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# THÉSE DE DOCTORAT
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**Spécialité** : MATHÉMATIQUES

## Titre de la thèse

« Processus Weyl presque périodique et équations différentielles stochastiques »

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# Résumé

La thèse est dédiée à l'étude de certaines équations différentielles à coefficients Weyl presque périodiques. Elle contient deux parties essentielles :

La première partie est consacrée à des problèmes déterministes. On y étudie l'existence et l'unicité d'une solution mild bornée Weyl presque périodique pour l'équation différentielle linéaire abstraite

$$u'(t) = Au(t) + f(t),\ \ t \in \mathbb{R},$$

dans un espace de Banach $\mathbb{X}$, où $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ est un opérateur linéaire (non borné) qui génère un $C_0$-semi-groupe exponentiellement stable et $f : \mathbb{R} \to \mathbb{X}$ est une fonction Weyl presque périodique. Finalement, toujours dans la première partie, nous étudions l'existence et l'unicité d'une solution mild bornée Weyl presque périodique pour l'équation différentielle semi-linéaire abstraite

$$u'(t) = Au(t) + f(t, u(t)),\ \ t \in \mathbb{R},$$

où $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}$ est une fonction Weyl presque périodique en $t \in \mathbb{R}$ uniformément par rapport aux compacts de $\mathbb{X}$.

Dans la deuxième partie, nous généralisons ces études au cas stochastique. Précisément, nous étudions l'existence et l'unicité de solution Weyl presque périodique en loi pour une classe d'équations différentielles stochastiques semi-linéaires, dans un espace de Hilbert séparable.

# Abstract

The thesis deals essentialy with a class of abstract differential equations with Weyl almost periodic coefficients, and comprises two part. The first part is devoted to the deterministic problems, in a first step, we study the existence and uniqueness of bounded Weyl almost periodic solution to the linear abstract differential equation

$$u'(t) = A\ u(t) + f(t),\ \ t \in \mathbb{R},$$

in a Banach space $\mathbb{X}$, where $A : D\,(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear (unbounded) operator which generates an exponentially stable $C_0$-semigroup on $\mathbb{X}$ and $f : \mathbb{R} \to \mathbb{X}$ is a Weyl almost periodic function. Finally, in a second step, always in the same frame, we consider the semi-linear diffential equation

$$u'(t) = Au(t) + f(t, u(t)),\ \ t \in \mathbb{R},$$

where $f(t,u)$ is a Weyl almost periodic in $t \in \mathbb{R}$ uniformly with respect compact subsets of $\mathbb{X}$.

The second part, is concerned with the stochastic case. Precisely, we examine the existence and uniqueness of Weyl almost periodic solution in law to the abstract semilinear stochastic evolution equation on a Hilbert separable space.

**Key words :** linear and semilinear differential equation, semilinear stochastic differential equation, mild solution, Weyl almost periodic, Weyl almost periodic in pth mean, Weyl almost periodic in quadratic mean, Weyl almost periodic in distribution, Weyl almost periodic in path distribution, Weyl almost periodic in probability, Nemytskii operator.

# Table des matières

# Introduction

Historiquement, la théorie des fonctions presque périodiques a été initiée par Harald Bohr en 1923 [Boh25] en tant que généralisation structurelle des fonctions périodiques continues. Cette théorie a été largement développée par plusieurs mathématiciens. En particulier, nous pouvons mentionner le nom de Stepanov [Ste26] qui a donné la première généralisation structurelle, réussissant à supprimer la restriction de continuité. D'autres généralisations ont suivi, celle de structurelle Weyl [Wey27], et celle plus large donnée par Besicovitch [Bes32], englobant toutes les autres classes. Cette dernière généralisation n'est cependant pas structurelle, au sens que la somme de deux fonctions Besicovitch presque périodiques n'est pas nécessairement Besicovitch presque périodique.

D'autres généralisations ont été considérées, notamment celle de Bochner [Boc27], où cette théorie a été étendue au cas des fonctions à valeurs dans espace de Banach.

Ensuite, Slutsky [Slu38] a introduit le concept de presque périodicité pour les processus aléatoires et a étudié sous certaines conditions la presque périodicité au sens de Besicovitch pour les trajectoires des processus aléatoires faiblement stationnaires. Puis, toujours dans le contexte des processus stochastiques, au cours des dernières décennies cette notion a été largement développée. Dans le cas des fonctions presque périodiques à valeurs dans un espace de toutes les mesures de probabilités sur l'espace Euclidien. Morozan et Tudor [MT89] ont introduit la presque périodicité en loi uni-dimensionnelle de processus. Dans le même temps, dans une note non publiée, Hurd, Russek and Surgailis [HS92] ont aussi introduit la presque périodicité en loi fini-dimensionnelle. Par la suite, Tudor [Tud92] a introduit la presque périodicité trajectorielle pour les processus à trajectoires continues. Il y a aussi Onicescu et Istratescu qui ont défini la presque périodicité en probabilité.

Par ailleurs, le concept de presque périodicité, pris dans différents sens, a été largement exploité dans l'étude des solutions de différents types d'équations d'évolution, en raison de ses nombreuses applications dans les phénomènes qui évoluent dans le temps, notamment en physique, mécanique céleste, systèmes dynamiques, etc. Il existe de nombreux travaux sur la nature des solutions des équations différentielles linéaires et semi-linéaires dans les cas déterministe et stochastique, où leurs coefficients sont presque périodiques dans différents sens.

La plupart des travaux sur les solutions d'équations à coefficients presque périodiques concernent le cas déterministe. Plusieurs auteurs ont montré l'existence et l'unicité des solutions presque périodiques pour des équations d'évolution abstraites à coefficients presque périodiques au sens de Bohr ou de Stepanov. En particulier, certains auteurs ont prouvé la non-existence de solution purement Stepanov presque périodique des équations différentielles ordinaires dans les espaces de Banach uniformément convexes. En ce qui concerne les solutions Weyl presque périodiques, la littérature n'est pas encore très fournie, les études existantes sont limitées à des équations différentielles particuliers.

Plus récemment, un nombre remarquable de travaux concerne le cas stochastique. Des auteurs ont montré la presque périodicité en loi pour les équations différentielles stochastiques (semi-)linéaires à coefficients Bohr et Stepanov presque périodiques. D'un autre coté, certains auteurs ont proclamé l'existence et l'unicité des solutions presque périodiques en $p$-moyenne ($p \geq 2$) pour les équations différentielles stochastiques semi-linéaires à coefficients presque périodiques de Bohr et de Stepanov. Malheureusement, il a été démontré notamment par des contre-exemples, que ce type de résultat est faux.

L'objectif de ce travail consiste à étudier le cas plus général des équations à coefficients Weyl presque périodiques. Plus exactement, nous montrons, sous certaines conditions, que la solution d'une équation différentielle semi-linéaire abstraite à coefficients Weyl presque périodiques est elle même Weyl presque périodique, et nous étendons ce résultat au cas d'une équation différentielle stochastique semi-linéaire abstraite. Nous montrons aussi que les solutions peuvent être purement Weyl presque périodiques.

La thèse est organisée en trois chapitres. Le premier chapitre est de nature introductif. Il contient des préliminaires et une brève présentation des résultats obtenus dans cette thèse.

Le deuxième chapitre est dédié à l'étude d'un problème déterministe. On y étudie l'existence et l'unicité d'une solution mild bornée Weyl presque périodique pour l'équation différentielle semi-linéaire abstraite

$$u'(t) = Au(t) + f(t, u(t)),\ \ t \in \mathbb{R},$$

dans un espace de Banach $\mathbb{X}$, où $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ est un opérateur linéaire (non borné) qui génère un $C_0$-semi-groupe exponentiellement stable et $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ est une fonction paramétrique Weyl presque périodique.

Dans le troisième chapitre, on examine la version stochastique du problème posé au chapitre précédent.

# CHAPITRE 1

# PRÉLIMINAIRES ET PRÉSENTATION DES RÉSULTATS

Dans ce chapitre, nous donnons quelques définitions et propriétés concernant les différentes notions de presque périodicité des fonctions à valeurs dans un espace de Banach. Nous rappelons aussi quelques résultats utiles pour les chapitre suivants. Nous terminons par la présentation des résultats principaux de ce travail.

## 1.1 Presque périodicité dans différents sens pour les fonctions déterministes

Dans tout le chapitre, $p \geq 0$ est un nombre réel, et $\mathbb{X}$ un espace de Banach muni de sa norme $\|.\|$.

**Définition 1.1.1** Un ensemble $A \subset \mathbb{R}$ est dit *relativement dense*, s'il existe un nombre réel strictement positif $l$ (longueur d'inclusion) tel que tout intervalle de longueur $l$ contient au moins un élément de $A$.

**Exemple 1.1.2** L'ensemble des nombres relatifs $\mathbb{Z}$ est *relativement dense* parce que tout intervalle de longueur supérieur ou égale à 2 contient un élément de $\mathbb{Z}$.

**Définition 1.1.3** Une fonction continue $f : \mathbb{R} \longrightarrow \mathbb{X}$ est dite *presque périodique au sens de Bohr* (ou simplement, *presque périodique*) si, pour tout $\varepsilon > 0$, l'ensemble

$$T(\varepsilon, f) := \left\{ \tau \in \mathbb{R}; \sup_{x \in \mathbb{R}} \|f(x+\tau) - f(x)\| < \varepsilon \right\},$$

est relativement dense. Le nombre $\tau$ est appelé $\varepsilon$-presque période de $f$. L'espace de toutes les fonctions presque périodiques de $\mathbb{R}$ dans $\mathbb{X}$ est noté par $\mathrm{AP}(\mathbb{R}, \mathbb{X})$.

**Exemple 1.1.4** 1. Toutes les fonctions continues périodiques sont Bohr presque périodiques.

2. La fonction $f : \mathbb{R} \longrightarrow \mathbb{R}$ définie par $f(x) = \cos x + \cos\sqrt{2}x$ est Bohr presque périodique mais n'est pas périodique.

Pour définir la presque périodicité au sens de Bohr nous nous plaçons dans la classe des fonctions continues et nous utilisons la norme uniforme. Il existe des généralisations des fonctions presque périodiques qui ne sont pas nécessairement continues et pour les définir nous avons besoin des normes suivantes.

### 1.1.1 La norme de Stepanov et la norme de Weyl

Soit $l > 0$. La *norme de Stepanov* associée à $l$ d'une fonction localement intégrable $f : \mathbb{R} \to \mathbb{X}$, i.e $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ $(p \geq 1)$, est définie par

$$\|f\|_{\mathbb{S}^p_l} = \sup_{\xi\in\mathbb{R}} \left( \frac{1}{l} \int_{\xi}^{\xi+l} \|f(t)\|^p \, dt \right)^{\frac{1}{p}}.$$

La *norme de Weyl* est donnée par

$$\|f\|_{\mathbb{W}^p} = \lim_{l\to+\infty} \|f\|_{\mathbb{S}^p_l}. \tag{1.1.1}$$

La distance induite par $\|.\|_{\mathbb{S}^p_l}$ (respectivement $\|f\|_{\mathbb{W}^p}$) est noté par $D_{\mathbb{S}^p_l}$ (respectivement $D_{\mathbb{W}^p}$).

**Proposition 1.1.1** *[Bes32]*

1. *Toutes les normes de Stepanov sont équivalentes. Cela provient de la propriété : pour tous $l_2 > l_1 > 0$ il existe toujours $n \in \mathbb{N}^*$ tel que $nl_1 < l_2 < (n+1)l_1$. Cette dernière permet d'avoir les inégalités suivantes :*

$$\left(\frac{l_1}{l_2}\right)^{\frac{1}{p}} \|f\|_{\mathbb{S}^p_{l_1}} \leq \|f\|_{\mathbb{S}^p_{l_2}} \leq \left( \left(\frac{l_1}{l_2}\right)^{\frac{1}{p}} + n^{\frac{1}{p}} \right) \|f\|_{\mathbb{S}^p_{l_1}}. \tag{1.1.2}$$

2. *La limite* (1.1.1) *existe toujours.*

**Remarque 1.1.5** En conséquence de l'équivalence des normes de Stepanov, nous ne considérons que $\mathbb{S}^p_1$ dans la suite.

Correspondant à ces normes, nous introduisons les définitions suivantes.

**Définition 1.1.6** ([AP71, p.76-77], [ABG06, ABL01], [Bes32, p.77], [GKL66, p.188]) Une fonction $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ est dite *presque périodique au sens de Stepanov* (où *p-Stepanov presque périodique* si nous voulons mettre en évidence le degré $p$) si, pour chaque $\varepsilon > 0$, l'ensemble

$$T_{\mathbb{S}^p}(\varepsilon, f) = \left\{ \tau \in \mathbb{R}; \sup_{\xi\in\mathbb{R}} \left( \int_{\xi}^{\xi+1} \|f(t+\tau) - f(t)\|^p \, dt \right)^{\frac{1}{p}} < \varepsilon \right\}$$

est relativement dense. Le nombre $\tau$ est appelé une *$\varepsilon$-Stepanov presque-période.* Nous désignons l'ensemble de toutes les fonctions Stepanov presque périodiques par $\mathbb{S}^p AP(\mathbb{R}, \mathbb{X})$.

**Exemple 1.1.7**
1. Toutes les fonctions Bohr presque périodiques sont aussi Stepanov presque périodiques.
2. La fonction $f : \mathbb{R} \longrightarrow \mathbb{R}$ définie par
$$f(t) = \begin{cases} \cos x & \text{pour} \quad x \neq k\pi; \\ k & \text{pour} \quad x = k\pi, \end{cases}$$
pour tout $k \in \mathbb{Z}$, est Stepanov presque périodique mais n'est pas Bohr presque périodique (voir[ABG06, Example 6.21]).

**Définition 1.1.8** ([ABG06, ABL01], [Bes32, p.78], [GKL66, p.190]) Une fonction $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ est appelée *Weyl presque périodique* si, pour chaque $\varepsilon > 0$, il existe $l = l(\varepsilon) > 0$ tel que, l'ensemble
$$T_{\mathbb{S}^p_l}(\varepsilon, f) = \left\{ \tau \in \mathbb{R};\ \|f(.+\tau) - f(.)\|_{\mathbb{S}^p_l} := \sup_{\xi \in \mathbb{R}} \left( \frac{1}{l} \int_{\xi}^{\xi+l} \|f(t+\tau) - f(t)\|^p \, dt \right)^{\frac{1}{p}} < \varepsilon \right\} \tag{1.1.3}$$
est relativement dense. Cela signifie que pour chaque $\varepsilon > 0$, il existe $l = l(\varepsilon) > 0$ tel que $f \in \mathbb{S}^p_l AP(\mathbb{R}, \mathbb{X})$. Le nombre $\tau$ est appelé une $\varepsilon$-*Weyl presque-période* de $f$. Nous désignons l'ensemble de toutes les fonctions Weyl presque périodiques par $\mathbb{W}^p AP(\mathbb{R}, \mathbb{X})$.

**Exemple 1.1.9**
1. Toutes les fonctions Stepanov presque périodiques sont aussi Weyl presque périodiques.
2. La fonction $f : \mathbb{R} \longrightarrow \mathbb{R}$ définie par
$$f(t) = \begin{cases} 1 & \text{pour} \quad 0 < t < 1; \\ 0 & \text{sinon}, \end{cases}$$
est Weyl presque périodique mais n'est pas Stepanov presque périodique (voir [BF45, p.68]).

**Remarque 1.1.10** D'après l'inégalité (1.1.2), nous pouvons remplacer dans l'ensemble (1.1.3) de la définition (1.1.8), la norme $\|f(.+\tau) - f(.)\|_{\mathbb{S}^p_l}$ par la norme $\|f(.+\tau) - f(.)\|_{\mathbb{S}^p_L}$, pour tout $L \geq l$.

**Remarque 1.1.11**
1. Les inclusions suivantes sont vraies : $AP(\mathbb{R}, \mathbb{X}) \subset \mathbb{S}^p AP(\mathbb{R}, \mathbb{X}) \subset \mathbb{W}^p AP(\mathbb{R}, \mathbb{X})$ pour tout $p \geq 1$. De plus elles sont strictes (voir [ABG06, ABL01]).
2. Les espaces $AP(\mathbb{R}, \mathbb{X})$, $\mathbb{S}^p AP(\mathbb{R}, \mathbb{X})$ et $\mathbb{W}^p AP(\mathbb{R}, \mathbb{X})$ sont des espaces vectoriels, contrairement à l'espace des fonctions périodiques.
3. Toute fonction Stepanov presque périodique et uniformément continue est Bohr presque périodique (voir par exemple [Bes32, Théorème de Bochner. page 81 ]).
4. Dans la définition de la presque périodicité au sens de Weyl, si $\limsup_{\varepsilon \to 0} l(\varepsilon) < \infty$, alors la fonction $f$ est aussi Stepanov presque périodique (voir [Bes32]).

5. Les fonctions Stepanov presque périodiques et les fonctions Weyl presque périodiques sont toujours bornées au sens de Stepanov ainsi qu'au sens de Weyl (voir [ABG06]). Plus précisément, les espaces

$$\mathbb{BS}^p(\mathbb{R},\mathbb{X}) := \left\{ f \in L^p_{loc}(\mathbb{R},\mathbb{X});\ \sup_{x\in\mathbb{R}} \left( \int_x^{x+1} \|f(t)\|^p\, dt \right)^{\frac{1}{p}} < \infty \right\},$$

et

$$\mathbb{BW}^p(\mathbb{R},\mathbb{X}) := \left\{ f \in L^p_{loc}(\mathbb{R},\mathbb{X});\ \lim_{l\to\infty} \sup_{x\in\mathbb{R}} \left( \frac{1}{l} \int_x^{x+l} \|f(t)\|^p\, dt \right)^{\frac{1}{p}} < \infty \right\},$$

coïncident.

### 1.1.2 Fonctions Weyl presque périodiques dépendant d'un paramètre

**Définition 1.1.12** 1. Nous disons qu'une fonction paramétrique $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ est *Weyl presque périodique* si, pour chaque $x \in \mathbb{X}$, la fonction $f(.,x)$ est Weyl presque périodique. Nous notons par $\mathbb{W}^p\mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ l'espace de toutes les fonctions paramétriques Weyl presque périodiques.

2. Une fonction $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}, (t,u) \mapsto f(t,u)$ avec $f(.,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ pour chaque $u \in \mathbb{X}$ est dite *Weyl presque périodique en* $t \in \mathbb{R}$ *uniformément par rapport aux compacts de* $\mathbb{X}$ si, pour chaque $\varepsilon > 0$, il existe $l = l(\varepsilon) > 0$ et pour tout compact $\mathbb{K}$ dans $\mathbb{X}$, l'ensemble

$$T_{\mathbb{S}^p_l}(\varepsilon,f,\mathbb{K}) = \left\{ \tau \in \mathbb{R}; \sup_{u\in\mathbb{K}} \sup_{\xi\in\mathbb{R}} \left( \frac{1}{l} \int_\xi^{\xi+l} \|f(t+\tau,u) - f(t,u)\|^p\, dt \right)^{\frac{1}{p}} < \varepsilon \right\}$$

est relativement dense. Nous notons par $\mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ l'espace de toutes les fonctions Weyl presque périodiques en $t \in \mathbb{R}$ uniformément par rapport aux compacts de $\mathbb{X}$.

Il est clair que $\mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X}) \subset \mathbb{W}^p\mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

Comme nous l'avons indiqué, parmi les objectifs de ce travail figure l'étude de l'existence et de l'unicité de la solution Weyl presque périodique pour une classe d'équations différentielles stochastiques semi-linéaires. Nous avons donc besoin de définir le concept de Weyl presque périodicité pour un processus stochastique.

