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ÉTUDE DES SOLUTIONS PRESQUE PÉRIODIQUES DES ÉQUATIONS DIFFÉRENTIELLES STOCHASTIQUES

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Présentation des résultats

0.1 Préliminaires

0.1.1 Presque périodicité des fonctions déterministes

Presque périodicité au sens de Bohr L'utlisation des fonctions presque périodiques remonte à l'époque grecque antique. Ptolémée dans son livre Almagest donne une méthode (scientifique) pour expliquer le comportement de la lune, de la terre et du soleil. Cette méthode s'appelle méthode d'épicycle, elle consiste à approcher la fonction qui donne le mouvement de la planète par un polynôme trigonométrique. Cette approximation des fonctions par des polynôme trigonométrique est une propriété importante des fonctions presque périodiques définies ci-dessous.

L'apparition formelle de la théorie des fonctions presque périodiques remonte aux travaux de Harald Bohr [15] publiés entre 1923 et 1926. Le Mathématicien Danois Harald Bohr s'interessa au problème (ouvert depuis les années 1850) de position des zéros de la fonction Zeta de Riemann définie par

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s};\ s \in \mathbb{C}.$$

Il remarqua que pour s de partie réelle constante, les series $\zeta(s)$ ont un comportement régulier proche de celui des fonctions périodiques mais moins restrictif (la somme et le produit de fonctions presque périodiques est presque périodique, limite uniforme d'une suite de fonctions presque périodiques est une fonction presque périodique). Ce ne sont pas des fonctions qui sont presque des fonctions périodiques, mais ce sont des fonctions qui possèdent de nombreuses presque périodes. Plus exactement :

Définition 0.1. (Presque périodicité au sens de Bohr) Une fonction f définie et continue sur $\mathbb{R}$ est dite *presque périodique au sens de Bohr* si pour tout $\varepsilon > 0$, il existe un nombre réel $l = l(\varepsilon) > 0$ tel que tout intervalle de longueur l de $\mathbb{R}$ contient au moins un point τ, appelé *ε-presque période*, qui vérifie

$$\sup_t |f(t+\tau) - f(t)| < \varepsilon.$$

Par la suite, entre les années 1920 et les années 1930, la théorie de Bohr s'est considérablement developpée. S. Bochner [12] a étendu cette notion au cas de fonctions à valeurs dans un espace (abstrait) de Banach. Il y a aussi V.V. Stepanov, H. Weil et A. Besicovitch qui ont défini la presque périodicité pour des fonctions non continues. Il faut aussi mentionner les noms de J. Favard, J. Von Newmann et N.N. Bogolyubov qui ont beaucoup contribué au développement de cette théorie.

Presque périodicité au sens de Bochner Beaucoup de résultats sur la presque périodicité des solutions de plusieurs classes d'équations différentielles ont été obtenus juste après l'apparition de la théorie. Dès les années 20, Bohr et Neugebauer [16] ont montré, en utilisant la définition, que les solutions bornées du système d'équations

$$x^{'} = Ax + f(t) \tag{0.1}$$

sont presque périodiques si la fonction f est presque périodique et l'application linéaire A ne dépend pas de t. Dans le cas où A est presque périodique, Bochner (dans son Article de 1933) a établi l'existence de solutions presque périodiques pour l'équation (0.1) en utilisant la normalité de la solution, notion équivalente à la presque périodicité au sens de Bohr.

Définition 0.2. (Presque périodicité au sens de Bochner) Une fonction f est *presque périodique au sens de Bochner* si et seulement si elle est *normale*, c-à-d si de toute suite (s_n') de $\mathbb{R}$, on peut extraire une sous suite (s_n) telle que $f(t+s_n)$ converge uniformément par rapport à t, ce qui est équivaut à dire que l'ensemble des fonctions translatées $\{f(t+.),\ t \in \mathbb{R}\}$ est précompact pour la topologie de la convergence uniforme.

La définition de Bochner donne un moyen plus efficace pour vérifier les propriétés algébriques et topologiques des fonctions presque périodiques. Mais elle est moins efficace pour trouver des solutions presque périodiques d'équations d'évolution d'une manière générale. Pour remédier à cette contrainte, Bochner [13] a trouvé un critère remarquable pour démontrer la presque périodicité des fonctions.

Définition 0.3. (Critère de suite double de Bochner) La fonction f vérifie le *critère de suite double de Bochner* si, de deux suites quelconques (α_n') et (β_n') de $\mathbb{R}$, on peut toujours extraire deux sous-suites de mêmes indices (α_n) et (β_n) telles que pour chaque $t \in I$ les limites

$$\lim_{n\to\infty} \lim_{m\to\infty} f(\alpha_n + \beta_m + t) \text{ et } \lim_{n\to\infty} f(\alpha_n + \beta_n + t)$$

existent et soient égales.

Bochner a démontré qu'une fonction est presque périodique si et seulement si elle vérifie le critère de suite double. La propriété remarquable de ce critère est que les limites existent selon les trois modes de convergence : uniforme, ponctuelle et uniforme sur les parties compactes de $\mathbb{R}$. Donc l'avantage de ce critère est que la convergence simple implique la convergence uniforme. Ce critère a été très exploité pour trouver des solutions presque périodiques par Bochner lui même, J. Favard et C. Tudor et ses collaborateurs. La plupart de nos résultats sont obtenus à l'aide de ce critère.

Presque périodicité des fonctions à paramètre Pour étudier la presque périodicité des solutions des équations différentielles de type

$$x' = f(t, x) \tag{0.2}$$

il est indispensable d'imposer la presque périodicité de la fonction $f(t, x)$ uniformement par rapport à x dans les compacts, sinon la fonction composée $f(t, \phi(t))$ (où ϕ est une solution de l'équation (0.2)) n'est pas presque périodique en général. Par exemple, la fonction $f(t, x) = \sin(tx)$ est presque périodique pour chaque x par contre la fonction composée $f(t, \sin(t)) = \sin(t \sin(t))$ n'est pas presque périodique. Pour plus de détail sur cet exemple, voir [36, page 13].

Presque périodicité asymptotique La notion de fonctions asymptotiquement presque périodiques a été introduite pour la première fois par Fréchet [38, 37], elle concerne les fonctions continues définies sur $\mathbb{R}_+$ et qui ont un comportement presque périodique à partir d'un certain rang, plus exactement :

Définition 0.4. (Presque périodicité asymptotique au sens de Bohr) Une fonction f définie et continue sur la demi-droite $\mathbb{R}_+$ est dite *asymptotiquement presque périodique* si pour tout $\varepsilon > 0$, il existe un nombre réel $l = l(\varepsilon) > 0$ et $T = T(\varepsilon) \geq 0$, tel que tout intervalle de longueur l de $\mathbb{R}$ contient au moins un point τ, qui vérifie

$$|f(t + \tau) - f(t)| < \varepsilon \text{ pour tout } t \geq T \text{ avec } t + \tau \geq T,$$

ce qui est équivaut à dire que $f = g + h$ où f est Bohr-presque périodique et h une fonction continue nulle à l'infini.

Par la suite Precupanu [62], Ruess et Summers [65, 66] et Vesentini [75] ont généralisé cette notion aux cas de fonctions à valeurs dans des espaces abstraits.

Comme dans le cas des fonctions presque périodiques, Beaucoup de chercheurs se sont interessés au problème de solutions asymptotiquement presque périodiques des différentes classes d'équations différentielles. Je cite entre autres, Arendt, Batty et Charles [3], Gheorghiu [39], Zaidman [80], Corduneanu [22], Fink [35].

Presque périodicité des semigroupes Le lien étroit entre les équations d'évolution et la théorie des semigroupes a encouragé l'introduction de la notion de presque périodicité des semigroupes pour étudier l'existence des solutions presque périodiques. Beaucoup de travaux ont été réalisés dans ce contexte, voir [5, 6, 1, 2, 3, 67, 68, 69]. On peut définir plusieurs types de presque périodicités des semigroupes (voir par exemple [6]). Dans notre cas on s'intéressera uniquement au type suivant :

Définition 0.5. (Presque périodicité des semigroupes) Un C_0-semigroupe $(T_t)_{t\geq 0}$ sur un espace de Banach $\mathbb{E}$ est presque périodique si, pout tout $x \in \mathbb{E}$, l'application $x \mapsto T_t(x)$ de $\mathbb{R}_+$ vers $\mathbb{E}$ est presque périodique.

0.1.2 Presque périodicité des fonctions aléatoires

Plusieurs phénomènes naturels ou dus à l'homme (comme le climat, la pollution, des phénomènes neurophysiologiques, etc) mathématiquement modélisables, sont aléatoires et sont régis par des équations différentielles stochastiques.

Presque périodicité des trajectoires Le premier à avoir introduit le concept de fonctions aléatoires presque périodiques est Eugene Slutsky, il a étudié dans son papier de 1938 [71] la presque périodicité des trajectoires d'un processus stochastique stationnaire. La théorie des fonctions aléatoires presque périodiques a connu un grand développement durant les années 1980 et les années 1990. T. Kawata [51] a étudié le concept d'uniforme presque périodicité des trajectoires et R. J. Swift [72] a obtenu, en utilisant les résultats de Kawata, quelques conditions suffisantes pour qu'un processus stochastique harmonisable soit presque périodique.

D'autres types de presque périodicité des processus stochastiques ont été introduits.

Presque périodicité en loi Le cas des fonctions presque périodiques à valeurs dans l'espace Polonais $\mathcal{P}\left(\mathbb{R}^d\right)$ de toutes les mesures de probabilités sur l'espace Euclidien $\mathbb{R}^d$ a été introduit par Morozan et Tudor [60], ce qui leur a permis de définir le concept de presque périodicité en loi unidimensionnelle (APOD, abréviation de C. Tudor) d'un processus stochastique.

Dans tout ce travail, nous avons adopté à plusieurs reprises les abréviations et les notations de C. Tudor [74].

Définition 0.6. (APOD) Un processus stochastique $\{X(t)\}_{t\in\mathbb{R}}$ à valeurs dans un espace Polonais $\mathbb{E}$ est dit *presque périodique en loi unidimensionnelle (APOD = with almost periodic one-dimentional distributions)* si l'application

$$\mu_t : \begin{cases} \mathbb{R} & \to & \mathcal{P}(E) \\ t & \mapsto & \mu_t \end{cases}$$

est presque périodique, où μ_t est la loi de $\big(X(t)\big)$.

Par la suite, H. Hurd, A. Russek et D. Surgailis [48] ont introduit la notion de presque périodicité en loi finidimensionnelle (APFD) d'un processus stochastique.

Définition 0.7. (APFD) Un processus stochastique $\{X(t)\}_{t\in\mathbb{R}}$ à valeurs dans un espace Polonais $\mathbb{E}$ est dit *presque périodique en loi finidimensionnelle (APFD= with almost periodic finite-dimentional distributions)* si, pour tout $n \in \mathbb{N}$ et $t_1 < \cdots < t_n$, l'application

$$\mu_t^{t_1,\ldots,t_n} : \begin{cases} \mathbb{R} & \rightarrow & \mathcal{P}(E^n) \\ t & \mapsto & \mu_t^{t_1,\ldots,t_n} : \end{cases}$$

est presque périodique, où $\mu_t^{t_1,\ldots,t_n}$ est la loi de $\big(X(t+t_1),\ldots,X(t+t_n)\big)$.

Dans le cas où le processus $\{X(t)\}_{t\in\mathbb{R}}$ est à trajectoires continues, C. Tudor [74] a défini la presque périodicité en loi infinidimensionnelle (APD).

Définition 0.8. (APD) *La presque périodicité en loi infinidimensionnelle (APD= with almost periodic distributions)* d'un processus stochastique continu $\{X(t)\}_{t\in\mathbb{R}}$ est caractérisée par la pesque périodicité de l'application

$$\widetilde{\mu}_t : \begin{cases} \mathbb{R} & \rightarrow & \mathcal{P}(\mathcal{C}_k) \\ t & \mapsto & \widetilde{\mu}_t \end{cases}$$

où $\widetilde{\mu}_t$ est la loi de $\widetilde{X}(t) := X(t+.)$ ($\mathcal{C}_k = \mathcal{C}(\mathbb{R}, E)$, l'espace des fonctions continues muni de la topologie de la convergence uniforme sur les compacts de $\mathbb{R}$).

Plusieurs résultats de C. Tudor et ses collaborateurs ont été obtenus grâce à une métrique notée d_{BL}, définie sur l'espace des probabilités $\mathcal{P}(\mathbb{E})$, compatible avec la topologie de la convergence étroite et complète. Cette distance est définie par :

$$|f|_L = \sup\Big\{\frac{f(x)-f(y)}{d_{\mathbb{E}}(x,y)};\, x \neq y\Big\}, \quad |f|_\infty = \sup_{x\in\mathbb{E}} |f(x)|\,,$$
$$\text{et } \ |f|_{BL} = \max\{|f|_\infty, |f|_L\}.$$

pour $\mu, \nu \in \mathcal{P}(E)$,

$$d_{BL}(\mu,\nu) = \sup_{|f|_{BL}\leq 1} \left| \int_{\mathbb{E}} f d(\mu-\nu) \right|,$$

où $f \in C_b(\mathbb{E})$, ($C_b(\mathbb{E})$ est l'espace de Banach des fonctions continues bornées définies sur $\mathbb{E}$ à valeurs dans $\mathbb{R}$). La plus part de nos résultat sont obtenus grâce à cette métrique.

Dans le cas général des fonctions à valeurs dans l'espace des mesures (signées), une étude exhaustive a eté réalisée par I. Danilov [27, 28, 29]

Dans le même sens, Gladyshev [41] a introduit la presque périodicité corrélée d'un processus stochastique et Hurd [49] la presque périodicité unitaire. Dans [74], C. Tudor a fait une étude comparative entre ces différentes notions de presque périodicité.

Presque périodicité en probabilité Le concept de la presque périodicité en probabilité semble avoir été introduit pour la première fois par O. Onicescu et V. Istratescu [61].

Définition 0.9. On dit que le processus X est *Bohr-presque périodique en probabilité* (resp. *asymptotiquement Bohr-presque périodique en probabilité*) si pour tout $\varepsilon > 0$ et $\eta > 0$, il existe $l = l(\varepsilon, \eta) > 0$ (resp. il existe $l = l(\varepsilon, \eta) > 0$ et $T = T(\varepsilon, \eta)$) tels que pour tout intervalle de longueur l il existe τ pour lequel

$$\mathrm{P}\left\{d_{\mathbb{E}}(X(t+\tau), X(t)) > \eta\right\} \leq \varepsilon, \text{ pour tout } t \in \mathbb{R} \, (\text{resp. pour tout } t \geq T, t+\tau \geq T)\,.$$

Precupanu [63] a démontré que la plupart des propriétés de base des fonctions presque périodiques restent vraies dans le cas des processus à valeurs réelles presque périodiques en probabilité.

On peut aussi définir *la presque périodicité en probabilité au sens de Bochner* d'un processus stochastique X par la relative compacité de l'ensemble $\left\{\widetilde{X}(t),\, t \in I\right\}$ par rapport à la topologie de la convergence uniforme en probabilité, ce qui revient à dire que X satisfait le critère de suite double de Bochner pour la convergence en probabilité.

Plusieurs résultats sur l'étude des solutions stationnaires, périodiques ou presque périodiques, des équations différentielles stochastiques ont été établis. Dorogovtsev [31] et Morozan [58, 59] ont étudié la périodicité des solutions des équations différentielles stochastiques affines (en dimension finie et infinie), la presque périodicité à été obtenue par Morozan, Halanay et Tudor [44]. Concernant la presque périodicité en loi unidimensionnelle (APOD) des solutions des équations différentielles stochastiques affines, elle a été obtenu par Morozan et Tudor [60]. Tudor et Da Prato [25] ont prouvé sous la condition de dissipativité l'existence des solutions presque périodiques en loi infinidimensionnnelle (APD) des équations différentielles stochastiques semi-linéaires (en particulier les équations affines) sur les espace de Hilbert.

0.2 Les principaux résultats

Le présent travail est une modeste contribution à l'étude des processus stochastiques presque périodiques d'une manière générale et en particulier à l'études des solutions presque périodiques d'EDS. Nous avons aussi proposé le principe de moyennisation faible des solutions d'EDS semilinéaires à coefficients presque périodiques dans l'espace de Hilbert. Nous terminons ce travail par un petit résultat sur les semigroupe presque périodiques.

0.2.1 Étude Comparative entre les différentes notions de presque périodicité

La première partie de ce travail est consacrée à une étude comparative entre les différentes notions de presque périodicité d'un processus stochastique à trajectoires continues et à valeurs dans un espace métrisable. Nous donnerons une extension et une généralisation des résultats de Tudor [74]. Le résultat sur la comparaison entre la presque périodicité en loi (APD) et la presque périodicité en probabilité est généralisé au cas de processus continus à valeurs dans un espace complètement régulier. Un supplément sur les critères de presque périodicité des fonctions à valeurs dans un espace métrique et à valeurs dans un espace complètement régulier est donné en appendice.

Dans ce qui suit, on se donne :

- un espace topologique complètement régulier $\mathbb{E}$ (espace des états), par exemple $\mathbb{E}$ est un espace métrisable séparable,
- un intervalle de temps $I = \mathbb{R}$ ou $\mathbb{R}^+$,
- un espace de probabilité $(\Omega, \mathcal{F}, \mathrm{P})$,
- un processus stochastique continu X défini sur $\Omega \times I$ à valeurs dans $\mathbb{E}$.

On utilise les notations suivantes.

- On note $\mathcal{C}(I;\mathbb{E})$ l'espace des fonctions continues définies sur I à valeurs dans $\mathbb{E}$.
- L'espace des trajectoires du processus X est noté par $\mathcal{C}_k = \mathcal{C}_k(I;\mathbb{E})$ qui est l'espace des fonctions continues définies de I à valeurs dans $\mathbb{E}$ muni de la topologie de la convergence uniforme sur les parties compactes de I. La topologie de l'espace $\mathcal{C}_k$ est indépendante du choix de l'uniformité sur $\mathbb{E}$ voir [33, Theorem 8.2.6].
- Pour tout espace uniforme F, on note $\mathcal{C}_u(I;F)$ l'espace des fonctions continues muni de la topologie de la convergence uniforme sur I.
- Pour toute fonction $x : I \to \mathbb{E}$, et pour tout $t \in I$, $\widetilde{x}(t)$ est la fonction translatée $x(t + .) : I \to \mathbb{E}$, $s \mapsto x(t+s)$.
- Pour tout espace topologique F, on note par $\mathcal{P}(F)$ l'espace de toutes les probabilités de Radon sur F muni de la topologie de la convergence etroite, c-à-d. la topologie la moins fine pour laquelle l'application $\mu \mapsto \mu(f)$ est continue pour toute fonction continue bornée $f : F \to \mathbb{R}$.
- La loi du processus X est la loi de la variable aléatoire $X = \widetilde{X}(0)$ qui à valeurs dans $\mathcal{C}_k$.

Comparaison entre APD, APFD et APOD En 1995, C. Tudor [74, Proposition 3.8] a montré que si X est un processus dont la fonction de covariance $K(t,s) : \mathbb{R}^2 \to \mathbb{R}$ est Hölderienne, alors X est APFD si et seulement si X est APD. Nous avons établi une généralisation de ce résultat au cas d'un processus stochastique continu quelconque sous la condition

$$\forall [a,b]\subset I,\ \forall\varepsilon>0,\ \forall\eta>0,\ \exists\delta>0,\ \forall r\in I,$$
$$\mathrm{P}\Big\{\sup_{\substack{|t-s|<\delta\\ t,s\in[a,b]}} d_{\mathbb{E}}\big(X(r+t),X(r+s)\big)>\eta\Big\}<\varepsilon. \quad (0.3)$$

Théorème 0.1. *Si le processus X vérifie (0.3), les propriétés suivantes sont équivalentes :*

(a) *X est APFD.*

(b) *X est APD.*

Pour démontrer ce résultat nous avons utilisé le critère double suite de Bochner et les properiétés de la métrique d_{BL}

C. Tudor [74, Proposition 3.9] a montré que si X est un processus de Feller réel homogène alors APFD $\Leftrightarrow$ APOD. Comme conséquence du théorème 0.1, nous avons obtenu l'équivalence entre les trois types de presque périodicité en loi pour un processus de feller homogène défini sur $I=\mathbb{R}^+$.

Corollaire 0.1. *On suppose que $I=\mathbb{R}^+$. Si X est un processus de Feller homogène et qu'il satisfait (0.3). Les properiétés suivantes sont équivalentes :*

(a) *X est APOD.*

(b) *X est APFD.*

(c) *X est APD.*

Presque périodicité en probabilité et en moyenne La presque périodicité en probabilité dépend uniquement de la topologie de la convergence en probabilité.

Dans le cas où $\mathbb{E}$ est un espace vectoriel normé, $\|.\|_{\mathbb{E}}$ sa norme et $I=\mathbb{R}$, on dit que le processus X est *presque périodique en moyenne d'ordre p* si l'application $X:\mathbb{R}\to \mathrm{L}^p(\Omega;\mathbb{E})$ est presque périodique pour la norme $\|.\|_p=\big(\int\|.\|_{\mathbb{E}}^p\,dP\big)^{\frac{1}{p}}$.

En utlisant le critère double suite de Bochner et le théorème de Vitali, nous avons obtenu un lien entre ces deux types de presque périodicité.

Proposition 0.2.1. *Posons $I=\mathbb{R}$. Si X est presque périodique en moyenne d'ordre p, alors il est presque périodique en probabilité. Inversement, si X est presque périodique en probabilité et la famille $\{\|X(t)\|_{\mathbb{E}}^p,\ t\in\mathbb{R}\}$ est uniformement intégrable, alors X est presque périodique en moyenne d'ordre p.*

Presque périodicité presque sûre Dans le cadre de la première partie de notre travail, nous avons introduit trois nouveaux concepts liés à la presque périodicité des processus stochastiques.

(a) Le processus stochastique X est *presque sûrement presque périodique* s'il existe un sous ensemble mesurable $\Omega_1 \subset \Omega$ tel que $\mathrm{P}(\Omega_1) = 1$ et pour tout $\omega \in \Omega_1$, la trajectoire $t \mapsto X(t)(\omega)$ est presque périodique.

(b) Le processus stochastique X satisfait *le critère de suite double uniforme de Bochner presque sûr* si, pour deux suites (α'_n) et (β'_n) dans I, il existe un sous ensemble mesurable Ω_0 de Ω tel que $\mathrm{P}(\Omega_0) = 1$ et il existe deux sous-suites $(\alpha_n) \subset (\alpha'_n)$ et $(\beta_n) \subset (\beta'_n)$ de même indice (indépendamment de ω) tels que, pour tout $t \in I$, les les limites
$$\lim_{n\to\infty} \lim_{m\to\infty} X(t + \alpha_n + \beta_m)(\omega) \text{ and } \lim_{n\to\infty} X(t + \alpha_n + \beta_n)(\omega) \tag{0.4}$$
existent elles sont égales pour tout les $\omega \in \Omega_0$. (dans ce cas, Ω_0 depend des deux suites (α'_n) et (β'_n).)

(c) Le processus stochastique X est dit *presque équi-presque périodique au sens de Bochner* si, pour chaque $\epsilon > 0$, il existe un sous ensemble mesurable Ω_ϵ de Ω avec $\mathrm{P}(\Omega_\epsilon) \geq 1-\epsilon$ tel que la famille des trajectoires $\{X(\omega)\}_{\omega\in\Omega_\epsilon}$ est équi-presque périodique au sens de Bochner, c-à-d, la famille $\{\widetilde{X}(t)(\omega);\, t \in I\}_{\omega\in\Omega_\epsilon}$ est précompact dans $\mathcal{C}_u(I, \mathbb{E})$.

Entre Tous ces concepts, nous avons établi les liens suivants.

Proposition 0.2.2. *Nous avons* (c) $\Rightarrow$ (b) *et* (c) $\Rightarrow$ (a).

Remarque 0.1. Nous avons aussi construit des contre-exemples pour démontrer que les implications (b) $\Rightarrow$ (a), (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c) et (b) $\Rightarrow$ (c) sont fausses.

Presque périodicité en probabilité et presque périodicité presque sûre Nous allons voir que le critère de suite double uniforme de Bochner presque sûr caractérise la presque périodicité en probabilité

Théorème 0.2. *Les properiétés suivantes sont équivalentes :*

(1) *X vérifie le critère double suite uniforme de Bochner presque sûr.*

(2) *X est presque périodique en probabilité.*

Pour démonter ce résultat, nous avons utilisé le lemme technique suivant.

Lemme 0.1. *Soit* $(X_n(t))$ *une suite de processus qui converge en probabilité uniformement par rapport à $t \in I$. Alors il existe une sous-suite* $(X_{n_k}(t)) \subset (X_n(t))$ *qui converge presque surement pour chaque $t \in I$.*

Remarque 0.2. A l'aide d'un contre-exemple, nous avons montré qu'en général il n'y a aucun lien entre presque périodicité en probabilité et presque périodicité presque sûre.

Presque périodicité en loi et les autres types de presque périodicités

Théorème 0.3. *On considère les propriétés suivantes de X :*

(i) *X est APD,*

(ii) *X est APFD,*

(iii) *X est presque périodique en probabilité.*

On a (iii) $\Rightarrow$ (ii)*. si de plus X satisfait* (0.3)*, alors* (iii) $\Rightarrow$ (i)*.*

Par des contre-exemples, nous avons montré que les autres implications sont fausses, dans le cas général il n'y a aucun lien entre (iii) et (i).

Théorème 0.4. *Si X est presque équi-presque periodique, alors X est APD et presque périodique en probabilité.*

Nous avons aussi généralisé le résultat du théorème 0.3 au cas de processus à valeurs dans un espace complètement régulier, et cela en considérant l'espace complètement régulier comme limite projective d'espace métriques, ce qui nous a permis de nous ramener au cas précédent.

A la fin de cette partie, nous avons ajouté un supplément sur les notions de presque périodicité et de presque périodicité asymptotique dans le cas où l'espace est métrique et dans le cas aussi où l'espace est complètement régulier.

Commentaire En plus des résultats cités ci-dessus, nous avons éclairci la notion de presque périodicité au sens de Bohr et celle au sens de Bochner ainsi que la presque périodicité asymptotique (dans les deux sens aussi). Comme nous avons pu voir que la presque périodicité au sens de Bochner est une properiété topologique, elle ne dépend pas de l'uniformité choisie.

0.2.2 Etude des solutions presque périodiques en moyenne quadratique des EDS

Récemment, Bezandry et Diagana [8, 9, 10] affirment que certaines EDS à coefficients presque périodiques admettent des solutions vérifiant la propriété plus forte de presque périodicité en moyenne quadratique. Le but de ce second travail que nous avons réalisé dans le cadre de cette thèse est de donner quelques contre-exemples aux affirmations de [8, 9].

Contre-exemple 1 Un processus d'Ornstein-Uhlenbeck stationnaire X est nécessairement presque périodique en loi, mais il n'est pas presque périodique en moyenne quadratique car sinon, il existerait une suite (t_n) convergeant vers ∞ telle (X_{t_n}) tende vers une variable aléatoire Y qui aurait même loi que les X_t. Or la variance de Y serait la limite de $\operatorname{cov}(X_{t_n}, Y)$ lorsque $n \to \infty$, alors que $\operatorname{cov}(X_{t_n}, Y)$ est toujours nul, d'où une contradiction car Y n'est pas constante.

