^{\aleph} \big(|E(\boldsymbol{z}_i)|^2 + |\mathrm{curl}\, E(\boldsymbol{z}_i)|^2\big)\right)^{\frac{1}{2}}. \tag{3.3.90}$$

The proof of Theorem 21 is similar to the anisotropic transmission problem with $\boldsymbol{c}_{L,k,\mu} := \boldsymbol{c}_{L,k}(|k|+1)\frac{4\mu^+}{8\mu^-}\boldsymbol{c}_{L,k}(|k|+1)$ and $\boldsymbol{c}_{L,k}$ describes the ratio of the largest Lipschitz constant that is involved in section 3.3 and the smallest one.

Appendix A

Appendix

A.0.1 Green function approximations

A simple application of mean value theorem gives

$$\begin{aligned}
\big(\Phi_k(x,y)-\Phi_k(\boldsymbol{z}_m,y)\big) &= \int_0^1 \nabla\Phi_k(tx+(1-t)\boldsymbol{z}_m,y)dt\circ(x-\boldsymbol{z}_m) = O\Big(\frac{1}{\boldsymbol{\delta}_{mj}}(\frac{1}{\boldsymbol{\delta}_{mj}}+|k|)\boldsymbol{a}\Big),\\
\nabla\big(\Phi_k(x,y)-\Phi_k(\boldsymbol{z}_m,y)\big) &= \int_0^1 D^2\Phi_k(tx+(1-t)\boldsymbol{z}_m,y)dt\circ(x-\boldsymbol{z}_m) = O\Big(\frac{1}{\boldsymbol{\delta}_{mj}}(\frac{1}{\boldsymbol{\delta}_{mj}}+|k|)^2\boldsymbol{a}\Big),\\
\nabla\nabla\big(\Phi_k(x,y)-\Phi_k(\boldsymbol{z}_m,y)\big) &= \int_0^1 D^3\Phi_k(tx+(1-t)\boldsymbol{z}_m,y)dt\circ(x-\boldsymbol{z}_m) = O\Big(\frac{1}{\boldsymbol{\delta}_{mj}}(\frac{1}{\boldsymbol{\delta}_{mj}}+|k|)^3\boldsymbol{a}\Big).
\end{aligned} \tag{A.0.1}$$

whenever $y\in D_j$ and $j\neq m$. We will also need the following first order expansion of the Green's function[1]

$$[\Phi_k-\Phi_{i\alpha}](x) = \frac{(ik-\alpha)}{4\pi}\int_{[0,1]} e^{\big((ik)t-(1-t)\alpha\big)|x|}\ dt, \tag{A.0.2}$$

$$\nabla[\Phi_k-\Phi_{i\alpha}](x) = \Big[\frac{(ik-\alpha)}{4\pi}\int_{[0,1]} e^{(ikt-(1-t)\alpha)|x|}\ \big(ikt+(1-t)\alpha\big)dt\Big]\ \frac{x}{|x|}, \tag{A.0.3}$$

$$[\otimes\nabla]^2[\Phi_k-\Phi_{i\alpha}](x) = \int_0^1 \frac{e^{(ikt-(1-t)\alpha)|x|}\big(ikt+(1-t)\alpha\big)}{4\pi\ (ik-\alpha)^{-1}|x|}\Big[I+\big(ikt+(1-t)\alpha-\frac{1}{|x|}\big)\frac{[\otimes x]^2}{|x|}\Big]dt. \tag{A.0.4}$$

A.0.2 Counting lemma

Lemma 43. *For any non negative function g we have*

$$\sum_{i\geq 1, i\neq j}^{\aleph} g(\boldsymbol{\delta}_{mj}) \leq 48 \sum_{\substack{1\leq l\leq\aleph\\ O\leq i\leq k\leq l}} g\Big(\big[\big(l^2+k^2+i^2\big)^{\frac{1}{2}}(\boldsymbol{c}_r+1)-1\big]\boldsymbol{a}\Big), \tag{A.0.5}$$

[1] By $(x)^{\overset{p}{\otimes}}$ we mean the p-times repeated tensor product of x.

and for any non negative sequence $(\alpha_m)_{m=1}^{\aleph}$

$$\sum_{m=1}^{\aleph}\Big(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{\alpha_m}{\boldsymbol{\delta}_{mj}^{q}}\Big)^2 \leq \Big(\frac{c_0}{\boldsymbol{\delta}^q}\sum_{l=1}^{\aleph^{\frac{1}{3}}} l^{2-q}\Big)^2\sum_{m=1}^{\aleph}\alpha_m^2. \tag{A.0.6}$$