## 1.2 Presque périodicité au sens de Weyl des fonctions aléatoires

Soient $(\Omega,\mathcal{F},\mathrm{P})$ un espace de probabilité, $(\mathbb{E},\|.\|)$ un espace de Banach séparable et $X = (X_t)_{t\in\mathbb{R}}$ un processus stochastique à valeurs dans $\mathbb{E}$. Nous notons par law $(X(t))$ la loi de la variable aléatoire $X(t)$, par $\mathrm{E}(X)$ son espérance et par $\mathcal{M}_1^+(\mathbb{E})$ l'espace de toutes les mesures de probabilités boréliennes sur $\mathbb{E}$ muni de la topologie de la convergence étroite. Pour $f \in C_b(\mathbb{E})$, (où $C_b(\mathbb{E})$ est l'espace de Banach des fonctions continues et bornées définies sur $\mathbb{E}$) et lipschitzienne, nous définissons

$$\|f\|_{\mathrm{L}} = \sup\left\{\frac{|f(x)-f(y)|}{d_{\mathbb{E}}(x,y)}; \qquad x \neq y\right\},$$

$$\|f\|_\infty = \sup_{x\in\mathbb{E}} |f(x)|\,.$$

et

$$\|f\|_{\mathrm{BL}} = \max\left\{\|f\|_{\mathrm{L}}, \|f\|_\infty\right\}.$$

Pour $\mu, \nu \in \mathcal{M}_1^+(\mathbb{E})$, nous définissons

$$d_{\mathrm{BL}}(\mu,\nu) = \sup_{\|f\|_{\mathrm{BL}}\leq 1} \Big|\int_{\mathbb{E}} f d(\mu-\nu)\Big|\,.$$

L'espace $(\mathcal{M}_1^+(\mathbb{E}), d_{\mathrm{BL}})$ est un espace polonais (voir [Dud89, Corollaire 11.5.5]). Nous rappelons une autre distance que nous pouvons utiliser pour la convergence en loi des variables aléatoires. Soient $X, Y$ deux variables aléatoires à valeurs dans $\mathbb{E}$ définies sur le même espace de probabilité. Rappelons que la distance de Wasserstein de degré $p$ entre la loi de $X$ et la loi de $Y$ est

$$\mathbf{W}^p(\mu,\nu) = \inf\left\{\left(\mathrm{E}\big(\|X-Y\|^p\big)\right)^{1/p};\ \ \mathrm{law}(X)=\mu,\, \mathrm{law}(Y)=\nu\right\}.$$

L'espace $(\mathcal{M}_1^+(\mathbb{X}), \mathbf{W}^p)$ est aussi un espace polonais. Comme indiqué dans [Vil08, Remarque 6.5], pour chaque $\mu, \nu \in \mathcal{M}_1^+(\mathbb{E})$, nous avons

$$d_{\mathrm{BL}}(\mu,\nu) \leq d_L(\mu,\nu) = \mathbf{W}^1(\mu,\nu) \leq \mathbf{W}^p(\mu,\nu)$$

pour tout $p \geq 1$.

Maintenant, nous donnons des définitions de Weyl presque périodicité pour un processus stochastique $X = (X_t)_{t\in\mathbb{R}}$ dans des sens différents. Ces définitions sont inspirées des références [HMT88, MT89].

**Définition 1.2.1**

1. Le processus $X$ est dit *Weyl presque périodique en loi uni-dimensionnelle* si l'application $t \to \mathrm{law}\,(X(t))$ de $\mathbb{R}$ dans $(\mathcal{M}_1^+(\mathbb{E}), d_{\mathrm{BL}})$ est Weyl presque périodique c'est-à-dire, si, pour chaque $\varepsilon > 0$, il existe $k, l > 0$ tels que tout intervalle de longueur $k$ contient au moins une $\varepsilon$-Weyl presque période $\tau$ qui vérifie

$$\sup_{x\in\mathbb{R}}\left(\frac{1}{l}\int_x^{x+l} d_{\mathrm{BL}}^p(\mathrm{law}\,(X(t+\tau))\,, \mathrm{law}\,(X(t)))dt\right)^{\frac{1}{p}} < \varepsilon.$$

2. Le processus $X$ est *Weyl presque périodique en loi fini-dimensionnelle* si, pour chaque suite finie $t_1, \ldots, t_m \in \mathbb{R}$ l'application $t \mapsto \mathrm{law}\,(X(t+t_1), X(t+t_2), \ldots, X(t+t_m))$ de $\mathbb{R}$ dans $(\mathcal{M}_1^+(\mathbb{E}^n), d_{\mathrm{BL}})$ est Weyl presque périodique.

3. Si $X$ a des trajectoires dans un espace de Banach $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ , nous disons que $X$ est *Weyl presque périodique trajectorielle* si l'application $t \mapsto \text{law}\,(X(t+.))$ de $\mathbb{R}$ dans $\mathcal{M}_1^+(\mathbb{Y})$ est Weyl presque périodique, où $\mathcal{M}_1^+(\mathbb{Y})$ est muni de la distance $d_{\text{BL}}$ associée à $\|.\|_{\mathbb{Y}}$.

Soit $\mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{E})$ l'espace des applications mesurables de $\Omega$ dans $\mathbb{E}$, muni de la topologie de la convergence en probabilité. Rappelons que la topologie de $\mathrm{L}^0(\Omega, P, \mathbb{E})$ est induite par la distance

$$d_{\text{prob}}(U, V) = \mathrm{E}(\|U - V\| \wedge 1),$$

qui est complète.

On introduit aussi $\mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{E})$ l'espace des variables aléatoires à valeurs dans $\mathbb{E}$ qui sont finies en $p$-moyenne ($p \geq 1$).

**Définition 1.2.2** Le processus $X$ est dit *Weyl presque périodique en probabilité* si l'application $X : t \to \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{E})$ est Weyl presque périodique.

**Définition 1.2.3** Le processus $X \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{E}))$ est dit *Weyl presque périodique en $p$-moyenne*, si l'application $t \mapsto X(t), \mathbb{R} \to \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{E})$ est Weyl presque périodique.

Quand $p = 2$, nous disons que $X$ est *Weyl presque périodique en moyenne quadratique.*

## 1.3 Semi-groupe de classe $C_0$

Dans cette partie, nous rappelons une classe d'opérateurs, dits semi-groupes, assez utiles dans l'étude des équations différentielles abstraites. Nous trouvons une présentation complète dans, par exemple, [Vra03]. Nous notons par $L(\mathbb{X})$ l'espace de Banach des opérateurs linéaires bornés dans $\mathbb{X}$ muni de la norme d'opérateur $\|.\|_{L(\mathbb{X})}$ définie par

$$\|B\|_{L(\mathbb{X})} = \sup_{\|x\| \leq 1} \|Bx\| \quad \forall B \in L(\mathbb{X}),$$

et par $I$ l'opérateur identité de $\mathbb{X}$.

**Définition 1.3.1** Un *$C_0$-semi-groupe* sur $\mathbb{X}$ est une application $T : \mathbb{R}_+ \longrightarrow L(\mathbb{X})$ telle que

1. $T(0) = I$
2. $T(t+s) = T(t)T(s)$, pour tout $t, s \geq 0$
3. $\lim_{t \to 0} \|T(t)x - x\|_{L(\mathbb{X})} = 0$, pour tout $x \in \mathbb{X}$.

**Théorème 1.3.2** *[Vra03, Théorème page.41] Soit $(T(t))_{t \geq 0}$ un $C_0$-semi-groupe. Alors il existe des constantes $\delta > 0$ et $M \geq 1$, telles que*

$$\|T(t)\|_{L(\mathbb{X})} \leq M e^{-\delta t}.$$

**Définition 1.3.3** *Le générateur infinitésimal* du $C_0$-semi-groupe $T(t)_{t\geq 0}$ est l'opérateur $A$ défini par

$$Ax = \lim_{t\longrightarrow 0} \frac{T(t)x - x}{t} \quad \forall x \in D(A),$$

où

$$D(A) := \left\{ x \in \mathbb{X};\ \lim_{t\longrightarrow 0} \frac{T(t)x - x}{t} \text{ existe} \right\}.$$

**Définition 1.3.4** Un opérateur $A : D(A) \subseteq \mathbb{X} \longrightarrow \mathbb{X}$ est dit *fermé* si son graphe $G(A) = \{(x, y) \in D(A) \times \mathbb{X};\ y = Ax\}$ est fermé dans $\mathbb{X} \times \mathbb{X}$.

**Théorème 1.3.5** *[Vra03, Théorème page.44] Soit $A : D(A) \subseteq \mathbb{X} \longrightarrow \mathbb{X}$ le générateur infinitésimal d'un $C_0$-semi-groupe $\{T(t);\ t \geq 0\}$. Alors $D(A)$ est dense dans $\mathbb{X}$, et $A$ est un opérateur fermé.*

## 1.4 Présentation des résultats

Ce travail comporte deux résultats principaux, donnés respectivement dans les chapitres 2 et 3.

### 1.4.1 Solution Weyl presque périodique des équations différentielles linéaires et semi-linéaires abstraites

Nous étudions d'abord l'existence et l'unicité de solutions mild bornées et Weyl presque périodiques pour les équations différentielles linéaires et semi-linéaires abstraites suivantes

$$u'(t) = Au(t) + f(t), \tag{1.4.1}$$

et

$$u'(t) = Au(t) + g(t, u(t)), \tag{1.4.2}$$

où $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ est un opérateur linéaire (éventuellement non borné) qui génére un $C_0$-semi-groupe $(T(t))_{t\geq 0}$ exponentiellement stable sur un espace de Banach $\mathbb{X}$ et $f : \mathbb{R} \to \mathbb{X}$, $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ sont Weyl presque périodiques.

#### Théorème de superposition des fonctions Weyl presque périodiques

Dans cette section, nous établissons un théorème de composition pour les fonctions Weyl presque périodiques. Notre résultat est basé sur une propriété de compacité donnée par Danilov [Dan06]. Dans ce qui suit, nous supposons que $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ $\left(\frac{p}{q} + \frac{p}{r} = 1\right)$ avec $p, q, r \geq 1$.

**Théorème 1.4.1** *Soit $f \in \mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X}, \mathbb{X})$. Supposons qu'il existe une fonction positive $L(.) \in B\mathbb{W}^r(\mathbb{R})$, telle que*

$$\|f(t, u) - f(t, v)\| \leq L(t)\, \|u - v\| \quad \forall t \in \mathbb{R}, \quad u, v \in \mathbb{X}.$$

*Alors, pour chaque $x(.) \in \mathbb{W}^q\mathrm{AP}(\mathbb{R}, \mathbb{X})$, on a*

$$f(., x(.)) \in \mathbb{W}^p\mathrm{AP}(\mathbb{R}, \mathbb{X}).$$

### Existence de solutions purement Weyl presque périodiques

Nous donnons deux exemples qui illustrent l'existence de solutions purement Weyl presque périodiques, lorsque les coefficients sont purement Weyl presque périodiques.

**Exemple 1.4.2** Soit $f : \mathbb{R} \to \mathbb{R}$ définie par

$$f(t) = \begin{cases} 1 & \text{pour} \quad 0 < t < \frac{1}{2}; \\ 0 & \text{sinon.} \end{cases} \tag{1.4.3}$$

La fonction $f$ est purement Weyl presque périodique (voir [ABG06, p.145]).

Soit $F : \mathbb{R} \to \mathbb{R}$ avec

$$F(t) = \int_{-\infty}^{t} f(s)ds = \begin{cases} 0 & \text{pour} \quad t \leq 0; \\ t & \text{pour} \quad 0 < t < \frac{1}{2}; \\ \frac{1}{2} & \text{sinon.} \end{cases}$$

La fonction $F$ est aussi purement Weyl presque périodique (voir [BIMRdF]).

**Exemple 1.4.3** *[BIMRdF]* Soit l'équation différentielle scalaire

$$x'(t) = -x(t) + f(t) \quad \forall t \in \mathbb{R},$$

où $f$ est donnée par (1.4.3). Son unique solution bornée est donnée par

$$x(t) = \begin{cases} 0 & \text{pour} \quad t \leq 0; \\ 1 - \exp(-t) & \text{pour} \quad 0 < t < \frac{1}{2}; \\ (\sqrt{\exp(1)} - 1)\exp(-t) & \text{sinon.} \end{cases} \tag{1.4.4}$$

La solution $x$ n'est pas ni Bohr presque periodique, ni même Stepanov presque périodique. Mais elle est Weyl presque périodique.

### Solution Weyl presque périodique de l'équation différentielle lineaire

Dans cette partie, nous généralisons un résultat obtenu par Radova [Rad04] au cas d'une équation différentielle linéaire abstraite. Plus précisément, nous montrons que l'équation (1.4.1) admet une unique solution mild et bornée et qui est (purement) Weyl presque périodique.

**Définition 1.4.4** *Une fonction continue $u : \mathbb{R} \longrightarrow \mathbb{X}$ est dite solution mild de l'équation (1.4.1) si*

$$u(t) = T(t-a)u(a) + \int_{a}^{t} T(t-s)f(s)ds,$$

*pour tout $t \geq a$ et pour chaque $a \in \mathbb{R}$.*

**Théorème 1.4.5** *Il existe une unique solution mild et bornée de l'équation (1.4.1). En plus cette solution est Weyl presque périodique donnée par*

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s)ds.$$

### Solution Weyl presque périodique de l'équation différentielle semi-linéaire

Finalement, dans la dernière partie du premier travail, nous généralisons l'étude précédente au cas des équations différentielles semi-linéaires abstraites. En fait, nous considérons les hypothèses suivantes :

(H1) L'opérateur $A$ génère un $C_0-$semi-groupe $(T(t))_{t\geq 0}$, exponentiellement stable.

(H2) $f \in \mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

(Lip) La fonction $f$ est L(.)-Lipschitz *i.e.*, il existe une fonction positive $L(.) \in \mathbb{BW}^p(\mathbb{R})$ telle que

$$\|f(t,u)-f(t,v)\| \leq L(t)\,\|u-v\|\,; \quad \forall t\in\mathbb{R}, \quad \forall\, u,v\in\mathbb{X}.$$

Nous désignons par $\mathrm{CB}(\mathbb{R},\mathbb{X})$ l'espace de Banach de toutes les fonctions continues et bornées de $\mathbb{R}$ vers $\mathbb{X}$, muni de la norme

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|\,.$$

**Définition 1.4.6** *Une fonction continue $u:\mathbb{R}\longrightarrow\mathbb{X}$ est dite solution mild de l'équation* (1.4.2) *si*

$$u(t) = T(t-a)u(a) + \int_a^t T(t-s)f(s,u(s))ds,$$

*pour tout $t\geq a$ et pour chaque $a\in\mathbb{R}$.*

**Théorème 1.4.7** *Supposons les hypothèses* (H1)-(H2) *vérifiées. Pour $p\geq 2$, et sous la condition*

$$\|L\|_{\mathbb{S}^p} < \left(\frac{\delta q}{M^q}\right)^{\frac{1}{q}}\left(\frac{e^\delta-1}{e^\delta}\right), \quad où\ \frac{1}{p}+\frac{1}{q}=1,$$

*l'équation* (1.4.2) *admet une unique solution mild dans* $\mathrm{CB}(\mathbb{R},\mathbb{X})$ *donnée par*

$$u(t) = \int_{-\infty}^t T(t-s)f(s,u(s))ds.$$

*Si de plus*

$$\|L\|_{\mathbb{S}^p} < \frac{p\delta}{M^p 2^{p+1}}\left(\frac{e^{\frac{p\delta}{4}}-1}{e^{\frac{p\delta}{4}}}\right)\left(\frac{p\delta}{2p-4}\right)^{p-2} \quad pour\ \ p>2,$$

*ou*

$$\|L\|_{\mathbb{S}^2} < \frac{p\delta}{8M^2}\left(\frac{e^{\frac{2\delta}{4}}-1}{e^{\frac{2\delta}{4}}}\right) \qquad pour\ \ p=2,$$

*alors, $u$ est Weyl presque périodique.*

Dans la démonstration de ce résultat, nous adaptons une méthode proposée par Kamenskii, Mellah et Raynaud de Fitte [KMRdF15], dans laquelle nous utilisons un théorème de point fixe dans l'espace de Banach $\mathrm{CB}(\mathbb{R},\mathbb{X})$ pour montrer l'existence et l'unicité et nous montrons manuellement que cette solution est Weyl presque périodique.

### 1.4.2 Solution Weyl presque périodique des équations différentielles stochastiques abstraites

Dans le troisième chapitre, nous généralisons l'étude précédente au cas des équations différentielles stochastiques semi-linéaires abstraites. Plus précisément, nous prenons deux espaces de Hilbert séparables $\mathbb{H}_1$ et $\mathbb{H}_2$ et nous considérons l'équation différentielle semi-linéaire suivante

$$dX_t = AX(t)dt + F(t, X(t))dt + G(t, X(t))dW(t), \quad t \in \mathbb{R} \tag{1.4.5}$$

où $A : D(A) \subset \mathbb{H}_2 \to \mathbb{H}_2$ est un opérateur linéaire dont le domaine de définition est dense dans $\mathbb{H}_2$ et $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$, $G : \mathbb{R} \times \mathbb{H}_2 \to L(\mathbb{H}_1, \mathbb{H}_2)$ sont des fonctions mesurables.

Nous considérons les hypothèses suivantes :

(H1) $W(t)$, $t \in \mathbb{R}$, est un processus de Wiener à valeur dans $\mathbb{H}_1$ dont l'opérateur de covariance $Q$ est nucléaire (nous désignons par $\operatorname{tr} Q$ la trace de $Q$), et $W$ est défini sur une base stochastique $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathrm{P})$.

(H2) $A : \mathrm{D}(A) \to \mathbb{H}_2$ est un générateur infinitésimal d'un $C_0$-semi-groupe $(T(t))_{t\geq 0}$ tel qu'il existe une constante $\delta > 0$ avec

$$\|T(t)\|_{L(\mathbb{H}_2)} \leq e^{-\delta t} \qquad t \geq 0.$$

(H3) Il existe une constante positive $M$ telle que

$$\|F(t,x)\|_{\mathbb{H}_2} + \|G(t,x)\|_{L_2(\mathbb{H}_1,\mathbb{H}_2)} \leq M(1 + \|x\|_{\mathbb{H}_2})$$

pour tout $t \in \mathbb{R}$.

(H4) Les fonctions $F$ et $G$ sont lipschitziennes, plus précisément, il existe une fonction positive $L(.) \in \mathbb{S}^q(\mathbb{R})$, $q \geq 2$, telle que

$$\|F(t,x) - F(t,y)\|_{\mathbb{H}_2} + \|G(t,x) - G(t,y)\|_{L_2(\mathbb{H}_1,\mathbb{H}_2)} \leq L(t)\|x - y\|_{\mathbb{H}_2}$$

pour tout $t \in \mathbb{R}$ et $x, y \in \mathbb{H}_2$.

(H5) $F \in \mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R} \times \mathbb{H}_2, \mathbb{H}_2)$ et $G \in \mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R} \times \mathbb{H}_2, L_2(\mathbb{H}_1, \mathbb{H}_2))$ $(p \geq 2)$.

**Définition 1.4.8** *Un Processus $X(t)_{t\in\mathbb{R}}$ $\mathcal{F}_t$-adapté est dit solution mild de l'équation* (1.4.5) *s'il satisfait l'équation d'intégrale suivante,*

$$X(t) = T(t-a)X(a) + \int_a^t T(t-s)F\big(s, X(s)\big)ds + \int_{-\infty}^t T(t-s)G\big(s, X(s)\big)dW(s),$$

*pour tout $t \geq a$ et pour chaque $a \in \mathbb{R}$*

Nous désignons par $\mathrm{UB}\big(\mathbb{R}, \mathrm{L}^p(\mathrm{P}, \mathbb{H}_2)\big)$ l'espace (de Banach) de tout les processus stochastiques uniformément bornés en $p$-moyenne muni de la norme

$$\|X\|_{\infty,p} = \left(\sup_{t\in\mathbb{R}} \left(\mathrm{E}\|X(t)\|^p_{\mathbb{H}_2}\right)\right)^{\frac{1}{p}}.$$

**Théorème 1.4.9** *Supposons les conditions* (H1) − (H5) *vérifiées. Pour* $p \geq 2$*, et sous les conditions* $\frac{2}{p} + \frac{2}{q} = 1$ *et*

$$\left( \frac{2^{\frac{pq+2p-q}{q}} \|L\|_{\mathbb{S}^p}^p}{\delta^{\frac{pq+2p}{2q}} \left(1 - \exp(\frac{p\delta}{4})\right)} + \frac{2^{\frac{pq+p-q}{q}} C_p \left(\operatorname{tr} Q\right)^{\frac{p}{2}} \|L\|_{\mathbb{S}^p}^p}{(q\delta)^{\frac{p}{p}} \left(1 - \exp(-\frac{p\delta}{2})\right)} \right)^{\frac{1}{p}} < 1,$$

*avec* $C_p = \frac{1}{(2c)^{\frac{p}{2}}} \left( \frac{2+2c}{p-1} - 2^{\frac{p}{2}} \right)$ *pour tout* $c > (p-1)2^{\frac{p}{2}} - 1$*, l'équation* (1.4.5) *a une unique solution mild dans* $\mathrm{UB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ *donnée par*

$$X(t) = \int_{-\infty}^{t} T(t-s) F\big(s, X(s)\big) ds + \int_{-\infty}^{t} T(t-s) G\big(s, X(s)\big) dW(s).$$

*Si de plus*

$$\frac{2^{2p} \|L\|_{\mathbb{S}^q}^p}{p\delta^{\frac{p}{2}+1} \left(1 - \exp(\frac{4}{q\delta})\right)^{\frac{p}{q}}} + \frac{2^{2p} \left(\operatorname{tr} Q\right)^{\frac{p}{2}} \|L\|_{\mathbb{S}^q}^p}{p\delta \left(1 - \exp(\frac{2}{q\delta})\right)^{\frac{p}{q}}} < 1,$$

*alors* $X$ *est Weyl presque périodique en loi fini-dimensionnelle.*

CHAPITRE 2

# Weyl almost periodic solutions to abstract linear and semilinear equations with Weyl almost periodic coefficients

## 2.1 Introduction

In 1927, Hermann Weyl [Wey27] introduced a generalization of Bohr and Stepanov almost periodic functions. Since then, generalized almost periodic functions bear his name and numerous contributions to theory of almost periodic functions have been brought, see for instance [ABG06, ABL01, BB31, Bes32, Dan06, GKL66, Urs31]. Weyl-almost periodic functions are very important and have several applications in dynamical systems, in particular in symbolic dynamics [Iwa88], where an important class of Weyl almost periodic trajectories are the regular Toeplitz sequences of Jacobs and Keane [JK69].