Contre-exemple 2 Le même raisonnement montre que le processus

$$X_t = e^{-t+\sin(t)} \int_{-\infty}^{t} e^{s-\sin(s)} \sqrt{1-\cos(s)}\, dW_s. \tag{0.5}$$

solution de l'équation

$$dX_t = (-1+\cos(t))X_t\, dt + \sqrt{1-\cos(t)}\, dW_t. \tag{0.6}$$

est périodique en loi mais n'est pas presque périodique en moyenne quadratique. Une manière de généraliser les contre-exemples précédents consiste à considérer une équation d'évolution stochastique de la forme

$$dX_t = A(t)X_t\, dt + g(t)\, dW_t \tag{0.7}$$

dans un cadre hilbertien, où $A(t)$ engendre un semi-groupe d'évolution $U(t,s)_{t\geq s}$ et g est presque périodique, et avec les hypothèses à la fois de [25, Theorem 4.3] et [9, Theorem 3.3]. Dans ce cas (0.7) n'a aucune solution presque périodique en moyenne quadratique, cependant, si la famille $(X_t)_{t\in\mathbb{R}}$ est tendue, l'unique solution d'évolution bornée dans L^2 de (0.7) est presque périodique en loi.

En conclusion, la proprieté de presque périodicité en moyenne quadratique pour des solutions d'EDS semble très forte comme propriété. Le problème de la caractérisation des EDS admettant de telles solutions reste ouvert.

0.2.3 Solution APD et moyennisation faible des EDS

Cette section est consacrée à deux résultats principaux. Le premier concerne la presque périodicité en loi infinidimensionnelle (APD) des solutions des EDS. Le second propose le principe de moyennisation dans le sens faible des solutions des EDS dans l'espace de Hilbert.

On considère l'équation différentielle semi-linéaire

$$dX_t = AX_t\, dt + F(t,X_t)\, dt + G(t,X_t)\, dW(t), t\in\mathbb{R} \tag{0.8}$$

où $A : \mathrm{Dom}(A) \subset \mathbb{H}_2 \to \mathbb{H}_2$ est un opérateur linéaire dont le domaine de définition est dense dans $\mathbb{H}_2$, $F : \mathbb{R}\times\mathbb{H}_2 \to \mathbb{H}_2$, et $G : \mathbb{R}\times\mathbb{H}_2 \to L(\mathbb{H}_1,\mathbb{H}_2)$ sont des fonctions continues.

On suppose que

(i) $A : \mathrm{Dom}(A) \to \mathbb{H}_2$ est un générateur infinitesimal d'un C_0-semigroupe $(S(t))_{t\geq 0}$ tel que il existe une constante $\delta > 0$ avec

$$\|S(t)\|_{L(\mathbb{H}_2)} \leq e^{-\delta t}, t \geq 0,$$

(ii) les applications $F : \mathbb{R}\times\mathbb{H}_2 \to \mathbb{H}_2$ et $G : \mathbb{R}\times\mathbb{H}_2 \to L(\mathbb{H}_1, \mathbb{H}_2)$ sont presque périodiques en $t \in \mathbb{R}$ uniformement par rapport à x dans des parties bornées de $\mathbb{H}_2$. De plus, il existe une constante K telle que

$$\|F(t,x)\|_{\mathbb{H}_2} + \|G(t,x)\|_{L(\mathbb{H}_1,\mathbb{H}_2)} \leq K(1 + \|x\|_{\mathbb{H}_2}),$$

(iii) les fonctions F et G sont lipschitziennes par rapport à la première variable uniformément par rapport à la deuxième variable, i.e. il existe une constante K telle que

$$\|F(t,x) - F(t,y)\|_{\mathbb{H}_2} + \|G(t,x) - G(t,y)\|_{L(\mathbb{H}_1,\mathbb{H}_2)} \leq K\|x - y\|_{\mathbb{H}_2}$$

pour tout $t \in \mathbb{R}$ et $x, y \in \mathbb{H}_2$,

(iv) $W(t)$ est un processus de Wiener à valeurs dans $\mathbb{H}_1$ dont l'opérateur de covariance Q est nucléaire et W est défini sur la base stochastique $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathrm{P})$.

Dans les deux résultats suivants, nous avons utilisé les conditions imposées par Bezandry et Diagana dans leur article [9].

Solution APD

Théorème 0.5. *On suppose que les hypothèses (i) - (iv) sont vérifiées et la constante* $\theta = \frac{K^2}{\delta^2}\Big(\frac{1}{2} + trQ\Big) < 1$. *Alors l'équation (0.8) admet une unique solution mild dans* $\mathrm{CUB}\big(\mathbb{R}, L^2(\mathrm{P}, \mathbb{H}_2)\big)$, *qui est presque périodique en loi (APD) et on peut l'écrire explicitement sous la forme suivante :*

$$X(t) = \int_{-\infty}^{t} S(t-s)F\big(s, X(s)\big)ds + \int_{-\infty}^{t} S(t-s)G\big((s), X(s)\big)dW(s)$$

pour tout $t \in \mathbb{R}$.

Pour démontrer ce résultat nous avons utilisé les mêmes techniques que celles de Bezandry et Diagana [9]. Pour l'unicité nous avons utilisé le théorème du point fixe ; la presque périodicité en loi, nous l'avons obtenue en exploitant la fameuse inégalité de Gronwall.

Moyennisation faible

Théorème 0.6. *On suppose que les hypothèses du théorème 0.5 sont satisfaites. Pour tout* $\varepsilon \in]0,1[$ *fixé, on note* X^ε *la solution mild de l'équation*

$$dX^\varepsilon(t) = AX^\varepsilon(t)\,dt + F(\frac{t}{\varepsilon}, X^\varepsilon(t))\,dt + G(\frac{t}{\varepsilon}, X^\varepsilon(t))\,dW(t), t \in \mathbb{R}. \tag{0.9}$$

alors $X^\varepsilon \to X^0$ *en loi quand* $\varepsilon \to 0$ *dans l'espace* $\mathcal{C}_k(\mathbb{R}, \mathbb{H}_2)$ *(muni de la topologie de la convergence uniforme sur des parties compactes de* $\mathbb{R}$*), où* $(X^0(t))$ *est la solution mild de l'équation*

$$dX^0(t) = AX^0(t)\,dt + F_0(X^0(t))\,dt + G_0(X^0(t))\,dW(t) \tag{0.10}$$

et qui est stationnaire.

0.2.4 Presque périodicité des semigroupes de transition

Dans cette dernière partie, on considère un espace de banach $\mathbb{E}$, $\mathfrak{B}(\mathbb{E})$ la σ-algèbre de toutes les parties boreliennes de $\mathbb{E}$ et $\mathcal{M}(\mathbb{E})$ l'espace linéaire de toutes les mesures boreliennes (signées). On considère aussi l'espace des fonctions continues bornées $C_b(\mathbb{E})$, définies sur $\mathbb{E}$ à valeurs dans $\mathbb{R}$, muni de la topologie de la convergence uniforme.

Définition 0.10. La function

$$K : \begin{cases} \mathbb{R}^+ \times \mathbb{E} \times \mathfrak{B}(\mathbb{E}) & \to \quad \mathbb{R} \\ (t, x, D) & \mapsto \quad K(t, x, D) \end{cases}$$

est dite *fonction markovienne de transition* si, pout tout $t, s \geq 0$, pour tout $D \in \mathfrak{B}(\mathbb{E})$ et pour tout $x \in \mathbb{E}$,

1. $K(t, x, .)$ est une mesure positive sur $\mathfrak{B}(\mathbb{E})$.
2. $K(t, ., D)$ est une fonction mesurable.
3. $K(0, x, D) = 1$ si $x \in D$ et $K(0, x, D) = 0$ si $x \in \mathbb{E} \setminus D$.
4. Il existe une constante $M > 0$ telle que $K(t, x, \mathbb{E}) \leq M$.
5. K vérifie l'équation de Chapman-Kolmogorov, c.à.d. $K(t + s, x, D)$ $= \int_{\mathbb{E}} K(t, y, D)K(s, x, dy)$, pout tout $t, s \geq 0$, pour tout $D \in \mathfrak{B}(\mathbb{E})$ et pour tout $x, y \in \mathbb{E}$.

A une fonction markovienne de transition K, on peut associer deux semigroupes d'opérateurs (appelés semigroupes de transition) :

(a)

$$S_t : \begin{cases} \mathcal{M}(\mathbb{E}) & \to \quad \mathcal{M}(\mathbb{E}) \\ \mu & \mapsto \quad S_t\mu \end{cases}$$

défini par

$$[S_t\mu](D) = \int_{\mathbb{E}} K(t, x, D)\mu(dx), \ \ \mu \in \mathcal{M}(\mathbb{E}) \text{ et } D \in \mathfrak{B}(\mathbb{E}),$$

(b)

$$T_t : \begin{cases} MB(\mathbb{E}) & \to \quad MB(\mathbb{E}) \\ f & \mapsto \quad T_t f \end{cases}$$

défini par

$$[T_t f](x) = \int_{\mathbb{E}} f(y)K(t, x, dy), \ \ f \in BM(\mathbb{E})$$

Où $BM(\mathbb{E})$ est l'espace de Banach des fonctions mesurables bornées définies sur $\mathbb{E}$ à valeurs dans $\mathbb{R}$.

En 1978, Harm Bart and Seymour Goldberg [6] ont caractérisé les C_0-semigroupes presque périodiques.

Théorème 0.7. *[6, Theorem 9] un C_0-semigroupe $(T_t)_{t\geq 0}$ presque périodique si et seulement si les trois conditions suivantes sont vérifiées*

1. *$(T_t)_{t\geq 0}$ est uniformement borné,*
2. *$\sigma(A) \subset i\mathbb{R}$,*
3. *l'ensemble des vecteurs propres de A engendrent un sous-espace dense dans $\mathbb{E}$.*

A l'aide de ce résultat et en supposant que $\mathbb{E}$ est un espace métrique compact, nous avons pu démontrer l'équivalence entre la presque périodicité des deux semigroupes ci-dessus définis.

Theorem 0.2.3. *Le C_0-semigroupe $(T_t)_{t\geq 0}$ est presque périodique si et seulement si $(S_t)_{t\geq 0}$ est presque périodique.*

Commentaire La thèse est rédigée de manière à ce que les quatre chapitres puissent être lus de manière indépendante. L'essentiel de la thèse reprend les résultats de nos articles [7, 57, 50], qui ont été concervés dans leur langue d'origine.

Chapitre 1

Bochner-almost periodicity for stochastic processes

1.1 Introduction

Since the creation of the theory of almost periodicity by Harald Bohr in the 1920s, many concepts of almost periodicity have proved useful, in particular those of Stepanov, Besicovitch, and Weyl. In the context of stochastic processes, each of these notions gives rise to several possible definitions, such as : almost periodicity in distribution, in probability, in quadratic mean, almost sure etc (not to mention e.g. processes with almost periodic covariance functions and many others, see [74]). In this paper we study the relationships between some of these definitions for Bochner-almost periodicity. Bochner-almost periodicity coincides with Bohr-almost periodicity when the time interval I is the whole line $\mathbb{R}$ [13], but it only amounts to asymptotic Bohr-almost periodicity when $I = [0, +\infty)$. It has the advantage over Bohr's definition that it does not depend on the uniform structure of the state space but only on it's topology (see Remark 1.1 in this paper).

Our work completes that of C. Tudor [74] who compared different types of Bohr (or Bochner) almost periodicity for continuous processes in the case when $I = \mathbb{R}$.

As an application of our study, it is possible to compare previous results proving almost periodicity in the Bohr sense of the solution of some stochastic differential equations. For example, in the works of C. Tudor and his collaborators (see in particular [60, 73, 4, 25]), almost periodicity in distribution of the solutions is proved. On the other hand, in a series of papers (in particular [8, 9, 10]) Bezandry and Diagana prove almost periodicity in quadratic mean, which amounts to almost periodicity in probability plus uniform integrability of the square of the norms. We show that Bochner-almost periodicty in distribution and Bochner-almost periodicty in probability are not equivalent actually neither notion of almost periodicity implies the other. However, under a tightness hypothesis, Bochner-almost periodicity in probability implies Bochner-almost periodicity in distribution.

The paper is organized as follows : The next section is devoted to the case when the state space is metrizable. An extension of the comparison between almost periodicity in distribution and in probability in the completely regular case is given in Section 1.3 for possible applications in nonmetrizable locally convex topological vector spaces. This extension is done by considering completely regular spaces as inverse limits of metrizable spaces and using the results of section 1.2. Some complements on criteria of almost periodicity and asymptotic almost periodicity on metric spaces and in completely regular spaces are given in the Appendix (Section 1.4).

Notations and terminology In all what follows, we are given

- a separable completely regular topological space $\mathbb{E}$ (the state space), e.g. $\mathbb{E}$ is a separable metrizable space,
- an interval I, which is $\mathbb{R}$ or $\mathbb{R}^+$ (the time interval),
- a probability space $(\Omega, \mathcal{F}, \mathrm{P})$,
- a continuous stochastic process X on $\mathbb{E}$ defined on $\Omega \times I$.

We use the following notations :

- We denote by $\mathcal{C}(I;\mathbb{E})$ the space of continuous functions from I to $\mathbb{E}$.
- The space of trajectories of the process X is the space of continuous functions from I to $\mathbb{E}$, which we endow with the topology of uniform convergence on compact subsets of I and denote by $\mathcal{C}_k = \mathcal{C}_k(I;\mathbb{E})$. The definition of $\mathcal{C}_k$ seems to rely on the choice of a uniformity on $\mathbb{E}$ (e.g. of a metric in the metrizable case), but actually the topology of $\mathcal{C}_k$ is independent of this choice, see [33, Theorem 8.2.6].
- For any uniform space F, we denote $\mathcal{C}_u(I;F)$ the space of continuous functions from I to F endowed with the topology of uniform convergence on I.
- For any function $x : I \to \mathbb{E}$, and for every $t \in I$, $\widetilde{x}(t)$ is the *translation mapping* $x(t+.) : I \to \mathbb{E}$, $s \mapsto x(t+s)$.
- For any topological space F, we denote by $\mathcal{P}(\mathcal{C}_k)$ the space of Radon probabilities on F endowed with the *topology of narrow convergence*, which is the coarsest topology such that the mappings $\mu \mapsto \mu(f)$ are continuous for all bounded continuous functions $f : F \to \mathbb{R}$.
- We denote by $\mathrm{law}(U)$ the law of a random variable U. The law of the process X is the law of the $\mathcal{C}_k$-valued random variable $X = \widetilde{X}(0)$.

As we concentrate mainly on the study of Bochner-almost periodicity for stochastic processes, *by default, in this paper, "almost periodic" means "Bochner-almost periodic"*.

1.2 Metrizable case

In this section, $\mathbb{E}$ is a Polish space i.e. $\mathbb{E}$ is separable and its topology can be defined by a distance $d_{\mathbb{E}}$ such that $(\mathbb{E}, d_{\mathbb{E}})$ is complete.

Before we give the different definitions of almost periodicity for stochastic processes, let us recall the corresponding definitions for deterministic functions.

1.2.1 Almost periodicity in the deterministic case

Let $d_{\mathbb{E}}$ be a distance on $\mathbb{E}$ which is compatible with the topology of $\mathbb{E}$, and let $x : I \to (\mathbb{E}; d_{\mathbb{E}})$ be a continuous function. We say that x is *Bochner-almost periodic* if the set $\{\widetilde{x}(t),\, t \in I\} = \{x(t + .),\, t \in I\}$ is totally bounded in the space $\mathcal{C}_u(I, \mathbb{E})$.

We say that x is *Bohr-almost periodic* (resp. *asymptotically Bohr-almost periodic*) if for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ (resp. there exists $l(\varepsilon) > 0$ and $T = T(\varepsilon) \geq 0$) such that any interval of length $l(\varepsilon)$ contains at least an ϵ*-almost period*, that is, a number τ for which

$$d_{\mathbb{E}}\left(x(t+\tau), x(t)\right) \leq \varepsilon, \text{ for all}\, t \in I = \mathbb{R}\, (\text{resp. for all } \ t \geq T\, \text{such that } t + \tau \geq T\,)\,.$$

In the first case (Bohr-almost periodicity), we say that the set of ϵ-almost periods of x is *relatively dense* in I. If $\mathbb{E}$ is a vector space, a function $x : \mathbb{R} \to \mathbb{E}$ is asymptotically Bohr-almost periodic iff it has the form $x = y + z$, where y is Bohr-almost periodic and z is a continuous function with $\lim_{\pm\infty} z = 0$, see e.g. [65].

The following statements are equivalent (see [13, 25] and Corollary 1.4.3) :

1. x is Bochner-almost periodic.
2. x satisfies *Bochner's double sequence criterion*, that is, for every pair of sequences (α'_n) and (β'_n) in I, there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ respectively with same indexes such that, for every $t \in I$, the limits
$$\lim_{n\to\infty} \lim_{m\to\infty} x(t + \alpha_n + \beta_m) \text{ and } \lim_{n\to\infty} x(t + \alpha_n + \beta_n), \tag{1.1}$$
exist and are equal.
3. x is almost periodic in Bohr's sense if $I = \mathbb{R}$, or asymptotically almost periodic in Bohr's sense if $I = \mathbb{R}^+$.

Remarque 1.1. (i) Bochner's double sequence criterion shows that Bochner-almost periodicity is topological, i.e. it doesn't depend on the choice of $d_{\mathbb{E}}$, provided that $d_{\mathbb{E}}$ is compatible with the topology of $\mathbb{E}$.

(ii) A striking property of Bochner's double sequence criterion is that the limits in (1.1) exist in any of the three modes of convergences : pointwise, uniform on compact intervals and uniform on I (with respect to $d_{\mathbb{E}}$). This criterion has thus the avantage that it allows to establish uniform convergence by checking pointwise convergence.

(iii) Asymptotic Bohr-almost periodicity is strictly weaker than Bohr-almost periodicity, as shows the following example provided by Ruess and Summers [65] :

$$x(t) = \frac{1}{1+t}\cos[\ln(1+t)],\ t \in \mathbb{R}^+.$$

1.2.2 Almost periodicity and asymptotic almost periodicity in distribution

Following Tudor's [74] terminology, we say that X is *Bochner-almost periodic in distribution*, APD for short, (resp. *Bohr-almost periodic in distribution*, resp. *asymptotically Bohr-almost periodic in distribution*) if the mapping $t \mapsto \text{law}\left(\widetilde{X}(t)\right)$, $I \to \mathcal{P}(\mathcal{C}_k)$ is Bochner-almost periodic (resp. Bohr-almost periodic in distribution, resp. asymptotically Bohr-almost periodic in distribution). Other definitions of almost periodicity in distribution are possible, see [4, 60, 74] : for example we say that X has *almost periodic finite dimensional distributions* (*APFD* for short) if, for all finite sequences $(t_1, \dots, t_n)$ in I, the mapping $t \mapsto \text{law}\,(X(t_1+t), \dots, X(t_n+t))$ is almost periodic. Likewise, if the mapping $t \mapsto \text{law}\,(X(t))$ is almost periodic, we say that X has *almost periodic one-dimensional distributions* (*APOD* for short).

Bochner-almost periodicity in distribution as defined here is proved for some solutions of stochastic differential equations in [60, 73, 4, 25].

Remarque 1.2. If X is almost periodic in distribution, the family $(\widetilde{X}(r))_{r\in I}$ is tight in $\mathcal{C}_k$. By [11, Theorem 7.3] or [77, Theorem 4], this implies that, for each interval $[a,b] \subset I$, for every $\varepsilon > 0$ and for every $\eta > 0$, there exists $\delta > 0$ such that

$$\mathrm{P}\{\sup_{\substack{|t-s|<\delta \\ t,s\in[a,b]}} d_{\mathbb{E}}\big(\widetilde{X}(r)(t), \widetilde{X}(r)(s)\big) > \eta\} < \varepsilon$$

for all $r \in I$. Equivalently :

$$\forall [a,b] \subset I,\ \forall \varepsilon > 0,\ \forall \eta > 0,\ \exists \delta > 0,\ \forall r \in I,$$
$$\mathrm{P}\Big\{\sup_{\substack{|t-s|<\delta \\ t,s\in[a,b]}} d_{\mathbb{E}}\big(X(r+t), X(r+s)\big) > \eta\Big\} < \varepsilon. \quad (1.2)$$

Theorem 1.2.1. *If X satisfies (1.2), the following properties are equivalent :*

(a) *X is APFD.*

(b) *X is APD.*

Before we give the proof of Theorem 1.2.1 we recall the following definition : For $f \in C_b(\mathbb{E})$, (where $C_b(\mathbb{E})$ is the Banach space of continuous and bounded functions defined on $\mathbb{E}$),

$$|f|_L = \sup\Big\{\frac{f(x)-f(y)}{d_{\mathbb{E}}(x,y)};\, x \neq y\Big\}, \quad |f|_\infty = \sup_{x\in\mathbb{E}} |f(x)|\,,$$
$$\text{and } \ |f|_{BL} = \max\{|f|_\infty\,, |f|_L\}.$$

For $\mu, \nu \in \mathcal{P}(E)$, we define

$$d_{BL}(\mu,\nu) = \sup_{|f|_{BL}\le 1} \left| \int_{\mathbb{E}} f d(\mu-\nu)\right|.$$

The metric d_{BL} on $\mathcal{P}(E)$ is complete and compatible with the topology of narrow convergence.

Proof of Theorem 1.2.1 Clearly (b)$\Rightarrow$(a). Assume that X is APFD. Let (α'_n) and (β'_n) be two sequences in I and for $t_1, t_2, \dots, t_k, t \in I$ define (using notations of [74])

$$\mu_t^{t_1,\dots,t_k} := \operatorname{law}\left(X(t_1+t),\dots,X(t_k+t)\right).$$

By a diagonal procedure we can find subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that, for every $k \geq 1$, for all $q_1, q_2, \dots, q_k \in \mathbb{Q} \bigcap I$ (where $\mathbb{Q}$ is the set of rational numbers), and for every $t \in I$, the limits

$$\lim_n \mu_{t+\alpha_n+\beta_n}^{q_1,\dots,q_k} \text{ and } \lim_n \lim_m \mu_{t+\alpha_n+\beta_m}^{q_1,\dots,q_k}$$

exist and are equal. We have, for all $t_1, t_2, \dots, t_k, t \in I$, and for all $q_1, q_2, \dots, q_k \in \mathbb{Q} \bigcap I$,

$$\begin{aligned} d_{BL}\Big(\mu_{t+\alpha_n+\beta_n}^{q_1,\dots,q_k}, \mu_{t+\alpha_n+\beta_n}^{t_1,\dots,t_k}\Big) &= \sup_{\|f\|_{BL}\le 1} \int_{\mathbb{E}^k} f d\big(\mu_{t+\alpha_n+\beta_n}^{q_1,\dots,q_k} - \mu_{t+\alpha_n+\beta_n}^{t_1,\dots,t_k}\big) \\ &\le \int_\Omega d_{\mathbb{E}^k}\Big[\big(X(q_1+t+\alpha_n+\beta_n),\dots,X(q_k+t+\alpha_n+\beta_n)\big), \\ &\qquad \big(X(t_1+t+\alpha_n+\beta_n),\dots,X(t_k+t+\alpha_n+\beta_n)\big)\Big]\, d\mathrm{P}\,. \end{aligned}$$

Thus, by (1.2), if $(q_1,\dots,q_k) \to (t_1,\dots,t_k)$, then

$$d_{BL}\Big(\mu_{t+\alpha_n+\beta_n}^{q_1,\dots,q_k}, \mu_{t+\alpha_n+\beta_n}^{t_1,\dots,t_k}\Big) \to 0$$

uniformly with respect to $t \in I$ and $n \geq 0$. By a classical result on inversion of limits, we deduce that, for all $k \geq 1$ and $t_1, \dots, t_k, t \in I$, the limits

$$\lim_n \mu_{t+\alpha_n+\beta_n}^{t_1,\dots,t_k} \text{ and } \lim_n \lim_m \mu_{t+\alpha_n+\beta_m}^{t_1,\dots,t_k}$$

exist and are equal. Therefore, to show that the limits

$$\lim_n \operatorname{law}\Big(\widetilde{X}(t+\alpha_n+\beta_n)\Big) \text{ and } \lim_n \lim_m \operatorname{law}\Big(\widetilde{X}(t+\alpha_n+\beta_m)\Big)$$

exist and are equal, it is enough to prove that $(\widetilde{X}(t))_{t\in I}$ is tight in $\mathcal{C}_k$. Since X is APFD, the family $(X(t))_{t\in I} = (\widetilde{X}(t)(0))_{t\in I}$ is tight, thus by (1.2) and [77, 11] we conclude that $(\widetilde{X}(t))_{t\in I}$ is tight in $\mathcal{C}_k$.

$\square$

Corollary 1.2.2. *Assume that $I = \mathbb{R}^+$. If X is an homogeneous Feller process and satisfies (1.2), the following properties of X are equivalent :*

(a) *X is APOD.*

(b) *X is APFD.*

(c) *X is APD.*

Proof The equivalence (b)⇔(c) follows from Theorem 1.2.1. Clearly (b)⇒(a). There remains to show that (a)⇒(b).

Using again Tudor's notations [74], let us set

$$\mu_t := \text{law}\,(X(t)) \ \text{ and } \ \mu_t^{t_1,\dots,t_k} := \text{law}\,((X(t_1+t),\dots,X(t_k+t)))\,.$$

Suppose that X is APOD, then for every sequence $(\alpha'_n) \subset \mathbb{R}^+$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that for each function $g \in C_b(\mathbb{E})$ ($C_b(\mathbb{E})$ is the Banach space of continuous and bounded functions, $g : \mathbb{E} \to \mathbb{R}$) the limit

$$\lim_n \mu_{t+\alpha_n}(g)$$

exists uniformly with respect to $t \in \mathbb{R}^+$. To show that the limit

$$\lim_n \mu_{t+\alpha_n}^{t_1,\dots,t_k}$$

exists uniformly with respect to $t \in \mathbb{R}^+$, it is enough to show it for $k = 2$ and $t_1 < t_2$. Let $f \in C_b(\mathbb{E}\times\mathbb{E})$, we have

$$\mu_{t+\alpha_n}^{t_1,t_2}(f) = \int_{\mathbb{E}} \mu_0(dx_0) \int_{\mathbb{E}} p_{t_1+t+\alpha_n}(x_0, dx_1) \int_{\mathbb{E}} p_{t_2-t_1}(x_1, dx_2) f(x_1, x_2),$$

where $p_t(x,.)$ is the transition function for a time-homogeneous Markov process X. Since X is a Feller process, the function

$$x_1 \mapsto g(x_1) := \int_{\mathbb{E}} p_{t_2-t_1}(x_1, dx_2) f(x_1, x_2)$$

is in $C_b(\mathbb{E})$. Therefore

$$\mu_{t+\alpha_n}^{t_1,t_2}(f) = \int_{\mathbb{E}} \mu_0(dx_0) \int_{\mathbb{E}} p_{t_1+t+\alpha_n}(x_0, dx_1) g(x_1) = \mu_{t+t_1+\alpha_n}(g)$$

converges uniformly with respect to $t \in \mathbb{R}^+$, hence X is APFD.

□

1.2.3 Almost periodicity and asymptotic almost periodicity in probability or in p-mean

We say that X is *Bochner-almost periodic in probability* if $\left\{\widetilde{X}(t),\, t \in I\right\}$ is totally bounded with respect to the topology of uniform convergence in probability, which amounts to say that X satisfies Bochner's double sequence criterion for the convergence in probability. Recall from Remark 1.1(ii) that this property depends only on the topology of convergence in probability, not on any metric which is compatible with this topology, thus it is independent of the metric $d_{\mathbb{E}}$ on $\mathbb{E}$.