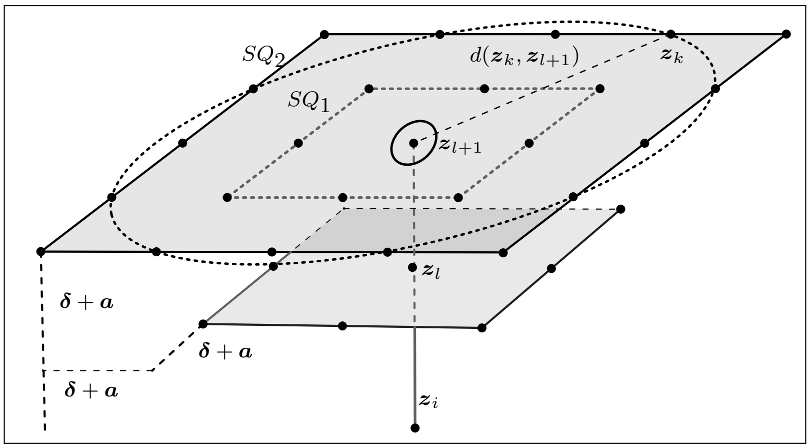

Figure A.1: Disposition of the faces F_l.

Proof. From a given position $\boldsymbol{z}_i$ we split the space into equidistant cubes (CU_l), centered at $\boldsymbol{z}_i$, such that each of its faces support some of the $(\boldsymbol{z}_j)_{\substack{j\geq 1\\ j\neq m}}^{\aleph}$, and each of its faces, (F_l) are distant from $F_{l\pm 1}$ with distance $\boldsymbol{\delta}+\boldsymbol{a}$, (see fig. A.1). Obviously the distance from a point $\boldsymbol{z}_j \in F_l$ to $\boldsymbol{z}_i$ is

$$\begin{aligned} d(\boldsymbol{z}_i, \boldsymbol{z}_j) &= \sqrt{(d(\boldsymbol{z}_i, F_l)^2 + d(\boldsymbol{z}_j, \boldsymbol{z}_l)^2)},\\ d(\boldsymbol{z}_i, F_l) &= l(\boldsymbol{\delta}+\boldsymbol{a}), \end{aligned} \tag{A.0.7}$$

z_l being the orthogonal projection of z_i on F_l. Repeating the same splitting on each faces F_l, we draw concentric squares $(SQ_k)_{k=1}^{l}$, centered at $\boldsymbol{z}_l$, and there is 4 or 8 location that are equidistant from a given square SQ_k to $\boldsymbol{z}_l$ which correspond to the intersection of a circle with a square sharing the same center, similarly, for a point $z_p \in SQ_k$, we get, with z_k standing for one of its orthogonal projection on SQ_k,

$$\begin{aligned} d(\boldsymbol{z}_p, \boldsymbol{z}_l) &= \sqrt{(d(\boldsymbol{z}_p, \boldsymbol{z}_k)^2 + d(SQ_k, \boldsymbol{z}_l)^2)},\\ d(SQ_k, \boldsymbol{z}_l) &= k(\boldsymbol{\delta}+\boldsymbol{a}), \end{aligned} \tag{A.0.8}$$

further

$$d(B_{\boldsymbol{z}_i}^{\boldsymbol{a}}, B_{\boldsymbol{z}_j}^{\boldsymbol{a}}) = d(\boldsymbol{z}_i, \boldsymbol{z}_j) + \boldsymbol{a}.$$

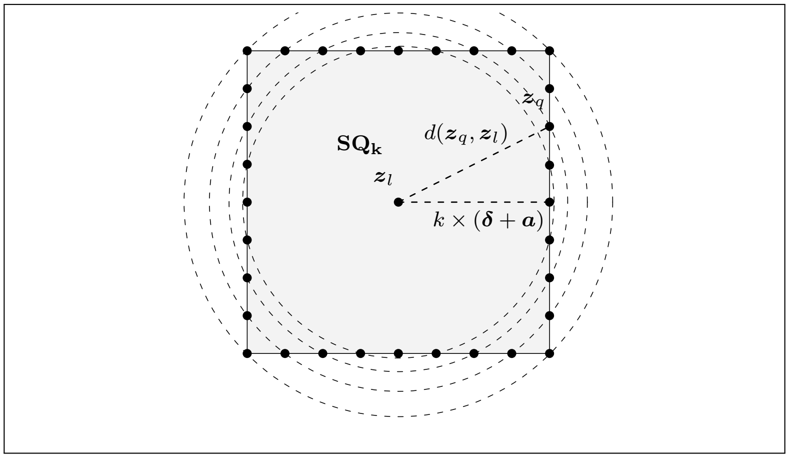

Figure A.2: Counting on the square SQ_k.