In recent years, many authors have been interested in the study of existence and uniqueness of almost periodic solutions to different types of differential equations with Stepanov almost periodic coefficients in both deterministic and stochastic cases. We can mention, in the deterministic case, the work by Andres and Pennequin [AP12a, AP12b], Long and Ding [LD11], Ding et al. [DLNG11], Hu and Mingarelli [HM08], Henriquez [Hen90], Rao [Rao75], Zaidman [Zai71]. Particularly, in [DLNG11] and [LD11], the authors show the existence and uniqueness of almost periodic solution for an abstract semilinear evolution equation with Stepanov almost periodic coefficients. They prove the existence and uniqueness of a Bohr almost periodic mild solution. Nevertheless, in the work [AP12b], Andres and Pennequin proved the nonexistence of purely Stepanov almost periodic solutions of ordinary differential equations in uniformly convex Banach spaces. Until now, to our knowledge, there are only few works dedicated to the study of existence and uniqueness of Weyl almost periodic solutions to some differential equations. The first investigation in this direction is due to Lenka Radova's [Rad04], but her study was limited to the scalar equation $x' = ax + f(t)$, where $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is an essentially bounded Weyl almost periodic function. The second one is due to Andres et al. [ABR06], and can be seen as an extension of Radova's work. Precisely, the authors investigate almost periodic solutions in various senses (Stepanov, Weyl, Besicovitch) of higher-order scalar differential equations with a nonlinear Lipschitz restoring term and a bounded Stepanov, Weyl or Besicovitch almost periodic forcing term. One can also mention the recent paper by Kostic [Kos], where Radova's study was addressed in the context of linear evolution equations with bounded and (asymptotically) Weyl-almost periodic functions coefficient.

Motivated by the previous papers, the aim of this paper is to generalize the previous investigations to the case of semilinear evolution equation, without any boundedness restriction on the forcing term. More precisely, we consider the following abstract semilinear evolution equation :

$$u'(t) = Au(t) + f(t, u(t)) \qquad \forall t \in \mathbb{R}, \tag{2.1.1}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear operator (possibly unbounded) which generates an exponentially stable $C_0$-semigroup on a Banach space $\mathbb{X}$, and $(t, x) \mapsto f(t, x)$ is a parametric Weyl almost periodic function of degree $p \geq 1$. We show that, under some conditions, (2.1.1) has a unique mild solution which is bounded and Weyl almost periodic. Note that, contrarily to the Stepanov case, the solution can be purely Weyl almost periodic.

The major problem encountered in our study is that the space of Weyl almost periodic functions endowed with the Weyl semi-norm is not complete [ABL01], which makes the approach and classical tools of functional analysis inapplicable, especially the fixed-point theorem in Banach spaces. To overcome this difficulty, we have opted for the method used by Kamenskii et al [KMRdF15], which consists in showing "manually" that the unique bounded mild solution to (2.1.1), obtained using the fixed-point theorem, is Weyl almost periodic.

This work is organized as follows : In the second section, we recall some definitions and results related to Stepanov almost periodic and Weyl almost periodic functions. In section 3, we present three essential parts. In the first part, based on a compactness property established by Danilov [Dan06], we give a new superposition result in the space of Weyl almost periodic functions. In the second part, after providing examples of bounded and purely Weyl almost periodic solutions, we tackle the problem of existence and uniqueness of a bounded Weyl almost periodic solution to an abstract linear differential equation. Finally, in the third part, inspired from [KMRdF15], we prove that equation (2.1.1) has a unique bounded mild solution which is Weyl almost periodic.

## 2.2 Notation and Preliminaries

First, we give some basic definitions and results on Stepanov almost periodic functions and Weyl almost periodic functions (for more details, see [ABG06, BB31, Bes32]).

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{X}$ a Banach space endowed with the norm $\|.\|$.

We denote also by $\mathrm{CB}(\mathbb{R}, \mathbb{X})$ the Banach space of continuous and bounded functions from $\mathbb{R}$ to $\mathbb{X}$, endowed with the norm

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\| .$$

### 2.2.1 Stepanov and Weyl norms

Given $l > 0$, the Stepanov norm associated with $l$ of a locally integrable function $f : \mathbb{R} \to \mathbb{X}$, i.e $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ $(p \geq 1)$, is defined by

$$\|f\|_{\mathbb{S}^p_l} = \sup_{\xi\in\mathbb{R}} \left( \frac{1}{l} \int_\xi^{\xi+l} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} .$$

The Weyl norm is defined by

$$\|f\|_{\mathbb{W}^p} = \lim_{l\to+\infty} \|f\|_{\mathbb{S}^p_l} .$$

The limit always exists for $l \to +\infty$ [Bes32, p.72-73]. The distance induced by $\|.\|_{\mathbb{S}^p_l}$ (respectively $\|f\|_{\mathbb{W}^p}$) is denoted by $D_{\mathbb{S}^p_l}$ (respectively $D_{\mathbb{W}^p}$).

### 2.2.2 Almost periodicity in Stepanov and Weyl senses

Let us recall some definitions of Stepanov and Weyl almost periodic functions. Recall that a set $T$ of real numbers is *relatively dense* if there exists a real number $k > 0$, such that $T \cap [a, a + k] \neq \emptyset$, for all $a$ in $\mathbb{R}$.

**Definition 2.2.1** ([AP71, p.76-77], [ABG06, ABL01], [Bes32, p.77], [GKL66, p.188]) *A function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be* almost periodic in the sense of Stepanov *(or p-*Stepanov almost periodic *if we want to highlight the degree p) if, for every $\varepsilon > 0$, the set*

$$T_{\mathbb{S}^p}(\varepsilon, f) = \left\{ \tau \in \mathbb{R};\ \sup_{\xi\in\mathbb{R}} \left( \int_\xi^{\xi+1} \|f(t+\tau) - f(t)\|^p \, dt \right)^{\frac{1}{p}} < \varepsilon \right\}$$

*is relatively dense. The number $\tau$ is called an $\varepsilon$-*Stepanov almost-period *(or* Stepanov $\varepsilon$-translation number *of $f$). We denote the set of all such functions by $\mathbb{S}^p\mathrm{AP}(\mathbb{R}, \mathbb{X})$.*

**Definition 2.2.2** ([ABG06, ABL01], [Bes32, p.78], [GKL66, p.190]) *A function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is called* Weyl almost periodic *if, for every $\varepsilon > 0$, there exist $l = l(\varepsilon) > 0$ such that, the set*

$$T_{\mathbb{S}^p_l}(\varepsilon, f) = \left\{ \tau \in \mathbb{R};\ \sup_{\xi\in\mathbb{R}} \left( \frac{1}{l} \int_\xi^{\xi+l} \|f(t+\tau) - f(t)\|^p \, dt \right)^{\frac{1}{p}} < \varepsilon \right\} \tag{2.2.1}$$

*is relatively dense. This means that for every $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that $f \in \mathbb{S}^p_l\mathrm{AP}(\mathbb{R}, \mathbb{X})$. The number $\tau$ is called a* Weyl $\varepsilon$-translation number *of $f$. We denote the set of all such functions by $\mathbb{W}^p\mathrm{AP}(\mathbb{R}, \mathbb{X})$.*

There is, however, another interesting definition given by Ursell [Urs31] :

**Definition 2.2.3 ([Urs31])** *A function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be* Weyl *-almost periodic *(or $\mathbb{W}$.a.p following Ursell's notations) if, for every $\varepsilon > 0$, there exists $l = l(\varepsilon)$ such that the set*

$$T^*_{\mathbb{S}^p_l}(\varepsilon, f) = \left\{ \tau \in \mathbb{R};\ \left( \frac{1}{l} \int_0^l \|f(t+\tau) - f(t)\|^p \, dt \right)^{\frac{1}{p}} < \varepsilon \right\}$$

*is relatively dense. We denote the set of all such functions by $\mathbb{W}^p_*\mathrm{AP}(\mathbb{R}, \mathbb{X})$.*

Ursell [Urs31] has shown that

**Theorem 2.2.4** *The classes $\mathbb{W}^p\mathrm{AP}(\mathbb{R}, \mathbb{X})$ and $\mathbb{W}^p_*\mathrm{AP}(\mathbb{R}, \mathbb{X})$ coincide.*

This identification is very important when establishing our main result.

**Remark 2.2.5** 1. *Some authors name the Weyl almost periodicity by equi-Weyl almost periodicity (e.g [ABG06, ABL01, Rad04]), distinguishing between the above definition and the following one introduced by Kovanko [Kov44] : A function $f \in L^p_{loc}(\mathbb{R}, \mathbb{X})$ is said to be* almost periodic in the sense of Weyl *according to [ABG06, ABL01, Rad04] if, for every $\varepsilon > 0$, the set*

$$T_{\mathbb{W}^p}(\varepsilon, f) = \left\{ \tau \in \mathbb{R};\ \lim_{l \to +\infty} \sup_{\xi \in \mathbb{R}} \left( \frac{1}{l} \int_\xi^{\xi+l} \|f(t+\tau) - f(t)\|^p \, dt \right)^{\frac{1}{p}} < \varepsilon \right\}$$

*is relatively dense. We denote the set of all such functions by $\mathbb{W}^p(\mathbb{R}, \mathbb{X})$.*
*Of course the space $\mathbb{W}^p\mathrm{AP}(\mathbb{R}, \mathbb{X})$ is an intermediate space between $\mathbb{S}^p\mathrm{AP}(\mathbb{R}, \mathbb{X})$ and $\mathbb{W}^p(\mathbb{R}, \mathbb{X})$, i.e. we have*

$$\mathbb{S}^p\mathrm{AP}(\mathbb{R}, \mathbb{X}) \subset \mathbb{W}^p\mathrm{AP}(\mathbb{R}, \mathbb{X}) \subset \mathbb{W}^p(\mathbb{R}, \mathbb{X}).$$

*According to [ABG06, ABL01], the previous inclusions are strict.*

2. *Every Weyl almost periodic function of degree $p$ is $\mathbb{S}^p$-bounded and also $\mathbb{W}^p$-bounded and also (see [ABG06]). Actually, the spaces*

$$\mathbb{BS}^p(\mathbb{R}, \mathbb{X}) := \left\{ f \in L^p_{loc}(\mathbb{R}, \mathbb{X});\ \sup_{\xi \in \mathbb{R}} \left( \frac{1}{L} \int_\xi^{\xi+L} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} < \infty \right\} \forall L > 0,$$

*and*

$$\mathbb{BW}^p(\mathbb{R}, \mathbb{X}) := \left\{ f \in L^p_{loc}(\mathbb{R}, \mathbb{X});\ \lim_{l \to \infty} \sup_{\xi \in \mathbb{R}} \left( \frac{1}{l} \int_\xi^{\xi+l} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} < \infty \right\},$$

*coincide, see [ABG06].*

### 2.2.3 Weyl almost periodic functions depending on a parameter

1. We say that a parametric function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is *Weyl almost periodic* if, for every $x \in \mathbb{X}$, the function $f(.,x)$ is Weyl almost periodic, we denote by $\mathbb{W}^p\mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ the space of such functions.
2. A function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}, (t,u) \mapsto f(t,u)$ with $f(.,u) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $u \in \mathbb{X}$ is said to be *Weyl almost periodic in $t \in \mathbb{R}$ uniformly with respect compact subsets of* $\mathbb{X}$ if, for each $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ and for all compacts $\mathbb{K}$ in $\mathbb{X}$, the set
$$T_{\mathbb{S}^p_l}(\varepsilon, f, \mathbb{K}) = \left\{\tau \in \mathbb{R};\ \sup_{u\in\mathbb{K}}\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l}\int_\xi^{\xi+l}\|f(t+\tau,u)-f(t,u)\|^p\,dt\right)^{\frac{1}{p}} < \varepsilon\right\}$$
is relatively dense. We denote by $\mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ the set of such functions.

It is clear that $\mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X}) \subset \mathbb{W}^p\mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{X})$.

Let us introduce some notations due to Danilov [Dan06]. We denote by $M^*_p(\mathbb{R},\mathbb{X})$ the set of functions $f \in \mathbb{BS}^p(\mathbb{R},\mathbb{X})$ such that
$$\lim_{\delta\to 0^+}\lim_{l_0\to\infty}\sup_{l>l_0}\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l}\sup_{T\subseteq[\xi,\xi+l]:\ \varkappa(T)\le\delta l}\int_T\|f(t)\|^p\,dt\right)^{\frac{1}{p}} = 0$$
holds, where $\varkappa$ is the Lebesgue measure on $\mathbb{R}$.

On the space $\mathbb{X}$ we also consider the truncated norm $\|.\|_* = \min\{1,\|.\|\}$. Let $\mathbb{W}^0\mathrm{AP}(\mathbb{R},\mathbb{X}) := \mathbb{W}^1\mathrm{AP}(\mathbb{R},(\mathbb{X},\|.\|_*))$ denote the space of Weyl almost periodic functions $f : \mathbb{R} \to \mathbb{X}$ (of degree 1), when $\mathbb{X}$ is endowed with $\|.\|_*$. From Danilov [Dan06], we have

**Theorem 2.2.6 (Danilov [Dan06])** *1. The following characterization holds true :*
$$\mathbb{W}^p\mathrm{AP}(\mathbb{R},\mathbb{X}) = \mathbb{W}^0\mathrm{AP}(\mathbb{R},\mathbb{X}) \cap M^*_p(\mathbb{R},\mathbb{X}). \tag{2.2.2}$$

*2. If $f \in \mathbb{W}^p\mathrm{AP}(\mathbb{R},\mathbb{X})$, then for every $\varepsilon,\delta > 0$ there exist a number $l > 0$ and a compact set $K_{\varepsilon,\delta} \subset \mathbb{X}$ such that*
$$\sup_{\xi\in\mathbb{R}}\frac{1}{l}\varkappa\{t\in[\xi,\xi+l] : d(f(t),K_{\varepsilon,\delta}) \ge \delta\} < \varepsilon.$$

Note that, as for almost periodicity in Stepanov sense, which is a metric property [BCM+17, Remark 2.4], it can be easily shown that almost periodicity in Weyl sense is a metric property too. In fact, using the arguments in [BCM+17, Remark 2.4], let $g = \exp\left(\sum_{n=2}^{+\infty} g_n\right)$, where $g_n$ is a $4n$-periodic function given by
$$g_n(t) = \beta_n\left(1-\frac{2}{\alpha_n}|t-n|\right)\mathbb{1}_{[n-\frac{2}{\alpha_n},n+\frac{2}{\alpha_n}]}(t), \quad t\in[-2n,2n],$$
where $\alpha_n \in ]0,\frac{1}{2}[$, and $\beta_n > 0$. As shown in [AP12b], for $\alpha_n = 1/n^5$ and $\beta_n = n^3$, we have $\|g\|_{\mathbb{S}^1} \ge \frac{\alpha_n\beta_n^2}{6} \to \infty$. Which means that $g$ is not $\mathbb{S}^1$-bounded. Hence $g$ cannot be in $\mathbb{W}^1\mathrm{AP}(\mathbb{R})$, . However, the function $g$ is almost periodic in Lebesgue measure, that is, it belongs to $\mathbb{S}^0\mathrm{AP}(\mathbb{R}) := \mathbb{S}^1\mathrm{AP}\big(\mathbb{R},(\mathbb{R},\|.\|_*)\big)$. In view of the inclusion $\mathbb{S}^0\mathrm{AP}(\mathbb{R}) \subset \mathbb{W}^0\mathrm{AP}(\mathbb{R})$, we deduce that $g$ belongs to $\mathbb{W}^0\mathrm{AP}(\mathbb{R})$. This shows also that the inclusion $\mathbb{W}^p\mathrm{AP}(\mathbb{R}) \subset \mathbb{W}^0\mathrm{AP}(\mathbb{R})$ is strict, and that $M^*_p(\mathbb{R},\mathbb{X})$ depends on the norm of the space $\mathbb{X}$.

## 2.3 Main results

### 2.3.1 A superposition theorem in $\mathbb{W}^p AP(\mathbb{R}, \mathbb{X})$

In this section, we establish a composition theorem for Weyl almost periodic functions. Our result is based on Theorem 2.2.6. In the following, we assume that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ $(\frac{p}{q} + \frac{p}{r} = 1)$ with $p$, $q$ and $r \geq 1$.

**Theorem 2.3.1** *Let $f \in \mathbb{W}^p AP_{\mathbb{K}}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$. Assume that there exists a positive function $L(.) \in \mathbb{BW}^r(\mathbb{R})$, such that*

$$\|f(t,u) - f(t,v)\| \leq L(t)\, \|u - v\| \quad \forall t \in \mathbb{R}, \quad u, v \in \mathbb{X}. \tag{2.3.1}$$

*Then, for every $x(.) \in \mathbb{W}^q AP(\mathbb{R}, \mathbb{X})$, we have*

$$f(., x(.)) \in \mathbb{W}^p AP(\mathbb{R}, \mathbb{X}).$$

**Proof** For a measurable set $A \subset \mathbb{R}$ and $l > 0$, let us introduce the notation

$$\overline{\varkappa}_l(A) = \sup_{\xi \in \mathbb{R}} \frac{1}{l} \varkappa \{t \in [\xi, \xi + l] \cap A\},$$

and let us denote by $A^c$ the complementary set of $A$. Fix $\varepsilon > 0$. Let $x(.) \in \mathbb{W}^q AP(\mathbb{R}, \mathbb{X})$. In view of (2.2.2), we can find $\delta_\varepsilon > 0$ and $l_0 = l_0(\varepsilon) > 0$ such that, for every measurable set $A$, if $\overline{\varkappa}_{l_0}(A) \leq \delta_\varepsilon$, then

$$\sup_{x \in \mathbb{R}} \left( \frac{1}{l_0} \int_{A \cap [x, x+l_0]} \|x(s)\|^q \, ds \right)^{\frac{1}{q}} \leq \varepsilon. \tag{2.3.2}$$

Using the second item of Theorem 2.2.6, for the above $\varepsilon > 0$ and the corresponding $\delta_\varepsilon > 0$, there exist a compact subset $\mathbb{K}_\varepsilon \subset \mathbb{X}$ and a positive number $l_1 = l_1(\varepsilon) > 0$ such that

$$\overline{\varkappa}_{l_1}(T_\varepsilon) < \delta_\varepsilon, \tag{2.3.3}$$

where

$$T_\varepsilon = \left\{ t \in \mathbb{R} : d(x(t), \mathbb{K}_\varepsilon) \geq \frac{\varepsilon}{24 \, \|L\|_{\mathbb{W}^r}} \right\}.$$

Hence, we obtain by taking $l_2 = \max(l_0, l_1)$,

$$\sup_{x \in \mathbb{R}} \left( \frac{1}{l_2} \int_{T_\varepsilon \cap [x, x+l_2]} \|x(s)\|^q \, ds \right)^{\frac{1}{q}} \leq \frac{\varepsilon}{24 \, \|L\|_{\mathbb{W}^r}}. \tag{2.3.4}$$

The compactness of $\mathbb{K}_\varepsilon$ yields that for every $t \in \mathbb{R}$, there exists $x_t^* \in \mathbb{K}_\varepsilon$ such that

$$d(x(t), \mathbb{K}_\varepsilon) = \|x(t) - x_t^*\|.$$

Now, since $x(.) \in M_q^*(\mathbb{R}, \mathbb{X})$ and $\{x_t^*;\ t \in \mathbb{R}\} \subset \mathbb{K}_\varepsilon$, we obtain

$$\sup_{\xi \in \mathbb{R}} \left( \frac{1}{l_2} \int_{T_\varepsilon \cap [\xi, \xi+l_2]} \|x(t) - x_t^*\|^q \, dt \right)^{\frac{1}{q}} < \frac{\varepsilon}{24 \, \|L\|_{\mathbb{W}^r}}. \tag{2.3.5}$$

We also have

$$\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l_2}\int_{T_\varepsilon^c\cap[\xi,\xi+l_2]}\|x(t)-x_t^*\|^q\,dt\right)^{\frac{1}{q}}<\frac{\varepsilon}{24\,\|L\|_{\mathbb{W}^r}}. \tag{2.3.6}$$

Using again the compactness of $\mathbb{K}_\varepsilon$, we can find a finite sequence $y_1, y_2, \cdots, y_n \in \mathbb{K}_\varepsilon$ such that

$$\mathbb{K}_\varepsilon\subset\bigcup_{i=1}^{n}B\left(y_i,\frac{\varepsilon}{24\,\|L\|_{\mathbb{W}^r}}\right). \tag{2.3.7}$$

Since $x_t^*\in\mathbb{K}_\varepsilon$, it follows that, for every $t\in\mathbb{R}$, there exists $i(t)\in\{1,2,\cdots,n\}$ such that

$$\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l_2}\int_{\xi}^{\xi+l_2}\left\|x_t^*-y_{i(t)}\right\|^q dt\right)^{\frac{1}{q}}<\frac{\varepsilon}{24\,\|L\|_{\mathbb{W}^r}}. \tag{2.3.8}$$