We say that X is *Bohr-almost periodic (resp. asymptotically Bohr-almost periodic) in probability* if for any $\varepsilon > 0$ and $\eta > 0$, there exists $l = l(\varepsilon, \eta) > 0$ (resp. there exists $l = l(\varepsilon, \eta) > 0$ and $T = T(\varepsilon, \eta)$) such that any interval of length l contains at least a number τ for which

$$\mathrm{P}\left\{d_{\mathbb{E}}(X(t+\tau), X(t)) > \eta\right\} \leq \varepsilon, \text{ for all } t \in \mathbb{R}\, (\text{resp. for all } t \geq T, t+\tau \geq T)\,.$$

Precupanu [61, 63] showed that most of the basic properties of Bohr-almost periodic functions can be formulated accordingly for $\mathbb{R}$-valued almost periodic in probability processes, particularly, the usual approximation theorem with random trigonometric polynomials is valid for stochastic processes.

Now, let us consider the case when $\mathbb{E}$ is a normed space and $I = \mathbb{R}$. We denote by $\|.\|_{\mathbb{E}}$ the norm of $\mathbb{E}$. Let $p \geq 1$. We say that X is *almost periodic in p-mean* if the mapping $X : \mathbb{R} \to \mathrm{L}^p(\Omega; \mathbb{E})$ is almost periodic with respect to the metric induced by $\|\cdot\|_p = \left(\int \|.\|_{\mathbb{E}}^p \, dP\right)^{\frac{1}{p}}$.

Proposition 1.2.3. *Let $I = \mathbb{R}$. If X is almost periodic in p-mean, then it is almost periodic in probability. Conversely, if X is almost periodic in probability and the familly $\{\|X(t)\|_{\mathbb{E}}^p,\ t \in \mathbb{R}\}$ is uniformly integrable, then X is p-mean almost periodic.*

Proof This is a direct consequence of Bochner's double sequence criterion and Vitali Theorem. □

Almost periodicity in quadratic mean (using Bohr's definition, but with $I = \mathbb{R}$) is proved for some solutions of stochastic differential equations in [8, 9, 10].

1.2.4 Almost sure almost periodicity properties

We can give at least three different definitions :

(a) The stochastic process X is *almost surely almost periodic* if there exists a measurable subset $\Omega_1 \subset \Omega$ such that $\mathrm{P}(\Omega_1) = 1$ and for every $\omega \in \Omega_1$, the trajectory $t \mapsto X(t)(\omega)$ is almost periodic.

(b) The stochastic process X satisfies *Bochner's almost sure uniform double sequence criterion* if, for every pair of sequences (α'_n) and (β'_n) in I, there exists a measurable subset Ω_0 of Ω such that $\mathrm{P}(\Omega_0) = 1$ and there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ respectively, with same indexes (independent of ω) such that, for every $t \in I$, the limits

$$\lim_{n\to\infty} \lim_{m\to\infty} X(t+\alpha_n+\beta_m)(\omega) \text{ and } \lim_{n\to\infty} X(t+\alpha_n+\beta_n)(\omega) \tag{1.3}$$

exist and are equal for all $\omega \in \Omega_0$. (In this case, Ω_0 depends on the pair of sequences (α'_n) and (β'_n).)

(c) We say that X is *almost equi-almost periodic in Bochner's sense* if, for each $\epsilon > 0$, there exists a measurable subset Ω_ϵ of Ω with $\mathrm{P}(\Omega_\epsilon) \geq 1-\epsilon$ such that the family of trajectories $\{X(\omega)\}_{\omega\in\Omega_\epsilon}$ is equi-almost periodic in Bochner's sense, that is, the family of translation mappings $\{\widetilde{X}(t)(\omega);\, t \in I\}_{\omega\in\Omega_\epsilon}$ is totally bounded on $\mathcal{C}_u(I, \mathbb{E})$.

Remarque 1.3. A stronger notion of almost equi-almost periodicity can be defined as follows :

(d) We say that X is *almost surely equi-almost periodic in Bochner's sense* if there exists a measurable subset $\Omega_0 \subset \Omega$ such that $\mathrm{P}(\Omega_0) = 1$ and the family of trajectories $\{X(\omega)\}_{\omega\in\Omega_0}$ is equi-almost periodic in Bochner's sense.

However this notion seems too strong for stochastic processes : For example, let U be an integer-valued random variable with $P(U = n) > 0$ for each integer n, and let $X(t) = U$ for every $t \in I$. Then X is almost surely equibounded, and X is also almost periodic in distribution (it is even stationary) and in probability, and it has almost periodic (constant) trajectories. But X is not almost surely equi-almost periodic, since the familly of its trajectories is not almost surely equibounded.

Proposition 1.2.4. *We have* (c) $\Rightarrow$ (b) *and* (c) $\Rightarrow$ (a).

Remarque 1.4. The implications (b) $\Rightarrow$ (a) and (a) $\Rightarrow$ (b) are false, this will be proved by Counterexamples 1.1 and 1.2. Consequently, the implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c) are false.

Proof of Proposition 1.2.4 Assume (c). Let (α'_n) and (β'_n) be two sequences in I. For each integer $k \geq 0$, we can apply Bochner's uniform double sequence criterion to the family $\{\widetilde{X}(t)(\omega);\, t \in I\}_{\omega\in\Omega_{2^{-k}}}$ (see Theorem 1.4.2), that is, we can find subsequences $(\alpha^k_n) \subset (\alpha'_n)$ and $(\beta^k_n) \subset (\beta'_n)$ respectively, with same indexes (independent of ω) such that, for every $t \in I$, the limits

$$\lim_{n\to\infty} \lim_{m\to\infty} X(t+\alpha^k_n+\beta^k_m)(\omega) \text{ and } \lim_{n\to\infty} X(t+\alpha^k_n+\beta^k_n)(\omega)$$

exist and are equal for all $\omega \in \Omega_{2^{-k}}$. By induction, we can take $(\alpha^{k+1}_n) \subset (\alpha^k_n)$ and $(\beta^{k+1}_n) \subset (\beta^k_n)$. Set $\alpha_n = \alpha^n_n$ and $\beta_n = \beta^n_n$ and $\Omega_0 = \liminf \Omega_{2^{-k}}$. Then $\mathrm{P}(\Omega_0) = 1$ and the limits (1.3) exist and are equal for all $\omega \in \Omega_0$. Thus (b) holds true. Furthermore, Ω_0 does not depend on (α'_n) and (β'_n). Thus, for every $\omega \in \Omega_0$, $X(\omega)$ is almost periodic, which proves (a). □

1.2.5 Comparison between almost periodicities

Almost periodicity in probability vs almost sure properties

Lemma 1.2.5. *Let $(X_n(t))$ be a sequence of processes which converges in probability uniformly with respect to $t \in I$. There exists a subsequence $(X_{n_k}(t)) \subset (X_n(t))$ which converges almost surely for each $t \in I$.*

Proof Assume that $(X_n(t))$ converges in probability to a process $X(t)$ uniformly with respect to $t \in I$, let us denote $Y_n(t) = d(X_n(t), X(t))$. Then for each $\varepsilon > 0$ and $\eta > 0$, there exists $N = N(\varepsilon, \eta)$ such that for all $t \in I$ and $n \geq N$, we have

$$\mathrm{P}(Y_n(t) > \varepsilon) < \eta.$$

Let (η_k) be a real positive sequence such that

$$\sum_{k \geq n} \eta_k \text{ converges to zero as } n \to \infty.$$

To each $\varepsilon > 0$ and $\eta_k > 0$, there corresponds $N_k = N(\varepsilon, \eta_k)$ such that, for all $t \in I$, $\mathrm{P}(Y_{N_k}(t) > \varepsilon) < \eta_k$. Let us show that $(Y_{N_k}(t))$ converges almost surely for every $t \in I$. We have

$$\mathrm{P}(\sup_{k \geq n} Y_{N_k}(t) > \varepsilon) = \mathrm{P}(\bigcup_{k \geq n} \{Y_{N_k}(t) > \varepsilon\}) \leq \sum_{k \geq n} \eta_k$$

and the right hand term converges to zero when $n \to \infty$. Thus $(Y_{N_k}(t))$ converges almost surely to zero for every $t \in I$. Consequently $(X_{N_k}(t))$ converges almost surely to $X(t)$ for every $t \in I$. □

Theorem 1.2.6. *The following properties of X are equivalent :*

(1) *X satisfies Bochner's almost sure uniform double sequence criterion.*

(2) *X is almost periodic in probability.*

Proof Obviously (1)$\Rightarrow$ (2).

Assume that X is almost periodic in probability. Let (α'_n) and (β'_n) be two sequences in I. There exist subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that for each $t \in I$, the limits in probability

$$\mathrm{P} - \lim_n X(\alpha_n + \beta_n + t) \text{ and } \mathrm{P} - \lim_n \lim_m X(\alpha_n + \beta_m + t)$$

exist and are equal. By Remark 1.1(ii) these limits exist uniformly with respect to $t \in I$. The previous Lemma implies that there exist further subsequences (still denoted (α_n) and (β_n)) with same indexes such that, for each $t \in I$, the limits

$$\lim_n X(\alpha_n + \beta_n + t) \text{ and } \lim_n \lim_m X(\alpha_n + \beta_m + t)$$

exist and are equal almost surely. $\square$

We now give counterexamples which show that X can be almost periodic in probability without any almost periodic trajectory, or almost surely almost periodic without being almost periodic in probability. By Theorem 1.2.6, these counterexamples show that the implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c) in Proposition 1.2.4 are false. Consequently, the converse implications to (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) in Proposition 1.2.4 are also false.

Counterexample 1.1. (almost periodicity in probability without almost periodic trajectories) Let (δ_n) be the sequence of random variables on the probability space $([0,1], dt)$ (dt is the Lebesgue measure) defined by $\delta_n = 1_{[\frac{n}{2^k}, \frac{n+1}{2^k}]}$ for $2^{k-1} \le n < 2^k$, $k \in \mathbb{N}$. For each $n \ge 1$, let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(t) = \begin{cases} nt & \text{if } \ t \in [0, \frac{1}{2}] \\ n(1-t) & \text{if } \ t \in [\frac{1}{2}, 1] \end{cases}$$

and let $(X_t)_{t \ge 0}$ be defined by

$$X_t(\omega) = \begin{cases} f_{[t]}(t - [t]) & \text{if } \ \delta_{[t]}(\omega) = 1 \\ 0 & \text{if } \ \delta_{[t]}(\omega) = 0 \end{cases}$$

where $[t]$ is the integer part of t. The trajectories of X are continuous on $\mathbb{R}^+$, but almost none is uniformly continuous, nor bounded, since $\mathrm{P}(\limsup \delta_n = 1) = 1$. We have, for every $\varepsilon > 0$,

$$\lim_{t \to \infty} \mathrm{P}(X_t > \varepsilon) \le \lim_{t \to \infty} \mathrm{P}(\delta_{[t]} = 1) = 0$$

which means that X is asymptotically Bohr-almost periodic in probability, thus it is Bochner almost periodic in probability.

A variant of this example, with uniformly bounded trajectories, is obtained by replacing X by $\min(X, 1)$.

Note that, for all integers $n, p \ge 0$ and for every $\eta > 0$ and every $\delta > 0$, we have

$$\begin{aligned} \mathrm{P}\Big\{ \sup_{\substack{|t-s|<\delta \\ t,s \in [n, n+p]}} d_{\mathbb{E}}\big(X(t), X(s)\big) > \eta \Big\} &\le \mathrm{P}\Big\{ \exists k \in \{n, \dots, n+p\};\ \delta_k = 1 \Big\} \\ &\le 2^{-n+1} \to 0 \text{ when } n \to \infty. \end{aligned}$$

As the trajectories of X are n-Lipschitz on $[0, n]$, this implies that X satisfies Condition (1.2). We shall see later that X is almost periodic in distribution, thus Counterexample 1.1 also provides an example of an almost periodic in distribution process without any almost periodic trajectory.

Counterexample 1.2. (almost sure almost periodicity does not imply almost periodicity in probability) Let X be defined by $X(t)(\omega) = e^{i\frac{t}{\omega}}$, for every $\omega \in]0,1]$ and $t \in I$. The process X is almost surely almost periodic, but it is not almost periodic in probability (thus it is not almost equi-almost periodic). Indeed, since $|X| \leq 1$ then to prove that X is not almost periodic in probability, il suffices to show that X is not almost periodic in quadratic mean. Let τ be an ε-almost period i.e. $E|X_{t+\tau} - X_t|^2 \leq \varepsilon$, for all $t \in I$. Let $A(\tau) := E|X_{t+\tau} - X_t|^2$,

$$\begin{aligned}
A(\tau) &= \int_0^1 |e^{i\frac{t+\tau}{\omega}} - e^{i\frac{t}{\omega}}|^2 d\omega = \int_0^1 |e^{i\frac{\tau}{\omega}} - 1|^2 d\omega = 4\int_0^1 |\frac{e^{i\frac{\tau}{2\omega}} - e^{-i\frac{\tau}{2\omega}}}{2i}|^2 d\omega \\
&= 2\int_0^1 \sin^2\left(\frac{\tau}{w}\right) d\omega = 2|\tau| \int_{|\tau|}^{\infty} \frac{1-\cos(2u)}{u^2}\, du \\
&= 2|\tau| \sum_{k=0}^{\infty} \int_{|\tau|+k\pi}^{|\tau|+(k+1)\pi} \frac{1-\cos(2u)}{u^2}\, du \geqslant 2|\tau| \int_{|\tau+\pi|}^{\infty} \frac{1}{u^2} du = 2\frac{|\tau|}{|\tau|+\pi}.
\end{aligned}$$

Thus if $|\tau| \geqslant \dfrac{\varepsilon\pi}{2-\varepsilon}$, we get $A(\tau) > \varepsilon$, which shows that the set of ε-almost periods is not relatively dense in I, thus X is not almost periodic in quadratic mean.

Almost periodicity in distribution vs other almost periodicity properties

Remarque 1.5. In all preceding definitions of almost periodicity, it is only in the definition of almost periodicity in distribution that the choice of a topology on the space $\mathcal{C}(I, \mathbb{E})$ of trajectories plays a role.

Theorem 1.2.7. (Almost periodicity in distribution vs almost periodicity in probability) *Consider the following properties of X :*

(a) *X is APD.*

(b) *X is APFD.*

(c) *X is almost periodic in probability.*

We have (c) $\Rightarrow$ (b). *If furthermore X satisfies* (1.2), *then* (c) $\Rightarrow$ (a).

Remarque 1.6. The implications (c) $\Rightarrow$ (a) and (a) $\Rightarrow$ (c) are false, see Counterexamples 1.3 and 1.4.

Proof of Theorem 1.2.7 Assume (c) and let (α'_n) and (β'_n) be two sequences in I. There exist subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indexes such that for each $t, s \in I$, the limits in probability

$$\mathrm{P} - \lim_n X(\alpha_n + \beta_n + t + s) \text{ and } \mathrm{P} - \lim_n \lim_m X(\alpha_n + \beta_m + t + s)$$

exist and are equal. Hence, for every $(t_1, t_2, \ldots, t_k) \in I^k$, and every $s \in I$

$$\mathrm{P} - \lim_n \big(X(\alpha_n + \beta_n + t_1 + s), X(\alpha_n + \beta_n + t_2 + s), \ldots, X(\alpha_n + \beta_n + t_k + s)\big)$$

and

$$\mathrm{P}-\lim_n \lim_m \big(X(\alpha_n+\beta_m+t_1+s), X(\alpha_n+\beta_m+t_2+s), \ldots, X(\alpha_n+\beta_m+t_k+s)\big)$$

exist and are equal, thus

$$\lim_n \mathrm{law}\,(X(\alpha_n+\beta_n+t_1+s), \ldots, X(\alpha_n+\beta_n+t_k+s))$$

and

$$\lim_n \lim_m \mathrm{law}\,((X(\alpha_n+\beta_m+t_1+s), \ldots, X(\alpha_n+\beta_m+t_k+s))$$

exist and are equal, that is, X is APFD, and by Theorem 1.2.1 we deduce (a).

□

Counterexample 1.3. (almost periodicity in distribution does not imply almost periodicity in probability) Let X be a strictly stationary (thus APD) continuous process such that $(X_n)_{n\in\mathbb{Z}}$ is independent and X_0 is nonconstant.

Assume that X is almost periodic in probability. Then we can extract from (X_n) a subsequence (X_{n_k}) which converges in probability to some random variable U. By stationarity, we have $\mathrm{law}\,(U) = \mathrm{law}\,(X_0)$. Extracting if necessary a further subsequence, we can assume that (X_{n_k}) converges a.e. to U. By Kolmogorov's zero-one law, U is constant a.e., a contradiction. Thus X is not almost periodic in probability.

To construct such a process X, we can use results of [30]. Let σ be an atomless probability measure on $[0, \pi[$, and let $\psi(u) = \inf\{x;\, \sigma([0, x]) \geq u\}$. Let γ be the symmetric measure on $[-\pi, \pi]$ defined by

$$\gamma(A) = \frac{1}{2}\left(\sigma(A \cap [0, \pi]) + \sigma(-A \cap [0, \pi])\right)$$

for every Borel subset of $[-\pi, \pi]$. Let $(B_u)_{0\leq u\leq 1}$ be a standard Brownian motion on the complex plane, and define a complex Gaussian centered process $(\xi_t)_{t\in\mathbb{R}}$ by

$$\xi_t = \int_0^1 e^{it\psi(u)}\, dB_u.$$

By the construction in [30], we have $\xi_t = B_1 \circ T^t$, where (T^t) is a stationary flow on the probability space on which B is defined, thus ξ is strictly stationary. Furthermore, by [30, Proposition 3.1], almost all trajectories of ξ are C^∞. Let X be the real part of ξ. The process X is Gaussian stationary continuous and centered, and, by [30, Proposition 2.2], we have

$$\mathrm{E}(X_n X_m) = \int_{-\pi}^{\pi} e^{it(n-m)}\, d\gamma(t)$$

for all $n, m \in \mathbb{Z}$. In particular, if σ is the normalized Lebesgue measure, $(X_n)_{n\in\mathbb{Z}}$ is independent.

Counterexample 1.4. (almost periodicity in probability does not imply almost periodicity in distribution) This example is constructed in the spirit of Counterexample 1.1, but the process constructed in Counterexample 1.1 is almost periodic in probability and satisfies Condition (1.2), thus by Theorem 1.2.7 it is almost periodic in distribution. The process constructed here is almost periodic in probability but does not satisfy Condition (1.2).

Let $\Omega = [0,1[$ endowed with the Lebesgue measure. For every integer $n \geq 0$ and every $\omega \in [0,1[$, let $\omega'_n = \min(\omega + 1/n, 1)$. Define a function f_n on $[0,1[\times[0,1[$ by

$$f_n(\omega,t) = \begin{cases} 0 & \text{if} \quad t \leq \omega \text{ or } t \geq \omega'_n \\ 2\frac{t-\omega}{\omega'_n-\omega} & \text{if} \quad \omega \leq t \leq \frac{\omega+\omega'_n}{2} \\ 2\frac{\omega'_n-t}{\omega'_n-\omega} & \text{if} \quad \frac{\omega+\omega'_n}{2} \leq t \leq \omega'_n. \end{cases}$$

For $t \geq 0$ and $\omega \in [0,1[$, set

$$X_t(\omega) = f_{[t]}(\omega, t - [t]),$$

where $[t]$ denotes the integer part of t. Then X has continuous trajectories, and we have, for every $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathrm{P}(X_t > \varepsilon) \leq \lim_{t\to\infty} \frac{1}{[t]} = 0$$

which means that X is asymptotically Bohr-almost periodic in probability thus it is Bochner almost periodic in probability. On the other hand, for each $n \geq 0$ and for each $\omega \in [0,1[$, $f_n(\omega,.)$ takes the value 1 at the point $\tau_n(\omega) = (\omega + \omega'_n)/2$ and the value 0 outside the interval $[\omega, \omega'_n]$. Thus, for each integer $n \geq 0$ and for each $\omega \in \Omega$, there exist $t = n + \tau_n(\omega) \in [n, n+1[$ such that $X_t(\omega) = 1$ and $s \in [n, n+1[$ such that $X_s(\omega) = 0$ and $|t - s| \leq 1/n$. For every $\delta > 0$, and for $n \geq 1/\delta$, we thus have

$$\mathrm{P}\Big\{ \sup_{\substack{|t-s|<\delta \\ t,s\in[n,n+1]}} d_{\mathbb{E}}\big(X(t), X(s)\big) > 1/2 \Big\} = 1,$$

which contradicts Condition (1.2). Thus X is not almost periodic in distribution.

Theorem 1.2.8. *If X is almost equi-almost periodic, then X is almost periodic in probability and in distribution.*

Proof By Proposition 1.2.4 and Theorem 1.2.6, X satisfies Bochner's almost sure uniform double sequence criterion which is equivalent to almost periodicity in probability.

Now, by Theorem 1.2.7, to prove that X is APD, we only need to show that X satisfies (1.2). Since X is almost equi-almost periodic, almost all its trajectories are uniformly continuous, thus we can find for each $\omega \in \Omega$ and for each $\eta > 0$, a number $\delta(\omega) > 0$, such that, for all $t, s \in I$,

$$\sup_{|t-s|<\delta(\omega)} d_{\mathbb{E}}\big(X(t)(\omega), X(s)(\omega)\big) < \eta. \tag{1.4}$$

Furthermore, by restricting to the rational numbers the variables t and s in (1.4), we can assume that the mapping $\omega \mapsto \delta(\omega)$ is measurable. For each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\mathrm{P}\{\delta > \delta_\varepsilon\} > 1 - \varepsilon. \tag{1.5}$$

Indeed, suppose that (1.5) is false, then there exists $\varepsilon_0 > 0$ such that for every $\delta^* > 0$, we have $\mathrm{P}\{\delta \leq \delta^*\} > \varepsilon_0$. In particular, for each $m \in \mathbb{N}$ $\mathrm{P}\{\delta \leq \frac{1}{m}\} > \varepsilon_0$. Let $A_m = \{\delta \leq \frac{1}{m}\}$, we have

$$\mathrm{P}\{\delta = 0\} = \inf_m \mathrm{P}\{\delta \leq \frac{1}{m}\} > \varepsilon_0 > 0,$$

which contradicts $\delta(\omega) > 0$ for every $\omega \in \Omega$. Thus, from (1.4) and (1.5), we deduce that there exists δ_ε such that

$$\mathrm{P}\{\sup_{|t-s|<\delta_\varepsilon} d_{\mathbb{E}}\big(X(t), X(s)\big) > \eta\} \leq \varepsilon.$$

Therefore, for every interval $[a,b] \subset I$ and every $r \in I$, we have

$$\mathrm{P}\{\sup_{\substack{|t-s|<\delta_\varepsilon \\ t,s\in[a,b]}} d_{\mathbb{E}}\big(\widetilde{X}(r)(t), \widetilde{X}(r)(s)\big) > \eta\} \leq \varepsilon$$

i.e. X satisfies (1.2). □

1.3 Extension to completely regular state spaces

Topological preliminaries We now assume that $\mathbb{E}$ is a separable completely regular space. Recall that a Hausdorff topological space is completely regular iff it is uniformizable, that is, iff its topology can be defined by a family of semi-metrics. Such a family defines a uniformity. The notions of uniform continuity, Cauchy sequences, Cauchy nets and totally bounded sets are defined relatively to a given uniformity. For more details see e.g. [33].

Let $\mathfrak{U}$ be a uniformity (a set of entourages) on $\mathbb{E}$ which is compatible with the topology of $\mathbb{E}$, and let $\mathfrak{D}$ be an upwards filtering set of semimetrics on $\mathbb{E}$ which induces this uniformity. For every $d \in \mathfrak{D}$, we denote by $\breve{\mathbb{E}}_d$ the quotient metric space $(\mathbb{E}/d, \breve{d})$, and by $\breve{\mathcal{C}}_k^d$ the space $\mathcal{C}_k(I; \breve{\mathbb{E}}_d)$. We denote by p_d the projection mapping :

$$p_d : \begin{cases} \mathbb{E} & \to \quad \breve{\mathbb{E}}_d \\ x & \mapsto \quad \breve{x}_d \end{cases}$$

which extends to $\mathcal{C}_k$:

$$p_d : \begin{cases} \mathcal{C}_k & \to \quad \breve{\mathcal{C}}_k^d \\ x & \mapsto \quad p_d \circ x =: \breve{x}_d(.). \end{cases}$$

These mappings are continuous and the uniform space $\mathbb{E}$ (resp. $\mathcal{C}_k$) is the limit of the inverse system $\left\{\breve{\mathbb{E}}_d\right\}_{d\in\mathfrak{D}}$ (resp. $\left\{\breve{\mathcal{C}}_k^d\right\}_{d\in\mathfrak{D}}$).

Almost periodicity for deterministic functions A function x in $\mathcal{C}(I;\mathbb{E})$ is said to be *Bochner-almost periodic*, if the set $\{x(t+.),\, t\in I\}$ of its translation mappings is totally bounded on $\mathcal{C}_u(I;\mathbb{E})$.

A function x in $\mathcal{C}(\mathbb{R};\mathbb{E})$ is said to be *Bohr-almost periodic* if for each entourage V in $\mathfrak{U}$, there exists a real number $l = l(V) > 0$ such that every interval $[a, a+l]$ contains at least one point τ satisfying

$$(x(t+\tau), x(t)) \in V, \text{ for all}\, t \in \mathbb{R}.$$

We say that a function $x \in \mathcal{C}(\mathbb{R}^+;\mathbb{E})$ is *asymptotically Bohr-almost periodic* if, for each entourage $V \in \mathfrak{U}$, there exist a real number $l = l(V) > 0$ and $T = T(V) > 0$ with the property that any interval $[a, a+l] \subset \mathbb{R}^+$ contains a number τ such that

$$(x(t+\tau), x(t)) \in V, \text{ for all}\, t \geq T \,\text{such that}\, t+\tau \geq T.$$

The relationships between these notions of almost periodicity are explained in Lemma 1.4.4 in the Appendix.

Almost periodicity in distribution vs in probability for stochastic processes Let $\mathcal{P}(\mathbb{E})$ (resp. $\mathcal{P}(\mathcal{C}_k)$) be the space of Radon probabilities on $\mathbb{E}$ (resp. $\mathcal{C}_k$) endowed with the initial topology of narrow convergence, that is the coarsest topology such that the mappings $\mu \to \mu(f)$ are continuous for all measurable bounded function $f : \mathbb{E} \to \mathbb{R}$ (resp. $f : \mathcal{C}_k \to \mathbb{R}$). For more details on this topology, see [70].

For each projection mapping p_d, we can define a mapping from $\mathcal{P}(\mathcal{C}_k)$ to $\mathcal{P}\left(\breve{\mathcal{C}}_k^d\right)$ (or from $\mathcal{P}(\mathbb{E})$ to $\mathcal{P}\left(\breve{\mathbb{E}}_d\right)$) by

$$\mu \to (p_d)_\sharp(\mu) = \breve{\mu}^d$$

where $(p_d)_\sharp(\mu)$ denotes the image of μ by p_d that is, $\breve{\mu}^d(f) = \mu(f \circ p_d)$ for every measurable bounded function $f : \breve{\mathcal{C}}_k^d \to \mathbb{R}$.

We say that the process X is *almost periodic in distribution* (resp. *asymptotically almost periodic in distribution*) in Bohr's or Bochner sense if the mapping

$$\begin{cases} I & \to & \mathcal{P}(\mathcal{C}_k) \\ t & \mapsto & \text{law}\left(\widetilde{X}(t)\right) \end{cases}$$

is almost periodic (resp. asymptotically almost periodic) in Bohr's or Bochner sense in $\mathcal{P}(\mathcal{C}_k)$.

We say that X is *Bochner-almost periodic in probability* if the set $\left\{\widetilde{X}(t),\ t \in I\right\}$ of translation mappings is totally bounded in probabilily.