So, for a non negative function g, it comes with (A.0.7) that

$$\begin{aligned}
g\big(d(B^{\boldsymbol{a}}_{\boldsymbol{z}_i}, B^{\boldsymbol{a}}_{\boldsymbol{z}_j})\big) = \sum_{j(\neq i)=1}^{\aleph} g\big(d(\boldsymbol{z}_i, \boldsymbol{z}_j) - \boldsymbol{a}\big) &= \sum_{l=1}^{\aleph} 6 \sum_{\boldsymbol{z}_j \in F_l} g\big(d(\boldsymbol{z}_j, \boldsymbol{z}_i) - \boldsymbol{a}\big), \\
&= 6 \sum_{l=1}^{\aleph} \sum_{z_j \in F_l} g\Big(\big(d(F_l, \boldsymbol{z}_i)^2 + d(\boldsymbol{z}_l, \boldsymbol{z}_j)^2\big)^{\frac{1}{2}} - \boldsymbol{a}\Big) \\
&\leq 6 \sum_{l=1}^{\aleph} \sum_{k=0}^{l} 8 \sum_{z_p \in SQ_k} g\Big(\big(l^2(\boldsymbol{\delta} + \boldsymbol{a})^2 + d(\boldsymbol{z}_l, \boldsymbol{z}_p)^2\big)^{\frac{1}{2}} - \boldsymbol{a}\Big),
\end{aligned}$$

and with (A.0.8)

$$\begin{aligned}
\sum_{j(\neq i)=1}^{\aleph} g\big(d(B^{\boldsymbol{a}}_{\boldsymbol{z}_i}, B^{\boldsymbol{a}}_{\boldsymbol{z}_j})\big) &\leq 6 \times 8 \sum_{l=1}^{\aleph} \sum_{k=0}^{l} \sum_{p=0}^{k} g\Big(\big(l^2(\boldsymbol{\delta} + \boldsymbol{a})^2 + d(\boldsymbol{z}_l, SQ_k)^2 + d(\boldsymbol{z}_p, \boldsymbol{z}_k)^2\big)^{\frac{1}{2}} - \boldsymbol{a}\Big), \\
&\leq 6 \times 8 \sum_{l \geq 1}^{\aleph} \sum_{k=0}^{l} \sum_{p=0}^{k} g\Big(\big(l^2(\boldsymbol{\delta} + \boldsymbol{a})^2 + k^2(\boldsymbol{\delta} + \boldsymbol{a})^2 + i^2(\boldsymbol{\delta} + \boldsymbol{a})^2\big)^{\frac{1}{2}} - \boldsymbol{a}\Big),
\end{aligned}$$

which guaranties that

$$\sum_{j(\neq i)=1}^{\aleph} g\big(d(B^{\boldsymbol{a}}_{\boldsymbol{z}_i}, B^{\boldsymbol{a}}_{\boldsymbol{z}_j})\big) \leq 6 \times 8 \sum_{\substack{1 \leq l \leq \aleph \\ O \leq i \leq k \leq l}} g\Big(\big[\big(l^2 + k^2 + i^2\big)^{\frac{1}{2}}(\boldsymbol{c}_r + 1) - 1\big]\boldsymbol{a}\Big).$$

For (A.0.6), using Hölder's inequality for the inner sum, and considering (A.0.5), we obtain

$$\begin{aligned}\sum_{m=1}^{\aleph}\Big(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{\alpha_j}{\boldsymbol{\delta}_{mj}^{q}}\Big)^2 &\leq \sum_{m=1}^{\aleph}\left(\left[\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big(\frac{\alpha_j}{\boldsymbol{\delta}_{mj}^{\frac{q}{2}}}\Big)^2\right]^{\frac{1}{2}}\left[\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big(\frac{1}{\boldsymbol{\delta}_{mj}^{\frac{q}{2}}}\Big)^2\right]^{\frac{1}{2}}\right)^2,\\ &\leq \sum_{m=1}^{\aleph}\left(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big(\frac{\alpha_j}{\boldsymbol{\delta}_{mj}^{q}}\Big)\ \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{1}{\boldsymbol{\delta}_{mj}^{q}}\right),\\ &\leq \frac{1}{\boldsymbol{\delta}^{q}}\sum_{l=1}^{\aleph^{\frac{1}{3}}} l^{(2-4)}\sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{\alpha_j}{\boldsymbol{\delta}_{mj}^{q}}.\end{aligned}$$

The proof ends with

$$\sum_{m=1}^{\aleph}\sum_{i\geq 1, m\neq j}^{\aleph}\frac{\alpha_j}{\boldsymbol{\delta}_{mj}^{q}} = \sum_{m=1}^{\aleph}\sum_{\substack{\aleph\geq j\geq 1\\ i\neq j}}\frac{\alpha_j}{\boldsymbol{\delta}_{mj}^{q}} = \sum_{j=1}^{\aleph}\alpha_j\sum_{\substack{m\geq 1\\ m\neq j}}\frac{1}{\boldsymbol{\delta}_{mj}^{q}}.$$

$\square$

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