On the other hand, as $f\in\mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and $x(.)\in\mathbb{W}^q\mathrm{AP}(\mathbb{R},\mathbb{X})$, for the above $\varepsilon>0$, we can chose $l_3=l_3(\varepsilon)>0$ and a common relatively dense set $T_{\mathbb{S}_l^p}(\varepsilon,x,f)\subset\mathbb{R}$ such that

$$\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l_3}\int_{\xi}^{\xi+l_3}\|x(t+\tau)-x(t)\|^q\,dt\right)^{\frac{1}{q}}<\frac{\varepsilon}{2\,\|L\|_{\mathbb{W}^r}}, \tag{2.3.9}$$

and

$$\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l_3}\int_{\xi}^{\xi+l_3}\|f(t+\tau,v)-f(t,v)\|^p\,dt\right)^{\frac{1}{p}}<\frac{\varepsilon}{12n}, \tag{2.3.10}$$

for all $\tau\in T_{\mathbb{S}_l^p}(\varepsilon,x,f)$ and $v\in\mathbb{K}_\varepsilon$. Furthermore, by triangular inequality, we have, for $l=\max(l_2,l_3)$ and for every $\tau\in T_{\mathbb{S}_l^p}(\varepsilon,x,f)$,

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}}\Big(\frac{1}{l}\int_{\xi}^{\xi+l}\|f(t+\tau,x(t+\tau))-f(t,x(t))\|^p\,dt\Big)^{\frac{1}{p}}\\
&\quad\le\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l}\int_{\xi}^{\xi+l}\|f(t+\tau,x(t+\tau))-f(t+\tau,x(t))\|^p\,dt\right)^{\frac{1}{p}}\\
&\qquad+\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l}\int_{\xi}^{\xi+l}\|f(t+\tau,x(t))-f(t,x(t))\|^p\,dt\right)^{\frac{1}{p}}\\
&\quad=:I_1+I_2.
\end{aligned}$$

Let us estimate $I_1$ and $I_2$. For $I_1$, by (2.3.1), (2.3.9) and Hölder's inequality $(\frac{r}{p},\frac{q}{p})$, we have, for every $\tau\in T_{\mathbb{S}_l^p}(\varepsilon,x,f)$,

$$\begin{aligned}
I_1&=\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l}\int_{\xi}^{\xi+l}\|f(t+\tau,x(t+\tau))-f(t+\tau,x(t))\|^p\,dt\right)^{\frac{1}{p}}\\
&\le\|L\|_{\mathbb{W}^r}\sup_{\xi\in\mathbb{R}}\left(\frac{1}{l}\int_{\xi}^{\xi+l}\|x(t+\tau)-x(t)\|^q\,dt\right)^{\frac{1}{q}}<\frac{\varepsilon}{2}.
\end{aligned}$$

For $I_2$, for all $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, x, f)$, we have

$$I_2 \leq \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{T_\varepsilon\cap[\xi,\xi+l]} \|f(t+\tau,x(t)) - f(t,x(t))\|^p \, dt\right)^{\frac{1}{p}} + \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{T_\varepsilon^c\cap[\xi,\xi+l]} \|f(t+\tau,x(t)) - f(t,x(t))\|^p \, dt\right)^{\frac{1}{p}} := \quad I_2^1 + I_2^2.$$

First, we consider $I_2^1$. Using triangular inequality, the condition (2.3.1), (2.3.5) and Hölder's inequality $(\frac{p}{r}, \frac{p}{q})$, we get, for all $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, x, f)$,

$$\begin{aligned} I_2^1 =& \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{T_\varepsilon\cap[\xi,\xi+l]} \|f(t+\tau,x(t)) - f(t,x(t))\|^p \, dt\right)^{\frac{1}{p}} \\ \leq& \frac{\varepsilon}{24} + \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \|f(t+\tau,x_t^*) - f(t,x_t^*)\|^p \, dt\right)^{\frac{1}{p}} + \frac{\varepsilon}{24} \\ :=& \frac{\varepsilon}{12} + I^*. \end{aligned}$$

By (2.3.10), for each $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, x, f)$ and $i = 1, 2, \cdots, n$, we obtain that

$$\sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \|f(t+\tau,y_i) - f(t,y_i)\|^p \, dt\right)^{\frac{1}{p}} < \frac{\varepsilon}{12n}. \tag{2.3.11}$$

Using (2.3.1) and (2.3.8), it follows that, for every $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, x, f)$,

$$\begin{aligned} I^* \leq& 2\,\|L\|_{\mathbb{W}^r} \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \left\|y_{i(t)} - x_t^*\right\|^q dt\right)^{\frac{1}{q}} \\ &+ \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \left\|f(t+\tau,y_{i(t)}) - f(t,y_{i(t)})\right\|^p dt\right)^{\frac{1}{p}} \\ \leq& \frac{\varepsilon}{12} + \sum_{j=1}^{n_l} \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \|f(t+\tau,y_j) - f(t,y_j)\|^p \, dt\right)^{\frac{1}{p}} < \frac{\varepsilon}{6}. \end{aligned}$$

We thus have, for all $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, x, f)$,

$$I_2^1 = \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{T_\varepsilon\cap[\xi,\xi+l]} \|f(t+\tau,x(t)) - f(t,x(t))\|^p \, dt\right)^{\frac{1}{p}} < \frac{\varepsilon}{4}.$$

For $I_2^2$, by (2.3.1), (2.3.6), (2.3.8), (2.3.11) and Hölder's inequality, we follow the same steps than for $I_2^1$, we obtain, for every $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, x, f)$,

$$I_2^2 = \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{T_\varepsilon^c\cap[\xi,\xi+l]} \|f(t+\tau,x(t)) - f(t,x(t))\|^p \, dt\right)^{\frac{1}{p}} < \frac{\varepsilon}{4}.$$

Then, for all $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, x, f)$, we get that

$$I_2 = \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \|f(t+\tau,x(t)) - f(t,x(t))\|^p \, dt\right)^{\frac{1}{p}} < \frac{\varepsilon}{2}.$$

Finally, combining $I_1$ and $I_2$, we obtain, for every $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, x, f)$,

$$\sup_{\xi \in \mathbb{R}} \left( \frac{1}{l} \int_{\xi}^{\xi+l} \|f(t+\tau, x(t+\tau)) - f(t, x(t))\|^p \, dt \right)^{\frac{1}{p}} < \varepsilon,$$

that is, $f(., x(.)) \in \mathbb{W}^p\mathrm{AP}(\mathbb{R}, \mathbb{X})$. $\square$

Now, we discuss the Weyl almost periodic solutions to linear differential equations.

### 2.3.2 Existence and uniqueness of bounded Weyl almost periodic solution to abstract linear differential equation

We investigate now the Weyl almost periodic solutions to

$$u'(t) = Au(t) + f(t), \tag{2.3.12}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear operator which generates an exponentially stable $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space $\mathbb{X}$, and $f : \mathbb{R} \to \mathbb{X}$ is Weyl almost periodic.

We recall that a function $u : \mathbb{R} \to \mathbb{X}$ is a *mild solution* to (2.3.12) if

$$u(t) = T(t-a)u(a) + \int_a^b T(t-s)f(s)ds, \quad t \geq s.$$

Let us also recall that a semigroup $(T(t))_{t \in \mathbb{R}}$ of linear operators is *exponentially stable* if there exist $\delta > 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{-\delta t}. \tag{2.3.13}$$

The main motivation of this study is that, contrarily the Stepanov case (see [AP12b] and [BCM+17]), where it is shown the nonexistence of purely Stepanov almost periodic solutions for ordinary differential equations with Stepanov almost periodic coefficients, we can find, in the Weyl case, a bounded and purely Weyl almost periodic mild solution to Equation (2.3.12). The problem is closely related to Bohl-Bohr theorem which fails in the Weyl case. Indeed, the following example shows that there exist Weyl almost periodic functions such that their integrals are bounded but not Bohr almost periodic, nor Stepanov almost periodic. However, they are (purely) Weyl almost periodic.

**Example 2.3.2** *We consider the following function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(t) = \begin{cases} 1 & \text{for} \quad 0 < t < \frac{1}{2}; \\ 0 & \text{elsewhere}, \end{cases} \tag{2.3.14}$$

*which is purely Weyl almost periodic (see [ABG06, p.145]). Let $F : \mathbb{R} \to \mathbb{R}$ with*

$$F(t) = \int_{-\infty}^{t} f(s)ds = \begin{cases} 0 & \text{for} \quad t \leq 0; \\ t & \text{for} \quad 0 < t < \frac{1}{2}; \\ \frac{1}{2} & \text{elsewhere}, \end{cases}$$

*be a primitive of $f$. Clearly $F$ is bounded on $\mathbb{R}$. Moreover, for every $t_1, t_2 \in \mathbb{R}$, we have*

$$|F(t_1) - F(t_2)| \leq |t_1 - t_2| .$$

*Thus $F$ is uniformly continuous. So by [Rad04, Proposition 4], $F$ is Weyl almost periodic.*

*On the other hand, for $\varepsilon < \frac{1}{4}$, there exists $t \in [0, \frac{1}{4}]$ such that, for every $\tau > \varepsilon$, we have*

$$|F(t+\tau) - F(t)| \geq \frac{1}{4} > \varepsilon.$$

*So, the function $F$ is not Bohr almost periodic. We can show more, that is $F$ is not Stepanov almost periodic. Indeed, if $F$ is Stepanov almost periodic, thanks to its uniform continuity, we get would have that $F$ was Bohr almost periodic (Bochner's Theorem, see. e.g [ABG06], [Bes32, p.82], [LZ, Lemme.4 p.34] and [FB00, Th 6.16]). A contradiction. Hence $F$ is not Stepanov almost periodic.*

**Example 2.3.3** *Let the scalar differential equation $x'(t) = -x(t) + f(t)$, $t \in \mathbb{R}$, with $f$ given by (2.3.14). Its unique bounded solution is given by*

$$x(t) = \begin{cases} 0 & \textit{for} \quad t \leq 0; \\ 1 - \exp(-t) & \textit{for} \quad 0 < t < \frac{1}{2}; \\ (\sqrt{\exp(1)} - 1)\exp(-t) & \textit{elsewhere.} \end{cases} \tag{2.3.15}$$

*The solution $x$ is clearly uniformly continuous. Moreover, by [Rad04, Proposition 4], $x$ is Weyl almost periodic. Now, we can get more, namely the solution $x$ is purely Weyl almost periodic. Indeed, for every $\varepsilon > 0$, the following inequality*

$$(\sqrt{\exp(1)} - 1)\exp(-t) < \varepsilon$$

*holds for all $t \geq -\ln\left(\frac{\varepsilon}{\sqrt{\exp(1)}-1}\right)$. Now, let us choose $\varepsilon = \frac{\sqrt{\exp(1)}}{2\left(\sqrt{\exp(1)}-1\right)}$. Then, every $\tau$ such that $\tau + \frac{1}{2} > -\ln\left(\frac{\varepsilon}{\sqrt{\exp(1)}-1}\right)$, satisfies*

$$x\left(\frac{1}{2}\right) - x\left(\tau + \frac{1}{2}\right) = 2\varepsilon - (\sqrt{\exp(1)} - 1)\exp(-\tau - \frac{1}{2}) \geq 2\varepsilon - \varepsilon = \varepsilon.$$

*Which means that $x$ is not almost periodic, nor Stepanov almost periodic.*

**Theorem 2.3.4** *There exists a unique bounded mild solution to (2.3.12). This solution is Weyl almost periodic of degree $p$ and is given by*

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s)ds. \tag{2.3.16}$$

**Proof** First, we show that the function $u(t) = \int_{-\infty}^{t} T(t-s)f(s)ds$ is well defined, i.e, $\lim_{a\to-\infty} \int_a^t T(t-s)f(s)ds$ exists. We set

$$G(a) = \int_a^t T(t-s)f(s)ds.$$

Let us show that $(G(a))_{a\in\mathbb{R}}$ is Cauchy at $-\infty$. If $a<b$, then by (2.3.13), we have

$$\|G(b)-G(a)\| \leq \int_a^b \|T(t-s)\| \, \|f(s)\| \, ds \leq M \int_a^b e^{-\delta(t-s)} \|f(s)\| \, ds.$$

Since $f$ is Weyl almost periodic, we have $f \in \mathbb{BS}_L^p(\mathbb{R},\mathbb{X})$, for every $L>0$. Thus, there exists $N \in \mathbb{N}$, such that, by Hölder's inequality $(\frac{1}{p},\frac{1}{q})$, we have

$$\begin{aligned}
\|G(b)-G(a)\| &\leq M \sum_{k=0}^{N} \int_{b-(k+1)L}^{b-kL} e^{-\delta(t-s)} \|f(s)\| \, ds \\
&\leq M \sum_{k=0}^{N} \left(\int_{b-(k+1)L}^{b-kL} e^{-\delta q(t-s)} ds\right)^{\frac{1}{q}} \left(\int_{b-(k+1)L}^{b-kL} \|f(s)\|^p \, ds\right)^{\frac{1}{p}} \\
&\leq ML \, \|f\|_{\mathbb{S}_L^p} \, e^{-\delta q(t-b)} \left(\frac{1-e^{-\delta q(N+1)L}}{1-e^{-\delta qL}}\right).
\end{aligned}$$

Consequently,

$$\lim_{b\to-\infty} \|G(b)-G(a)\| \leq \lim_{b\to-\infty} ML \, \|f\|_{\mathbb{S}_L^p} \, e^{-\delta q(t-b)} \left(\frac{1-e^{-\delta q(N+1)L}}{1-e^{-\delta qL}}\right) = 0.$$

Thus the limit

$$\lim_{a\to-\infty} \int_a^t T(t-s) f(s) ds$$

exists. This means that $u$ is well-defined. Moreover, it is easy to see that $u$ given by (2.3.16) is a mild solution to (2.3.12).

In the following, we show that $u$ is bounded. By Hölder's inequality and (2.3.13), we obtain

$$\|u(t)\| \leq \int_{-\infty}^t \|T(t-s)\| \, \|f(s)\| \, ds \leq M \sum_{k=0}^{\infty} e^{-\delta k} \left(\int_{t-k-1}^{t-k} \|f(s)\|^p \, ds\right)^{\frac{1}{p}}.$$

Then, we have

$$\sup_{t\in\mathbb{R}} \|u(t)\| \leq M \sum_{k=0}^{\infty} e^{-\delta k} \sup_{t\in\mathbb{R}} \left(\int_{t-k-1}^{t-k} \|f(s)\|^p \, ds\right)^{\frac{1}{p}}.$$

Since Weyl almost periodic functions of degree $p$ are $\mathbb{S}^p$-bounded, we have

$$\sup_{t\in\mathbb{R}} \|u(t)\|_\infty \leq M \, \|f\|_{\mathbb{S}^p} \sum_{k=0}^{\infty} e^{-\delta k} < \infty.$$

Thus $u$ is bounded.

Now, we prove that the solution to (2.3.12) is Weyl almost periodic. Using a change of variable and Hölder's inequality $(\frac{1}{p}+\frac{1}{q}=1)$, we have any $\tau \in \mathbb{R}$,

$$\begin{aligned}
\|u(t+\tau)-u(t)\| &\leq M \int_{-\infty}^0 e^{\delta s} \|f(t+s+\tau)-f(t+s)\| \, ds \\
&\leq M \left(\frac{2}{\delta q}\right)^{\frac{1}{q}} \left(\int_{-\infty}^0 e^{\frac{\delta ps}{2}} \|f(t+s+\tau)-f(t+s)\|^p \, ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Let $l > 0$. Using Fubini's theorem, we get, for every $\tau \in \mathbb{R}$,

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \|u(t+\tau)-u(t)\|^p\,dt\right)^{\frac{1}{p}} \\
&\leq M\left(\frac{2}{\delta q}\right)^{\frac{1}{q}} \sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \left(\left(\int_{-\infty}^{0} e^{\frac{\delta ps}{2}} \|f(t+s+\tau)-f(t+s)\|^p\,ds\right)^{\frac{1}{p}}\right)^{p} dt\right)^{\frac{1}{p}} \\
&\leq M\left(\frac{2}{\delta q}\right)^{\frac{1}{q}} \left(\int_{-\infty}^{0} e^{\frac{\delta ps}{2}} \sup_{\xi\in\mathbb{R}} \frac{1}{l}\int_{\xi}^{\xi+l} \|f(t+s+\tau)-f(t+s)\|^p\,dt\,ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $f$ is Weyl almost periodic, for any $\varepsilon > 0$, we can choose positive number $l = l(\varepsilon)$ and there exists a relatively dense set $T_{\mathbb{S}_l^p}(\varepsilon, f)$ as in (2.2.1), such that

$$\left(\int_{-\infty}^{0} e^{\frac{\delta ps}{2}} \sup_{\xi\in\mathbb{R}} \frac{1}{l}\int_{\xi}^{\xi+l} \|f(t+s+\tau)-f(t+s)\|^p\,dt\,ds\right)^{\frac{1}{p}} \leq \left(\frac{2}{\delta p}\right)^{\frac{1}{p}} \varepsilon \quad \forall\tau \in T_{\mathbb{S}_l^p}(\varepsilon, f).$$

Hence,

$$\sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \|u(t+\tau)-u(t)\|^p\,dt\right)^{\frac{1}{p}} \leq C\varepsilon,$$

where $C = M\left(\frac{2}{\delta q}\right)^{\frac{1}{q}}\left(\frac{2}{\delta p}\right)^{\frac{1}{p}}$. Summing up, we have shown that, for every $\varepsilon > 0$, there corresponds a positive number $l = l(\varepsilon)$ and there exists a relatively dense set $T_{\mathbb{S}_l^p}(\varepsilon, u) = T_{\mathbb{S}_l^p}(\frac{\varepsilon}{C}, f)$ such that

$$\sup_{\xi\in\mathbb{R}} \left(\frac{1}{l}\int_{\xi}^{\xi+l} \|u(t+\tau)-u(t)\|^p\,dt\right)^{\frac{1}{p}} < \varepsilon; \qquad \forall\tau \in T_{\mathbb{S}_l^p}(\varepsilon, u).$$

Which implies the Weyl-almost-periodicity of the solution $u$.

Uniqueness of the solution follows from the same arguments as those in [DLNG11, LD11]. We omit the details. $\square$

### 2.3.3 Existence and uniqueness of bounded Weyl almost periodic solution to abstract semilinear differential equation

In this section we study the existence and uniqueness of the Weyl almost periodic mild solution to equation

$$u'(t) = Au(t) + f(t, u(t)), \tag{2.3.17}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is unbounded linear, of domain $D(A)$ dense in $\mathbb{X}$, and $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ measurable function.

We consider the following hypothesis :

(H1) The operator $A$ generates a $C_0-$semigroup $(T(t))_{t\geq 0}$, exponentially stable.

(H2) $f \in \mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X}, \mathbb{X})$.

(H3) The function $f$ is L(.)-Lipschitz i.e., there exists a nonnegative function $L(.) \in \mathbb{BW}^p(\mathbb{R})$ such that

$$\|f(t,u)-f(t,v)\| \leq L(t)\,\|u-v\|\,; \quad \forall t \in \mathbb{R}, \quad \forall\, u,v \in \mathbb{X}.$$

**Theorem 2.3.5** *Let the assumptions* (H1)-(H3) *be fulfilled. For $p \geq 2$, the equation* (2.1.1) *has a unique mild solution in* $\mathrm{CB}(\mathbb{R},\mathbb{X})$ *given by*

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s,u(s))ds, \tag{2.3.18}$$

*provided that*

$$\|L\|_{\mathbb{S}^p} < \left(\frac{\delta q}{M^q}\right)^{\frac{1}{q}} \left(\frac{e^{\delta}-1}{e^{\delta}}\right), \tag{2.3.19}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*If, furthermore*

$$\|L\|_{\mathbb{S}^p} < \frac{p\delta}{M^p 2^{p+1}} \left(\frac{e^{\frac{p\delta}{4}}-1}{e^{\frac{p\delta}{4}}}\right) \left(\frac{p\delta}{2p-4}\right)^{p-2} \text{for } p>2, \tag{2.3.20}$$

*or*

$$\|L\|_{\mathbb{S}^2} < \frac{p\delta}{8M^2} \left(\frac{e^{\frac{2\delta}{4}}-1}{e^{\frac{2\delta}{4}}}\right) \qquad \text{for } p=2, \tag{2.3.21}$$

*then, $u$ is Weyl almost periodic.*

Before giving the proof of Theorem 2.3.5, we need the following technical results.