We say that X is *Bohr-almost periodic (resp. asymptotically Bohr-almost periodic) in probability* if for any entourage V and $\eta > 0$, there exists $l = l(V, \eta) > 0$ (resp. there exist

$l = l(V, \eta) > 0$ and $T = T(V, \eta))$ such that any interval of length l contains at least a number τ for which

$$P\{(X(t+\tau), X(t)) \notin V\} \leq \eta, \text{ for all } t \in \mathbb{R} \,(\text{resp. for all } t \geq T, t+\tau \geq T).$$

In order to establish a result similar to Theorem 1.2.7, we need to characterize total boundedness in inverse limits of uniform spaces.

Lemma 1.3.1. *Let $\{\mathbb{F}_\rho\}_{\rho \in \Sigma}$ be an inverse system of uniform spaces where Σ is directed set, and let*

$$\mathbb{F} = \varprojlim \{\mathbb{F}_\rho\}$$

the projective limit satisfying $\mathbb{F}_\rho = p_\rho(\mathbb{F})$, for every projection mapping p_ρ from $\mathbb{F}$ to $\mathbb{F}_\rho$. Then, $\mathbb{F}$ is totally bounded if and only if for each $\rho \in \Sigma$, $\mathbb{F}_\rho$ is totally bounded.

Proof It is straightforward that, if $\mathbb{F}$ is totally bounded, $\mathbb{F}_\rho$ is also totally bounded for each $\rho \in \Sigma$.

Conversely, assume that for each $\rho \in \Sigma$, $\mathbb{F}_\rho$ is totally bounded. Let $\widehat{\mathbb{F}_\rho}$ be the completion of $\mathbb{F}_\rho$, then $\widehat{\mathbb{F}_\rho}$ is a compact space and $\left\{\widehat{\mathbb{F}_\rho}, \rho \in \Sigma\right\}$ is an inverse system (see [70, second proof of Prokhorov theorem, page 78]). Hence the projective limit

$$\widehat{\mathbb{F}} := \varprojlim \widehat{\mathbb{F}_\rho}$$

is a compact space (see [17, page I.64, Proposition 8]). On the other hand, using [33, Proposition 2.5.6. page 100] one can see that $\overline{\mathbb{F}_\rho}, \rho \in \Sigma$, is an inverse system satisfying

$$\varprojlim \overline{\mathbb{F}_\rho} = \overline{\mathbb{F}},$$

where $\overline{\mathbb{F}_\rho}$ is the closure of $\mathbb{F}_\rho$ in $\widehat{\mathbb{F}_\rho}$ and $\overline{\mathbb{F}}$ is the closer of $\mathbb{F}$ in $\widehat{\mathbb{F}}$, which yields $\overline{\mathbb{F}} = \widehat{\mathbb{F}}$, thus $\mathbb{F}$ is totally bounded. $\square$

Now we can state and prove a version of Theorem 1.2.7 in the case of a process with values in a completely regular space $\mathbb{E}$.

Theorem 1.3.2. *Consider the following properties of X :*

(a) *X is almost periodic in distribution .*

(b) *X is almost periodic in probability.*

(c) *for each interval $[a, b] \subset I$, for every $\varepsilon > 0$ and for every $\eta > 0$, there exists $\delta > 0$ such that for every $d \in \mathfrak{D}$*

$$\mathrm{P}\{\sup_{\substack{|t-s|<\delta \\ t,s \in [a,b]}} d\big(X(r+t), X(r+s)\big) > \eta\} < \varepsilon$$

for all $r \in I$.

Then $\big((\mathrm{c})$ *and* $(\mathrm{b})\big) \Rightarrow (\mathrm{a})$.

Proof Assume that X is almost periodic in probability and satisfies Property (c). Then for each $d \in \mathfrak{D}$ the process $X^d = p_d \circ X$ satisfies (1.2) (see Section 1.2) and by Lemma 1.4.4 X^d is almost periodic in probability in $\breve{\mathbb{E}}_d$ for each $d \in \mathfrak{D}$, hence X^d is almost periodic in probability in the completion $\widehat{\mathbb{E}_d}$ (of $\breve{\mathbb{E}}_d$) for each $d \in \mathfrak{D}$. In view of Theorem 1.2.7 we get that X^d is almost periodic in distribution in $\widehat{\mathbb{E}_d}$, i.e. the set $\left\{\widetilde{\mu}^d_{t+.};\, t \in I\right\}$ of translation mappings is totally bounded on the space of functions on I with values in the space $\mathcal{P}\left(\breve{\mathcal{C}}^d_k\right)$. By Prokhorov Theorem on inverse systems of Radon measures (see [70, Theorem 21, page 74]), it is easy to see that the set $\left\{\widetilde{\mu}^d_{t+.};\, t \in I\right\}$ is an exact inverse system and that its projective limit is the set $\{\widetilde{\mu}_{t+.};\, t \in I\}$ of translation mappings. Now, applying Lemma 1.3.1 to this set, we obtain that it is totally bounded, that is, X is almost periodic in distribution. □

1.4 Appendix : Almost periodicity and asymptotic almost periodicity in completely regular spaces

First we consider the case when $\mathbb{E}$ is a metric space.

Bochner's criteria of almost periodicity for functions with values in a metric space Let $(\mathbb{E}, d)$ be a metric space (not necessarily complete). A subset $\mathcal{A} \subset I$ is said to be *relatively dense* if there exists a real number $l > 0$ such that for each number $a \in I$, $[a, a+l] \cap \mathcal{A} \neq \emptyset$. Any such number l is called an *inclusion length* of $\mathcal{A}$.

A subset $\mathcal{H}$ of $\mathcal{C}(I;\mathbb{E})$ is said to be *equi-almost periodic* (respectively *equi-asymptotically almost periodic*) *in Bohr's sense* if, for every $\varepsilon > 0$, there exists a relatively dense set $\mathcal{A} = \mathcal{A}(\mathcal{H}, \varepsilon)$ in I (resp. there exist a relatively dense set $\mathcal{A} = \mathcal{A}(\mathcal{H}, \varepsilon)$ in I and a number $r \geq 0$) such that, for every $\tau \in \mathcal{A}$ and $h \in \mathcal{H}$,

$$d(h(t+\tau), h(t)) < \varepsilon \text{ for all } t \in \mathbb{R} \text{ (resp. for all } t \geq r, t+\tau \geq r).$$

A subset $\mathcal{H}$ of $\mathcal{C}(I;\mathbb{E})$ is said to be *equinormal* if, from every sequence (α'_n) in I, one may extract a subsequence (α_n) such that the sequence $(h(t+\alpha_n))$ converges uniformly with respect to $t \in I$ and $h \in \mathcal{H}$.

The following results are adaptations of those obtained by Ruess and Summers [66] in the case of locally convex spaces. We only consider the case $I = \mathbb{R}^+$, the almost periodic case follows exactly in the same way with obvious minor changes.

Lemma 1.4.1. *Let* $\mathcal{H}$ *be a subset of* $\mathcal{C}(\mathbb{R}^+;\mathbb{E})$. *Suppose that* $\mathcal{H}$ *is a totally bounded in* $\mathcal{C}_k(\mathbb{R}^+;\mathbb{E})$ *and equi-asymptotically almost periodic. Then,* $\mathcal{H}$ *is uniformly equicontinuous on* $\mathbb{R}^+$ *and* $\mathcal{H}(\mathbb{R}^+)$ *is totally bounded in* $\mathbb{E}$.

Proof Let $\varepsilon > 0$, since $\mathcal{H}$ is equi-asymptotically almost periodic in Bohr's sense, there exist a number $r = r(\varepsilon) > 0$ and a relatively dense set $\mathcal{A} = \mathcal{A}(\mathcal{H}, \varepsilon)$ in $[r, +\infty[$ such that, for each $\tau \in \mathcal{A}$ and for every $h \in \mathcal{H}$,

$$d(h(t+\tau), h(t)) < \varepsilon \text{ for all } t \geq r \text{ such that } t + \tau \geq r.$$

Let $N = \max(r, l)$, where l is an inclusion length of $\mathcal{A}$. Choose $\tau_k \in [kN, (k+1)N]$, $k = 1, 2, \ldots$. Let us denote by $\mathcal{H}/_{[0,5N]}$ the restriction of all functions of $\mathcal{H}$ to the interval $[0, 5N]$. Then $\mathcal{H}/_{[0,5N]}$ is totally bounded in $\mathcal{C}([0, 5N]; \mathbb{E})$. Hence, by Ascoli Theorem, we get that $\mathcal{H}/_{[0,5N]}$ is uniformly equicontinuous. Let $\delta \in [0, \frac{N}{2}]$ such that $d(h(t_1), h(t_2)) < \frac{\varepsilon}{3}$ whenever $h \in \mathcal{H}$ and $t_1, t_2 \in [0, 5N]$ with $|t_1 - t_2| < \delta$. Now, suppose that $t_1, t_2 > 4N$, with $|t_1 - t_2| < \delta$. Taking $k \in \mathbb{N}$ such that $t_1, t_2 \in [kN, (k+2)N]$, let $s_i = t_i - \tau_{k-2}$, $i = 1, 2$. Since $s_1, s_2 \in [N, 4N]$ and $|s_1 - s_2| < \delta$, we have

$$\begin{aligned} d(h(t_1), h(t_2)) < d(h(s_1 + \tau_{k-2}), h(s_1)) + d(h(s_1), h(s_2)) \\ + d(h(s_2), h(s_2 + \tau_{k-2})) < \varepsilon \end{aligned}$$

for any h in $\mathcal{H}$, which implies that $\mathcal{H}$ is indeed uniformly equicontinuous on $\mathbb{R}^+$. To verify that $\mathcal{H}(\mathbb{R}^+)$ is totally bounded in $\mathbb{E}$, we start again from the equicontinuity of $\mathcal{H}$ to obtain a finite cover $\{T_i\}_{i=1}^n$ of $[0, 3N]$ and $t_i \in T_i$, $i = 1, \ldots, n$ such that for every $h \in \mathcal{H}$

$$d(h(t), h(t_i)) \leq \frac{\varepsilon}{2} \text{ whenever } t \in T_i,\ i = 1, \ldots, n.$$

If $t > 3N$, let us choose $k \in \mathbb{N}$ so that $t \in [kN, (k+1)N]$. Setting $s = t - \tau_{k-2}$, we then have that $s \in [N, 3N]$ whence $s \in T_i$ for some $i \in 1, \ldots, n$. Therefore, given any $h \in \mathcal{H}$, we get

$$d(h(t), h(t_i)) < d(h(s + \tau_{k-2}), h(s)) + d(h(s), h(t_i)) < \varepsilon. \tag{1.6}$$

Since for each $i = 1, \ldots, n$ the set $\mathcal{H}(t_i)$ is totally bounded, we deduce in view of inequality (1.6) that $\mathcal{H}(\mathbb{R}^+)$ is totally bounded.

$\square$

Theorem 1.4.2. *Let $\mathcal{H}$ be a subset of $\mathcal{C}(\mathbb{R}^+; \mathbb{E})$. The following conditions are equivalent :*

1. *The set $\widetilde{\mathcal{H}}^+ = \left\{\widetilde{h}(t),\ h \in \mathcal{H},\ t \in \mathbb{R}^+\right\}$ is a totally bounded subset of $\mathcal{C}_u(\mathbb{R}^+; \mathbb{E})$ i.e. $\mathcal{H}$ is Bochner-equi-almost periodic.*
2. *(a) $\mathcal{H}$ is a totally bounded subset in $\mathcal{C}_k(\mathbb{R}^+; \mathbb{E})$, and*

   (b) $\mathcal{H}$ satisfies Bochner's uniform double sequence criterion.
3. *(a) $\mathcal{H}$ is a totally bounded subset in $\mathcal{C}_k(\mathbb{R}^+; \mathbb{E})$, and*

   (b) $\mathcal{H}$ is equinormal.
4. *(a) $\mathcal{H}$ is a totally bounded subset in $\mathcal{C}_k(\mathbb{R}^+; \mathbb{E})$, and*

   (b) $\mathcal{H}$ is equi-asymptotically almost periodic in Bohr's sense.

Proof From [13], we have the equivalence between 2 and 3. From [79, Theorem 5, and Theorem 6 page 56] and Lemma 1.4.1, we deduce the equivalence between 3 and 4.

There remains to show that 4 is equivalent to 1.

Assume 4, then in view of Lemma 1.4.1 the subset $\mathcal{H}(\mathbb{R}^+)$ is totally bounded in $\mathbb{E}$, thus $\widetilde{\mathcal{H}}^+(t)$ is totally bounded in $\mathbb{E}$ for each $t \in \mathbb{R}^+$. We have also in view of Lemma 1.4.1 that $\widetilde{\mathcal{H}}^+$ is uniformly equicontinuous. Thus by Ascoli Theorem [18, Theorem 2, page X.17], it is totally bounded in $\mathcal{C}_u(\mathbb{R}^+;\mathbb{E})$.

Conversely, it is obvious that if $\widetilde{\mathcal{H}}^+$ is a totally bounded subset of $\mathcal{C}_u(\mathbb{R}^+;\mathbb{E})$ then $\mathcal{H}$ is a totally bounded subset of $\mathcal{C}_k(\mathbb{R}^+;\mathbb{E})$. There remains to show that $\mathcal{H}$ is equi-asymptotically almost periodic in Bohr's sense. For every $\varepsilon > 0$, we can construct a finite cover $\{T_i\}_{i=1}^n$ of $\mathbb{R}^+$ and $t_i \in T_i,\ i = 1, \ldots, n$ such that, for every $\omega \in \mathbb{R}^+$, we have

$$d(h(\omega + t), h(\omega + t_i)) \le \varepsilon, \text{ whenever } t \in T_i,\ i = 1, \ldots, n.$$

Let $r = l > \max\{t_1, t_2, \ldots, t_n\}$, and let us set

$$\mathcal{A} = \left[\bigcup_1^n (T_i - t_i)\right] \bigcap [r, +\infty[.$$

Let us check that $\mathcal{A} \cap [t, t + l] \neq \emptyset$ for all $t \ge r$, i.e. that $\mathcal{A}$ is a relatively dense set in $[r, +\infty[$. Let $t \ge r$, since $\{T_i\}_{i=1}^n$ is a cover of $\mathbb{R}^+$ and $t + l \in \mathbb{R}^+$, there exists $i \in 1, \ldots, n$ such that $t + l \in T_i$. Observe that $t \le t + l - t_i \le t + l$ and then $t + l - t_i \in \mathcal{A} \cap [t, t + l]$. Now, for a given $t \ge r$ and $\tau \in \mathcal{A}$, let us choose $i \in \{1, \ldots, n\}$ and $s \in T_i$ such that $\tau = s - t_i$. Since $t - t_i \ge 0$ we have, for all $t \in [r, +\infty[$ and for all $h \in \mathcal{H}$

$$d(h(t + \tau), h(t)) = d(h(t - t_i + s), h((t - t_i) + t_i)) < \varepsilon.$$

Hence $\mathcal{H}$ is equi-asymptotically almost periodic in Bohr's sense.

□

Corollary 1.4.3. *Let $x \in \mathcal{C}(\mathbb{R}^+;\mathbb{E})$. The following two statements are equivalent :*

1. *The set $\widetilde{\mathcal{H}}^+(x) = \{\widetilde{x}(t),\ t \in \mathbb{R}^+\}$ is totally bounded in $\mathcal{C}_u(\mathbb{R}^+;\mathbb{E})$.*
2. *x is asymptotically almost periodic in Bohr's sense.*

Almost periodicity and asymptotic almost periodicity in completely regular spaces Let $\mathbb{E}$ be a separable completely regular space, let $\mathfrak{U}$ be a uniformity on $\mathbb{E}$ which is compatible with the topology of $\mathbb{E}$, and $\mathfrak{D}$ a famili of semimetrics on $\mathbb{E}$ which induce the uniformity. We denote by $\mathbb{E}_d$ the semimetric space $(\mathbb{E}; d)$.

Let $\mathbb{E}$ be a separable completely regular space. Let $\mathfrak{U}$ be a uniformity on $\mathbb{E}$ and let $\mathfrak{D}$ be a famili of semimetrics on $\mathbb{E}$ which induces the uniformity $\mathfrak{U}$. For each $d \in \mathfrak{D}$, we denote by $\mathbb{E}_d$ the semimetric space $(\mathbb{E}; d)$.

Lemma 1.4.4. *Let x in $\mathcal{C}(I;\mathbb{E})$. For each $d \in \mathfrak{D}$, let $x_d = \mathrm{i}_d \circ x$, where $\mathrm{i}_d : \mathbb{E} \to \mathbb{E}_d$ is the canonical injection. Then,*

1. *x is Bohr-almost periodic (resp. Bohr-asymptotically almost periodic) if and only for each semimetric d in $\mathfrak{D}$, the function x_d is Bohr-almost periodic (resp. Bohr-asymptotically almost periodic).*
2. *x is Bochner-almost periodic if and only for each semimetric d in $\mathfrak{D}$, the function x_d is Bochner-almost periodic.*

Proof

1. Assume that x is Bohr-almost periodic. Let d be a semimetric in $\mathfrak{D}$ and let $\varepsilon > 0$. The set $V = \{(x,y) \in \mathbb{E} \times \mathbb{E};\ d(x,y) < \varepsilon\}$ is in $\mathfrak{U}$. Since x is almost periodic, there exists a real number $l = l(V) > 0$ such that every interval $[a, a+l]$ contains at least one point τ satisfying

$$(x(t+\tau), x(t)) \in V, \text{ for every } t \in I$$

hence

$$d(x(t+\tau), x(t)) < \varepsilon, \text{ for every } t \in I$$

which means that x_d is almost periodic.

Conversely, assume that for each d in $\mathfrak{D}$, the function f_d is almost periodic i.e. for each $\varepsilon > 0$ there exists a real number $l = l(\varepsilon, d) > 0$ such that every interval $[a, a+l]$ contains at least one point τ such that

$$d(x(t+\tau), x(t)) < \varepsilon, \text{ for every } t \in I.$$

Let V in $\mathfrak{U}$, then there exists a semimetric d in $\mathfrak{D}$ such that

$$\{(x,y); d(x,y) < \varepsilon\} \subset V.$$

In particular, we have

$$(x(t+\tau), x(t)) \in V, \text{ for every } t \in I,$$

and thus x is almost periodic.

2. In view of Lemma 1.3.1, the equivalence we wish to prove is a direct consequence of the fact that the set $\{\widetilde{x}(t),\ t \in I\}$ of translation mappings associated to x is totally bounded in the space $\mathcal{C}_u(I;\mathbb{E})$ if and only if, for each $d \in \mathfrak{D}$, the set $\{x_d(t+.),\ t \in I\}$ of translation mappings associated to x_d is totally bounded in the space $\mathcal{C}_u(I;\mathbb{E}_d)$.

$\square$

Remarque 1.7. By passing to the quotient spaces $\breve{\mathbb{E}}_d$, the previous Lemma reduces the study of almost periodicity and asymptotic almost periodicty in uniform spaces to that of almost periodicity and asymptotic almost periodicty in metric spaces. Thanks to Lemma 1.4.4, we can extend to uniform spaces the equivalence between Bohr's almost periodicity (resp. asymptotic almost periodicity) and Bochner's almost periodicity. This result is well known in the case when $I = \mathbb{R}$, see [40]. In the case when $I = \mathbb{R}^+$ and $\mathbb{E}$ is a locally convex space, it is also well-known, see [66].

Theorem 1.4.5. *Let $\mathbb{E}$ be a uniform space (nonnecessarily complete) and $f : I \to \mathbb{E}$ a continuous function. Then*

1. *If $I = \mathbb{R}$, the following statements are equivalent :*
   (a) *x is Bohr-almost periodic.*
   (b) *x is Bochner-almost periodic.*
2. *If $I = \mathbb{R}^+$, the following conditions are equivalent :*
   (a) *x is Bohr-asymptotically almost periodic.*
   (b) *x is Bochner-almost periodic.*

Proof This is a direct consequence of Lemma 1.4.4 and of the corresponding result in (semi)metric spaces. $\square$

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Chapitre 2

Counterexamples to mean square almost periodicity of the solutions of some SDEs with almost periodic coefficients

2.1 Introduction

Almost periodicity for stochastic processes and in particular for solutions of stochastic differential equations is investigated in an increasing number of papers since the works of C. Tudor and his collaborators [60, 73, 4, 25], who proved almost periodicity in distribution of solutions of some SDEs with almost periodic coefficients. More recently, Bezandry and Diagana [8, 9, 10] claimed that some SDEs with almost periodic coefficients have solutions which satisfy the stronger property of mean square almost periodicity. These claims are repeated in some subsequent papers and a book by different authors.

The aim of this short note is to give counterexamples to the results of [8, 9, 10].

Notations and definitions We denote by law (Y) the distribution of a random variable Y. If $\mathbb{X}$ is a metrizable topological space, we denote by $\mathcal{M}^{1,+}(\mathbb{X})$ the set of Borel probability measures on $\mathbb{X}$, endowed with the topology of narrow (or weak) convergence, i.e. the coarsest topology such that the mappings $\mu \mapsto \mu(\varphi)$, $\mathcal{M}^{1,+}(\mathbb{X}) \to \mathbb{R}$ are continuous for all bounded countinuous $\varphi : \mathbb{X} \to \mathbb{R}$.

Let $(\mathbb{X}, d)$ be a metric space. A continuous mapping $f : \mathbb{R} \to \mathbb{X}$ is said to be *almost periodic* (in Bohr's sense) if, for every $\epsilon > 0$, there exists a number $l(\epsilon) > 0$ such that every interval I of length greater than $l(\epsilon)$ contains an ϵ-*almost period*, that is, a number $\tau \in I$ such that $d(f(t+\tau), f(t)) \leq \epsilon$ for all $t \in \mathbb{R}$. Equivalently, by a criterion of Bochner, f is almost periodic if and only if the set $\{x(t + .),\ t \in \mathbb{R}\}$ is totally bounded in the space $\mathrm{C}(\mathbb{R}, \mathbb{X})$ endowed with the topology of uniform convergence. Thanks to another criterion of Bochner [13], almost periodicity of f does not depend on the metric d nor on the uniform structure of $(\mathbb{X}, d)$, but only on f and the topology generated by d (see [7] for details). We refer to e.g. [23, 79] for beautiful expositions of almost periodic functions and their many properties.

Let $X = (X_t)_{t\in\mathbb{R}}$ be a continuous stochastic process with values in a separable Banach space $\mathbb{E}$:

- We say that X is *mean square almost periodic* if X_t is square integrable for each t and the mapping $t \mapsto X_t$, $\mathbb{R} \to \mathrm{L}^2(\mathbb{E})$ is almost periodic.
- We say that X is *almost periodic in distribution* (in Bohr's sense) if the mapping $t \mapsto \operatorname{law}(X_{t+.})$, $\mathbb{R} \mapsto \mathcal{M}^{1,+}(\mathrm{C}(\mathbb{R}, \mathbb{E}))$ is almost periodic, where $\mathrm{C}(\mathbb{R}, \mathbb{E})$ is endowed with the topology of uniform convergence on compact subsets.

It is shown in [7] that, if X is mean square almost periodic, then X is almost periodic in distribution. The counterexamples of this paper also show that the converse implication is false (actually, it is proved in [7] that the converse implication is true under a tightness condition).

2.2 Two explicit counterexamples

The following very simple counterexample, inspired by [7, Counterexample 2.16], was suggested to us by Adam Jakubowski. It contradicts [8, Theorem 3.2], [9, Theorem 3.3], and [10, Theorem 4.2].

Exemple 2.1. (stationary Ornstein-Uhlenbeck process) Let $W = (W_t)_{t\in\mathbb{R}}$ be a standard Brownian motion on the real line. Let $\alpha, \sigma > 0$, and let X be the stationary Ornstein-Uhlenbeck process (see [56]) defined by

$$X_t = \sqrt{2\alpha\sigma} \int_{-\infty}^{t} e^{-\alpha(t-s)} dW_s. \tag{2.1}$$

Then X is the only L^2-bounded solution of the following SDE, which is a particular case of Equation (3.1) in [8] :

$$dX_t = -\alpha X_t \, dt + \sqrt{2\alpha}\sigma \, dW_t.$$

The process X is Gaussian with mean 0, and we have, for all $t \in \mathbb{R}$ and $\tau \geq 0$,

$$\operatorname{Cov}(X_t, X_{t+\tau}) = \sigma^2 e^{-\alpha\tau}.$$

Assume that X is mean square almost periodic, and let (t_n) be any increasing sequence of real numbers which converges to ∞. By Bochner's characterization, we can extract a sequence (still denoted by (t_n) for simplicity) such that (X_{t_n}) converges in L^2 to a random variable Y. Necessarily Y is Gaussian with law $\mathcal{N}(0, 2\alpha\sigma^2)$, and Y is $\mathcal{G}$-measurable, where

$\mathcal{G} = \sigma(X_{t_n};\, n \geq 0)$. Moreover (X_{t_n}, Y) is Gaussian for every n, and we have, for any integer n,

$$\mathrm{Cov}(X_{t_n}, Y) = \lim_{m\to\infty} \mathrm{Cov}(X_{t_n}, X_{t_{n+m}}) = 0.$$

because $(X_t^2)_{t\in\mathbb{R}}$ is uniformly integrable. This proves that Y is independent of X_{t_n} for every n, thus Y is independent of $\mathcal{G}$. Thus Y is constant, a contradiction.

Thus (2.1) has no mean square almost periodic solution.

A similar reasoning applies to the next counterexample, which also contradicts [8, Theorem 3.2], [9, Theorem 3.3], and [10, Theorem 4.2] :

Exemple 2.2. Again, $W = (W_t)_{t\in\mathbb{R}}$ is a standard Brownian motion on the real line. Let X be defined by

$$X_t = e^{-t+\sin(t)} \int_{-\infty}^{t} e^{s-\sin(s)} \sqrt{1-\cos(s)}\, dW_s.$$

Then X statisfies the SDE with periodic coefficients

$$dX_t = (-1+\cos(t))X_t\, dt + \sqrt{1-\cos(t)}\, dW_t.$$

The process X is Gaussian, with $\mathrm{E}\, X_t = 0$ and

$$\begin{aligned}\mathrm{Cov}(X_t, X_{t+\tau}) &= e^{-t-\tau+\sin(t+\tau)} e^{-t+\sin(t))} \int_{-\infty}^{t} e^{2(s-\sin(s))}(1-\cos(s))\, ds \\ &= \frac{1}{2}\, e^{-\tau+\sin(t+\tau)-\sin(t)} \to 0 \text{ when } \tau \to +\infty\end{aligned}$$

in particular $\mathrm{E}\, X_t^2 = \frac{1}{2}\, e^{2\sin(t)} \geq \frac{1}{2}\, e^{-2}$ thus the same reasoning as in Example 2.1 shows that X is not mean square almost periodic, because if X_{t_n} converges in L^2 to Y, with $t_n \to \infty$, then $Y = 0$ and $\mathrm{E}\, Y^2 \geq e^{-2}/2$.

By [25, Theorem 4.1], the process X is periodic in distribution.

The argument in the previous counterexamples can be slightly generalized for non necessarily Gaussian processes as follows :

Lemme 2.1. *Let X be a continuous square integrable stochastic process with values in a Banach space $\mathbb{E}$. Assume that $(\|X_t\|^2)_{t\in\mathbb{R}}$ is uniformly integrable and that there exists a sequence (t_n) of real numbers, $t_n \to \infty$, such that for any $x^* \in \mathbb{E}^*$ and any integer $n \geq 0$,*

$$\lim_{m\to\infty} \mathrm{Cov}\left(\langle x^*, X_{t_n}\rangle, \langle x^*, X_{t_m}\rangle\right) = 0, \tag{2.2}$$

$$\lim_{m\to\infty} \mathrm{Var}\left(\|X_{t_m}\|\right) > 0. \tag{2.3}$$

Then X is not mean square almost periodic.

Proof Assume that X is mean square almost periodic. Then, for some subsequence (t'_n) of (t_n), $X_{t'_n}$ converges in L^2 to some random vector Y. By (2.3) and the uniform integrability hypothesis, Y is not constant. On the other hand, by (2.2) and the uniform integrability hypothesis, we have

$$\mathrm{Cov}\left(\left\langle x^*, X_{t'_n}\right\rangle, \langle x^*, Y\rangle\right) = 0$$

for every $x^* \in \mathbb{E}^*$ and every integer n. Then

$$\mathrm{Var}\,\langle x^*, Y\rangle = \lim_n \mathrm{Cov}\left(\left\langle x^*, X_{t'_n}\right\rangle, \langle x^*, Y\rangle\right) = 0,$$

thus Y is constant, a contradiction. ■

2.3 Generalization

We present a generalization of Counterexamples 2.1 and 2.2 in a Hilbert space setting. Other generalizations in the same setting are possible.

From now on, $\mathbb{H}$ and $\mathbb{U}$ are separable Hilbert spaces, Q is a symmetric nonnegative operator on $\mathbb{U}$ with finite trace, and $(W_t)_{t\in\mathbb{R}}$ is a Q-Brownian motion with values in $\mathbb{U}$. We denote $\mathbb{U}_0 = Q^{1/2}\mathbb{U}$ and $L_2^0 = L_2(\mathbb{U}_0, \mathbb{H})$ the space of Hilbert-Schmidt operators from $\mathbb{U}_0$ to $\mathbb{H}$, endowed with the Hilbertian norm

$$\|\Psi\|_{L_2^0}^2 = \left\|\Psi Q^{1/2}\right\|_{L_2}^2 = \mathrm{Tr}\left(\Psi Q \Psi^*\right).$$

It is well known that, if Φ is a predictable stochastic process with values in L_2^0 such that $\int_0^t \|\Phi_s\|_{L_2^0}^2\, ds < +\infty$, then we have the Ito isometry

$$\mathrm{E}\left(\left\|\int_0^t \Phi_s\, dW_s\right\|^2\right) = \int_0^t \|\Phi_s\|_{L_2^0}^2\, ds.$$

Recall (see e.g. [43, Definitions 1.4.1 and 1.4.2]) that a linear operator $A(t)$ on $\mathbb{H}$ with domain $D(A(t))$ generates an *evolution semigroup* $(U(t,s))_{t\geq s}$ on $\mathbb{H}$, if $(U(t,s))_{t\geq s}$ is a family of bounded linear operators on $\mathbb{H}$ such that

(i) $U(t,r)\,U(r,s) = U(t,s)$ for all $t, r, s \in \mathbb{R}$ such that $s \leq r \leq t$, and, for every $t \in \mathbb{R}$, $U(t,t) = I$ the identity operator on $\mathbb{H}$,

(ii) for every $x \in \mathbb{H}$, the mapping $(t,s) \mapsto U(t,s)$ from $\{(t,s);\, t \geq s\}$ to $\mathbb{H}$ is continuous,

(iii) for every $T > 0$, there exists $K_T < \infty$ such that $\|U(t,s)\| \leq K_T$ for $0 \leq s \leq t \leq T$,

(iv) for all $t, s \in \mathbb{R}$ such that $s \leq t$, the domain $D(A(t))$ is dense in $\mathbb{H}$, $U(t,s)\,D(A(s)) \subset D(A(t))$, and

$$\frac{\partial}{\partial t} U(t,s)\,x = A(t)\,U(t,s)\,x \text{ for } t > s \text{ and } x \in D(A(s)).$$

The following theorem contains Counterexamples 2.1 and 2.2. For example, Counterexample 2.2 can be seen as a particular case of Equation (2.4) below, with $A(t) = -1+\cos(t)$ which generates the evolution semigroup $U(t,s) = e^{-(t-s)+\sin(t)-\sin(s)}$.

Théorème 2.1. (linear evolution equations with almost periodic noise) *Let us consider the stochatic evolution equation*

$$dX_t = A(t)X_t\,dt + g(t)\,dW_t \tag{2.4}$$

where $A(.)$ generates an evolution semigroup $(U(t,s))_{t\geq s}$ on $\mathbb{H}$. We assume that

(a) (see Hypothesis 1 in [25]) the Yosida approximations $A_n(t) = nA(t)(nI - A(t))^{-1}$ of $A(t)$, $t \in \mathbb{R}$, generate corresponding evolution operators $(U_n(t,s))_{t\geq s}$ such that, for every $x \in \mathbb{H}$ and for all $t, s \in \mathbb{R}$ such that $s \leq t$,

$$\lim_{n\to\infty} U_n(t,s)x = U(t,s)x$$

(b) A is uniformly dissipative *(see Hypothesis 3 in [25]), i.e. there exists $\beta > 0$ such that*

$$\langle A(t)x, x\rangle \leq -\beta \|x\|^2,\ t \in \mathbb{R},\ x \in D(A(t)),$$

(c) U is exponentially stable *(see Hypothesis H0 in [9]), i.e.*

$$\|U(t,s)\| \leq Me^{-\delta(t-s)},\ t \geq s \tag{2.5}$$

(d) $g : \mathbb{R} \to L_2^0$ is almost periodic and satisfies

$$0 < \int_{-\infty}^{+\infty} \|U(t,s)g(s)\|^2_{L_2^0}\,ds < +\infty. \tag{2.6}$$

Then (2.4) *has no mean square almost periodic solution. However, if the family $(X_t)_{t\in\mathbb{R}}$ is tight, the only* L^2*-bounded solution of* (2.4) *is almost periodic in distribution.*

Note that, if A and g are T-periodic, then by [25, Theorem 4.1] the L^2-bounded solution is T-periodic in distribution, that is, the mapping $t \mapsto \text{law}\,(X_{t+.})$, $\mathbb{R} \mapsto \mathcal{M}^{1,+}\,(\mathrm{C}(\mathbb{R}, \mathbb{E}))$, is periodic.

Proof The only L^2-bounded (mild) solution to (2.4) is given by

$$X_t = \int_{-\infty}^{t} U(t,s)g(s)\,dW_s, \tag{2.7}$$

see the proof of [25, Theorem 3.3]. Note that X is Gaussian because the integrand in (2.7) is deterministic. By [43, Theorem 1.4.5], X has a continuous version (actually Theorem 1.4.5 of [43] is given for processes defined on the half line $\mathbb{R}^+$, but we can repeat the argument on any interval $[-R, \infty)$). By [25, Theorem 4.3], if the family $(X_t)_{t\in\mathbb{R}}$ is tight, X is almost periodic in distribution.

Let $p > 2$. Applying Burkholder-Davis-Gundy inequalities to the process $t \mapsto \int_{-\infty}^t U(t_0, s)g(s)\, dW_s$ for fixed t_0, and then setting $t = t_0$ yields, for some constant c_p,

$$\mathrm{E}\, \|X_t\|^p \le c_p \left(\int_{-\infty}^t \|U(t,s)g(s)\|_{L_2^0}^2 \, ds \right)^{p/2} \le c_p \left(\int_{-\infty}^{+\infty} \|U(t,s)g(s)\|_{L_2^0}^2 \, ds \right)^{p/2} < +\infty$$

(see e.g. [43, Theorems 1.2.1, 1.2.3-(e) and Proposition 1.3.3-(f)]). Thus (X_t) is bounded in L^p, which proves that $(\|X_t\|^2)_{t\in\mathbb{R}}$ is uniformly integrable.

We have $\mathrm{E}(X_t) = 0$ for all $t \in \mathbb{R}$. Let $x \in \mathbb{H}$, $t \in \mathbb{R}$ and $\tau \ge 0$, and let us compute the covariance $\mathrm{Cov}\left(\langle x, X_t\rangle, \langle x, X_{t+\tau}\rangle\right)$: We get

$$\begin{aligned}
&\mathrm{Cov}\left(\langle x, X_t\rangle, \langle x, X_{t+\tau}\rangle\right) \\
&\qquad = \mathrm{E}\Bigg(\left\langle x, \int_{-\infty}^t U(t,s)g(s)\, dW_s \right\rangle \\
&\qquad\qquad \times \left\langle x, \left(\int_{-\infty}^t U(t+\tau,s)g(s)\, dW_s + \int_t^{t+\tau} U(t+\tau,s)g(s)\, dW_s \right) \right\rangle \Bigg) \\
&\qquad = \mathrm{E}\left(\left\langle x, \int_{-\infty}^t U(t,s)g(s)\, dW_s \right\rangle \left\langle U(t+\tau,t)^* x, \int_{-\infty}^t U(t,s)g(s)\, dW_s \right\rangle \right).
\end{aligned}$$

We deduce, using (2.5) and (2.6),

$$\lim_{\tau\to+\infty} \left|\mathrm{Cov}\left(\langle x, X_t\rangle, \langle x, X_{t+\tau}\rangle\right)\right| \le \lim_{\tau\to+\infty} \|U(t+\tau,t)\| \, \|x\|^2 \, \mathrm{E} \int_{-\infty}^t \|(U(t,s)g(s)\|_{L_2^0}^2 \, ds = 0.$$

On the other hand, we have, using (2.6),

$$\mathrm{Var}\left(\|X_t\|\right) = \mathrm{E}\left(\int_{-\infty}^t \|U(t,s)g(s)\|_{L_2^0}^2 \, ds \right) \to \int_{-\infty}^{+\infty} \|U(t,s)g(s)\|_{L_2^0}^2 \, ds > 0$$

We conclude by Lemma 2.1 that X is not mean square almost periodic. ■

2.4 Conclusion

A close look at the proofs of [8, 9, 10] shows the same error in each of those papers, which besides are clever at other places. Let us use the notations of the Hilbert setting of Section 2.3, and assume that all processes are defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. The error lies in the proof of the (untrue) assertion that, if $G : \mathbb{R} \times \mathrm{L}^2(\mathrm{P}; \mathbb{H}) \to \mathrm{L}^2(\mathrm{P}; L_2^0)$ is almost periodic in the first variable, uniformly with respect to the second on compact subsets of $\mathrm{L}^2(\mathrm{P}; \mathbb{H})$, then the stochastic convolution

$$\Psi(Y)_t := \int_{-\infty}^t U(t,s)G(s, Y_s)\, dW_s$$

is mean square almost periodic for any continous square integrable stochastic process Y. If this were true, then with $G(t, Y) = g(t)$ an almost periodic function, and assuming the hypothesis of Theorem 2.1, the process $X = \psi(1)$ of Equation (2.7), which is solution of (2.4), would be mean square almost periodic, but we know from Theorem 2.1 that this is not the case. The error consists in a wrong identification between integrals of the form $\int_{-\infty}^{t} Z_s \, dW_s$ and $\int_{-\infty}^{t} Z_s \, d\widetilde{W}_s$, where $\widetilde{W}$ has the same distribution as W.

Actually, mean square almost periodicity appears to be a very strong property for solutions of SDEs. Our counter-examples suggest that there are "very few" examples of SDEs with non trivial mean square almost periodic solutions. The question of their characterization remains open.

Chapitre 3

Weak Averaging of Semilinear Stochastic differential Equations with Almost Periodic Coefficients

3.1 Introduction

Since the classical work of N.M. Krylov and N.N. Bogolyubov [54] devoted to the analysis, by the method of averaging, of the problem of the dependence on a small parameter $\varepsilon > 0$ of almost periodic solutions of ordinary differential equation containing terms of frequency of order $\frac{1}{\varepsilon}$, several articles and books have appeared, which develop this method for different kinds of differential equations. See the bibliography in the book of V.Sh. Burd [19], where a list of books related to this problem for deterministic differential equations is presented. We note here that the authors of these papers are greatly influenced by the books of N.N. Bogolyubov and A.Yu. Mitropolskii [14] and M.A. Krasnosel′skiĭ, V.Sh. Burd and Yu.S. Kolesov [53].

The method of averaging has been applied of course to stochastic differential equations, but in general it was applied to the initial problem in a finite interval, see for example [52]. Even in this we can see a great difference with the deterministic case. To ensure the strong convergence in a space of stochastic processes, we must assume such convergence of the stochastic term when $\varepsilon \to 0$, which virtually excludes the consideration of high frequency oscillation of this term. R.Z. Khasminskii [52] has shown, in a finite dimensional setting, that it is possible to overcome this problem if one only looks for convergence *in distribution* to the solution of the averaged equation. Later Ivo Vrkoč [76] generalized this result in a Hilbert space setting, for which the initial problem was at this time already well developed (see for example the book of Da Prato and Zabczyck [26]).

During the last 20 years an intensive study of the problem of existence of almost periodic solutions of stochastic differential equations was performed by A. Arnold, C. Tudor,

G. Da Prato and later by P.H. Bezandry and T. Diagana [8, 9, 10]. For the first group, an almost periodic solution means that the stochastic process generates an almost periodic measure on the paths space. The second group claims the existence of square mean almost periodic solutions, but square mean almost periodicity seems to be a too strong property for solutions of SDEs, see counterexamples in [57].

In this paper we propose the averaging principle for solutions to a family of semilinear stochastic differential equations in Hilbert space which are almost periodic in distribution. The second member of these equations contains a high frequency term. Under the Bezandry-Diagana conditions, we establish the convergence in distribution of the solutions of these equations to the solution of the averaged equation in the sense of Khasminskii-Vrkoč.

The paper is organized as follows : The next section is devoted to the notations and preliminaries. We then prove in Section 3.3 that the solutions of the equations we consider are almost periodic in distribution, when their coefficients are almost periodic. In section 3.4, we prove the fondamental averaging result of this paper.

3.2 Notations and Preliminaries

In the sequel, $(\mathbb{H}_1, \|.\|_{\mathbb{H}_1})$ and $(\mathbb{H}_2, \|.\|_{\mathbb{H}_2})$ denote separable Hilbert spaces and $L(\mathbb{H}_1, \mathbb{H}_2)$ (or $L(\mathbb{H}_1)$ if $\mathbb{H}_1 = \mathbb{H}_2$) is the space of all bounded linear operators from $\mathbb{H}_1$ to $\mathbb{H}_2$, whose norm will be denoted by $\|.\|_{L(\mathbb{H}_1,\mathbb{H}_2)}$. If $A \in L(\mathbb{H}_1)$ then A^* denotes its adjoint operator and if A is a nuclear operator,

$$|A|_{\mathcal{N}} = \sup\left\{ \sum_i | < Ae_i, f_i > |, \{e_i\}, \{f_i\} \text{ orthonormal bases of } \mathbb{H}_1 \right\}$$

is the nuclear norm of A.

3.2.1 Almost periodic functions

Let $(\mathbb{E}, d)$ be a separable metric space, we denote by $C_b(\mathbb{E})$ the Banach space of continuous and bounded functions $f : \mathbb{E} \to \mathbb{R}$ with $\| f \|_\infty = \sup_{x\in\mathbb{E}} |f(x)|$ and by $\mathcal{P}(\mathbb{E})$ the set of all probability measures onto σ-Borel field of $\mathbb{E}$. For $f \in C_b(\mathbb{E})$ we define

$$\| f \|_L = \sup\Big\{ \frac{f(x) - f(y)}{d_{\mathbb{E}}(x, y)} : x \neq y \Big\}$$
$$\| f \|_{\mathrm{BL}} = \max\{\| f \|_\infty, \| f \|_L\}$$

and we define

$$\mathrm{BL}(\mathbb{E}) = \big\{ f \in C_b(\mathbb{E}); \| f \|_{\mathrm{BL}} < \infty \big\}.$$

For $\mu, \nu \in \mathcal{P}(\mathbb{E})$ we define

$$d_{\mathrm{BL}}(\mu,\nu) = \sup_{\|f\|_{\mathrm{BL}}\leq 1}\Big|\int_{\mathbb{E}} f d(\mu-\nu)\Big|$$

which is a complete metric on $\mathcal{P}(\mathbb{E})$ and generates the narrow (or weak) topology, i.e. the coarsest topology on $\mathcal{P}(\mathbb{E})$ such that the mappings $\mu \mapsto \mu(f)$ are continuous for all bounded continuous $f : \mathbb{E} \to \mathbb{R}$.

Let $(\mathbb{E}_1, d_1)$ and $(\mathbb{E}_2, d_2)$ be separable and complete metric spaces. Let f be a continuous mapping from $\mathbb{R}$ to $\mathbb{E}_2$ (resp. from $\mathbb{R} \times \mathbb{E}_1$ to $\mathbb{E}_2$). Let $\mathcal{K}$ be a set of subsets of $\mathbb{E}_1$. The function f is said to be *almost periodic* (respectively *uniformly with respect to x in elements of $\mathcal{K}$*) if for every $\varepsilon > 0$ (respectively for every $\varepsilon > 0$ and every subset $K \in \mathcal{K}$), there exists a constant $l(\varepsilon, K) > 0$ such that any interval of length $l(\varepsilon, K)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} d_2(f(t+\tau), f(t)) < \varepsilon$$

$$\text{(respectively } \sup_{t\in\mathbb{R}}\sup_{x\in K} d_2(f(t+\tau, x), f(t,x)) < \varepsilon).$$

A characterization of almost periodicity is given in the following result, due to Bochner :

Theorem 3.2.1. *Let $f : \mathbb{R} \to \mathbb{H}_1$ be continuous. Then the following statements are equivalent*

- *f is almost periodic.*
- *The set of translated functions $\{f(t+.)\}_{t\in\mathbb{R}}$ is relatively compact in $\mathcal{C}(\mathbb{R};\mathbb{E}_2)$ with respect to the uniform norm.*
- *f satisfies* Bochner's double sequence criterion*, that is, for every pair of sequences $\{\alpha_n'\} \subset \mathbb{R}$ and $\{\beta_n'\} \subset \mathbb{R}$, there are subsequences $(\alpha_n) \subset (\alpha_n')$ and $(\beta_n) \subset (\beta_n')$ respectively with same indexes such that, for every $t \in \mathbb{R}$, the limits*

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t+\alpha_n+\beta_m) \text{ and } \lim_{n\to\infty} f(t+\alpha_n+\beta_n), \tag{3.1}$$

exist and are equal.

Remarque 3.1.

(i) A striking property of Bochner's double sequence criterion is that the limits in (1.1) exist in any of the three modes of convergences : pointwise, uniform on compact intervals and uniform on $\mathbb{R}$ (with respect to $d_{\mathbb{E}}$). This criterion has thus the avantage that it allows to establish uniform convergence by checking pointwise convergence.

(ii) The previous result holds for the metric spaces $(\mathcal{P}(\mathbb{E}), d_{\mathrm{BL}})$ and $(\mathcal{P}(\mathcal{C}(\mathbb{R},\mathbb{E})), d_{\mathrm{BL}})$

3.2.2 Almost periodic stochastic processes

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let $X : \mathbb{R} \times \Omega \to \mathbb{H}_2$ be a stochastic process. We denote by $\mu(t)$ the distribution of the random variable $X(t)$. Following Tudor's terminology [74], we say that X has *almost periodic one-dimensional distributions* if the mapping $t \mapsto \mu(t)$ from $\mathbb{R}$ to $(\mathcal{P}(\mathbb{H}_2), d_{\mathrm{BL}})$ is almost periodic.

If X has continuous trajectories, we say that X is *almost periodic in distribution* if the mapping $t \mapsto \mathrm{law}\,(X(t + .)$ from $\mathbb{R}$ to $\mathcal{P}(C(\mathbb{R}; \mathbb{H}_2))$ is almost periodic, where $C(\mathbb{R}; \mathbb{H}_2)$ is endowed with the uniform convergence on compact intervals and $\mathcal{P}(C(\mathbb{R}; \mathbb{H}_2))$ is endowed with the distance d_{BL}.

Let $L^2(\mathrm{P}, \mathbb{H}_2)$ be the space of $\mathbb{H}_2$-valued random variables with a finite quadratic-mean. We say that a stochastic process $X : \mathbb{R} \to L^2(\mathrm{P}, \mathbb{H}_2)$ is *square-mean continuous* if, for every $s \in \mathbb{R}$,

$$\lim_{t \to s} E\|X(t) - X(s)\|^2_{\mathbb{H}_2} = 0.$$

We denote by $\mathrm{CUB}\big(\mathbb{R}, L^2(\mathrm{P}, \mathbb{H}_2)\big)$ the Banach space of square-mean continuous and uniformly bounded stochastic processes, endowed with the norm

$$\|X\|^2_\infty = \sup_{t \in \mathbb{R}} (E\|X(t)\|^2_{\mathbb{H}_2}).$$

A square-mean continuous stochastic process $X : \mathbb{R} \to L^2(\mathrm{P}, \mathbb{H}_2)$ is said to be *square-mean almost periodic* if, for each $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t \in \mathbb{R}} E\|X(t + \tau) - X(t)\|^2_{\mathbb{H}_2} < \varepsilon.$$

The next theorem is interesting for itself, but we shall not use it in the sequel.

Theorem 3.2.2. *Let $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$ be an almost periodic function uniformly with respect to x in compact subsets of $\mathbb{H}_2$ such that*

$$\|F(t, x)\|_{\mathbb{H}_2} \le C_1(1 + \|x\|_{\mathbb{H}_2}) \text{ and } \|F(t, x) - F(t, y)\|_{\mathbb{H}_2} \le C_2\|x - y\|_{\mathbb{H}_2}.$$

Then the function

$$\widetilde{F} : \mathbb{R} \times L^2(\mathrm{P}, \mathbb{H}_2) \to L^2(\mathrm{P}, \mathbb{H}_2)$$

(where $\widetilde{F}(t, Y)(\omega) = F(t, Y(\omega))$ for every $\omega \in \Omega$) is square-mean almost periodic uniformly with respect to Y in compact subsets of $L^2(\mathrm{P}, \mathbb{H}_2)$.

Proof Let us prove that for each $Y \in L^2(\mathrm{P}, \mathbb{H}_2)$ the process $\widetilde{F}_Y : \mathbb{R} \to L^2(\mathrm{P}, \mathbb{H}_2)$, $t \mapsto \widetilde{F}(t, Y)$ is almost periodic.

For every $\delta > 0$, there exists a compact subset S of $\mathbb{H}_2$ such that

$$\mathrm{P}\{Y \notin S\} \le \delta.$$

Let $\varepsilon > 0$, then there exist $\delta > 0$ and a compact subset S of $\mathbb{H}_2$ such that $\mathrm{P}\{Y \notin S\} \leq \delta$ and

$$\int_{\{Y \notin S\}} \left(1 + \|Y\|_{\mathbb{H}_2}^2\right) d\,\mathrm{P} < \frac{\varepsilon}{4C_1}.$$

Since F is almost periodic uniformly with respect to x in the compact subset S, there exists a constant $l(\varepsilon, S) > 0$ such that any interval of length $l(\varepsilon, S)$ contains at least a number τ for which

$$\sup_t \|F(t+\tau, x) - F(t, x)\|_{\mathbb{H}_2} < \frac{\sqrt{\varepsilon}}{\sqrt{2}} \quad \text{for all } x \in S.$$

We have

$$\begin{aligned} E(\|F(t+\tau, Y) - F(t, Y)\|_{\mathbb{H}_2}^2) &= \int_{\{Y \in S\}} \|F(t+\tau, Y) - F(t, Y)\|_{\mathbb{H}_2}^2 d\,\mathrm{P} \\ &\quad + \int_{\{Y \notin S\}} \|F(t+\tau, Y) - F(t, Y)\|_{\mathbb{H}_2}^2 d\,\mathrm{P} \\ &< \frac{\varepsilon}{2} + 2C_1 \int_{\{Y \notin S\}} \left(1 + \|Y\|_{\mathbb{H}_2}^2\right) d\,\mathrm{P} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore the process $\widetilde{F}_Y$ is almost periodic. Since $\widetilde{F}$ is Lipschitz, it is almost periodic uniformly with respect to Y in compact subsets of $L^2(\mathrm{P}, \mathbb{H}_2)$ (see [36, Theorem 2.10 page 25]). $\square$

Proposition 3.2.3. *([24, 36]) For any almost periodic function $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$, there exists a continuous function $F_0 : \mathbb{H}_2 \to \mathbb{H}_2$ such that for each $x \in \mathbb{H}_2$ the mean value*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T F(t, x) dt = F_0(x). \tag{3.2}$$

Furthermore, if $F(t, x)$ is Lipschitz in $x \in \mathbb{H}_2$ uniformly with respect to $t \in I$, the mapping F_0 is Lipschitz too.

Let $Q \in L(\mathbb{H}_1)$ be a linear operator. Then Q is a bijection from $\mathrm{range}(Q) = Q(\mathbb{H}_1)$ to $(\ker Q)^\perp$. We denote by Q^{-1} the *pseudo-inverse* of Q (see [64, Appendix C] or [26, Appendix B.2]), that is, the inverse of the mapping $(\ker Q)^\perp \to \mathrm{range}(Q)$, $x \mapsto Q(x)$. Note that $\mathrm{range}(Q)$ is a Hilbert space for the scalar product $\langle x, y \rangle_{\mathrm{range}(Q)} = \langle Q^{-1}(x), Q^{-1}(y) \rangle$.

Proposition 3.2.4. *Let $G : \mathbb{R} \times \mathbb{H}_2 \to L(\mathbb{H}_1, \mathbb{H}_2)$ be an almost periodic function and let $Q \in L(\mathbb{H}_1)$ be a self-adjoint nonnegative operator. Let $\mathbb{H}_0 = \mathrm{range}(Q^{1/2})$, endowed with the scalar product $\langle x, y \rangle_{\mathrm{range}(Q^{1/2})} = \langle Q^{-1/2}(x), Q^{-1/2}(y) \rangle$. There exists a continuous function $G_0 : \mathbb{H}_2 \to L(\mathbb{H}_0, \mathbb{H}_2)$ such that, for every $x \in \mathbb{H}_1$,*

$$\lim_{t \to \infty} \Big| \frac{1}{t} \int_0^t G(s, x) Q G^*(s, x) ds - G_0(x) Q G_0^*(x) \Big|_{\mathcal{N}} = 0, \tag{3.3}$$

where $G^*(s, x) = (G(s, x))^*$ and $G_0^*(x) = (G_0(x))^*$.

Proof Observe first that $G_0(x)QG_0^*(x) = (G_0(x)Q^{1/2})(G_0(x)Q^{1/2})^*$, thus $G_0(x)$ does not need to be defined on the whole space $\mathbb{H}_1$, it is sufficient that it be defined on $\mathbb{H}_0$.

Since G is almost periodic, the function $H(s,x) = G(s,x)QG^*(s,x)$ is almost periodic too, with positive self-adjoint nuclear values in $L(\mathbb{H}_2)$. Thus there exists a mapping $H_0 : \mathbb{H}_2 \to L(\mathbb{H}_2)$ such that, for every $x \in \mathbb{H}_2$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t G(s,x)QG^*(s,x)\,ds = H_0(x).$$

By e.g. [36, Theorem 3.1], H_0 is continuous. Thus the mapping

$$H_0^{1/2} : \begin{cases} \mathbb{H}_2 & \to & L(\mathbb{H}_2) \\ x & \mapsto & (H_0(x))^{1/2} \end{cases}$$

is continuous with positive self-adjoint values.