**Lemma 2.3.6** *[KMRdF15] Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function such that, for every $t\in\mathbb{R}$,*

$$0 \leq g(t) \leq \alpha(t) + \beta_1 \int_{-\infty}^{t} e^{-\delta_1(t-s)}g(s)ds + ... + \beta_n \int_{-\infty}^{t} e^{-\delta_1(t-s)}g(s)ds, \tag{2.3.22}$$

*for some locally integrable function $\alpha:\mathbb{R}\to\mathbb{R}$, and for some constants $\beta_1,...,\beta_n \geq 0$, and some constants $\delta_1,...,\delta_n > \beta$, where $\beta = \sum_{i=1}^{i=n}\beta_i$. We assume that the integrals in the right hand side of* (2.3.22) *are convergent. Let $\delta = \min_{1\leq i\leq n}\delta_i$. Then, for every $\gamma \in ]0,\delta-\beta]$ such that $\int_{-\infty}^{0} e^{\gamma s}\alpha(s)ds$ converges, we have, for every $t\in\mathbb{R}$,*

$$g(t) \leq \alpha(t) + \beta \int_{-\infty}^{t} e^{-\gamma(t-s)}\alpha(s)ds.$$

*In particular, if $\alpha$ is constant, we have*

$$g(t) \leq \alpha\frac{\delta}{\delta-\beta}.$$

**Proposition 2.3.7** *Let $f$ be in $\mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ $(2\leq p<\infty)$ and satisfying* (H1)*. Let $u\in \mathrm{CB}(\mathbb{R},\mathbb{X})$. Then, for every $\varepsilon>0$ there exists $l=l(\varepsilon)>0$ such that for every $\tau\in T_{\mathbb{S}^p_l}(\varepsilon,f,\mathbb{K})$, we have*

$$\int_{-\infty}^{0} e^{\delta_1 s}\frac{1}{l}\int_s^{s+l}\|f(t+\tau,u(t))-f(t,u(t))\|^p\,dt\,ds<\varepsilon^p \tag{2.3.23}$$

*and*

$$\int_{-\infty}^{0} e^{\gamma r}\int_{-\infty}^{r} e^{-\delta_1(r-s)}\frac{1}{l}\int_s^{s+l}\|f(t+\tau,u(t))-f(t,u(t))\|^p\,dt\,ds\,dr<\varepsilon^p, \tag{2.3.24}$$

*for all $\delta_1>0$ and $\gamma>0$.*

**Proof** [Proposition (2.3.7)] Let fix $\delta_1>0$ and $\gamma>0$. For $s\in\mathbb{R}$, $\xi\in\mathbb{R}$, $l>0$ and $\tau\in\mathbb{R}$, let

$$\begin{aligned}h_{\tau,l}(s) &= \frac{1}{l}\int_s^{s+l}\|f(t+\tau,u(t))-f(t,u(t))\|^p\,dt,\\ \alpha_{\tau,l}(\xi) &= \int_{-\infty}^{\xi} e^{-\delta_1(\xi-s)}h_{\tau,l}(s)ds.\end{aligned}$$

Inequalities (2.3.23) and (2.3.24) become $\alpha_{\tau,l}(0)<\varepsilon^p$ and $\int_{-\infty}^{0} e^{\gamma r}\alpha_{\tau,l}(r)dr<\varepsilon^p$, respectively.

First, we deal with estimation (2.3.23). Using the boundedness of both $u$ and $f$ in the Weyl sense ($f$ is $\mathbb{W}^p$-bounded), we have for every $\tau\in\mathbb{R}$, $l>0$ and for every $s\in]-\infty,0]$,

$$\begin{aligned}h_{\tau,l}(s) =&\frac{1}{l}\int_s^{s+l}\|f(t+\tau,u(t))-f(t,u(t))\|^p\,dt\\ \leq&2^{2p-2}\left(\frac{1}{l}\int_s^{s+l}\|f(t+\tau,u(t))-f(t+\tau,0)\|^p\,dt+\frac{1}{l}\int_s^{s+l}\|f(t+\tau,0)\|^p\,dt\right)\\ &+2^{2p-2}\left(\frac{1}{l}\int_s^{s+l}\|f(t,0)\|^p\,dt+\frac{1}{l}\int_s^{s+l}\|f(t,0)-f(t,u(t))\|^p\,dt\right)\\ \leq&2^{2p-1}\left(\|L(.)\|^p_{\mathbb{W}^p}\|u(.)\|^p_\infty+\|f(.,0)\|^p_{\mathbb{W}^p}\right)<+\infty.\end{aligned}$$

The function $s\mapsto e^{\delta_1 s}h_{\tau,l}(s)$ is bounded and continuous on $]-\infty,0]$, hence

$$\alpha_{\tau,l}(0)=\int_{-\infty}^{0} e^{\delta_1 s}h_{\tau,l}(s)ds=\sum_{n=0}^{\infty}\int_{-n-1}^{-n}e^{\delta_1 s}h_{\tau,l}(s)ds\leq\sum_{n=0}^{\infty}e^{-\delta_1 n}\left(\max_{s\in[-n-1,-n]}h_{\tau,l}(s)\right). \tag{2.3.25}$$

Due to the uniform boundedness of the sequence $\left(\max_{s\in[-n-1,-n]}h_{\tau,l}(s)\right)_n$, the series in (2.3.25) is uniformly convergent. Thus, for every $\varepsilon>0$, we can find an integer $N_1=N_1(\varepsilon)\geq 1$ such that for every $\tau\in\mathbb{R}$, $l>0$, and $\delta_1>0$, we have

$$\sum_{n>N_1}e^{-\delta_1 n}\left(\max_{s\in[-n-1,-n]}h_{\tau,l}(s)\right)<\frac{\varepsilon^p}{2}. \tag{2.3.26}$$

Moreover, we claim that there exists $l = l(\varepsilon) > 0$ such that every $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, f, \mathbb{K})$ satisfies

$$\sum_{n=0}^{N_1} e^{-\delta_1 n} \left( \max_{s \in [-n-1,-n]} h_{\tau,l}(s) \right) < \frac{\varepsilon^p}{2}. \tag{2.3.27}$$

In fact, for the above $\varepsilon > 0$, let $l = l(\varepsilon) > 0$ provided by the condition $T_{\mathbb{S}_l^p}(\varepsilon, f, K)$ is relatively dense. In view of the continuity of the mapping $s \mapsto h_{\tau,l}(s)$ on the compact set $[-n-1, -n]$, there exists $s_n^* \in [-n-1, -n]$ such that, for every $\tau \in \mathbb{R}$ and $\delta_1 > 0$,

$$\max_{s \in [-n-1,-n]} h_{\tau,l}(s) = h_{\tau,l}(s_n^*).$$

Put $s^* = s_\varepsilon^* := \underset{n \in \{0,\cdots,N_1\}}{\arg\max}\, h_{\tau,l}(s_n^*)$. We have then

$$\sum_{n=0}^{N_1} e^{-\delta_1 n} \max_{s \in [-n-1,-n]} h_{\tau,l}(s) \leq \sum_{n=0}^{N_1} e^{-\delta_1 n} h_{\tau,l}(s^*).$$

Now, since $u$ is continuous, we have that $\mathbb{K}^* = u\left([s^*, s^* + l]\right)$ is a compact set that depends only on $\varepsilon$. Hence, there exist $u_1, \cdots, u_k \in \mathbb{K}^*$ such that

$$\mathbb{K}^* \subset \bigcup_{i=1}^{i=k} B\left(u_i, \frac{\varepsilon}{2\sqrt[p]{6}\, \|L\|_{\mathbb{W}^p}}\right).$$

It follows that, for each $t \in [s^*, s^* + l]$, there exists $i(t) \in \{1, 2, \cdots, k\}$ such that

$$\left\|u(t) - u_{i(t)}\right\| < \frac{\varepsilon}{2\sqrt[p]{6}\, \|L\|_{\mathbb{W}^p}}. \tag{2.3.28}$$

Note that, due to (2.3.28) and the continuity of $u$, the mapping $t \to u_{i(t)}$ is measurable as a continuous function. On the other hand, since $f \in \mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, every $\tau \in T_{\mathbb{S}_l^p}(\varepsilon, f, \mathbb{K})$ is such that for every $j = 1, \ldots, k$

$$\frac{1}{l} \int_{s^*}^{s^*+l} \|f(t+\tau, u_j) - f(t, u_j)\|^p \, dt < \frac{\varepsilon^p}{2^{p+1} 3k}. \tag{2.3.29}$$

To combine the previous arguments, we use the triangle inequality :

$$\begin{aligned} h_{\tau,l}(s^*) \leq & 2^p \frac{1}{l} \int_{s^*}^{s^*+l} \left\|f(t+\tau, u(t)) - f(t+\tau, u_{i(t)})\right\|^p dt + 2^p \frac{1}{l} \int_{s^*}^{s^*+l} \left\|f(t+\tau, u_{i(t)}) - f(t, u_{i(t)})\right\|^p dt \\ & + 2^p \frac{1}{l} \int_{s^*}^{s^*+l} \left\|f(t, u_{i(t)}) - f(t, u(t))\right\|^p dt \\ =: & 2^p \left(A_1 + A_2 + A_3\right). \end{aligned} \tag{2.3.30}$$

Let us estimate $A_1$. By (H2) and (2.3.28), we have

$$\begin{aligned} A_1 &= \frac{1}{l} \int_{s^*}^{s^*+l} \left\|f(t+\tau, u(t)) - f(t+\tau, u_{i(t)})\right\|^p dt \\ &\leq \frac{1}{l} \int_{s^*}^{s^*+l} \|L(t+\tau)\|^p \left\|u(t) - u_{i(t)}\right\|^p dt \leq \left(\frac{\varepsilon}{2\sqrt[p]{6}\, \|L\|_{\mathbb{W}^p}}\right)^p \|L\|_{\mathbb{W}^p}^p < \frac{\varepsilon^p}{2^{p+1} 3}. \end{aligned} \tag{2.3.31}$$

We follow the same steps as for $A_1$, we obtain

$$A_3 \leq \left( \frac{\varepsilon}{2\sqrt[p]{6}\,\|L\|_{\mathbb{W}^p}} \right)^p \|L\|^p_{\mathbb{W}^p} < \frac{\varepsilon^p}{2^{p+1}3}. \tag{2.3.32}$$

Now, we estimate $A_2$. By (2.3.29), we have

$$A_2 \leq \sum_{j=1}^{k} \frac{1}{l} \int_{s^*}^{s^*+l} \|f(t+\tau, u_j) - f(t, u_j)\|^p \, dt \leq \frac{\varepsilon^p}{2^{p+1}3}. \tag{2.3.33}$$

Taking into account (2.3.30)-(2.3.33), we obtain that

$$h_{\tau,l}(s^*) \leq \frac{\varepsilon^p}{2}, \text{ for every } \tau \in T_{\mathbb{S}^p_l}(\varepsilon, f, \mathbb{K}). \tag{2.3.34}$$

This implies that

$$\sum_{n=0}^{N_1} e^{-\delta_1 n} \left( \max_{s \in [-n-1, -n]} h_{\tau,l}(s) \right) \leq (1 - e^{-\delta_1}) \frac{\varepsilon^p}{2} \leq \frac{\varepsilon^p}{2}.$$

Having disposed the previous preliminaries steps, we now return to estimation (2.3.25), we get in view of (2.3.26) and (2.3.27) that for every $\varepsilon > 0$, there is $l = l(\varepsilon) > 0$ (provided by the condition $f \in \mathbb{S}^p_l\mathrm{AP}_{\mathbb{K}}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) such that any $\tau \in T_{\mathbb{S}^p_l}(\varepsilon, f, \mathbb{K})$ verifies

$$\alpha_{\tau,l}(0) = \int_{-\infty}^{0} e^{\frac{p\delta}{4}s} \frac{1}{l} \int_{s}^{s+l} \|f(t+\tau, u(t)) - f(t, u(t))\|^p \, dt \, ds \leq \varepsilon^p. \tag{2.3.35}$$

Next, let us consider $\int_{-\infty}^{0} e^{\gamma r} \alpha_{\tau,l}(r) dr$. The arguments we use are in part the same as those mentioned above. The function $r \mapsto \alpha_{\tau,l}(r)$ is continuous and uniformly bounded with respect to both $\tau$ and $l$, thus the series $\sum_{n \geq 0} e^{-\gamma n} \alpha_{\tau,l}(r^*_n)$ is uniformly convergent, where $r^*_n = \underset{r \in [-n-1, -n]}{\arg\max}\, \alpha_{\tau,l}(r)$. Therefore,

$$\int_{-\infty}^{0} e^{\gamma r} \alpha_{\tau,l}(r) dr \leq \sum_{n \geq 0} e^{-\gamma n} \alpha_{\tau,l}(r^*_n). \tag{2.3.36}$$

Now, let $\varepsilon > 0$. We can find an integer $N_2 = N_2(\varepsilon) > 1$ such that

$$\sum_{n > N_2} e^{-\gamma n} \alpha_{\tau,l}(r^*_n) < \frac{\varepsilon^p}{2}. \tag{2.3.37}$$

Furthermore, by putting $r^* = \underset{n \in \{0, \ldots, N_2\}}{\arg\max}\, \alpha_{\tau,l}(r^*_n)$, we have

$$\sum_{n=0}^{N_2} e^{-\gamma n} \alpha_{\tau,l}(r^*_n) \;\; \leq \;\; (1 - e^{-\gamma})^{-1} \alpha_{\tau,l}(r^*). \tag{2.3.38}$$

Repeating the same argument as for $\alpha_{\tau,l}(0)$ enables us to write that for the positive number $l = l(\varepsilon)$ provided by the condition $f \in \mathbb{S}_l^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and for every $\tau$ belonging to the relatively dense set $\in T_{\mathbb{S}_l^p}(\varepsilon, f, \mathbb{K})$ we have

$$\alpha_{\tau,l}(r^*) \leq (1-e^{-\gamma})\frac{\varepsilon^p}{2}. \tag{2.3.39}$$

Consequently, combining (2.3.36), (2.3.37) and (2.3.39), we obtain for the above $l$ and $\tau$ the desired estimation :

$$\int_{-\infty}^{0} e^{\gamma r}\alpha_\tau(r)dr \leq \varepsilon^p.$$

Which achieves the proof of Proposition 2.3.7.

**Proof of Theorem 2.3.5** We can break down the demonstration into two part. In the first part, we that, assuming (2.3.19), equation (2.1.1) has a unique mild solution belonging to $\mathrm{UB}(\mathbb{R},\mathbb{X})$, and given by (2.3.18). In the second, using the technique by Kaminski et al. [KMRdF15], we show that solution (2.3.18) is Weyl almost periodic.

## First part

Clearly, following the same steps as in the proof of Theorem 2.3.4, $u$ is solution to

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s,u(s))ds$$

if, and only if,

$$u(t) = T(t-a)u(a) + \int_{a}^{t} T(t-s)f(s,u(s))ds,$$

for all $t \geq a$ for each $a \in \mathbb{R}$, which means that $u$ is a mild solution to (2.1.1).

We introduce an operator $\Gamma$ by

$$\Gamma u(t) = \int_{-\infty}^{t} T(t-s)f(s,u(s))ds.$$

Let us show that, assuming (2.3.19), the operator $\Gamma$ maps $\mathrm{CB}(\mathbb{R},\mathbb{X})$ into $\mathrm{CB}(\mathbb{R},\mathbb{X})$ and has a unique fixed point.

**First step.** Let us show that $\Gamma u$ is bounded, for any $u \in \mathrm{CB}(\mathbb{R},\mathbb{X})$. From the hypotheses (H1), Hölder's inequality $(\frac{1}{p},\frac{1}{q})$ and triangular inequality, we have

$$\begin{aligned}\sup_{t\in\mathbb{R}}\|\Gamma u(t)\| \leq & M\sup_{t\in\mathbb{R}}\int_{-\infty}^{t} e^{-\delta(t-s)}\|f(s,u(s))\|\,ds\\ \leq & M\sup_{t\in\mathbb{R}}\sum_{k=1}^{\infty}\left(\int_{t-k}^{t-k+1} e^{-q\delta(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{t-k}^{t-k+1}\|f(s,u(s))-f(s,0)\|^p\,ds\right)^{\frac{1}{p}}\\ & + M\sup_{t\in\mathbb{R}}\sum_{k=1}^{\infty}\left(\int_{t-k}^{t-k+1} e^{-q\delta(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{t-k}^{t-k+1}\|f(s,0)\|^p\,ds\right)^{\frac{1}{p}}.\end{aligned}$$

Since the singleton $\{0\}$ is compact, $f(.,0)$ is bounded with respect to Stepanov's norm. By (H2) and Hölder's inequality, we have

$$\begin{aligned}
\sup_{t\in\mathbb{R}}\|\Gamma u(t)\| \leq & M\sup_{t\in\mathbb{R}}\sum_{k=1}^{\infty}\left(\int_{t-k}^{t-k+1} e^{-q\delta(t-s)}ds\right)^{\frac{1}{q}}\left(\left(\int_{t-k}^{t-k+1}\|L(s)\|^p\|u(s)\|^p\,ds\right)^{\frac{1}{p}}+\|f(.,0)\|_{\mathbb{S}^p}\right)\\
\leq & M\frac{e^{\delta}}{e^{\delta}-1}\sup_{t\in\mathbb{R}}\left(\left(\int_{t-k}^{t-k+1}\|L(s)\|^p\,ds\right)^{\frac{1}{p}}\|u(.)\|_{\infty}+\|f(.,0)\|_{\mathbb{S}^p}\right)\\
\leq & M\frac{e^{\delta}}{e^{\delta}-1}\Big(\|L(.)\|_{\mathbb{S}^p}\|u(.)\|_{\infty}+\|f(.,0)\|_{\mathbb{S}^p}\Big)<\infty.
\end{aligned}$$

Thus $\Gamma u$ is bounded.

**Second step** Let us show that the mapping $t\mapsto\Gamma u$ is continuous on $\mathbb{R}$, for every $u\in \mathrm{CB}(\mathbb{R},\mathbb{X})$. For, let fix $u\in\mathrm{CB}(\mathbb{R},\mathbb{X})$. We have

$$\Gamma u(t)=\int_{-\infty}^{t}T(t-s)f(s,u(s))ds=\sum_{n=0}^{\infty}\int_{t-n-1}^{t-n}T(t-s)f(s,u(s))ds.$$

Let us set, for $n\in\mathbb{N}$,

$$\Gamma_n u(t)=\int_{t-n-1}^{t-n}T(t-s)f(s,u(s))ds,\ t\in\mathbb{R}.$$

Let us check that $\Gamma_n(u)\in\mathrm{CB}(\mathbb{R},\mathbb{X})$. To this end, let $t_0\in\mathbb{R}$. Put $g(s)=:f(s,u(s))$. Clearly $g\in L^p_{loc}(\mathbb{R},\mathbb{X})$. Hence, in view of [KJF77, Teorem 2.4.2 p.70]

$$\lim_{h\to 0}\int_{t_0-n-1}^{t_0-n}\|g(s+h)-g(s)\|^p\,ds=0. \tag{2.3.40}$$

Using Hölder's inequality, one has

$$\begin{aligned}
&\lim_{h\to 0}\|\Gamma_n u(t_0+h)-\Gamma_n u(t_0)\|\\
&\qquad\leq\lim_{h\to 0}M\int_{t_0-n-1}^{t_0-n}e^{-\delta(t_0-s)}\|g(s+h)-g(s)\|\,ds\\
&\qquad\leq M\left(\int_{t_0-n-1}^{t_0-n}e^{-q\delta(t_0-s)}ds\right)^{\frac{1}{q}}\left(\lim_{h\to 0}\int_{t_0-n-1}^{t_0-n}\|g(s+h)-g(s)\|^p\,ds\right)^{\frac{1}{p}}.
\end{aligned}$$

The continuity of $\Gamma_n u$ is then a straightforward consequence of (2.3.40).

The boundedness of $\Gamma_n u$ is a simple consequence of the following estimation

$$\begin{aligned}
\|\Gamma_n u\|_{\infty}\leq&\sup_{t\in\mathbb{R}}\left(\int_{t-n-1}^{t-n}e^{-q\delta(t-s)}ds\right)^{\frac{1}{q}}\Big(\|L(.)\|_{\mathbb{S}^p}\|u(.)\|_{\infty}+\|f(.,0)\|_{\mathbb{S}^p}\Big)\\
\leq&e^{-n\delta}\Big(\|L(.)\|_{\mathbb{S}^p}\|u(.)\|_{\infty}+\|f(.,0)\|_{\mathbb{S}^p}\Big),
\end{aligned}$$

from which deduce in addition that the sense $\sum_{n=0}^{\infty}\|\Gamma_n u(t)\|$ is uniformly convergent. The continuity of $\Gamma u$ is provided by the continuity of $\Gamma_n u$ for each $n$ and the uniform convergence of the series $\sum_{n\geq 0}^{\infty}\|\Gamma_n u(t)\|$. Thus $\Gamma$ maps $\mathrm{CB}(\mathbb{R},\mathbb{X})$ into $\mathrm{CB}(\mathbb{R},\mathbb{X})$.

**Third step** Let us show that, assuming (2.3.19), $\Gamma$ has a fixed point in CB$(\mathbb{R},\mathbb{X})$. For every $t\in\mathbb{R}$, we have from (H1)

$$\begin{aligned}\|\Gamma u(t)-\Gamma v(t)\| &\leq M\int_{-\infty}^{t} e^{-\delta(t-s)}\,\|f(s,u(s))-f(s,v(s))ds\|\\ &\leq M\sum_{k=1}^{+\infty}\int_{t-k}^{t-k+1} e^{-\delta(t-s)}\,\|f(s,u(s))-f(s,v(s))\|\,ds.\end{aligned}$$

Using Hölder's inequality and (H2), we get for every $t\in\mathbb{R}$,

$$\begin{aligned}\|\Gamma u(t)-\Gamma v(t)\| \leq& M\sum_{k=1}^{+\infty}\left(\int_{t-k}^{t-k+1} e^{-\delta q(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{t-k}^{t-k+1}\|L(s)\|^{p}\,\|u(s)-v(s)\|^{p}\,ds\right)^{\frac{1}{p}}\\ &\leq M\sum_{k=1}^{+\infty}\left(\int_{t-k}^{t-k+1} e^{-\delta q(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{t-k}^{t-k+1}\|L(s)\|^{p}\,ds\right)^{\frac{1}{p}}\|u-v\|_{\infty}\\ &\leq\left(\frac{M^{q}}{\delta q}\right)^{\frac{1}{q}}\left(\frac{e^{\delta}}{e^{\delta}-1}\right)\|L\|_{\mathbb{S}^{p}}\,\|u-v\|_{\infty}\,.\end{aligned}$$

Hence, it follows that, for each $t\in\mathbb{R}$,

$$\|\Gamma u-\Gamma v\|_{\infty}\leq k\,\|u-v\|_{\infty}\,.$$

Since $k<1$, we conclude that $\Gamma$ is a contraction operator. We conclude that, there exists a unique mild solution to (2.1.1) in CB$(\mathbb{R},\mathbb{X})$.

## Second part

In this part, we show that, under (2.3.19) and (2.3.20), the solution $u\in$ CB$(\mathbb{R},\mathbb{X})$ to (2.1.1) is Weyl almost periodic.

By (H1), we obtain that, for any $\tau\in\mathbb{R}$ and any $t\in\mathbb{R}$,

$$\begin{aligned}\|u(t+\tau)-u(t)\| &\leq\int_{-\infty}^{t}\|T(t-s)\|\,\|f(s+\tau,u(s+\tau))-f(s,u(s))\|\,ds\\ &\leq M\int_{-\infty}^{t} e^{-\delta(t-s)}\,\|f(s+\tau,u(s+\tau))-f(s,u(s))\|\,ds.\end{aligned}$$

By Hölder's inequality $(\frac{2}{p},\frac{p-2}{p})$ and triangular inequality, we obtain

$$\begin{aligned}\|u(t+\tau)-u(t)\| \leq& M\left(\frac{2p-4}{p\delta}\right)^{\frac{p-2}{p}}\left(\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)}\,\|f(s+\tau,u(s+\tau))-f(s+\tau,u(s))\|^{\frac{p}{2}}\,ds\right)^{\frac{2}{p}}\\ &+M\left(\frac{2p-4}{p\delta}\right)^{\frac{p-2}{p}}\left(\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)}\,\|f(s+\tau,u(s))-f(s,u(t))\|^{\frac{p}{2}}\,ds\right)^{\frac{2}{p}}\\ =:& M\left(\frac{2p-4}{p\delta}\right)^{\frac{p-2}{p}}\Big(I_1(t)+I_2(t)\Big).\end{aligned}$$

For $I_1(t)$, by (H2) and Hölder's inequality $(\frac{1}{2}, \frac{1}{2})$, we have

$$\begin{aligned}
I_1(t) &= \left( \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \left\| f(s+\tau, u(s+\tau)) - f(s+\tau, u(s)) \right\|^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \\
&\leq \left( \sum_{k=0}^{+\infty} e^{-\frac{p\delta}{4}k} \int_{t-k-1}^{t-k} \| L(s+\tau) \|^p \, ds \right)^{\frac{1}{p}} \left( \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \| u(s+\tau) - u(s) \|^p \, ds \right)^{\frac{1}{p}} \\
&\leq \left( \frac{e^{\frac{p\delta}{4}}}{e^{\frac{p\delta}{4}} - 1} \right)^{\frac{1}{p}} \| L \|_{\mathbb{S}^p} \left( \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \| u(s+\tau) - u(s) \|^p \, ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Next, let us consider $I_2(t)$. By Hölder's inequality $(\frac{1}{2}, \frac{1}{2})$, we have

$$\begin{aligned}
I_2(t) &= \left( \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \| f(s+\tau, u(s)) - f(s, u(s)) \|^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \\
&\leq \left( \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} ds \right)^{\frac{1}{p}} \left( \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \| f(s+\tau, u(s)) - f(s, u(s)) \|^p \, ds \right)^{\frac{1}{p}} \\
&\leq \left( \frac{4}{p\delta} \right)^{\frac{1}{p}} \left( \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \| f(s+\tau, u(s)) - f(s, u(s)) \|^p \, ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Thus, for all $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$ and $l > 0$, we have

$$\begin{aligned}
\frac{1}{l} \int_{\xi}^{\xi+l} \| u(t+\tau) - u(t) \|^p \, dt \leq & M^p \left( \frac{2p-4}{p\delta} \right)^{p-2} \frac{1}{l} \int_{\xi}^{\xi+l} \| I_1(t) + I_2(t) \|^p \, dt \\
\leq & M^p 2^{p-1} \left( \frac{2p-4}{p\delta} \right)^{p-2} \left( \frac{1}{l} \int_{\xi}^{\xi+l} \| I_1(t) \|^p \, dt + \frac{1}{l} \int_{\xi}^{\xi+l} \| I_2(t) \|^p \, dt \right).
\end{aligned}$$

Let us estimate the first term of the last inequality. We have, for every $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$ and $l > 0$

$$\begin{aligned}
\frac{1}{l} \int_{\xi}^{\xi+l} \| I_1(t) \|^p \, dt &\leq \| L \|_{\mathbb{S}^p}^p \left( \frac{e^{\frac{p\delta}{4}}}{e^{\frac{p\delta}{4}} - 1} \right) \frac{1}{l} \int_{\xi}^{\xi+l} \left( \left( \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \| u(s+\tau) - u(s) \|^p \, ds \right)^{\frac{1}{p}} \right)^p dt \\
&\leq \| L \|_{\mathbb{S}^p}^p \left( \frac{e^{\frac{p\delta}{4}}}{e^{\frac{p\delta}{4}} - 1} \right) \frac{1}{l} \int_{\xi}^{\xi+l} \int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \| u(s+\tau) - u(s) \|^p \, ds \, dt.
\end{aligned}$$

By a change of variables and Fubini's theorem, it follows that

$$\frac{1}{l} \int_{\xi}^{\xi+l} \| I_1(t) \|^p \, dt \leq \| L \|_{\mathbb{S}^p}^p \left( \frac{e^{\frac{p\delta}{4}}}{e^{\frac{p\delta}{4}} - 1} \right) \int_{-\infty}^{\xi} e^{-\frac{p\delta}{4}(\xi-s)} \left( \frac{1}{l} \int_{s}^{s+l} \| u(t+\tau) - u(t) \|^p \, dt \right) ds. \tag{2.3.41}$$

For the second term, using again a change of variables and Fubini's theorem, we have, for all $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$ and $l > 0$,

$$\begin{aligned}\frac{1}{l}\int_{\xi}^{\xi+l}\|I_2(t)\|^p\,dt &\leq \left(\frac{4}{p\delta}\right)\frac{1}{l}\int_{\xi}^{\xi+l}\left(\left(\int_{-\infty}^{t}e^{-\frac{p\delta}{4}(t-s)}\|f(s+\tau,u(s))-f(s,u(s))\|^p\,ds\right)^{\frac{1}{p}}\right)^p dt\\ &\leq \left(\frac{4}{p\delta}\right)\int_{-\infty}^{\xi}e^{-\frac{p\delta}{4}(\xi-s)}\frac{1}{l}\int_{s}^{s+l}\|f(t+\tau,u(t))-f(t,u(t))\|^p\,dt\,ds\\ &= \left(\frac{4}{p\delta}\right)\int_{-\infty}^{\xi}e^{-\frac{p\delta}{4}(\xi-s)}h_{\tau,l}(s)ds,\end{aligned}$$

where

$$h_{\tau,l}(s) := \frac{1}{l}\int_{s}^{s+l}\|f(t+\tau,u(t))-f(t,u(t))\|^p\,dt.$$

Summing up, for all $\xi, \tau \in \mathbb{R}$ and $l > 0$, we obtain

$$0 \leq g_{\tau,l}(\xi) := \frac{1}{l}\int_{\xi}^{\xi+l}\|u(t+\tau)-u(t)\|^p\,dt \leq C\alpha_{\tau,l}(\xi) + \beta\int_{-\infty}^{\xi}e^{-\delta_1(\xi-s)}g_{\tau,l}(s)ds,$$

with

$$C := \frac{M^p 2^{p+1}}{p\delta}\left(\frac{2p-4}{p\delta}\right)^{p-2},\quad \alpha_{\tau,l}(\xi) := \int_{-\infty}^{\xi}e^{-\frac{p\delta}{4}(\xi-s)}h_{\tau,l}(s)ds,\quad \delta_1 := \frac{p\delta}{4},$$

and

$$\beta := M^p 2^{p-1}\|L\|_{\mathbb{S}^p}^p\left(\frac{2p-4}{p\delta}\right)^{p-2}\left(\frac{e^{\frac{p\delta}{4}}}{e^{\frac{p\delta}{4}}-1}\right).$$

It should be noted that the hypothesis (2.3.20) is equivalent $\delta_1 > \beta$. We conclude by Lemma 2.3.6 that, for all $\gamma \in ]0, \delta_1 - \beta]$ such that $\int_{-\infty}^{0} e^{\gamma r}\alpha_{\tau,l}(r)dr$ converge, we have, for every $\xi \in \mathbb{R}$,

$$0 \leq g_{\tau,l}(\xi) \leq C\alpha_{\tau,l}(\xi) + C\beta\int_{-\infty}^{\xi}e^{-\gamma(\xi-r)}\alpha_{\tau,l}(r)dr.$$

We use Theorem 2.2.4 to get the Weyl almost periodicity of $u$. We take $\xi = 0$, then for all $\tau \in \mathbb{R}$ and $l > 0$, we have

$$0 \leq g_{\tau,l}(0) = \frac{1}{l}\int_{0}^{l}\|u(t+\tau)-u(t)\|^p\,dt \leq C\alpha_{\tau,l}(0) + C\beta\int_{-\infty}^{0}e^{\gamma r}\alpha_{\tau,l}(r)dr.$$

Therefore, in view of Proposition 2.3.7, for all $\varepsilon > 0$, there exist $l = l(\varepsilon) > 0$ and a relatively dense set $T_{\mathbb{S}_l^p}(\varepsilon, f, \mathbb{K})$ such that

$$\frac{1}{l}\int_{0}^{l}\|u(t+\tau)-u(t)\|^p\,dt \leq C(1+\beta)\varepsilon^p,\quad \forall\tau \in T_{\mathbb{S}_l^p}(\varepsilon, f, \mathbb{K}).$$

Weyl almost periodicity of $u$ is then proved thanks to the inclusion

$$T_{\mathbb{S}_l^p}\left(\varepsilon^p C(1+\beta), u\right) \supset T_{\mathbb{S}_l^p}\left(\varepsilon^p, f, \mathbb{K}\right).$$

For $p = 2$, the proof is similar to that of $p > 2$. The details are omitted. □

**Remark 2.3.8** *The results obtained here can also be applied to the more general case $p \geq 1$ provided that $L(.)$ in (H1) is independent of $t$.*

CHAPITRE 3

# Weyl almost periodic solution in distribution for stochastic differential equation with Weyl almost periodic coefficients

## 3.1 Introduction

The concept of almost periodicity for random processes was introduced for the first time by Slutsky [Slu38] in the late 1930s. In recent decades, this concept has been extensively developed and studied in connection with problems of stochastic differential equations by many authors. Among these authors there are those who were interested in studying almost periodicity in various senses (in one-dimentional distributions, in finite-dimentional distribution and in path distribution) of the solutions to some stochastic differential equations with almost periodic coefficients. For example, we can mention, the work of Tudor and his collaborators [AT98, DPT95, Hal87, HMT88, MT89, Tud92], Kamenskii et al. [KMRdF15]. Others have asserted the existence of almost periodic solutions in quadratic mean and in $p$th-mean ($p \geq 2$) for semi-linear stochastic differential equations with Bohr and Stepanov almost periodic coefficients, however, in [BCM$^+$15, MRdF13], counter-examples were given to these latter results.

The question of the existence of almost periodic solutions in distribution for semi-linear stochastic differential equations with Stepanov almost periodic coefficients was examined in [BCM$^+$17] by Bedouhene et al., where they generalized the study made by Andres and Pennequin [AP12b] to the stochastic case. Regarding the case of Weyl, the only results we are aware of concern the deterministic case (see [ABR06, BIMRdF, Kos18, Rad04]).

The present work aims to generalize the study made in the deterministic case by Bedouhene et al. [BIMRdF] in the stochastic case. More precisely, we study the existence of Weyl almost periodic in distribution solutions of the following stochastic differential equation

$$dX(t) = AX(t)dt + F(t, X(t))dt + G(t, X(t))dW(t), \tag{3.1.1}$$

where $A$ is a linear operator (possibly unbounded) which generates an exponentially stable $C_0$ -semigroup on a Hilbert separable space, $W(t)$ is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathrm{P})$, and $F$ and $G$ are Weyl almost periodic functions satisfying some conditions.

The paper is organized as follows. In the second section we recall some notions of almost periodicity and normality in Weyl's sense and we introduce different types of almost periodicity in Weyl's sense for random processes. Finally, in the third section we prove the existence and uniqueness of the mild Weyl almost periodic solution in finite-dimensional distribution to (3.1.1).

## 3.2 Notations and Preliminaries

Let $(\mathbb{X}, d)$ be a complete metric space. When $\mathbb{X}$ is a Banach space, its norm is denoted by $\|.\|$ and $d$ is the distance induced by $\|.\|$. We denote also by $\mathrm{UB}(\mathbb{R},\ \mathbb{X})$ the Banach space of continuous and uniformly bounded functions from $\mathbb{R}$ to $\mathbb{X}$, endowed with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|\,.$$

Let $p \geq 1$. We remind that the *$p$-Stepanov distance* is given by

$$D_{\mathbb{S}^p_l}(f, g) = \sup_{t\in\mathbb{R}} \left(\frac{1}{l}\int_t^{t+l} d^p(f(s), g(s))ds\right)^{\frac{1}{p}} \qquad f, g \in \mathrm{L}^p_{loc}(\mathbb{R}, \mathbb{X}).$$

The metrics $D_{\mathbb{S}^p_l}$, $l > 0$ are equivalent and $l \mapsto D_{\mathbb{S}^p_l}(f, g)$ is nonincreasing [Bes32, p.72], thus the following limit always exists :

$$D_{\mathbb{W}^p}(f, g) = \lim_{l\to\infty} D_{\mathbb{S}^p_l}(f, g) \qquad f, g \in \mathrm{L}^p_{loc}(\mathbb{R}, \mathbb{X}).$$

This limit is called the *$p$-Weyl distance*.

For a Banach space $(\mathbb{X}, \|.\|)$ and $f : \mathbb{R} \to \mathbb{X}$, we denote

$$\|f\|_{\mathbb{S}^p_l} = \sup_{t\in\mathbb{R}} \left(\frac{1}{l}\int_t^{t+l} \|f(s)\|^p\, ds\right)^{\frac{1}{p}}$$

and

$$\|f\|_{\mathbb{W}^p} = \lim_{l\to\infty} \|f\|_{\mathbb{S}^p_l}\,.$$

**Definition 3.2.1** *A set $T \subset \mathbb{R}$ is said to be* relatively dense *if there exists a number $k > 0$ such that any interval of length $k$ contains at least one number of $T$.*

**Definition 3.2.2** [Bes32] *Let $p \geq 1$. We say that a locally $p$-integrable function $f : \mathbb{R} \to \mathbb{X}$ (i.e., $f \in \mathrm{L}^p_{loc}(\mathbb{R}, \mathbb{X})$) is* $p$-Weyl almost periodic *if, for each $\varepsilon > 0$, there exists a number $l = l(\varepsilon) > 0$ and a relatively dense set $T_{\mathbb{S}^p_l}(\varepsilon, f) \subset \mathbb{R}$ such that*

$$\sup_{t\in I} \left( \frac{1}{l} \int_t^{t+l} d^p \left( f(s), f(s+\tau) \right) ds \right)^{\frac{1}{p}} < \varepsilon \quad \forall \tau \in T_{\mathbb{S}^p_l}(\varepsilon, f), \tag{3.2.1}$$

*where $I = \mathbb{R}$. We denote the set of all such functions by $\mathbb{W}^p\mathrm{AP}(\mathbb{R}, \mathbb{X})$.*

To study the existence and uniqueness of solution to semi-linear differential equations, we also need to recall the definition of almost periodicity in Weyl's sense for parametric functions.

**Definition 3.2.3** 1. *We say that a function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is* Weyl almost periodic *if, for every $x \in \mathbb{X}$, the function $f(.,x)$ is Weyl almost periodic. We denote by $\mathbb{W}^p\mathrm{AP}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ the space of such functions.*

2. *A function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}, (t,u) \mapsto f(t,u)$ with $f(.,u) \in \mathrm{L}^p_{loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{X}$ is said to be* Weyl almost periodic in $t \in \mathbb{R}$ uniformly with respect compact subsets of $\mathbb{X}$ *(respectively uniformly with respect bounded subset of $\mathbb{X}$) if, $f(.,x)$ is $\mathbb{W}^p$-almost periodic uniformly with respect to $x \in \mathbb{K}$ (respectively $x \in \mathbb{B}$) for all compact sets $\mathbb{K}$ in $\mathbb{X}$ (respectively for all bounded $\mathbb{B} \in \mathbb{X}$). We denote by $\mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ (respectively $\mathbb{W}^p\mathrm{AP}_{\mathbb{B}}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$) the set of such functions.*

Our aim is to study Weyl almost periodicity for stochastic processes, for this we also need to introduce the concept of almost periodic random functions (almost periodic random processes).

## Weyl almost periodicity for stochastic processes

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, let $(\mathbb{X}, \|.\|)$ be a separable Banach space, and let $X = (X_t)_{t\in\mathbb{R}}$ be an $\mathbb{X}$-valued stochastic process. We denote by law $(X(t))$ the distribution of the random variable $X(t)$, by $\mathrm{E}(X)$ its expectation, and by $\mathcal{M}_1^+(\mathbb{X})$ the space of Borel probability measures on $\mathbb{X}$ endowed which the topology of narrow convergence, which is the coarsest topology such that the mappings $\mu \to \mu(f)$ are continuous for all bounded continuous functions $f : \mathbb{X} \to \mathbb{R}$. For $f \in C_b(\mathbb{X})$, (where $C_b(\mathbb{X})$ is the Banach space of continuous and bounded functions defined on $\mathbb{X}$), we define

$$\|f\|_{\mathrm{L}} = \sup \left\{ \frac{|f(x) - f(y)|}{d_{\mathbb{X}}(x,y)}; \qquad x \neq y \right\},$$

$$\|f\|_{\infty} = \sup_{x\in\mathbb{X}} |f(x)|.$$

and

$$\|f\|_{\mathrm{BL}} = \max \left\{ \|f\|_{\mathrm{L}}, \|f\|_{\infty} \right\}.$$

For $\mu, \nu \in \mathcal{M}_1^+(\mathbb{X})$, we define

$$d_{\mathrm{BL}}(\mu, \nu) = \sup_{\|f\|_{\mathrm{BL}}\leq 1} \left| \int_{\mathbb{X}} f d(\mu - \nu) \right|.$$

The space $(\mathcal{M}_1^+(\mathbb{X}), d_{\text{BL}})$ is a Polish space (see [Dud89, Corollary 11.5.5, P.317])). We recall another distance that we can use for convergence in law of random variables. Let $X, Y$ be $\mathbb{X}$-valued random variables defined on some probability space. Recall that the Wassertein distance betwen the distribution of $X$ and $Y$ is

$$\mathbf{W}^p(\mu,\nu) = \inf\left\{ \left(\mathrm{E}(\|X-Y\|^p)\right)^{1/p};\ \text{law}(X)=\mu,\, \text{law}(Y)=\nu \right\}. \tag{3.2.2}$$

The space $(\mathcal{M}_1^+(\mathbb{X}), \mathbf{W}^p)$ is also a Polish space.

**Proposition 3.2.4** [Vil08, Remark 6.5] *For $\mu, \upsilon \in \mathcal{M}_1^+(\mathbb{X})$, we have*

$$d_{\text{BL}}(\mu,\nu) \le d_L(\mu,\nu) = \mathbf{W}^1(\mu,\nu) \le \mathbf{W}^p(\mu,\nu),$$

*for all $p \ge 1$.*

Now, we give the definitions of Weyl almost periodicity for a stochastic process $X = (X_t)_{t\in\mathbb{R}}$ in different senses. These definitions are inspired by Tudor [HMT88, MT89].