Let $G_0(x) = H_0^{1/2}(x)Q^{-1/2} : \mathbb{H}_0 \to \mathbb{H}_2$. We then have, for every $x \in \mathbb{H}_2$,

$$H_0(x) = G_0(x)Q(G_0(x))^*$$

and G_0 is continuous, which proves (3.3). $\square$

3.3 Solutions almost periodic in distribution

We consider the semilinear stochastic differential equation,

$$dX_t = AX_t dt + F(t,X_t)dt + G(t,X_t)dW(t), t \in \mathbb{R} \tag{3.4}$$

Where $A : \mathrm{Dom}(A) \subset \mathbb{H}_2 \to \mathbb{H}_2$ is a densely defined closed (possibly unbounded) linear operator, $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$, and $G : \mathbb{R} \times \mathbb{H}_2 \to L(\mathbb{H}_1, \mathbb{H}_2)$ are continuous functions. In what follows we assume that :

(i) $W(t)$ is an $\mathbb{H}_1$-valued Wiener process with nuclear covariance operator Q (we denote by $\operatorname{tr} Q$ the trace of Q), defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathrm{P})$.

(ii) $A : \mathrm{Dom}(A) \to \mathbb{H}_2$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ such that there exists a constant $\delta > 0$ with

$$\|S(t)\|_{L(\mathbb{H}_2)} \leq e^{-\delta t}, t \geq 0.$$

(iii) There exists a constant K such that the mappings $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$ and $G : \mathbb{R} \times \mathbb{H}_2 \to L(\mathbb{H}_1, \mathbb{H}_2)$ satisfy

$$\|F(t,x)\|_{\mathbb{H}_2} + \|G(t,x)\|_{L(\mathbb{H}_1,\mathbb{H}_2)} \leq K(1 + \|x\|_{\mathbb{H}_2})$$

(iv) The functions F and G are Lipschitz, more precisely there exists a constant K such that

$$\|F(t,x)-F(t,y)\|_{\mathbb{H}_2}+\|G(t,x)-G(t,y)\|_{L(\mathbb{H}_1,\mathbb{H}_2)}\le K\|x-y\|_{\mathbb{H}_2}$$

for all $t\in\mathbb{R}$ and $x,y\in\mathbb{H}_2$.

(v) The mappings F and G are almost periodic in $t\in\mathbb{R}$ uniformly with respect to x in bounded subsets of $\mathbb{H}_2$.

The assumptions in the following theorem are contained in those of Bezandry and Diagana [8, 9]. The result is similar to [25, Theorem 4.3], with different hypothesis and a different proof.

Theorem 3.3.1. *Let the assumptions (i) - (v) be fulfiled and the constant* $\theta=\dfrac{K^2}{\delta^2}\Big(\dfrac{1}{2}+\operatorname{tr}Q\Big)<1$. *Then there exists a unique mild solution X to (3.4) in* $\mathrm{CUB}\big(\mathbb{R},L^2(\mathrm{P},\mathbb{H}_2)\big)$. *Furthermore, X has a.e. continuous trajectories, is almost periodic in distribution, and $X(t)$ can be explicitly expressed as follows, for each $t\in\mathbb{R}$:*

$$X(t)=\int_{-\infty}^{t}S(t-s)F\big(s,X(s)\big)ds+\int_{-\infty}^{t}S(t-s)G\big((s),X(s)\big)dW(s).$$

To prove Theorem 3.3.1, we will use the following result, which is given in a more general form in ([25] :

Proposition 3.3.2. *([25, Proposition 3.1-(c)]) Let $\tau\in\mathbb{R}$. Let $(\xi_n)_{0\le n\le\infty}$ be a sequence of square integrable $\mathbb{H}_2$-valued random variables. Let $(F_n)_{0\le n\le\infty}$ and $(G_n)_{0\le n\le\infty}$ be sequences of mappings from $\mathbb{R}\times\mathbb{H}_2$ to $\mathbb{H}_2$ and $L(\mathbb{H}_1,\mathbb{H}_2)$ respectively, satisfying* (iii) *and* (iv) *(replacing F and G by F_n and G_n respectively, and the constant K being independent of n). For each n, let X_n denote the solution of*

$$X_n(t)=S(t-\tau)\xi_n+\int_{\tau}^{t}S(t-s)F\big(s,X_n(s)\big)ds+\int_{\tau}^{t}S(t-s)G\big((s),X(s)\big)dW(s).$$

Assume that, for every $(t,x)\in\mathbb{R}\times\mathbb{H}_2$,

$$\lim_{n\to\infty}F_n(t,x)=F_\infty(t,x),\ \lim_{n\to\infty}G_n(t,x)=G_\infty(t,x),$$
$$\lim_{n\to\infty}d_{\mathrm{BL}}(\mathrm{law}\,(\xi_n,W)\,,\mathrm{law}\,(\xi_\infty,W))=0,$$

(the last equality takes place in $\mathcal{P}\left(\mathbb{H}_2\times C(\mathbb{R},\mathbb{H}_1)\right)$). Then we have in $\mathcal{C}([\tau,T];\mathbb{H}_2)$, for any $T>\tau$,

$$\lim_{n\to\infty}d_{\mathrm{BL}}(\mathrm{law}\,(X_n)\,,\mathrm{law}\,(X_\infty))=0.$$

Proof of Theorem 3.3.1 Note that

$$X(t)=\int_{-\infty}^{t}S(t-s)F\big(s,X(s)\big)ds+\int_{-\infty}^{t}S(t-s)G\big((s),X(s)\big)dW(s)$$

satisfies

$$X(t) = S(t-s)X(s) + \int_s^t S(t-s)F\big(s, X(s)\big)ds + \int_s^t S(t-s)G\big((s), X(s)\big)dW(s)$$

for all $t \geq s$ for each $s \in \mathbb{R}$, and hence X is a mild solution to (3.4).

We introduce an operator L by

$$LX(t) = \int_{-\infty}^t S(t-s)F\big(s, X(s)\big)ds + \int_{-\infty}^t S(t-s)G\big((s), X(s)\big)dW(s).$$

It can be see easily that the operator L maps $\mathrm{CUB}\big(\mathbb{R}, L^2(\mathrm{P}, \mathbb{H}_1)\big)$ into itself.

First step. Let us show that L has a unique fixed point.

$$\begin{aligned}
&\|(LX)(t) - (LY)(t)\|_{\mathbb{H}_2} \\
&\qquad = \|\int_{-\infty}^t S(t-s)[F(s, X(s)) - F(s, Y(s))]ds \\
&\qquad + \int_{-\infty}^t S(t-s)[G(s, X(s)) - G(s, Y(s))]dW(s)\|_{\mathbb{H}_2} \\
&\qquad \leq \int_{-\infty}^t e^{-\delta(t-s)}\|F(s, X(s)) - F(s, Y(s))\|_{\mathbb{H}_2}ds \\
&\qquad + \|\int_{-\infty}^t S(t-s)[G(s, X(s)) - G(s, Y(s))]dW(s)\|_{\mathbb{H}_2}.
\end{aligned}$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ we obtain

$$\begin{aligned}
&E\|(LX)(t) - (LY)(t)\|^2_{\mathbb{H}_2} \\
&\qquad \leq 2E\Big(\int_{-\infty}^t e^{-\delta(t-s)}\|F(s, X(s)) - F(s, Y(s))\|_{\mathbb{H}_2}ds\Big)^2 \\
&\qquad + 2E\Big(\|\int_{-\infty}^t S(t-s)[G(s, X(s)) - G(s, Y(s))]dW(s)\|_{\mathbb{H}_2}\Big)^2 \\
&\qquad = I_1 + I_2.
\end{aligned}$$

We have

$$\begin{aligned}
I_1 &\leq 2\int_{-\infty}^t e^{-\delta(t-s)}ds \int_{-\infty}^t e^{-\delta(t-s)}E\|F(s, X(s)) - F(s, Y(s))\|^2_{\mathbb{H}_2}ds \\
&\leq 2K^2\int_{-\infty}^t e^{-\delta(t-s)}ds \int_{-\infty}^t e^{-\delta(t-s)}E\|X(s)) - Y(s))\|^2_{\mathbb{H}_2}ds \\
&\leq 2K^2\big(\int_{-\infty}^t e^{-\delta(t-s)}ds\big)^2 \sup_{s\in\mathbb{R}} E\|X(s)) - Y(s))\|^2_{\mathbb{H}_2} \\
&\leq \frac{K^2}{2\delta^2}\sup_{s\in\mathbb{R}} E\|X(s)) - Y(s))\|^2_{\mathbb{H}_2}.
\end{aligned}$$

For I_2, using the isometry identity we obtain

$$\begin{aligned} I_2 &\leq 2\operatorname{tr} Q \int_{-\infty}^{t} e^{-2\delta(t-s)} E\|G(s,X(s)) - G(s,Y(s))\|^2_{L(\mathbb{H}_1,\mathbb{H}_2)} ds \\ &\leq 2\operatorname{tr} Q K^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} E\|X(s) - Y(s)\|^2_{\mathbb{H}_2)} ds \\ &\leq 2K^2 \operatorname{tr} Q \big(\int_{-\infty}^{t} e^{-2\delta(t-s)} ds \big) \sup_{s\in\mathbb{R}} E\|X(s) - Y(s)\|^2_{\mathbb{H}_2} \\ &\leq \frac{K^2 \operatorname{tr} Q}{\delta} \sup_{s\in\mathbb{R}} E\|X(s) - Y(s)\|^2_{\mathbb{H}_2}. \end{aligned}$$

Thus

$$(LX)(t) - E\|(LY)(t)\|^2_{\mathbb{H}_2} \leq I_1 + I_2 \leq \theta \sup_{s\in\mathbb{R}} E\|X(s) - Y(s)\|^2_{\mathbb{H}_2}.$$

Consequently, as $\theta < 1$, we deduce that L is a contraction operator, hence there exists a unique mild solution to (3.4) in $\mathrm{CUB}\big(\mathbb{R}, L^2(\mathrm{P}, \mathbb{H}_1)\big)$.

Furthermore, by [26, Theorem 7.4], almost all trajectories of this solution are continuous.

Second step. Let show that X is almost periodic in distribution. We use Bochner's double sequences criterion. Let (α_n') and (β_n') be two sequences in $\mathbb{R}$. We show that there are subsequences $(\alpha_n) \subset (\alpha_n')$ and $(\beta_n) \subset (\beta_n')$ with same indexes such that, for every $t \in \mathbb{R}$, the limits

$$\lim_{n\to\infty} \lim_{m\to\infty} \mu(t + \alpha_n + \beta_m) \text{ and } \lim_{n\to\infty} \mu(t + \alpha_n + \beta_n), \tag{3.5}$$

exist and are equal, where $\mu(t) := \operatorname{law}(X)(t)$ is the law or distribution of $X(t)$.

Since F and G are almost periodic, there are subsequences $(\alpha_n) \subset (\alpha_n')$ and $(\beta_n) \subset (\beta_n')$ with same indexes such that

$$\lim_{n\to\infty} \lim_{m\to\infty} F(t + \alpha_n + \beta_m, x) = \lim_{n\to\infty} F(t + \alpha_n + \beta_n, x) =: F_0(t, x) \tag{3.6}$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} G(t + \alpha_n + \beta_m, x) = \lim_{n\to\infty} G(t + \alpha_n + \beta_n, x) =: G_0(t, x). \tag{3.7}$$

These limits exist uniformly with respect to $t \in \mathbb{R}$ and x in bounded subsets of $\mathbb{H}_2$.

Set now $(\gamma_n) = (\alpha_n + \beta_n)$. For each fixed integer n, we consider

$$X^n(t) = \int_{-\infty}^{t} S(t-s)F(s + \gamma_n, X^n(s))ds + \int_{-\infty}^{t} S(t-s)G(s + \gamma_n, X^n(s))dW(s)$$

the mild solution of

$$dX^n(t) = AX^n(t)dt + F(t + \gamma_n, X^n(t))dt + G(t + \gamma_n, X^n(t))dW(t) \tag{3.8}$$

and

$$X^0(t) = \int_{-\infty}^{t} S(t-s)F_0(s, X^0(s))ds + \int_{-\infty}^{t} S(t-s)G_0(s, X^0(s))dW(s)$$

the mild solution of

$$dX^0(t) = AX^0(t)dt + F_0(t, X^0(t))dt + G_0(t, X^0(t))dW(t). \tag{3.9}$$

Make the change of variable $\sigma - \gamma_n = s$, the process

$$X(t+\gamma_n) = \int_{-\infty}^{t+\gamma_n} S(t+\gamma_n - s)F(s, X(s))ds + \int_{-\infty}^{t+\gamma_n} S(t+\gamma_n - s)G(s, X(s))dW(s)$$

becomes

$$X(t+\gamma_n) = \int_{-\infty}^{t} S(t-s)F(s+\gamma_n, X(s+\gamma_n))ds + \int_{-\infty}^{t} S(t-s)G(s+\gamma_n, X(s+\gamma_n))d\tilde{W}_n(s),$$

where $\tilde{W}_n(s) = W(s+\gamma_n) - W(\gamma_n)$ is a Brownian motion with the same distribution as $W(s)$. Thus the process $X(t+\gamma_n)$ has the same distribution as $X^n(t)$.

Let us show that $X^n(t)$ converges in mean to $X^0(t)$ for each fixed $t \in \mathbb{R}$. Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ we obtain

$$\begin{aligned}
&E\|X^n(t) - X^0(t)\|^2 \\
&\qquad = E\|\int_{-\infty}^{t} S(t-s)[F(s+\gamma_n, X^n(s)) - F_0(s, X^0(s))]ds \\
&\qquad\quad + \int_{-\infty}^{t} S(t-s)[G(s+\gamma_n, X^n(s)) - G_0(s, X^0(s))]dW(s)\|^2 \\
&\qquad \leq 2E\|\int_{-\infty}^{t} S(t-s)[F(s+\gamma_n, X^n(s)) - F_0(s, X^0(s))]ds\|^2 \\
&\qquad\quad + 2E\int_{-\infty}^{t} S(t-s)[G(s+\gamma_n, X^n(s)) - G_0(s, X^0(s))]dW(s)\|^2 \\
&\qquad \leq 4E\|\int_{-\infty}^{t} S(t-s)[F(s+\gamma_n, X^n(s)) - F(s+\gamma_n, X^0(s))]ds\|^2 \\
&\qquad\quad + 4E\|\int_{-\infty}^{t} S(t-s)[F(s+\gamma_n, X^0(s)) - F_0(s, X^0(s))]ds\|^2 \\
&\qquad\quad + 4E\|\int_{-\infty}^{t} S(t-s)[G(s+\gamma_n, X^n(s)) - G(s+\gamma_n, X^0(s))]dW(s)\|^2 \\
&\qquad\quad + 4E\|\int_{-\infty}^{t} S(t-s)[G(s+\gamma_n, X^0(s)) - G_0(s, X^0(s))]dW(s)\|^2 \\
&\qquad \leq I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now, using (ii), (iv) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
I_1 &= 4E\|\int_{-\infty}^{t} S(t-s)[F(s+\gamma_n, X^n(s)) - F(s+\gamma_n, X^0(s))]ds\|^2 \\
&\leq 4E\big(\int_{-\infty}^{t} \|S(t-s)\|\|F(s+\gamma_n, X^n(s)) - F(s+\gamma_n, X^0(s))\|ds\big)^2 \\
&\leq 4E\big(\int_{-\infty}^{t} e^{-\delta(t-s)}\|F(s+\gamma_n, X^n(s)) - F(s+\gamma_n, X^0(s))\|ds\big)^2 \\
&\leq 4\big(\int_{-\infty}^{t} e^{-2\delta(t-s)}ds\big)\big(\int_{-\infty}^{t} E\|F(s+\gamma_n, X^n(s)) - F(s+\alpha_n, X^0(s))\|^2 ds\big) \\
&\leq \frac{2K^2}{\delta}\int_{-\infty}^{t} E\|X^n(s) - X^0(s)\|^2 ds.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
I_2 &= 4E\|\int_{-\infty}^{t} S(t-s)[F(s+\gamma_n, X^0(s)) - F_0(s, X^0(s))]ds\|^2 \\
&\leq 4E\big(\int_{-\infty}^{t} e^{-\delta(t-s)}\|F(s+\gamma_n, X^0(s)) - F_0(s, X^0(s))\|ds\big)^2 \\
&\leq 4E\big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\big)\big(\int_{-\infty}^{t} e^{-\delta(t-s)}\|F(s+\gamma_n, X^0(s)) - F_0(s, X^0(s))\|^2 ds\big) \\
&\leq 4\big(\int_{-\infty}^{t} e^{-\delta(t-s)}ds\big)^2 \sup_s E\|F(s+\gamma_n, X^0(s)) - F_0(s, X^0(s))\|^2 \\
&\leq \frac{4}{\delta^2}\sup_s E\|F(s+\gamma_n, X^0(s)) - F_0(s, X^0(s))\|^2,
\end{aligned}$$

which converges to 0 as $n \to \infty$ because $\sup_{t\in\mathbb{R}} E\|X^0(t)\|^2 < \infty$ which implies that $\{X^0(t)\}_t$ is tight relatively to bounded sets.

Applying Itô's isometry, we get

$$\begin{aligned}
I_3 &= 4E\|\int_{-\infty}^{t} S(t-s)[G(s+\gamma_n, X^n(s)) - G(s+\gamma_n, X^0(s))]dW(s)\|^2 \\
&\leq 4\,\mathrm{tr}\,QE\int_{-\infty}^{t} \|S(t-s)\|^2\|G(s+\gamma_n, X^n(s)) - G(s+\gamma_n, X^0(s))\|^2 ds \\
&\leq 4\,\mathrm{tr}\,Q\int_{-\infty}^{t} E\|G(s+\gamma_n, X^n(s)) - G(s+\gamma_n, X^0(s))\|^2 ds \\
&\leq 4^2 K\,\mathrm{tr}\,Q\int_{-\infty}^{t} E\|X^n(s) - X^0(s)\|^2 ds.
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= 4E\|\int_{-\infty}^{t} S(t-s)[G(s+\gamma_n, X^0(s)) - G_0(s, X^0(s))]dW(s)\|^2 \\
&\leq 4\operatorname{tr} QE\Big(\int_{-\infty}^{t} \|S(t-s)\|^2 \|G(s+\gamma_n, X^0(s)) - G_0(s, X^0(s))\|^2 ds\Big) \\
&\leq 4\operatorname{tr} Q\Big(\int_{-\infty}^{t} e^{-2\delta(t-s)} ds\Big) \sup_{s\in\mathbb{R}} E\|G(s+\gamma_n, X^0(s)) - G_0(s, X^0(s))\|^2 \\
&\leq \frac{2\operatorname{tr} Q}{\delta} \sup_{s\in\mathbb{R}} E\|G(s+\gamma_n, X^0(s)) - G_0(s, X^0(s))\|^2.
\end{aligned}$$

For the same reason as for I_2, the right hand term goes to 0 as $n \to \infty$.
Thus, applying Gronwall's inequality, we obtain

$$\lim_{n\to\infty} E\|X^n(t) - X^0(t)\|^2 = 0,$$

hence $X^n(t)$ converges in distribution to $X^0(t)$, but since the distribution of $X^n(t)$ is the same as that of $X(t+\gamma_n)$ we deduce that $X(t+\gamma_n)$ converges in distribution to $X^0(t)$, i.e.

$$\lim_{n\to\infty} \mu(t+\alpha_n+\beta_n) = \operatorname{law}\big(X^0(t)\big) =: \mu_t^0.$$

By analogy and using (3.6), (3.7) we can easily deduce that

$$\lim_{n\to\infty} \lim_{m\to\infty} \mu(t+\alpha_n+\beta_m) = \mu_t^0.$$

We have thus proved that X has almost periodic one-dimensional distributions. To prove that X is almost periodic in distribution, we apply Proposition 3.3.2 : for fixed $\tau \in \mathbb{R}$, let $\xi_n = X(\tau+\alpha_n)$, $F_n(t,x) = F(t+\alpha_n)$, $G_n(t,x) = G(t+\alpha_n)$. By the foregoing, (ξ_n) converges in distribution to some variable $Y(\tau)$. We can choose $Y(\tau)$ such that (ξ_n, W) converges in distribution to (Y, W). Then, for every $T \geq \tau$, $X(.+\alpha_n)$ converges in distribution on $C([\tau, T]; \mathbb{H}_2)$ to the (unique in distribution) solution to

$$Y(t) = S(t-\tau)Y(\tau) + \int_{\tau}^{t} S(t-s)F\big(s, Y(s)\big)ds + \int_{\tau}^{t} S(t-s)G\big((s), Y(s)\big)dW(s).$$

Note that Y does not depend on the chosen interval $[\tau, T]$, thus the convergence takes place on $C(\mathbb{R}; \mathbb{H}_2)$. Similarly, $Y_n := Y(.+\beta_n)$ converges in distribution on $C(\mathbb{R}; \mathbb{H}_2)$ to a continuous process Z such that, for $t \geq \tau$,

$$Z(t) = S(t-\tau)Z(\tau) + \int_{\tau}^{t} S(t-s)F\big(s, Z(s)\big)ds + \int_{\tau}^{t} S(t-s)G\big((s), Z(s)\big)dW(s).$$

But, by (3.6) and (3.7), $X(.+\gamma_n)$ converges in distribution to the same process Z. Thus X is almost periodic in distribution. $\square$

3.4 Weak averaging

In this section we assume Conditions (i) – (iv) of Section 3.3, but we replace Condition (v) by Condition (v') below, which is weaker thanks to Propositions 3.2.3 and 3.2.4. Let us define the Hilbert space $\mathbb{H}_0 = \text{range}(Q^{1/2})$ as in Proposition 3.2.4, where Q is the covariance operator of the Wiener process W. We assume that the mappings $F : \mathbb{R}\times\mathbb{H}_2 \to \mathbb{H}_2$ and $G : \mathbb{R} \times \mathbb{H}_2 \to L(\mathbb{H}_1, \mathbb{H}_2)$ satisfy

(v') There exists continuous functions $F_0 : \mathbb{H}_2 \to \mathbb{H}_2$ and $G_0 : \mathbb{H}_2 \to L(\mathbb{H}_0, \mathbb{H}_2)$ satisfying (3.2) and (3.3) for every $x \in \mathbb{H}_2$.

Theorem 3.4.1. *Let the assumptions* (i) – (iv) *and (v') be fulfilled and the constant* $\theta = \dfrac{K^2}{\delta^2}\Big(\dfrac{1}{2} + \operatorname{tr} Q\Big) < 1$. *For each fixed* $\varepsilon \in]0, 1[$, *let* X^ε *be the mild solution of the equation*

$$dX^\varepsilon(t) = AX^\varepsilon(t)dt + F(\frac{t}{\varepsilon}, X^\varepsilon(t))dt + G(\frac{t}{\varepsilon}, X^\varepsilon(t))dW(t), \ t \in \mathbb{R}. \tag{3.10}$$

Then $(X^\varepsilon(t)) \to (X^0(t))$ *in distribution as* $\varepsilon \to 0+$ *on the space* $\mathcal{C}(\mathbb{R}, \mathbb{H}_2)$ *endowed with the topology of uniform convergence on compacts subsets of* $\mathbb{R}$, *where* $(X^0(t))$ *is the mild solution to*

$$dX^0(t) = A(X^0(t))dt + F_0(X^0(t))dt + G_0(X^0(t))dW(t) \tag{3.11}$$

which is a stationary process.

Before we give the proof of this theorem, let us recall some well-known results.

Proposition 3.4.2. *([20]) Let* $(X_n)_{n\geq 0}$ *be a sequence of centered Gaussian random variable on a separable Hilbert space* $\mathbb{H}$ *with sequence of covariance operators* $(Q_n)_{n\geq 0}$. *Then* $(X_n)_{n\geq 0}$ *converges in distribution to* X_0 *in* $\mathbb{H}$ *if and only if*

$$| Q_n - Q_0 |_{\mathcal{N}} \to 0, n \to \infty$$

Let $\mathbb{U}, \mathbb{V}, \mathbb{H}$ be real separable Hilbert spaces, let W be a $\mathbb{U}$-valued $(\mathcal{F}_t)$-adapted Wiener process with nuclear covariance operator Q.

Proposition 3.4.3. *([76, Proposition 2.2]) Let* $\alpha : \mathbb{H} \to \mathbb{V}$ *be a Lipschitz mapping and* $\sigma : \mathbb{R} \times \mathbb{H} \to L(\mathbb{U}, \mathbb{V})$ *be a measurable mapping such that* $\|\sigma(r, x)\|_{L(\mathbb{U},\mathbb{V})} \leq M(1 + \|x\|_{\mathbb{H}})$ *and* $\|\sigma(r, x) - \sigma(r, y)\|_{L(\mathbb{U},\mathbb{V})} \leq M\|x - y\|_{\mathbb{H}}$ *for a constant* M *and every* $r \in [s, t], x, y \in \mathbb{H}$. *Let* $g \in BL(\mathbb{V})$, *we define*

$$\psi(y) = Eg\Big(\alpha(y) + \int_s^t \sigma(r, y)dW(r)\Big), y \in \mathbb{H}.$$

Let $u : \Omega \to \mathbb{H}$ *be a* $(\mathcal{F}_s)$*-measurable random variable with* $E\|u\|^2_{\mathbb{H}} < \infty$. *Then*

$$E\Big[g\Big(\alpha(u) + \int_s^t \sigma(r, u)dW(r)\Big)|\mathcal{F}_s\Big] = \psi(u) \ \text{ P -}a.s.$$

Proof of Theorem 3.4.1 We denote $F_\varepsilon(s,x) := F\big(\dfrac{s}{\varepsilon},x\big)$, $G_\varepsilon(s,x) := G\big(\dfrac{s}{\varepsilon},x\big)$, and for every $X \in \mathrm{CUB}\big(\mathbb{R}, L^2(\mathrm{P},\mathbb{H}_2)\big)$,

$$\begin{aligned} L_\varepsilon(X)(t) &:= \int_{-\infty}^t S(t-s)F_\varepsilon(s,X(s))ds + \int_{-\infty}^t S(t-s)G_\varepsilon(s,X(s))dW(s), \\ L_0(X)(t) &:= \int_{-\infty}^t S(t-s)F_0(X(s))ds + \int_{-\infty}^t S(t-s)G_0(X(s))dW(s). \end{aligned}$$

First step. Let us show that $L_\varepsilon(X) \longrightarrow L_0(X)$ in distribution, as $\varepsilon \to 0$, in the space $\mathcal{C}(\mathbb{R},\mathbb{H}_2)$ endowed with the topology of uniform convergence on the compact subsets of $\mathbb{R}$. This amounts to prove that, for any $\tau \in \mathbb{R}$ and any $T \in \mathbb{R}$ such that $\tau \le T$, $L_\varepsilon(X) \longrightarrow L_0(X)$ in distribution in the space $\mathcal{C}([\tau,T],\mathbb{H}_2)$ (see [77, Theorem 5]). Since the previous integral exists, then for every $\eta > 0$, there exists τ such that for each $\sigma < \tau$, we have $E\|Y^n(\sigma)\|^2_{\mathbb{H}_2} < \eta$, thus for the proof that Y^n converges in distribution to Y^0 in $\mathcal{C}(\mathbb{R},H_2)$, it suffices to show the convergence in distribution on $\mathcal{C}([\tau,T],\mathbb{H}_2)$ of

$$\int_{-\tau}^t S(t-s)F_\varepsilon(s,X(s))ds + \int_{-\tau}^t S(t-s)G_\varepsilon(s,X(s))dW(s)$$

to

$$\int_{-\tau}^t S(t-s)F_0(X(s))ds + \int_{-\tau}^t S(t-s)G_0(X(s))dW(s).$$

Let (ε_n) be an arbitrary sequence in $]0,1[$ such that $\varepsilon_n \longrightarrow 0$ and $X \in \mathrm{CUB}\big(\mathbb{R}, L^2(\mathrm{P},\mathbb{H}_2)\big)$. We denote $F_n(s,x) := F(\frac{s}{\varepsilon_n},x)$, $G_n(s,x) := G(\frac{s}{\varepsilon_n},x)$, we simplify the notation L_{ε_n} in L_n, and we denote

$$\begin{aligned} Y^n(t) &:= L_n(X)(t) \\ &:= \int_{-\infty}^t S(t-s)F_n(s,X(s))ds + \int_{-\infty}^t S(t-s)G_n(s,X(s))dW(s). \end{aligned}$$

We have $\sup E\|X_t\|^2_{\mathbb{H}_2} < \infty$, thus the process X satisfies the following condition : for every $\eta > 0$, there exist a partition

$$\{\tau = t_o < t_1 < \cdots < t_k = T\} \text{ of } [\tau,T]$$

and a process

$$\widetilde{X}(t) = \Sigma_{i=1}^k \widetilde{X}(t_{i-1})1_{[t_{i-1},t_i[}(t)$$

such that

$$\sup E\|X(t) - \widetilde{X}(t)\|^2_{\mathbb{H}_2} < \eta.$$

Using the fact that L_ε is Lipschitz, we obtain

$$\sup E\|L_\varepsilon X(t) - L_\varepsilon\widetilde{X}(t)\|^2_{\mathbb{H}_2} < \eta$$

uniformly with respect to ε.