**Definition 3.2.5** — *We say that $X$ is* Weyl almost periodic in one-dimensional distribution *if the mapping $t \to \text{law}\,(X(t))$ from $\mathbb{R}$ to $(\mathcal{M}_1^+(\mathbb{X}), d_{\text{BL}})$ is Weyl almost periodic i.e., if, for every $\varepsilon > 0$, there exist $k = k(\varepsilon)$, $l = l(\varepsilon) > 0$ such that any interval of length $k$ contains at least an $\varepsilon$-Weyl almost period (i.e a number $\tau$) for which*

$$\sup_{x\in\mathbb{R}} \left( \frac{1}{l} \int_x^{x+l} d^p_{\text{BL}}(\text{law}\,(X(t+\tau))\,, \text{law}\,(X(t)))dt \right)^{\frac{1}{p}} < \varepsilon.$$

— *We say that $X$ is* Weyl almost periodic in finite-dimensional distribution *if, for every finite sequence $t_1, ..., t_m \in \mathbb{R}$ the mapping $t \mapsto \text{law}\,(X(t+t_1), X(t+t_2), X(t+t_m))$ from $\mathbb{R}$ to $(\mathcal{M}_1^+(\mathbb{X}^n), d_{\text{BL}})$ is Weyl almost periodic.*

— *If $X$ has a version (shol dented by $X$ for simplicity) where trajectories lie in some complete metric space $(\mathbb{Y}, d_{\mathbb{Y}})$ of functions, we say that $X$ is* Weyl almost periodic in path distribution *if the mapping $t \mapsto \text{law}\,(X(t+.))$ from $\mathbb{R}$ to $\mathcal{M}_1^+(\mathbb{Y})$ is Weyl almost periodic, where $\mathcal{M}_1^+(\mathbb{Y})$ is endowed with the distance $d_{\text{BL}}$ associated with $d_{\mathbb{Y}}$.*

Obviously Weyl almost periodicity in path distribution implies Weyl presque periodicity in finite-dimensional distribution, which implies Weyl almost periodicity one dimensional distribution.

We shall prove Weyl almost periodicity in finite-dimensional distribution for solutions of semilinear stochastic differential in Hilbert spaces.

**Definition 3.2.6** *Let $\mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X})$ be the space of measurable mappings from $\Omega$ to $\mathbb{X}$, endowed with the topology of convergence in probability. Recall that the topology of $\mathrm{L}^0(\Omega, P, \mathbb{X})$ is induced by the distance*

$$d_{\text{prob}}(U,V) = \mathrm{E}(\|U-V\| \wedge 1),$$

*which is complete.*

*The process $X$ is said to be* Weyl almost periodic in probability *if the mapping $X : t \to \mathrm{L}^0(\Omega, \mathrm{P}, \mathbb{X})$ is Weyl almost periodic.*

Let $\mathrm{L}^p(\mathrm{P}, \mathbb{X})$ be the space of $\mathbb{X}$-valued random variables with a finite $p$-moment ($p \geq 1$).

**Definition 3.2.7** *The process* $X \in \mathrm{L}^p_{loc}(\mathbb{R}, \mathrm{L}^p(\Omega, \mathbb{X}))$ *is said to be* Weyl almost periodic in $p$th mean*, if the mapping* $t \mapsto X(t), \mathbb{R} \to \mathrm{L}^p(\Omega, \mathrm{P}, \mathbb{X})$ *is Weyl almost periodic.*

When $p = 2$, we say that $X$ is *Weyl almost periodic in square-mean.*

# 3.3 Weyl almost periodicity in distribution for solutions of SDEs

Let $(\mathbb{H}_1, \|.\|_{\mathbb{H}_1})$ and $(\mathbb{H}_2, \|.\|_{\mathbb{H}_2})$ be separable Hilbert spaces, and let us denote by $L(\mathbb{H}_1, \mathbb{H}_2)$ the space of bounded linear operators from $\mathbb{H}_1$ to $\mathbb{H}_2$, and by $L_2(\mathbb{H}_1, \mathbb{H}_2)$ the space of Hilbert-Schmidt operators from $\mathbb{H}_1$ to $\mathbb{H}_2$.

We consider the semilinear stochastic differential equation (SDEs)

$$dX_t = AX(t)dt + F(t, X(t))dt + G(t, X(t))dW(t), \quad t \in \mathbb{R}, \tag{3.3.1}$$

where $A : D(A) \subset \mathbb{H}_2 \to \mathbb{H}_2$ is a densely defined closed (possibly unbounded) linear operator and $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$, and $G : \mathbb{R} \times \mathbb{H}_2 \to L(\mathbb{H}_1, \mathbb{H}_2)$ are measurable functions. We consider the following hypotheses :

(H1) $W(t)$, $t \in \mathbb{R}$, is an $\mathbb{H}_1$-valued Wiener process with nuclear covariance operator $Q$ (we denote by $\operatorname{tr} Q$ the trace of $Q$), defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathrm{P})$.

(H2) $A : \mathrm{D}(A) \to \mathbb{H}_2$ is the infinitesimal generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ such that there exists a constant $\delta > 0$ with

$$\|T(t)\|_{L(\mathbb{H}_2)} \leq e^{-\delta t}, t \geq 0.$$

(H3) There exists a positive constant $M$ such that the mappings $F : \mathbb{R} \times \mathbb{H}_2 \to \mathbb{H}_2$ and $G : \mathbb{R} \times \mathbb{H}_2 \to L_2(\mathbb{H}_1, \mathbb{H}_2)$ satisfy

$$\|F(t, x)\|_{\mathbb{H}_2} + \|G(t, x)\|_{L_2(\mathbb{H}_1,\mathbb{H}_2)} \leq M(1 + \|x\|_{\mathbb{H}_2})$$

for all $t \in \mathbb{R}$.

(H4) The functions $F$ and $G$ are Lipschitz, more precisely there exists a positive function $L(.) \in \mathbb{S}^q(\mathbb{R})$, $q \geq 2$, such that

$$\|F(t, x) - F(t, y)\|_{\mathbb{H}_2} + \|G(t, x) - G(t, y)\|_{L_2(\mathbb{H}_1,\mathbb{H}_2)} \leq L(t)\|x - y\|_{\mathbb{H}_2}$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{H}_2$.

(H5) $F \in \mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R} \times \mathbb{H}_2, \mathbb{H}_2)$ and $G \in \mathbb{W}^p\mathrm{AP}_{\mathbb{K}}(\mathbb{R} \times \mathbb{H}_2, L_2(\mathbb{H}_1, \mathbb{H}_2))$ ($p \geq 2$).

The Banach space $\mathrm{UB}\big(\mathbb{R}, \mathrm{L}^p(\mathrm{P}, \mathbb{H}_2)\big)$ of $p$-mean uniformly bounded stochastic processes is endowed with the norm

$$\|X\|_{\infty,p} = \left(\sup_{t\in\mathbb{R}} \left(\mathrm{E}\|X(t)\|^p_{\mathbb{H}_2}\right)\right)^{\frac{1}{p}}.$$

**Theorem 3.3.1** *Let the conditions* (H1) − (H5) *be fulfilled. For* $p \geq 2$, *the equation* (3.3.1) *admits a unique mild solution in* $\mathrm{UB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ *given by*

$$X(t) = \int_{-\infty}^{t} T(t-s)F\big(s, X(s)\big)ds + \int_{-\infty}^{t} T(t-s)G\big(s, X(s)\big)dW(s), \qquad (3.3.2)$$

*provided that* $\frac{2}{p} + \frac{2}{q} = 1$ *and*

$$\left( \frac{2^{\frac{pq+2p-q}{q}} \|L\|^p_{\mathbb{S}^p}}{\delta^{\frac{pq+2p}{2q}} \left(1 - \exp(\frac{p\delta}{4})\right)} + \frac{2^{\frac{pq+p-q}{q}} C_p \,(\mathrm{tr}\, Q)^{\frac{p}{2}} \|L\|^p_{\mathbb{S}^p}}{(q\delta)^{\frac{p}{p}} \left(1 - \exp(-\frac{p\delta}{2})\right)} \right)^{\frac{1}{p}} < 1, \qquad (3.3.3)$$

*with* $C_p = \frac{1}{(2c)^{\frac{p}{2}}} \left( \frac{2+2c}{p-1} - 2^{\frac{p}{2}} \right)$ *for any* $c > (p-1)2^{\frac{p}{2}} - 1$.

*If furthermore* $p \geq q \geq 2$ *with* $\frac{2}{p} + \frac{2}{q} = 1$ *and*

$$\frac{2^{2p} \|L\|^p_{\mathbb{S}^q}}{p\delta^{\frac{p}{2}+1} \left(1 - \exp(\frac{4}{q\delta})\right)^{\frac{p}{q}}} + \frac{2^{2p} (\mathrm{tr}\, Q)^{\frac{p}{2}} \|L\|^p_{\mathbb{S}^q}}{p\delta \left(1 - \exp(\frac{2}{q\delta})\right)^{\frac{p}{q}}} < 1, \qquad (3.3.4)$$

*then* $X$ *is Weyl almost periodic in finite-dimensional distribution.*

**Proof**

Let $\Gamma$ be the operator defined by

$$\Gamma X(t) = \int_{-\infty}^{t} T(t-s)F(s, X(s))ds + \int_{-\infty}^{t} T(t-s)G(s, X(s))dW(s) \quad \text{for } X \in \mathrm{UB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big).$$

In the first part, we show the existence and uniqueness of a mild solution in $\mathrm{UB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$.

## First part

This part is divided into two stages; in the first one, we show that the operator $\Gamma$ is well-defined, *i.e.*, the operator $\Gamma$ maps $\mathrm{UB}\big(\mathbb{R}, \mathrm{L}^p(\mathrm{P}, \mathbb{H}_2)\big)$ into itself, and in the second one, we show that $\Gamma$ is a contraction operator.

### Part 1-a

Let us show that $\Gamma$ is well-defined and that $\Gamma X$ has a continuous version.

Let us denote $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 X(t) = \int_{-\infty}^{t} T(t-s)F(s, X(s))ds$$

and

$$\Gamma_2 X(t) = \int_{-\infty}^{t} T(t-s)G(s, X(s))dW(s).$$

First, let us show that, $\Gamma(X(t))$ is well-defined and $\mathrm{L}^p$-bounded. Using Hölder inequality [KMRdF15, Lemma 3.4], we have, for any $X \in \mathrm{UB}\big(\mathbb{R}, \mathrm{L}^p(\mathrm{P}, \mathbb{H}_2)\big)$,

$$\begin{aligned}
\mathrm{E}\|\Gamma X(t)\|^p \leq & 2^{p-1}\left(\mathrm{E}\|\Gamma_1 X(t)\|^p + \mathrm{E}\|\Gamma_2 X(t)\|^p\right) \\
\leq & 2^{p-1}\,\mathrm{E}\left\|\int_{-\infty}^t T(t-s)F(s,X(s))ds\right\|^p + 2^{p-1}\,\mathrm{E}\left\|\int_{-\infty}^t T(t-s)G(s,X(s))dW(s)\right\|^p \\
\leq & 2^{p-1}\,\mathrm{E}\left(\left(\int_{-\infty}^t e^{-\delta(t-s)}ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^t e^{-\delta(t-s)}\left\|F(s,X(s))\right\|^2 ds\right)^{\frac{1}{2}}\right)^p \\
& + 2^{p-1}\,\mathrm{E}\left\|\int_{-\infty}^t T(t-s)G(s,X(s))dW(s)\right\|^p \\
\leq & \frac{2^{p-1}}{\delta^{\frac{p}{2}}}\,\mathrm{E}\left(\int_{-\infty}^t e^{-\delta(t-s)}\left\|F(s,X(s))\right\|^2 ds\right)^{\frac{p}{2}} + 2^{p-1}C_p\,\mathrm{E}\left(\operatorname{tr} Q\int_{-\infty}^t e^{-2\delta(t-s)}\left\|G(s,X(s))\right\|^2 ds\right)
\end{aligned}$$

where $C_p$ a constant given in the hypothesis (3.3.3). Using Hölder's inequality $(\frac{2}{p}, \frac{2}{q})$, we have

$$\begin{aligned}
\mathrm{E}\|\Gamma X(t)\|^p \leq & \frac{2^{p-1}}{\delta^{\frac{p}{2}}}\left(\int_{-\infty}^t e^{-\frac{q\delta}{4}(t-s)}ds\right)^{\frac{p}{q}}\mathrm{E}\int_{-\infty}^t e^{-\frac{p\delta}{4}(t-s)}\left\|F(s,X(s))\right\|^p ds \\
& + 2^{p-1}C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}\left(\int_{-\infty}^t e^{-\frac{q\delta}{2}(t-s)}ds\right)^{\frac{p}{q}}\mathrm{E}\int_{-\infty}^t e^{-\frac{p\delta}{2}(t-s)}\left\|G(s,X(s))\right\|^p ds \\
\leq & \frac{2^{\frac{pq+2p-q}{q}}}{\delta^{\frac{pq+2p}{2q}}\,(q)^{\frac{p}{q}}}\int_{-\infty}^t e^{-\frac{p\delta}{4}(t-s)}\,\mathrm{E}\|F(s,X(s))\|^p ds \\
& + \frac{2^{\frac{pq+p-q}{q}}C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}}{(q\delta)^{\frac{p}{q}}}\int_{-\infty}^t e^{-\frac{p\delta}{2}(t-s)}\,\mathrm{E}\|G(s,X(s))\|^p ds.
\end{aligned}$$

Then, using (H3), we have, since $1 + \mathrm{E}\|X(t)\|^p$ $(t \in \mathbb{R})$ is bounded

$$\begin{aligned}
\mathrm{E}\|\Gamma X(t)\|^p \leq & \frac{2^{\frac{2pq+2p-2q}{q}}M^p}{\delta^{\frac{pq+2p}{2q}}\,(q)^{\frac{p}{q}}}\int_{-\infty}^t e^{-\frac{p\delta}{4}(t-s)}\left(1+\mathrm{E}\|X(s)\|^p\right)ds \\
& + \frac{2^{\frac{2pq+p-2q}{q}}C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}M^p}{(q\delta)^{\frac{p}{q}}}\int_{-\infty}^t e^{-\frac{p\delta}{2}(t-s)}\left(1+\mathrm{E}\|X(s)\|^p\right)ds \\
\leq & \left(\frac{2^{\frac{2pq+2p}{q}}M^p}{p\,(q)^{\frac{p}{q}}\,\delta^{\frac{pq+2p+2q}{2q}}} + \frac{2^{\frac{2pq+p-2q}{q}}C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}M^p}{p\,(q)^{\frac{p}{q}}\,\delta^{\frac{p+q}{q}}}\right)\left(1+\mathrm{E}\|X(s)\|^p\right) < \infty.
\end{aligned}$$

This shows that the integrals in $\Gamma_1 X(t)$ and $\Gamma_2 X(t)$ are convergent and that $\Gamma(X)$ is $\mathrm{L}^p$-bounded. Furthermore, the continuity of $\Gamma_1 X$ can be proved as in [BIMRdF]. The continuity of $\Gamma_2 X$ follows from Da Prato-Zabczyk [DPZ14, Proposition 7.3], which show, in addition, that $X$ has a continuous version.

**Part 1-b**

Assume now (3.3.3), and let us show that $\Gamma$ has a fixed point in $\mathrm{UB}\big(\mathbb{R}, \mathrm{L}^p(\mathrm{P}, \mathbb{H}_2)\big)$. For every $t \in \mathbb{R}$ and $X, Y \in \mathrm{UB}\big(\mathbb{R}, \mathrm{L}^p(\mathrm{P}, \mathbb{H}_2)\big)$, we have from (H1)

$$\begin{aligned}
\mathrm{E}\,\|(\Gamma X)(t) - (\Gamma Y)(t)\|_{\mathbb{H}_2}^p \leq & 2^{p-1}\,\mathrm{E}\left(\int_{-\infty}^t e^{-\delta(t-s)}\|F(s,X(s)) - F(s,Y(s))\|_{\mathbb{H}_2} ds\right)^p \\
& + 2^{p-1}\,\mathrm{E}\left\|\int_{-\infty}^t T(t-s)[G(s,X(s)) - G(s,Y(s))]dW(s)\right\|_{\mathbb{H}_2}^p \\
= & 2^{p-1}\left(I_1(t) + I_2(t)\right).
\end{aligned}$$

Let us estimate $I_1(t)$. Using (H4) and Hölder's inequality, we obtain :

$$\begin{aligned}
I_1(t) &= \mathrm{E}\left(\int_{-\infty}^t e^{-\delta(t-s)}\|F(s,X(s)) - F(s,Y(s))\|_{\mathbb{H}_2} ds\right)^p \\
&\leq \left(\int_{-\infty}^t e^{-\delta(t-s)} ds\right)^{\frac{p}{2}} \mathrm{E}\left(\int_{-\infty}^t e^{-\delta(t-s)}\|F(s,X(s)) - F(s,Y(s))\|_{\mathbb{H}_2}^2 ds\right)^{\frac{p}{2}} \\
&\leq \frac{1}{\delta^{\frac{p}{2}}}\,\mathrm{E}\left(\int_{-\infty}^t e^{-\delta(t-s)}\|F(s,X(s)) - F(s,Y(s))\|_{\mathbb{H}_2}^2 ds\right)^{\frac{p}{2}}.
\end{aligned}$$

Then, using Hölder's inequality $(\frac{2}{p}, \frac{2}{q},$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2})$, we have

$$\begin{aligned}
I_1(t) &\leq \frac{1}{\delta^{\frac{p}{2}}}\left(\int_{-\infty}^t e^{-\frac{q\delta}{4}(t-s)} ds\right)^{\frac{p}{q}} \int_{-\infty}^t e^{-\frac{p\delta}{4}(t-s)} L^p(s)\,\mathrm{E}\,\|X(s) - Y(s)\|_{\mathbb{H}_2}^p\, ds \\
&\leq \frac{2^{\frac{2p}{q}}}{\delta^{\frac{pq+2p}{2q}}}\left(\int_{-\infty}^t e^{-\frac{p\delta}{4}(t-s)} L^p(s) ds\right) \sup_{s\in\mathbb{R}} \mathrm{E}\,\|X(s) - Y(s)\|^p \\
&\leq \frac{2^{\frac{2p}{q}}}{\delta^{\frac{pq+2p}{2q}}}\left(\sum_{n=0}^{\infty}\int_{t-n-1}^{t-n} e^{-\frac{p\delta}{4}(t-s)} L^p(s) ds\right) \sup_{s\in\mathbb{R}} \mathrm{E}\,\|X(s) - Y(s)\|^p \\
&\leq \frac{2^{\frac{2p}{q}}}{\delta^{\frac{pq+2p}{2q}}}\left(\sum_{n=0}^{\infty} e^{-\frac{np\delta}{4}}\int_{t-n-1}^{t-n} L^p(s) ds\right) \sup_{s\in\mathbb{R}} \mathrm{E}\,\|X(s) - Y(s)\|^p \\
&\leq \frac{2^{\frac{2p}{q}}\,\|L\|_{\mathbb{S}^p}^p}{\delta^{\frac{pq+2p}{2q}}\left(1 - \exp(\frac{p\delta}{4})\right)} \sup_{s\in\mathbb{R}} \mathrm{E}\,\|X(s) - Y(s)\|^p.
\end{aligned}$$

For $I_2(t)$, using Itô's isometry, (H4) and Hölder's inequality, we get :

$$\begin{aligned}
I_2(t) =& \mathrm{E}\left\|\int_{-\infty}^{t} T(t-s)[G(s,X(s))-G(s,Y(s))]dW(s)\right\|_{\mathbb{H}_2}^{p}\\
\leq& C_p\,\mathrm{E}\left(\operatorname{tr} Q\int_{-\infty}^{t} e^{-\delta(t-s)}\left\|G(s,X(s))-G(s,Y(s))\right\|^2 ds\right)^{\frac{p}{2}}\\
\leq& C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}\left(\int_{-\infty}^{t} e^{-\frac{q\delta}{2}(t-s)}\right)^{\frac{p}{q}}\int_{-\infty}^{t} e^{-\frac{p\delta}{2}(t-s)}L^p(s)\,\mathrm{E}\left\|X(s)-Y(s)\right\|^p ds\\
\leq& \frac{C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}}{(q\delta)^{\frac{p}{q}}}\left(\sum_{n=0}^{+\infty}\int_{t-n-1}^{t-n} e^{-\frac{p\delta}{2}(t-s)}L^p(s)ds\right)\sup_{s\in\mathbb{R}}\mathrm{E}\left\|X(s)-Y(s)\right\|_{\mathbb{H}_2}^p\\
\leq& \frac{2^{\frac{p}{q}}C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}\left\|L\right\|_{\mathbb{S}^p}^p}{(q\delta)^{\frac{p}{p}}\left(1-\exp(-\frac{p\delta}{2})\right)}\sup_{s\in\mathbb{R}}\mathrm{E}\left\|X(s))-Y(s))\right\|_{\mathbb{H}_2}^p.
\end{aligned}$$

We thus have

$$\left\|\Gamma X-\Gamma Y\right\|_{\infty,p}^p\leq\left(\frac{2^{\frac{pq+2p-q}{q}}\left\|L\right\|_{\mathbb{S}^p}^p}{\delta^{\frac{pq+2p}{2q}}\left(1-\exp(\frac{p\delta}{4})\right)}+\frac{2^{\frac{pq+p-q}{q}}C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}\left\|L\right\|_{\mathbb{S}^p}^p}{(q\delta)^{\frac{p}{p}}\left(1-\exp(-\frac{p\delta}{2})\right)}\right)\left\|X-Y\right\|_{\infty,p}^p.$$

It implies that

$$\left\|\Gamma X-\Gamma Y\right\|_{\infty,p}\leq\left(\frac{2^{\frac{pq+2p-q}{q}}\left\|L\right\|_{\mathbb{S}^p}^p}{\delta^{\frac{pq+2p}{2q}}\left(1-\exp(\frac{p\delta}{4})\right)}+\frac{2^{\frac{pq+p-q}{q}}C_p\left(\operatorname{tr} Q\right)^{\frac{p}{2}}\left\|L\right\|_{\mathbb{S}^p}^p}{(q\delta)^{\frac{p}{p}}\left(1-\exp(-\frac{p\delta}{2})\right)}\right)^{\frac{1}{p}}\left\|X-Y\right\|_{\infty,p}.$$

Then, we deduce from hypothesis (3.3.3) that $\Gamma$ is a contraction operator, hence there exists a unique mild solution to equation (3.3.1) in $\mathrm{UB}\big(\mathbb{R},\mathrm{L}^2(\mathrm{P},\mathbb{H}_2)\big)$. Furthermore $X$ has a continuous version by Part 1-a.