We set, with a slight abuse of notation :

$$\begin{aligned}\widetilde{X}^n(t) = L_n\widetilde{X}(t) = \sum_{i=1}^{k}\Big(\int_{t_{i-1}\wedge t}^{t_i\wedge t} & S(t-s)F_n(s,\widetilde{X}_{t_{i-1}})ds \\ &+ \int_{t_{i-1}\wedge t}^{t_i\wedge t} S(t-s)G_n(s,\widetilde{X}_{t_{i-1}})dW(s)\Big)\end{aligned}$$

To prove that for each $t\in[\tau,T]$, $L_nX(t)$ converges in distribution to $L_0X(t)$, it suffices to show that for each $l\in\{0,1,\dots,k\}$, $\widetilde{X}^n(t_l)\longrightarrow\widetilde{X}^0(t_l)$ in distribution as $n\to\infty$, where

$$\begin{aligned}\widetilde{X}^n(t_l) =& \sum_{i=1}^{l}\Big(\int_{t_{i-1}}^{t_i} S(t_l-s)F_n(s,\widetilde{X}(t_{i-1}))ds \\ &+ \int_{t_{i-1}}^{t_i} S(t_l-s)G_n(s,\widetilde{X}(t_{i-1}))dW(s)\Big) \\ =& S(t_l-t_{l-1})\widetilde{X}^n(t_{l-1}) + \int_{t_{l-1}}^{t_l} S(t_l-s)F_n(s,\widetilde{X}(t_{l-1}))ds \\ &+ \int_{t_{l-1}}^{t_l} S(t_l-s)G_n(s,\widetilde{X}(t_{l-1}))dW(s).\end{aligned}$$

Indeed, we just use the following two properties : For any $\gamma>0$ there exists a partition $\{\tau=t_0<\cdots<t_k=T\}$ of the interval $[\tau,T]$ such that

$$\sup_n E\Big(\max_{l=1,\dots,k+1}\max_{t_{l-1}\le t\le t_l}\Big|\widetilde{X}^n(t)-\widetilde{X}^n(t_{l-1})\Big|\Big)<\gamma$$

and

$$\begin{aligned}d_{\mathrm{BL}}\Big(\mathrm{law}\,(L_nX(t))\,,\mathrm{law}\,(L_0X(t))\Big) \le& d_{\mathrm{BL}}\Big(\mathrm{law}\,(L_nX(t))\,,\mathrm{law}\,\Big(\widetilde{X}^n(t)\Big)\Big) \\ &+ d_{\mathrm{BL}}\Big(\mathrm{law}\,\Big(\widetilde{X}^n(t)\Big)\,,\mathrm{law}\,\Big(\widetilde{X}^0(t)\Big)\Big) \\ &+ d_{\mathrm{BL}}\Big(\mathrm{law}\,\Big(\widetilde{X}^0(t)\Big)\,,\mathrm{law}\,(L_0X(t))\Big) \\ \le& E\|L_nX(t)-\widetilde{X}^n(t)\| \\ &+ d_{\mathrm{BL}}\Big(\mathrm{law}\,\Big(\widetilde{X}^n(t)\Big)\,,\mathrm{law}\,\Big(\widetilde{X}^0(t)\Big)\Big) \\ &+ E\|L_0X(t)-\widetilde{X}^0(t)\|.\end{aligned}$$

We define a mapping

$$\gamma_n:\mathbb{H}_2^l\longrightarrow L^1(\Omega,\mathbb{H}_2)$$

by

$$\gamma_n(y_0, y_1, \ldots, y_{l-1}) = \sum_{i=1}^{l}\Big(\int_{t_{i-1}}^{t_i} S(t_l - s)F_n(s, y_{i-1})\, ds$$
$$+ \int_{t_{i-1}}^{t_i} S(t_l - s)G_n(s, y_{i-1})\, dW(s)\Big)$$

Obviously,

$$\gamma_n(\widetilde{X}_{t_0}, \widetilde{X}_{t_1}, \ldots, \widetilde{X}_{t_{l-1}}) = \widetilde{X}^n_{t_l}$$

Let

$$\mu_{t_0,t_1,\ldots,t_{l-1}} = \text{law}\left(\widetilde{X}_{t_0}, \widetilde{X}_{t_1}, \ldots, \widetilde{X}_{t_{l-1}}\right).$$

Let $g \in BL(\mathbb{H}_2)$, and $h_n(y) = E[g(\gamma_n(y))];\ y \in \mathbb{H}_2^l$. From Proposition 3.4.3, we have

$$E[g(\widetilde{X}^n_{t_l})] = E[h_n(\widetilde{X}_{t_0}, \widetilde{X}_{t_1}, \ldots, \widetilde{X}_{t_{l-1}})] = \int_{\mathbb{H}_2^l} h_n(y) d\mu_{t_0,t_1,\ldots,t_{l-1}}(y)$$

and

$$|E[g(\widetilde{X}^n_{t_l})] - E[g(\widetilde{X}^0_{t_l})]| = |\int_{\mathbb{H}_2^l} h_n(y) - h_0(y) d\mu_{t_0,t_1,\ldots,t_{l-1}}(y)|$$
$$\leq \int_{\mathbb{H}_2^l} |h_n(y) - h_0(y)| d\mu_{t_0,t_1,\ldots,t_{l-1}}(y).$$

Let us show that $h_n(y) \longrightarrow h_0(y)$ as $n \to \infty$ for every $y \in \mathbb{H}_2^l$:

$$\begin{aligned}\gamma_n(y) - \gamma_0(y) =& \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} S(t_l - s)\big(F_n(s, y_{i-1}) - F_0(y_{i-1})\big)ds \\ &+ \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} S(t_l - s)\big(G_n(s, y_{i-1}) - G_0(y_{i-1})\big)\, dW(s) \\ =& I_n + J_n.\end{aligned}$$

Assumption (v') implies that $I_n \longrightarrow 0$ as $n \longrightarrow \infty$, and since

$$\sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} S(t_l - s)G_n(s, y_{i-1})\, dW(s)$$

is a centered Gaussian random variable in $\mathbb{H}_2$, we deduce by Assumption (v') and Proposition 3.4.2 that $J_n \longrightarrow 0$ in distribution as $n \longrightarrow \infty$ hence $\gamma_n(y) \longrightarrow \gamma_0(y)$ in distribution as $n \longrightarrow \infty$, consequently

$$h_n(y) \longrightarrow h_0(y) \text{ for any } y \in \mathbb{H}_2^l. \tag{3.12}$$

For every $\eta' > 0$, there exists a compact set $\mathcal{K} \subset \mathbb{H}_2^l$ such that

$$\mu_{t_0,t_1,\ldots,t_{l-1}}(\mathbb{H}_2^l \setminus \mathcal{K}) < \eta'.$$

We have

$$h_n \in BL(\mathbb{H}_2^l) \text{ and } \sup_n \|h_n\|_{\text{BL}} < \infty \tag{3.13}$$

because, for all $y, z \in \mathbb{H}_2^l$, and for some constant K_1,

$$|h_n(y) - h_n(z)| \leq \|g\|_{\text{BL}} E\|\gamma_n(y) - \gamma_n(z)\|_{\mathbb{H}_2} \leq K_1 \|g\|_{\text{BL}} \|y - z\|_{\mathbb{H}_2^l}.$$

From (3.12), (3.13) and the compactness of $\mathcal{K}$, we deduce that h_n converges to h_0 uniformly on $\mathcal{K}$, hence

$$\lim_{n\to\infty} \int_{\mathcal{K}} |h_n(y) - h_0(y)| d\mu_{t_0,t_1,\ldots,t_{l-1}}(y) = 0$$

and, since g is a bounded function,

$$\int_{\mathbb{H}_2^l \setminus \mathcal{K}} |h_n(y) - h_0(y)| d\mu_{t_0,t_1,\ldots,t_{l-1}}(y)$$
$$\leq 2 \sup_n \sup_y |h_n(y)| \eta' = 2 \sup_n \sup_y |E[g(\gamma_n(y))]| \eta'.$$

Thus $\widetilde{X}^n(t_l) \longrightarrow \widetilde{X}^0(t_l)$ in distribution as $n \to \infty$.

Let us now show that $(\widetilde{X}^n_{t_0}, \widetilde{X}^n_{t_1}, \ldots, \widetilde{X}^n_{t_k}) \longrightarrow (\widetilde{X}^0_{t_0}, \widetilde{X}^0_{t_1}, \ldots, \widetilde{X}^0_{t_k})$ in distribution as $n \longrightarrow \infty$. We proceed by induction. By the foregoing, we have $\widetilde{X}^n_{t_0} \longrightarrow \widetilde{X}^0_{t_0}$ in distribution. Assume that for $0 \leq l \leq k-1$, $(\widetilde{X}^n_{t_0}, \widetilde{X}^n_{t_1}, \ldots, \widetilde{X}^n_{t_l})$ converges in distribution in $\mathbb{H}_2^{l+1}$. Let us define $\alpha_n : \mathbb{H}_2^{l+1} \longrightarrow \mathbb{H}_2^{l+2}$ by

$$\alpha_n(y_0, y_1, \ldots, y_l) = \Big(y_0, y_1, \ldots, y_l, S(t_{l+1} - t_l) y_l + \int_{t_l}^{t_{l+1}} S(t_{l+1} - s) F_n(s, y_l) ds\Big)$$

and $\beta_n : \mathbb{H}_2^{l+1} \longrightarrow L^1(\Omega, \mathbb{H}_2^{l+2})$ by

$$\beta_n(y_0, y_1, \ldots, y_l) = \Big(0, \ldots, 0, \int_{t_l}^{t_{l+1}} S(t_{l+1} - s) G_n(s, y_l) dW(s)\Big)$$

so that

$$(\alpha_n + \beta_n)(\widetilde{X}^n_{t_0}, \widetilde{X}^n_{t_1}, \ldots, \widetilde{X}^n_{t_l}) = (\widetilde{X}^n_{t_0}, \widetilde{X}^n_{t_1}, \ldots, \widetilde{X}^n_{t_l}, \widetilde{X}^n_{t_{l+1}}).$$

We denote $u_n = (\widetilde{X}^n_{t_0}, \widetilde{X}^n_{t_1}, \ldots, \widetilde{X}^n_{t_l})$ and $\mu_n = \operatorname{law}(u_n)$. Let $g \in BL(\mathbb{H}_2^{l+2})$, and

$$h_n(y) = Eg\big(\alpha_n(y) + \beta_n(y)\big), \; y \in \mathbb{H}_2^{l+1}.$$

Proposition 3.4.3 yields

$$Eg(\widetilde{X}^n_{t_0}, \widetilde{X}^n_{t_1}, \ldots, \widetilde{X}^n_{t_{l+1}}) = Eh_n(u_n) = \int_{\mathbb{H}_2^{l+1}} h_n(y) d\mu_n(y).$$

It follows that

$$|Eg(\widetilde{X}^n_{t_0}, \widetilde{X}^n_{t_1}, \ldots, \widetilde{X}^n_{t_{l+1}}) - Eg(\widetilde{X}^0_{t_0}, \widetilde{X}^0_{t_1}, \ldots, \widetilde{X}^0_{t_{l+1}})|$$
$$\leq \int_{\mathbb{H}_2^{l+1}} |h_n(y) - h_0(y)| d\mu_n(y) + \Big| \int_{\mathbb{H}_2^{l+1}} h_0(y) d\mu_n(y) - \int_{\mathbb{H}_2^{l+1}} h_0(y) d\mu_0(y) \Big|$$
$$\leq J_1(n) + J_2(n).$$

As in the above reasoning, we can prove that

$$h_n \in BL(\mathbb{H}_2^{l+1}),\ \sup_n \|h_n\|_{\text{BL}} < \infty$$

and

$$\alpha_n(y) + \beta_n(y) \longrightarrow \alpha_0(y) + \beta_0(y)$$

in distribution, for every $y \in \mathbb{H}_2^{l+1}$ thus $h_n(y) \longrightarrow h_0(y)$ as $n \longrightarrow \infty$, for any $y \in \mathbb{H}_2^{l+1}$. On the other hand, by the induction hypothesis we have $\mu_n \longrightarrow \mu_0$ and since $h_0 \in BL(\mathbb{H}_2^{l+1})$ we have $J_2(n) \longrightarrow 0$ as $n \longrightarrow 0$. The convergence of μ_n implies that $\{\mu_n\}$ is tight, i.e. for each $\eta' > 0$ there exists a compact set $\mathcal{K} \in \mathbb{H}_2^{l+1}$ such that

$$\sup_n \mu_n(\mathbb{H}_2^{l+1} \setminus \mathcal{K}) < \eta' \tag{3.14}$$

Since for every $y \in \mathbb{H}_2^{l+1}$, $h_n(y) \longrightarrow h_0(y)$, from (3.14) and the compactness of $\mathcal{K}$ the function h_n converges to h_0 uniformly on $\mathcal{K}$, hence

$$\lim_{n\to\infty} \int_{\mathcal{K}} |h_n(y) - h_0(y)| d\mu_n(y) = 0$$

and

$$\int_{\mathbb{H}_2^l \setminus \mathcal{K}} |h_n(y) - h_0(y)| d\mu_n(y) \leq 2 \sup_n \sup_y |h_n(y)| \eta$$

So $J_1(n) \longrightarrow 0$ as $n \longrightarrow 0$, consequently $(\widetilde{X}^n_{t_0}, \widetilde{X}^n_{t_1}, \ldots, \widetilde{X}^n_{t_{l+1}})$ converges in distribution in $\mathbb{H}_2^{l+2}$, which implies that for every $q \in \mathbb{N}$, we have

$$(Y^n_{t_0}, Y^n_{t_1}, \ldots, Y^n_{t_q}) \to (Y^0_{t_0}, Y^0_{t_1}, \ldots, Y^0_{t_q}) \tag{3.15}$$

in distribution on $\mathbb{H}_2^{q+1}$.

It remains to show that for each positive η, ν there exist α, $0 < \alpha < 1$ and an integer n_0, such that

$$\frac{1}{\alpha} \mathrm{P}\{ \sup_{s \leq t \leq s+\alpha} \|Y^n(s) - Y^n(t)\| > \nu\} < \eta,\ n \geq n_0 \tag{3.16}$$

for every s in $[\tau, T]$. We have

$$\begin{aligned} &Y^n(t) - Y^n(s) \\ &\qquad = \int_\tau^t S(t-\sigma) F_n(\sigma, X(\sigma)) d\sigma - \int_\tau^s S(s-\sigma) F_n(\sigma, X(\sigma)) d\sigma \\ &\qquad\quad + \int_\tau^t S(t-\sigma) G_n(\sigma, X(\sigma)) dW(\sigma) - \int_\tau^s S(s-\sigma) G_n(\sigma, X(\sigma)) dW(\sigma) \\ &\qquad = \int_s^t S(t-\sigma) F_n(\sigma, X(\sigma)) d\sigma + \int_s^t S(t-\sigma) G_n(\sigma, X(\sigma)) dW(\sigma). \end{aligned}$$

We have

$$\frac{1}{\alpha} \mathrm{P}\{ \sup_{s \leq t \leq s+\alpha} \|Y^n(s) - Y^n(t)\| > \nu\} \leq \frac{1}{\alpha\nu^4} E(\sup_{s \leq t \leq s+\alpha} \|Y^n(s) - Y^n(t)\|^4)$$

To check Inequality (3.16) it suffices to show that we can choose α such that

$$E(\sup_{s\le t\le s+\alpha}\|Y^n(s)-Y^n(t)\|^4)\le \alpha\nu^4\eta$$

for every integer n and every $s\in[\tau,T]$.

Using the inequality $(a+b)^4\le 8a^4+8b^4$, we obtain

$$\begin{aligned}E(\sup_{s\le t\le s+\alpha}\|Y^n(s)-Y^n(t)\|^4) \le & 8E(\sup_{s\le t\le s+\alpha}\|\int_s^t S(t-\sigma)F_n(\sigma,X(\sigma))d\sigma\|^4)\\ &+8E(\sup_{s\le t\le s+\alpha}\|\int_s^t S(t-\sigma)G_n(\sigma,X(\sigma))dW(\sigma)\|^4)\\ \le & I_1+I_2.\end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}I_1&=8E(\sup_{s\le t\le s+\alpha}\|\int_s^t S(t-\sigma)F_n(\sigma,X(\sigma))d\sigma\|^4)\\ &\le 8E\Big(\sup_{s\le t\le s+\alpha}\Big(\int_s^t\|S(t-\sigma)F_n(\sigma,X(\sigma)\|d\sigma\Big)^4\Big)\\ &\le 8E\sup_{s\le t\le s+\alpha}\Big[\Big(\int_s^t e^{-2\delta(t-\sigma)}d\sigma\Big)^2\Big(\int_s^t\|F_n(\sigma,X(\sigma)\|^2d\sigma\Big)^2\Big]\\ &\le 8K^2\sup_{s\le t\le s+\alpha}\Big(\frac{1}{2\delta}(1-e^{-2\delta(t-s)})\Big)^2E\Big(\int_s^{s+\alpha}(1+\|X(\sigma)\|)^2d\sigma\Big)^2\end{aligned}$$

For I_2, using the stochastic convolution inequality, we obtain, for some constant C_{conv},

$$\begin{aligned}I_2&=8E(\sup_{s\le t\le s+\alpha}\|\int_s^t S(t-\sigma)G_n(\sigma,X(\sigma))dW(\sigma)\|^4)\\ &\le 8C_{\text{conv}}E\Big(\int_s^{s+\alpha}\|G_n(\sigma,X(\sigma))\|^2d\sigma\Big)^2\\ &\le 8C_{\text{conv}}K^2\int_s^{s+\alpha}(1+\|X(\sigma)\|)^2d\sigma.\end{aligned}$$

Thus, for α small enough,

$$E\big(\sup_{s\le t\le s+\alpha}\|Y^n(s)-Y^n(t)\|^2\big)\le\eta.$$

Therefore $L_\varepsilon(X)\to L_0(X)$ in distribution as $\varepsilon\to 0+$ on the space $\mathcal{C}([\tau,T],\mathbb{H}_2)$ for all τ,T such that $T>\tau$. Hence $L_\varepsilon(X)\to L_0(X)$ in distribution as $\varepsilon\to 0+$ on the space $\mathcal{C}(\mathbb{R},\mathbb{H}_2)$.

Second step. Now, let us show that $L_\varepsilon(X^\varepsilon)\to L_0(X^0)$ in distribution as $\varepsilon\to 0+$ on the space $\mathcal{C}(\mathbb{R},\mathbb{H}_2)$, which means that $X^\varepsilon\to X^0$ in distribution as $\varepsilon\to 0+$ on the space $\mathcal{C}(\mathbb{R},\mathbb{H}_2)$.

We denote by : $\mu_\varepsilon^k(X):=\text{law}\big(L_\varepsilon^k(X)\big)$ $(k\ge 1)$ where $L_\varepsilon^k=L_\varepsilon\circ L_\varepsilon\circ\cdots\circ L_\varepsilon$, $\mu(X):=\text{law}\,(X)$ and $\mu_0^k(X):=\text{law}\big(L_0^k(X)\big)$ where $L_0^k=L_0\circ L_0\circ\cdots\circ L_0$.

Observe that :

(a) Since, for every $k \in \mathbb{N}$, L_ε^k and L_0^k are contraction operators (see the proof of Theorem 1.2.7), we deduce that, for every $f \in BL(\mathcal{C}(\mathbb{R}, \mathbb{H}_2)$ such that $\|f\|_{\mathrm{BL}} \leq 1$, for each $\omega \in \Omega$, $f \circ L_\varepsilon^k(\omega) \in BL(\mathcal{C}(\mathbb{R}, \mathbb{H}_2)$ and $\|f \circ L_\varepsilon^k(w)\|_{\mathrm{BL}} \leq 1$.

(b) Since X^ε and X^0 are solutions, we have $\mu_\varepsilon^k(X^\varepsilon) = \mu(X^\varepsilon)$ and $\mu_0^k(X^0) = \mu(X^0)$ for every $k \in \mathbb{N}^*$

(c) Since L_ε is θ-Lipschitz,

$$\Big(E\|L_\varepsilon^k(X) - L_\varepsilon^k(Y)\|\Big)^2 \leq E\|L_\varepsilon^k(X) - L_\varepsilon^k(Y)\|^2 \leq \theta^k E\|X - Y\|^2$$

Now, using the properties of the metric d_{BL} on the probability space $\mathcal{P}\left(\mathcal{C}(\mathbb{R}, \mathbb{H}_2)\right)$ and (c) we obtain, for $X, Y \in \mathrm{CUB}\big(\mathbb{R}, L^2(\mathrm{P}, \mathbb{H}_1)\big)$,

$$\begin{aligned} d_{\mathrm{BL}}(\mu_\varepsilon^k(X), \mu_\varepsilon^k(Y)) &= \sup_{\|f\|_{\mathrm{BL}} \leq 1} \Big| \int_{\mathcal{C}(\mathbb{R}, \mathbb{H}_2)} f d(\mu_\varepsilon^k(X) - \mu_\varepsilon^k(Y)) \Big| \\ &= \sup_{\|f\|_{\mathrm{BL}} \leq 1} \Big| \int_\Omega f[L_\varepsilon^k(X)] - f[L_\varepsilon^k(Y)] d\,\mathrm{P} \Big| \\ &\leq \int_\Omega \|L_\varepsilon^k(X) - L_\varepsilon^k(Y)\| d\,\mathrm{P} \leq \theta^k E\|X - Y\|^2 \end{aligned}$$

Furthermore, from (a) we obtain

$$\begin{aligned} d_{\mathrm{BL}}(\mu_\varepsilon^k(X), \mu_\varepsilon^k(Y)) &= \sup_{\|f\|_{\mathrm{BL}} \leq 1} \Big| \int_\Omega f[L_\varepsilon^k(X)] - f[L_\varepsilon^k(Y)] d\,\mathrm{P} \Big| \\ &\leq \sup_{\|f\|_{\mathrm{BL}} \leq 1} \Big| \int_\Omega f(X) - f(Y) d\,\mathrm{P} \Big| = d_{\mathrm{BL}}\big(\mu(X), \mu(Y)\big) \end{aligned} \tag{3.17}$$

From above we deduce that there exist $k \in \mathbb{N}^*$ and $0 < \theta' < 1$ such that

$$d_{\mathrm{BL}}(\mu_\varepsilon^k(X), \mu_\varepsilon^k(Y)) \leq \theta' d_{\mathrm{BL}}\big(\mu(X), \mu(Y)\big) \tag{3.18}$$

Indeed, assume that Inequality (3.18) is false, for every $0 < \theta' < 1$ there exist $X, Y \in \mathrm{CUB}\big(\mathbb{R}, L^2(\mathrm{P}, \mathbb{H}_1)\big)$ such that $\mu(X) \neq \mu(Y)$ and

$$\theta^k \Big(E\|X - Y\|^2\Big)^{\frac{1}{2}} \geq d_{\mathrm{BL}}(\mu_\varepsilon^k(X), \mu_\varepsilon^k(Y)) > \theta' d_{\mathrm{BL}}\big(\mu(X), \mu(Y)\big); \ \forall k \in \mathbb{N}^*$$

therefore taking k elarge enough, we see that $d_{\mathrm{BL}}\big(\mu(X), \mu(Y)\big) = 0$, a contradiction with $\mu(X) \neq \mu(Y)$.

Now, by using (b), (3.18), and (3.17), it is easy to conclude that

$$\begin{aligned} d_{\mathrm{BL}}\big(\mu(X^\varepsilon), \mu(X^0)\big) &= d_{\mathrm{BL}}(\mu_\varepsilon^k(X^\varepsilon), \mu_0^k(X^0) \\ &\leq \frac{1}{1 - \theta'} d_{\mathrm{BL}}(\mu_\varepsilon^k(X^0), \mu_0^k(X^0) \\ &\leq \frac{k - 1}{1 - \theta'} d_{\mathrm{BL}}(\mu_\varepsilon^1(X^0), \mu_0^1(X^0). \end{aligned}$$

Indeed, we have

$$
\begin{aligned}
&(1-\theta^{'})d_{\mathrm{BL}}\big(\mu(X^{\varepsilon}),\mu(X^{0})\big)\\
&\qquad= d_{\mathrm{BL}}\big(\mu(X^{\varepsilon}),\mu(X^{0})\big)-\theta^{'} d_{\mathrm{BL}}\big(\mu(X^{\varepsilon}),\mu(X^{0})\big)\\
&\qquad\le d_{\mathrm{BL}}\big(\mu(X^{\varepsilon}),\mu(X^{0})\big)-d_{\mathrm{BL}}(\mu_{\varepsilon}^{k}(X^{\varepsilon}),\mu_{\varepsilon}^{k}(X^{0}))\\
&\qquad= d_{\mathrm{BL}}\big(\mu_{\varepsilon}^{k}(X^{\varepsilon}),\mu(X^{0})\big)-d_{\mathrm{BL}}(\mu_{\varepsilon}^{k}(X^{\varepsilon}),\mu_{\varepsilon}^{k}(X^{0}))\\
&\qquad\le d_{\mathrm{BL}}(\mu_{\varepsilon}^{k}(X^{0}),\mu(X^{0})) = d_{\mathrm{BL}}(\mu_{\varepsilon}^{k}(X^{0}),\mu_{0}^{1}(X^{0}))\\
&\qquad\le d_{\mathrm{BL}}(\mu_{\varepsilon}^{k}(X^{0}),\mu_{\varepsilon}^{k-1}(X^{0})) + d_{\mathrm{BL}}(\mu_{\varepsilon}^{k-1}(X^{0}),\mu_{0}^{1}(X^{0})).
\end{aligned}
\tag{3.19}
$$

We thus have

$$
\begin{aligned}
&(1-\theta^{'})d_{\mathrm{BL}}\big(\mu(X^{\varepsilon}),\mu(X^{0})\big)\\
&\qquad\le d_{\mathrm{BL}}(\mu_{\varepsilon}^{k-1}(L_{\varepsilon}(X^{0})),\mu_{\varepsilon}^{k-1}(X^{0})) + d_{\mathrm{BL}}(\mu_{\varepsilon}^{k-1}(X^{0}),\mu_{0}^{1}(X^{0}))\\
&\qquad\le d_{\mathrm{BL}}(\mu_{\varepsilon}^{1}(X^{0}),\mu(X^{0})) + d_{\mathrm{BL}}(\mu_{\varepsilon}^{k-1}(X^{0}),\mu_{0}^{1}(X^{0}))\\
&\qquad= d_{\mathrm{BL}}(\mu_{\varepsilon}^{1}(X^{0}),\mu_{0}^{1}(X^{0})) + d_{\mathrm{BL}}(\mu_{\varepsilon}^{k-1}(X^{0}),\mu_{0}^{1}(X^{0}))\\
&\qquad\le (k-1)d_{\mathrm{BL}}(\mu_{\varepsilon}^{1}(X^{0}),\mu_{0}^{1}(X^{0}))
\end{aligned}
$$

the last inequality being obtained by finite induction, repeating the calculation of (3.19). Thus, using the result of the first step,

$$\lim_{\varepsilon\to 0} d_{\mathrm{BL}}\big(\mu(X^{\varepsilon}),\mu(X^{0})\big)= 0.$$

Finally, by [25, Theorem 4.1] we deduce that the mild solution (X^{0}) of (3.11) is a stationary process. $\square$

Chapitre 4

Strongly Continuous Markov Semigroups and Almost Periodicity

4.1 Introduction

Let $K(t, x, D)$ be a Markov transition function. Here x is a point in a state space $\mathbb{E}$, D is an element of a σ-algebra $\mathfrak{B}(\mathbb{E})$ of subsets of $\mathbb{E}$ and t can be interpreted as time. We can associate with K a semigroup of operators in at least two different ways : we can define a semigroup $(S_t)_{t\geq 0}$ on the Banach space of measures on $\mathfrak{B}(\mathbb{E})$ with bounded variation, $\mathcal{M}(\mathbb{E})$ by

$$[S_t\mu](D) = \int_{\mathbb{E}} K(t, x, D)\mu(dx), \ \mu \in \mathcal{M}(\mathbb{E}) \text{ and } D \in \mathfrak{B}(\mathbb{E}),$$

[34, X.8]. Secondly, we can define a semigroup $(T_t)_{t\geq 0}$ on the Banach space of bounded measurable functions on $\mathbb{E}$, $BM(\mathbb{E})$ by

$$[T_t f](x) = \int_{\mathbb{E}} f(y)K(t, x, dy), \ f \in BM(\mathbb{E}),$$

[42, Sec.1.9]. Both semigroups satisfy a duality relation

$$\int_{\mathbb{E}} [T_t f](x)\mu(dx) = \int_{\mathbb{E}} f(x)(S_t\mu(dx)).$$

In 1978, H. Bart and S. Goldberg [6] characterized the almost periodicity of C_0-semigroups. By using this characterization, we study here the link between almost periodicity of $(T_t)_{t\geq 0}$ and that of $(S_t)_{t\geq 0}$. But for this, it is necessary that both semigroups be C_0. Unfortunately, neither $S_t\mu$ nor $T_t f$ are continuous in t in general. However, some conditions for $S_t\mu$ and $T_t f$ to be continuous can be found [78, 55, 46, 47], under which we undertake this study.

4.2 Definitions and Preliminaries

Almost periodic C_0-semigroup
Let $\mathbb{E}$ be a (complex) Banach space and $\|.\|$ its norm, and let $f : \mathbb{R}^+ \to \mathbb{E}$ be a continuous function. We say that f is *almost periodic* if for every $\varepsilon > 0$, there exists a number $l(\varepsilon) > 0$ such that every subinterval of $\mathbb{R}^+$ of length $l(\varepsilon) > 0$ contains at least a real number τ for which

$$\|f(t+\tau) - f(t)\| \leq \varepsilon \text{ for all } t \in \mathbb{R}^+.$$

The function f is weakly almost periodic if, for each x^* in the conjugate space $\mathbb{E}^*$, the function $x^* \circ f$, mapping $\mathbb{R}^+$ into the complex plane $\mathbb{C}$, is almost periodic.

A C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space $\mathbb{E}$ is *almost periodic* if for each $x \in \mathbb{E}$, the map $t \to T_t(x)$ from $\mathbb{R}^+$ to $\mathbb{E}$ is almost periodic. We say that $(T_t)_{t\geq 0}$ is *weakly almost periodic* if for each x^* in the conjugate space $\mathbb{E}^*$ and for each $x \in \mathbb{E}$ the map $t \to x^*(T_t(x))$ from $\mathbb{R}^+$ into $\mathbb{C}$ is almost periodic.

Topologies on the space of measures on $\mathbb{E}$
Let $\mathfrak{B}(\mathbb{E})$ be the σ-algebra of Borel subsets of $\mathbb{E}$, and let $\mathcal{M}(\mathbb{E})$ to be the linear space of Borel measures of variable sign with bounded variation. Let $\|.\|_{TV}$ denote the total variation norm on $\mathcal{M}(\mathbb{E})$. It is known that $\mathcal{M}(\mathbb{E})$ endowed with $\|.\|_{TV}$ is a Banach space.

Let $BL(\mathbb{E})$ be the Banach space of bounded Lipschitz functions from $\mathbb{E}$ to $\mathbb{R}$, endowed with norm $\|.\|_{BL}$, where, for $f \in BL(\mathbb{E})$, $\|f\|_{BL}$ is defined by : $\|f\|_{BL} := \max(\|f\|_\infty, \|f\|_L)$, $\|f\|_\infty = \sup_{x\in\mathbb{E}} |f(x)|$ and $\|f\|_L = \sup\Big\{\dfrac{|f(x)-f(y)|}{\|x-y\|}, ;\, x \neq y\Big\}$. We shall write $\|.\|^*_{BL}$ to denote the dual norm on $BL(\mathbb{E})^*$. Each $\mu \in \mathcal{M}(\mathbb{E})$ defines a unique element I_μ in $BL(\mathbb{E})^*$ ($I_\mu(f) := \int_{\mathbb{E}} f d\mu$, for each $f \in BL(\mathbb{E})$). The Dirac functions $\delta_x(f) = f(x)$ for $x \in \mathbb{E}$ are in $BL(\mathbb{E})^*$. Let

$$D := \Big\{\sum_{k=1}^{n} \alpha_k \delta_{x_k} : n \in \mathbb{N}, \alpha_k \in \mathbb{R}, x_k \in \mathbb{E}\Big\}.$$

We define $\mathcal{S}_{BL}$ as the closure of the linear subspace D in $BL(\mathbb{E})^*$ with respect to $\|.\|^*_{BL}$. The dual space $\mathcal{S}^*_{BL}$ of $\mathcal{S}_{BL}$ is isometrically isomorphic to $BL(\mathbb{E})$.

By [47, Corollary 3.10], $\mathcal{M}(\mathbb{E})$ is a $\|.\|^*_{BL}$-dense subspace of $\mathcal{S}_{BL}$ ($\mathcal{M}(\mathbb{E})$ endowed with $\|.\|^*_{BL}$ is not a complete space and $\mathcal{S}_{BL}$ is his completion). In the sequel when $\mathcal{M}(\mathbb{E})$ is equipped with the $\|.\|^*_{BL}$-topology, we write $\mathcal{M}(\mathbb{E})_{BL}$ and when we use the $\|.\|_{TV}$-topology, we write $\mathcal{M}(\mathbb{E})_{TV}$. Since $\|\mu\|^*_{BL} \leq \|\mu\|_{TV}$, $\mathcal{M}(\mathbb{E})_{TV}$ embeds continuously into $\mathcal{M}(\mathbb{E})_{BL}$ (for more details see [46, 47]).

For each $b \geq 0$ and each family $(K_\delta)_{\delta>0}$ of compact subsets of $\mathbb{E}$ let

$$\mathcal{K}(b,(K_\delta)) = \big\{\mu \in \mathcal{M}(\mathbb{E}) : |\mu|\,[\mathbb{E}] \leq b \text{ and } |\mu|\,[\mathbb{E}\setminus K_\delta] \leq \delta \text{ for all } \delta > 0\big\}.$$

The sets $\mathcal{K}(b,(K_\delta)) \subset \mathcal{M}(\mathbb{E})_{BL}$ are convex and compact. We denote by $\mathcal{K}_{\mathbb{E}}$ the collection of all sets $\mathcal{K}(b,(K_\delta))$.

Lemma 4.2.1. *[29, Lemma 3] Let $B \subset \mathcal{M}(\mathbb{E})$ be a non-empty subset with the following properties :*

(i)

$$\sup_{\mu \in B} |\mu(f)| < +\infty \text{ for all } f \in C_b(\mathbb{E}),$$

where $C_b(\mathbb{E})$ is the Banach space of continuous and bounded functions defined on $\mathbb{E}$.

(ii) For each sequence of positive functions $f_n \in C_b^+(\mathbb{E}), n \in \mathbb{N}$, such that $f_n(x) \downarrow 0$ for all $x \in \mathbb{E}$ as $n \to +\infty$ the relation

$$\sup_{\mu \in B} |\mu(f_n)| \to 0 \text{ as } n \to +\infty$$

holds.

Then $B \subset K$ for some set $K \in \mathcal{K}_{\mathbb{E}}$.

4.3 Markov transition semigroups

From now on we will assume that the space $\mathbb{E}$ is compact separable metric space. Let $C = C(\mathbb{E})$ be the space of continuous functions with supremum norm.

Markov transition functions A function

$$K : \begin{cases} \mathbb{R}^+ \times \mathbb{E} \times \mathfrak{B}(\mathbb{E}) & \to \mathbb{R} \\ (t, x, D) & \mapsto K(t, x, D) \end{cases}$$

is called a *Markov transition function* if, for all $t, s \geq 0$, for all $D \in \mathfrak{B}(\mathbb{E})$ and for all $x \in \mathbb{E}$,

1. $K(t, x, .)$ is a non-negative measure on $\mathfrak{B}(\mathbb{E})$.
2. $K(t, ., D)$ is a $\mathfrak{B}(\mathbb{E})$-measurable function.
3. $K(0, x, D) = 1$ if $x \in D$ and $K(0, x, D) = 0$ if $x \in \mathbb{E} \setminus D$.
4. There exists a positive constant M such that $K(t, x, \mathbb{E}) \leq M$.
5. K satisfies the Chapman-Kolmogorov equation, i.e. $K(t+s, x, D) = \int_{\mathbb{E}} K(t, y, D) K(s, x, dy)$ for all.

The Markov transition function K induces two operators semigroups :

(a)

$$S_t : \begin{cases} \mathcal{M}(\mathbb{E}) & \to \mathcal{M}(\mathbb{E}) \\ \mu & \mapsto S_t\mu \end{cases}$$

which is defined by

$$[S_t\mu](D) = \int_{\mathbb{E}} K(t, x, D)\mu(dx), \ \mu \in \mathcal{M}(\mathbb{E}) \text{ and } D \in \mathfrak{B}(\mathbb{E}),$$

(b)

$$T_t : \begin{cases} MB(\mathbb{E}) & \rightarrow & MB(\mathbb{E}) \\ f & \mapsto & T_t f \end{cases}$$

which is defined by

$$[T_t f](x) = \int_{\mathbb{E}} f(y)K(t,x,dy), \;\; f \in BM(\mathbb{E})$$

where $BM(\mathbb{E})$ is the Banach space of bounded complex valued measurable functions defined on $\mathbb{E}$.

Strongly continuous Markov transition semigroup To study the almost periodicity of $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$, it is necessary that both semigroups be strongly continuous (i.e. C_0-semigroups). Unfortunately, neither $(T_t)_{t\geq 0}$ nor $(S_t)_{t\geq 0}$ are strongly continuous in t. However, some conditions for $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ to be continuous in t concerning K can be found :

Remarque 4.1. The restriction of $(T_t)_{t\geq 0}$ to C is a C_0-semigroup if and only if the following are satisfied.

– For $t > 0$ and $x_0 \in \mathbb{E}$ fixed, $K(t,x,.) \rightarrow K(t,x_0,.)$ weakly as $x \rightarrow x_0$.
– For every $x \in \mathbb{E}$ and for every neighborhood U of x, $K(t,x,U) \rightarrow 1$ as $t \downarrow 0$.

In the sequel we assume that the Markov transition function K satisfies the conditions of Remark 4.1 and let $(T_t)_{t\geq 0}$ still denote the C_0-semigroup on C.

Lemma 4.3.1. *For each fixed measure $\mu \in \mathcal{M}(\mathbb{E})$ there exists $B \in \mathcal{K}_{\mathbb{E}}$ such that the family $\{S_t\mu\}_{t\geq 0}$ is contained in B.*

Proof Let us show that both conditions of Lemma 4.2.1 are satisfied.

Let $f \in C$. We have

$$\begin{aligned} \sup_t |[S_t\mu](f)| &= \sup_t \left| \int_{\mathbb{E}} \Big(\int_{\mathbb{E}} f(y)K(t,x,dy) \Big) \mu(dx) \right| \\ &\leq \sup_t \int_{\mathbb{E}} \left| \int_{\mathbb{E}} f(y)K(t,x,dy) \right| |\mu|(dx) \\ &\leq \sup_t \int_{\mathbb{E}} M' K(t,x,\mathbb{E}) |\mu|(dx) \leq M'M |\mu|(\mathbb{E}) < \infty, \end{aligned}$$

where $M' = \sup f(x)$. By the uniform boundedness principle, we deduce that

$$\sup_t |[S_t\mu](f)| < \infty \text{ for all } f \in C.$$

Let $(f_n)_{n\in\mathbb{N}} \in C^+$ be a decreasing sequence such that $f_n(x) \downarrow 0$ for all $x \in \mathbb{E}$ as $n \rightarrow \infty$ i.e. for every $\varepsilon > 0$, there exists $N(\varepsilon, x) \in \mathbb{N}$ such that for each $n \geq N(\varepsilon, x)$, $f_n(x) < \varepsilon$. By Dini's theorem, the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent in $\mathbb{E}$.

Let us show that $\sup_t |[S_t\mu](f_n)| \to 0$ as $n \to \infty$.

$$|[S_t\mu](f_n)| = \left| \int_{\mathbb{E}} f_n(y)[S_t\mu](dy) \right| \leq \varepsilon M\, |\mu|\, (\mathbb{E}).$$

Thus both conditions of Lemma 4.2.1 are satisfied. □

Theorem 4.3.2. *If $(T_t)_{t\geq 0}$ is a C_0-semigroup on C, the map $t \mapsto S_t\mu$ is continuous from $\mathbb{R}^+$ to $\mathcal{M}_{BL}(\mathbb{E})$ for every $\mu \in \mathcal{M}(\mathbb{E})$.*

Proof Through

$$< \mu, f >= \int_{\mathbb{E}} f d\mu, f \in C, \mu \in \mathcal{M}(\mathbb{E}),$$

the space C can be identified with a subspace of $\mathcal{M}^*(\mathbb{E})$. We see that the semigroup of dual operators $(S_t^*)_{t\geq 0}$ leaves C invariant and the restriction of $(S_t^*)_{t\geq 0}$ to C coincides with $(T_t)_{t\geq 0}$. The map $t \mapsto T_t(f)$ is continous for every $f \in C$ thus it is weakly continuous, and since the dual space C^* can be identified with the Banach space $\mathcal{M}(\mathbb{E})_{TV}$ [21], the map $t \mapsto< \mu, T_t f >=< \mu, S_t^* f >=< S_t\mu, f >$ is continuous for every $\mu \in \mathcal{M}(\mathbb{E})$ and every $f \in C$. From [46, Lemma 3.5], we deduce that the map $t \mapsto S_t\mu$ is continuous from $\mathbb{R}^+$ to $\mathcal{M}_{BL}(\mathbb{E})$ for every $\mu \in \mathcal{M}(\mathbb{E})$. □

Remarque 4.2.
- The subspace, denoted $\mathcal{M}(\mathbb{E})^0_{TV}$, of $\mathcal{M}(\mathbb{E})$ (which is the set of all measure $\mu \in \mathcal{M}(\mathbb{E})$ for which the map $t \to S_t\mu$ is continuous for $\|.\|_{TV}$) is dense in $\mathcal{M}(\mathbb{E})$ for the $\mathcal{S}_{BL}$-topology. The restriction of (S_t) to $\mathcal{M}(\mathbb{E})^0_{TV}$ is a C_0-semigroup, denoted by $(S_t^0)_{t\geq 0}$.
- The semigroup $(S_t)_{t\geq 0}$ is not in general extendable to a semigroup on $\mathcal{S}_{BL}$, see e.g. [78]. The following result (which is a direct consequence of Proposition 3.3.3 of [78]) gives necessary and sufficient conditions for $(S_t)_{t\geq 0}$ to be extendable to a C_0-semigroup on $\mathcal{S}_{BL}$. We denote this semigroup by $\widetilde{(S_t)}_{t\geq 0}$.

Proposition 4.3.3. *The family of bounded linear operators $(S_t)_{t\geq 0}$ ($S_t : \mathcal{M}_{BL}(\mathbb{E}) \to \mathcal{M}_{BL}(\mathbb{E})$) can be extended to a C_0-semigroup $\widetilde{(S_t)}_{t\geq 0}$ on $\mathcal{S}_{BL}$ if and only if the semigroup $(T_t)_{t\geq 0}$ leaves $BL(\mathbb{E})$ invariant and there exists a constant $L > 0$ such that*

$$\|T_t f\|_{BL} \leq L \|f\|_{BL}, \tag{4.1}$$

for all $f \in BL(\mathbb{E})$ and $t \geq 0$

Proof Suppose that $(S_t)_{t\geq 0}$ is extendable to a C_0-semigroup $\widetilde{(S_t)}_{t\geq 0}$ on $\mathcal{S}_{BL}$. Let $\mu \in \mathcal{M}(\mathbb{E})$, then for all $f \in BL(\mathbb{E}) \cong \mathcal{S}_{BL}^*$ we have

$$< S_t\mu, f >=< \widetilde{(S_t)}\mu, f >=< \mu, \widetilde{S_t^*} f >=< \mu, T_t f > .$$

Since this holds for all $\mu \in \mathcal{M}(\mathbb{E})$, $\widetilde{S_t^*}$ is the restriction of T_t to $BL(\mathbb{E})$, consequently, T_t leaves $BL(\mathbb{E})$ invariant and (4.1) is satisfied with $L = \sup_t \left\| \widetilde{S_t^*} \right\|$.

Assume now that $(T_t)_{t\geq 0}$ leaves $BL(\mathbb{E})$ invariant and that (4.1) is satisfied. Let $\mu \in \mathcal{S}_{BL}$, then there exists $(\mu_n)_n \subset \mathcal{M}(\mathbb{E})$ such that $\|\mu_n - \mu\|_{BL}^* \to 0$ as $n \to \infty$. For all $m, n \in \mathbb{N}$ we have

$$\|S_t\mu_n - S_t\mu_m\|_{BL}^* = \sup_{\|f\|_{BL}\leq 1} |< \mu_n - \mu_m, T_t f >| \leq L \|\mu_n - \mu_m\|_{BL}^*,$$

for all $t \geq 0$, hence $(S_t\mu_n)$ is Cauchy in $\mathcal{S}_{BL}$ uniformly with respect to t. Thus we can define $\widetilde{S}_t\mu := \lim S_t\mu_n$ and this limit exists uniformly with respect to t. Since the map $t \to S_t\mu$ is continuous from $\mathbb{R}^+$ to $\mathcal{M}(\mathbb{E})_{BL}$ (Theorem 4.3.2), the map $t \mapsto \widetilde{(S_t)}\mu$ is continuous from $\mathbb{R}^+$ into $\mathcal{S}_{BL}$ for every $\mu \in \mathcal{S}_{BL}$ and $\left\|\widetilde{S}_t\mu\right\|_{BL}^* \leq L\|\mu\|_{BL}^*$ for all $\mu \in \mathcal{S}_{BL}$ and $t \geq 0$ which means that $\widetilde{(S_t)}$ is a C_0-semigroup. □

4.4 Almost periodicity of Markov transition semigroups

In this section we suppose that $(S_t)_{t\geq 0}$ is extendable to a C_0-semigroup $\widetilde{(S_t)}_{t\geq 0}$ on $\mathcal{S}_{BL}$. The almost periodicity of $(S_t)_{t\geq 0}$ and $\widetilde{(S_t)}_{t\geq 0}$ is defined relatively to the norm $\|.\|_{BL}^*$

Proposition 4.4.1. *The C_0-semigroup $\widetilde{(S_t)}_{t\geq 0}$ is almost periodic if and only if the function $t \to S_t\mu$ (from $\mathbb{R}^+$ into $\mathcal{M}(\mathbb{E})_{BL}$) is almost periodic for each $\mu \in \mathcal{M}(\mathbb{E})$*

Proof Obviously, if $\widetilde{(S_t)}_{t\geq 0}$ is almost periodic then the function $t \to S_t\mu$ is almost periodic for each $\mu \in \mathcal{M}(\mathbb{E})$.

Assume that the function $t \to S_t\mu$ is almost periodic for each $\mu \in \mathcal{M}(\mathbb{E})$. Let $\mu \in \mathcal{S}_{BL}$, there exists $(\mu_n)_n \subset \mathcal{M}(\mathbb{E})$ such that $\|\mu_n - \mu\|_{BL}^* \to 0$ as $n \to \infty$. Using the same reasoning as in the proof of Proposition 4.3.3, we show that the sequence of functions $\{S_t\mu_n; n \in \mathbb{N}\}$ converges to $\widetilde{S}_t\mu$ uniformly with respect to t. Since, for each $n \in \mathbb{N}$, the function $t \to S_t\mu_n$ is almost periodic, then the map $t \to \widetilde{S}_t\mu$ is also almost periodic for each $\mu \in \mathcal{S}_{BL}$, which means that the C_0-semigroup $\widetilde{(S_t)}_{t\geq 0}$ is almost periodic. □

In the sequel, A denotes the infinitesimal generator of $(T_t)_{t\geq 0}$, $\widetilde{A}$ the infinitesimal generator of $\widetilde{(S_t)}_{t\geq 0}$ and $\widetilde{A}_{TV}$ the infinitesimal generator of $(S_t^0)_{t\geq 0}$.

Theorem 4.4.2. *The C_0-semigroup $(T_t)_{t\geq 0}$ is almost periodic if and only if $(S_t)_{t\geq 0}$ is almost periodic .*

Proof To start with, let us recall a few well-known results.

The following result due to Bart and Goldberg [6] gives a characterization of almost periodic C_0-semigroups. The spectrum of the generator A is denoted $\sigma(A)$.

Theorem 4.4.3. *[6, Theorem 9] A C_0-semigroup $(T_t)_{t\geq 0}$ is almost periodic if and only if the following three conditions are satisfied*

1. *$(T_t)_{t\geq 0}$ is uniformly bounded,*
2. *$\sigma(A) \subset i\mathbb{R}$,*
3. *the set of eigenvectors of A spans a dense subspace in $\mathbb{E}$*

The following theorems are well known from ergodic theory. Let $\mathcal{N}(A)$ denote the kernel of the infinitesimal generator A of the C_0-semigroup $(T_t)_{t\geq 0}$ and let $\mathcal{R}(A)$ denote the range of A.

A C_0-semigroup $(T_t)_{t\geq 0}$ on Banach space $\mathbb{E}$ is said to be *mean ergodic* if the limit

$$P(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t T_s(x)ds$$

exists for all $x \in \mathbb{E}$

Theorem 4.4.4. *[32, Theorem 4.5] Let $(T_t)_{t\geq 0}$ be a C_0-semigroup with infinitesimal generator A on a Banach space $\mathbb{E}$. If $(T_t)_{t\geq 0}$ is uniformly bounded, then the following assertions are equivalent*

1. *$(T_t)_{t\geq 0}$ is mean ergodic.*
2. *The mean $\frac{1}{t}\int_0^t T_s ds$ converges in the weak operator topology as $t \to \infty$.*

Theorem 4.4.5. *[32, Lemma 4.4] or [45, theorem 18.4.3 and 18.6.2] Let $(T_t)_{t\geq 0}$ be a mean ergodic semigroup and uniformly bounded. Then P is a projection onto $\mathcal{N}(A)$ along $\overline{\mathcal{R}(A)}$.*

Proof of Theorem 4.4.2

Necessary condition. Assume that $(S_t)_{t\geq 0}$ is almost periodic, then, by Proposition 4.3.3, the C_0-semigroup $\widetilde{(S_t)}_{t\geq 0}$ is almost periodic. Let us show that the three conditions of Theorem 4.4.3 are satisfied.

By definition of a Markov transition function we deduce that $(T_t)_{t\geq 0}$ is bounded and the uniform boundedness principle implies that $(T_t)_{t\geq 0}$ is uniformly bounded, thus Condition 1 of Theorem 4.4.3 is satisfied.

Since the dual space C^* can be identified with the Banach space $\mathcal{M}(\mathbb{E})_{TV}$ and $\widetilde{(S_t)}_{t\geq 0}$ is almost periodic, we have

$$\sigma(A) = \sigma(A^*) = \sigma(\widetilde{A}_{TV}) \subset \sigma(\widetilde{A}) \subset i\mathbb{R}$$

(see [32, Proposition 2.18]). Thus Condition 2 of Theorem 4.4.3 is satisfied.

Let us now show that the set of eigenvectors V of A spans a subspace dense in C. Let $\mu \in C^*$ ($\cong \mathcal{M}(\mathbb{E})_{TV}$) and let $f \in C$, we have

$$\mu(\frac{1}{t}\int_0^t (T_s f)ds) = \frac{1}{t}\int_0^t (S_s\mu(f))ds.$$

Since $\widetilde{(S_t)}_{t\geq 0}$ is almost periodic, the limit

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (S_s\mu f)ds$$

exists. By Theorem 4.4.4, the limit

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (T_s(f))ds$$

exists also. Using the same reasoning, we can show that

$$M(r,f)=\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-irs}(T_s(f))ds$$

exists. It is straightforward to see that the operator $(A-ir)$ is the infinitesimal generator of the C_0-semigroup $(e^{-irt}T_t)_{t\geq 0}$. By [32, Lemma 4.4], we obtain

$$\mathcal{N}(A-ir)=\{M(r,f), f\in C\}, r\in\mathbb{R}.$$

Let $\mu\in C^*$ ($\cong \mathcal{M}(\mathbb{E})_{TV}$) such that $\mu(V)=0$, since $M(r,f)\in V$ for all $r\in\mathbb{R}$, we conclude

$$0=\mu(M(r,f))=\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-irs}(S_s\mu(f))ds.$$

Since the map $s\to S_s\mu$ is almost periodic for $\|.\|_{BL}$, by Lemma 4.3.1 and applying [29, Theorem 2], we deduce that the function $s\to S_s\mu(f)$ is almost periodic for each $f\in C$, therefore the Fourier coefficients of the function $s\to S_s\mu(f)$ vanish. By the uniqueness of these coefficients, we have $S_s\mu(f)=0$ for all $s\in\mathbb{R}$, in particular for $s=0, \mu(f)=S_0\mu(f)=0$ for all $f\in C$, thus $\mu\equiv 0$, which implies that V spans a subspace dense in C. Thus Condition 3 of Theorem 4.4.3 is satisfied.

Sufficient condition. Assume that $(T_t)_{t\geq 0}$ is almost periodic. We show that for each $\mu\in\mathcal{S}_{BL}$ the map $t\to\widetilde{(S_t)}\mu$ is almost periodic.

If $\mu=\delta_{x_0},\ x_0\in\mathbb{E}$, then $\widetilde{(S_t)}\mu(f)=S_t\mu(f)=T_tf(x_0)$ is almost periodic for all $f\in C$. Thus, If $\mu=\sum_1^n\alpha_i\delta_{x_i}$, then $\widetilde{(S_t)}\mu(f)=S_t\mu(f)=\sum_1^n\alpha_iT_tf(x_i)$ is almost periodic for all $f\in C$.

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of linear combinations of Dirac measures such that $\mu_n\xrightarrow{\|\cdot\|_{BL}}\mu$. As seen above, the sequence $(\widetilde{(S_t)}\mu_n)$ converge uniformly to $\widetilde{(S_t)}\mu$, therefore the map $t\to\widetilde{(S_t)}\mu$ is almost periodic.

$\square$

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