## Second part

In this part, we show that the solution $X$ given by Equation (3.3.2) is Weyl almost periodic in one-dimensional distribution.

Let $\tau$ be a real number. We consider the mild solution $X^\tau$ to

$$dX^\tau(t)=AX^\tau(t)+F(t+\tau,X^\tau(t))dt+G(t+\tau,X^\tau(t))dW(t). \tag{3.3.5}$$

Make the change of variable $s=r-\tau$, the process

$$X(t+\tau)=\int_{-\infty}^{t+\tau}T(t+\tau-r)F(r,X(r))dr+\int_{-\infty}^{t+\tau}T(t+\tau-r)G(r,X(r))dW(r) \tag{3.3.6}$$

becomes

$$X(t+\tau)=\int_{-\infty}^{t}T(t-s)F(s+\tau,X(s+\tau))ds+\int_{-\infty}^{t}T(t-s)G(s+\tau,X(s+\tau))d\tilde{W}_\tau(s),$$

where $\tilde{W}_\tau(s) = W(s+\tau) - W(\tau)$ is a Brownian motion with the same distribution as $W(s)$. We deduce that the process $X(.+\tau)$ has the same distribution as $X^\tau$.

So, to show that $X$ is $\mathbb{W}^p$ almost periodic in one-dimensional distribution, it is enough to show that, for every $\varepsilon > 0$, there exist $l = l(\varepsilon) > 0$ and $k = k(\varepsilon) > 0$, such that any interval of lenght $k$ contains at least a number $\tau$ for which

$$\sup_{x\in\mathbb{R}} \left( \frac{1}{l} \int_x^{x+l} \big(\mathrm{E}\|X^\tau(t) - X(t)\|^2\big)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} < \varepsilon. \tag{3.3.7}$$

Note that, since $X \in \mathrm{UB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$, the mapping $t \longmapsto (\mathrm{E}\|X(t)\|^2)^{\frac{p}{2}}$ is locally integrable, thus the left hand side of (3.3.7) is always finite.

Thus, for every $t, \tau \in \mathbb{R}$ and by the triangular inequality, we have

$$\begin{aligned}
\mathrm{E}\|X^\tau(t) - X(t)\|^2 =& \mathrm{E}\|\int_{-\infty}^t T(t-s)[F(s+\tau, X^\tau(s)) - F(s, X(s))]ds \\
&+ \int_{-\infty}^t T(t-s)[G(s+\tau, X^\tau(s)) - G(s, X(s))]dW(s)\|^2 \\
\leq& 4\,\mathrm{E}\|\int_{-\infty}^t T(t-s)[F(s+\tau, X^\tau(s)) - F(s+\tau, X(s))]ds\|^2 \\
&+ 4\,\mathrm{E}\|\int_{-\infty}^t T(t-s)[G(s+\tau, X^\tau(s)) - G(s+\tau, X(s))]dW(s)\|^2 \\
&+ 4\,\mathrm{E}\|\int_{-\infty}^t T(t-s)[F(s+\tau, X(s))) - F(s, X(s))]ds\|^2 \\
&+ 4\,\mathrm{E}\|\int_{-\infty}^t T(t-s)[G(s+\tau, X(s)) - G(s, X(s))]dW(s)\|^2 \\
:=& I_1^\tau(t) + I_2^\tau(t) + I_3^\tau(t) + I_4^\tau(t).
\end{aligned}$$

Now, using (H1), (H4) and Hölder's inequality, we obtain

$$\begin{aligned}
I_1^\tau(t) &= 4\,\mathrm{E}\|\int_{-\infty}^t T(t-s)[F(s+\tau, X^\tau(s)) - F(s+\tau, X(s))]ds\|^2 \\
&\leq 4\,\mathrm{E}\left( \left( \int_{-\infty}^t e^{-\delta(t-s)} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^t e^{-\delta(t-s)} \|F(s+\tau, X^\tau(s)) - F(s+\tau, X(s))\|^2 ds \right)^{\frac{1}{2}} \right)^2 \\
&\leq \frac{4}{\delta} \int_{-\infty}^t e^{-\delta(t-s)} L^2(s+\tau)\,\mathrm{E}\|X^\tau(s) - X(s)\|^2 ds.
\end{aligned}$$

Using Hölder's inequality $(\frac{p}{2}, \frac{q}{2})$, we have

$$
\begin{aligned}
I_1^\tau(t) &\le \frac{4}{\delta}\left(\int_{-\infty}^{t} e^{-\frac{q\delta}{4}(t-s)} L^q(s+\tau)ds\right)^{\frac{2}{q}} \left(\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \left(\mathrm{E}\|X^\tau(s)-X(s)\|^2\right)^{\frac{p}{2}} ds\right)^{\frac{2}{p}} \\
&\le \frac{4}{\delta}\left(\sum_{n=0}^{\infty}\int_{t-n-1}^{t-n} e^{-\frac{q\delta}{4}(t-s)} L^q(s+\tau)ds\right)^{\frac{2}{q}} \left(\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \left(\mathrm{E}\|X^\tau(s)-X(s)\|^2\right)^{\frac{p}{2}} ds\right)^{\frac{2}{p}} \\
&\le \frac{4}{\delta}\left(\sum_{n=0}^{\infty} e^{-\frac{q\delta}{4}n}\int_{t-n-1}^{t-n} L^q(s+\tau)ds\right)^{\frac{2}{q}} \left(\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \left(\mathrm{E}\|X^\tau(s)-X(s)\|^2\right)^{\frac{p}{2}} ds\right)^{\frac{2}{p}} \\
&\le \frac{4\,\|L\|_{\mathbb{S}^q}^2}{\delta\left(1-\exp(\frac{4}{q\delta})\right)^{\frac{2}{q}}} \left(\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \left(\mathrm{E}\|X^\tau(s)-X(s)\|^2\right)^{\frac{p}{2}} ds\right)^{\frac{2}{p}}
\end{aligned}
$$

For $I_2^\tau(t)$, using Itô's isometry and Hölder's inequality, we get

$$
\begin{aligned}
I_2^\tau(t) &= 4\,\mathrm{E}\|\int_{-\infty}^{t} T(t-s)[G(s+\tau,X^\tau(s)) - G(s+\tau,X(s))]dW(s)\|^2 \\
&\le 4\,\mathrm{tr}\,Q\,\mathrm{E}\int_{-\infty}^{t} \|T(t-s)\|^2 \|G(s+\tau,X^\tau(s)) - G(s+\tau,X(s))\|^2_{L_2(\mathbb{H}_1,\mathbb{H}_2)} ds \\
&\le 4\,\mathrm{tr}\,Q\int_{-\infty}^{t} e^{-2\delta(t-s)} L^2(s+\tau)\,\mathrm{E}\|X^\tau(s)-X(s)\|^2 ds \\
&\le 4\,\mathrm{tr}\,Q\left(\int_{-\infty}^{t} e^{-\frac{q\delta}{2}(t-s)} L^q(s+\tau)ds\right)^{\frac{2}{q}} \left(\int_{-\infty}^{t} e^{-\frac{p\delta}{2}(t-s)} \left(\mathrm{E}\|X^\tau(s)-X(s)\|^2\right)^{\frac{p}{2}} ds\right)^{\frac{2}{p}} \\
&\le \frac{4\,\mathrm{tr}\,Q\,\|L\|_{\mathbb{S}^q}^2}{\left(1-\exp(\frac{2}{q\delta})\right)^{\frac{2}{q}}} \left(\int_{-\infty}^{t} e^{-\frac{p\delta}{2}(t-s)} \left(\mathrm{E}\|X^\tau(s)-X(s)\|^2\right)^{\frac{p}{2}} ds\right)^{\frac{2}{p}}.
\end{aligned}
$$

For $I_3^\tau(t)$, using Hölder's inequality, we obtain

$$
\begin{aligned}
I_3^\tau(t) &= 4\,\mathrm{E}\|\int_{-\infty}^{t} T(t-s)[F(s+\tau,X(s))) - F(s,X(s))]ds\|^2 \\
&\le \frac{4}{\delta}\int_{-\infty}^{t} e^{-\delta(t-s)}\,\mathrm{E}\|F(s+\tau,X(s)) - F(s,X(s))\|^2 ds \\
&\le \frac{4}{\delta}\left(\int_{-\infty}^{t} e^{-\frac{q\delta}{4}(t-s)}ds\right)^{\frac{2}{q}} \left(\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \left(\mathrm{E}\|F(s+\tau,X(s)) - F(s,X(s))\|^2\right)^{\frac{p}{2}} ds\right)^{\frac{2}{p}} \\
&\le \frac{2^{\frac{2q+4}{q}}}{\delta^{\frac{q+2}{q}}\,q^{\frac{2}{q}}} \left(\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)} \left(\mathrm{E}\|F(s+\tau,X(s)) - F(s,X(s))\|^2\right)^{\frac{p}{2}} ds\right)^{\frac{2}{p}}
\end{aligned}
$$

Finally, for $I_4^\tau(t)$, using Itô's isometry and Hölder's inequality, we get

$$\begin{aligned}
I_4^\tau(t) &= 4\,\mathrm{E}\|\int_{-\infty}^{t} T(t-s)[G(s+\tau,X(s))) - G(s,X(s))]dW(s)\|^2 \\
&\leq 4\operatorname{tr}Q\int_{-\infty}^{t} e^{-2\delta(t-s)}\,\mathrm{E}\|G(s+\tau,X(s)) - G(s,X(s))\|^2 ds \\
&\leq 4\operatorname{tr}Q\left(\int_{-\infty}^{t} e^{-\frac{q\delta}{2}(t-s)}ds\right)^{\frac{2}{q}}\left(\int_{-\infty}^{t} e^{\frac{-p\delta}{2}(t-s)}\left(\mathrm{E}\|G(s+\tau,X(s)) - G(s,X(s))\|^2\right)^{\frac{p}{2}}ds\right)^{\frac{2}{p}} \\
&\leq \frac{2^{\frac{2q+2}{q}}\operatorname{tr}Q}{(q\delta)^{\frac{2}{q}}}\left(\int_{-\infty}^{t} e^{-\frac{p\delta}{2}(t-s)}\left(\mathrm{E}\|G(s+\tau,X(s)) - G(s,X(s))\|^2\right)^{\frac{p}{2}}ds\right)^{\frac{2}{p}}.
\end{aligned}$$

Then for all $x\in\mathbb{R}$, $\tau\in\mathbb{R}$ and $l>0$, we have

$$\begin{aligned}
g_{\tau,l}(x) :=& \frac{1}{l}\int_{x}^{x+l}\left(\mathrm{E}\|X^\tau(t) - X(t)\|^2\right)^{\frac{p}{2}}dt \\
\leq& 2^{p-2}\left(\frac{1}{l}\int_{x}^{x+l}\left((I_1^\tau(t))^{\frac{p}{2}} + (I_2^\tau(t))^{\frac{p}{2}}\right)dt + \frac{1}{l}\int_{x}^{x+l}\left((I_3^\tau(t))^{\frac{p}{2}} + (I_4^\tau(t))^{\frac{p}{2}}\right)dt\right).
\end{aligned} \tag{3.3.8}$$

Using a change of variables and Fubini's theorem, we get

$$\begin{aligned}
&\frac{1}{l}\int_{x}^{x+l}\left((I_1^\tau(t))^{\frac{p}{2}} + (I_2^\tau(t))^{\frac{p}{2}}\right)dt \\
&\quad\leq \frac{2^p\,\|L\|_{\mathbb{S}^q}^p}{\delta^{\frac{p}{2}}\left(1-\exp(\frac{4}{q\delta})\right)^{\frac{p}{q}}}\frac{1}{l}\int_{x}^{x+l}\int_{-\infty}^{t} e^{-\frac{p\delta}{4}(t-s)}\left(\mathrm{E}\|X^\tau(s) - X(s)\|^2\right)^{\frac{p}{2}}dsdt \\
&\qquad+ \frac{2^p\,(\operatorname{tr}Q)^{\frac{p}{2}}\,\|L\|_{\mathbb{S}^q}^p}{\left(1-\exp(\frac{2}{q\delta})\right)^{\frac{p}{q}}}\frac{1}{l}\int_{x}^{x+l}\int_{-\infty}^{t} e^{-\frac{p\delta}{2}(t-s)}\left(\mathrm{E}\|X^\tau(s) - X(s)\|^2\right)^{\frac{p}{2}}dsdt \\
&\quad\leq \frac{2^p\,\|L\|_{\mathbb{S}^q}^p}{\delta^{\frac{p}{2}}\left(1-\exp(\frac{4}{q\delta})\right)^{\frac{p}{q}}}\frac{1}{l}\int_{x}^{x+l}\int_{-\infty}^{0} e^{\frac{p\delta}{4}s}\left(\mathrm{E}\|X^\tau(t+s) - X(t+s)\|^2\right)^{\frac{p}{2}}dsdt \\
&\qquad+ \frac{2^p\,(\operatorname{tr}Q)^{\frac{p}{2}}\,\|L\|_{\mathbb{S}^q}^p}{\left(1-\exp(\frac{2}{q\delta})\right)^{\frac{p}{q}}}\frac{1}{l}\int_{x}^{x+l}\int_{-\infty}^{0} e^{\frac{p\delta}{2}s}\left(\mathrm{E}\|X^\tau(t+s) - X(t+s)\|^2\right)^{\frac{p}{2}}dsdt \\
&\quad\leq \frac{2^p\,\|L\|_{\mathbb{S}^q}^p}{\delta^{\frac{p}{2}}\left(1-\exp(\frac{4}{q\delta})\right)^{\frac{p}{q}}}\int_{-\infty}^{0} e^{\frac{p\delta}{4}s}\frac{1}{l}\int_{x}^{x+l}\left(\mathrm{E}\|X^\tau(t+s) - X(t+s)\|^2\right)^{\frac{p}{2}}dtds \\
&\qquad+ \frac{2^p\,(\operatorname{tr}Q)^{\frac{p}{2}}\,\|L\|_{\mathbb{S}^q}^p}{\left(1-\exp(\frac{2}{q\delta})\right)^{\frac{p}{q}}}\int_{-\infty}^{0} e^{\frac{p\delta}{2}s}\frac{1}{l}\int_{x}^{x+l}\left(\mathrm{E}\|X^\tau(t+s) - X(t+s)\|^2\right)^{\frac{p}{2}}dtds.
\end{aligned}$$

Using again a change of variables

$$\begin{aligned}
&\frac{1}{l}\int_x^{x+l}\left((I_1^\tau(t))^{\frac{p}{2}}+(I_2^\tau(t))^{\frac{p}{2}}\right)dt\\
&\quad\leq\frac{2^p\,\|L\|_{\mathbb{S}^q}^p}{\delta^{\frac{p}{2}}\left(1-\exp(\frac{4}{q\delta})\right)^{\frac{p}{q}}}\int_{-\infty}^0 e^{\frac{p\delta}{4}s}\frac{1}{l}\int_{x+s}^{x+s+l}\left(\mathrm{E}\|X^\tau(t)-X(t)\|^2\right)^{\frac{p}{2}}dtds\\
&\qquad+\frac{2^p\,(\operatorname{tr}Q)^{\frac{p}{2}}\,\|L\|_{\mathbb{S}^q}^p}{\left(1-\exp(\frac{2}{q\delta})\right)^{\frac{p}{q}}}\int_{-\infty}^0 e^{\frac{p\delta}{2}s}\frac{1}{l}\int_{x+s}^{x+s+l}\left(\mathrm{E}\|X^\tau(t)-X(t)\|^2\right)^{\frac{p}{2}}dtds.\\
&\quad\leq\frac{2^p\,\|L\|_{\mathbb{S}^q}^p}{\delta^{\frac{p}{2}}\left(1-\exp(\frac{4}{q\delta})\right)^{\frac{p}{q}}}\int_{-\infty}^x e^{-\frac{p\delta}{4}(x-s)}\frac{1}{l}\int_{s}^{s+l}\left(\mathrm{E}\|X^\tau(t)-X(t)\|^2\right)^{\frac{p}{2}}dtds\\
&\qquad+\frac{2^p\,(\operatorname{tr}Q)^{\frac{p}{2}}\,\|L\|_{\mathbb{S}^q}^p}{\left(1-\exp(\frac{2}{q\delta})\right)^{\frac{p}{q}}}\int_{-\infty}^x e^{-\frac{p\delta}{2}(x-s)}\frac{1}{l}\int_{s}^{s+l}\left(\mathrm{E}\|X^\tau(t)-X(t)\|^2\right)^{\frac{p}{2}}dtds.
\end{aligned}$$

With the same steps, we also find

$$\begin{aligned}
&\frac{1}{l}\int_x^{x+l}\left((I_3^\tau(t))^{\frac{p}{2}}+(I_4^\tau(t))^{\frac{p}{2}}\right)dt\\
&\quad\leq\frac{2^{\frac{pq+2p}{q}}}{\delta^{\frac{pq+2p}{2q}}q^{\frac{p}{q}}}\int_{-\infty}^x e^{-\frac{p\delta}{4}(x-s)}\frac{1}{l}\int_s^{s+l}\left(\mathrm{E}\|F(t+\tau,X(t))-F(t,X(t))\|^2\right)^{\frac{p}{2}}dtds\\
&\qquad+\frac{2^{\frac{pq+p}{q}}(\operatorname{tr}Q)^{\frac{p}{2}}}{(q\delta)^{\frac{p}{q}}}\int_{-\infty}^x e^{-\frac{p\delta}{2}(x-s)}\frac{1}{l}\int_s^{s+l}\left(\mathrm{E}\|G(t+\tau,X(t))-G(t,X(t))\|^2\right)^{\frac{p}{2}}dtds\\
&\quad:=C_1J_{\tau,l}^1(x)+C_2J_{\tau,l}^2(x),
\end{aligned}$$

where

$$C_1=\frac{2^{\frac{pq+2p}{q}}}{\delta^{\frac{pq+2p}{2q}}q^{\frac{p}{q}}}\quad\text{and}\quad C_2=\frac{2^{\frac{pq+p}{q}}(\operatorname{tr}Q)^{\frac{p}{2}}}{(q\delta)^{\frac{p}{q}}}.$$

Then, for all $x\in\mathbb{R}$, $\tau\in\mathbb{R}$ and $l>0$, we have, using (3.3.8),

$$0\leq g_{\tau,l}(x)\leq CJ_{\tau,l}(x)+\beta_1\int_{-\infty}^x e^{-\delta_1(x-s)}g_{\tau,l}(s)ds+\beta_2\int_{-\infty}^x e^{-\delta_2(x-s)}g_{\tau,l}(s)ds,$$

where

$$J_{\tau,l}(x)=J_{\tau,l}^1(x)+J_{\tau,l}^2(x),\quad C=2^{p-2}\max\{C_1,C_2\},\quad \beta_1=\frac{2^{2p-2}\,\|L\|_{\mathbb{S}^q}^p}{\delta^{\frac{p}{2}}\left(1-\exp(\frac{4}{q\delta})\right)^{\frac{p}{q}}},$$

$$\beta_2=\frac{2^{2p-2}\,(\operatorname{tr}Q)^{\frac{p}{2}}\,\|L\|_{\mathbb{S}^q}^p}{\left(1-\exp(\frac{2}{q\delta})\right)^{\frac{p}{q}}},\quad\delta_1=\frac{p\delta}{4}\ \text{ and }\ \delta_2=\frac{p\delta}{2}$$

The hypothesis (3.3.4) is equivalent to $\delta' > \beta$, with $\delta' = \min\{\delta_1, \delta_2\}$ and $\beta = \beta_1 + \beta_2$. We conclude by a variant of Gronwall's lemma [KMRdF15, Lemma 3.3], that, for all $\gamma \in ]0, \delta' - \beta]$ such that $\int_{-\infty}^{0} e^{\gamma s} J_{\tau,l}(s)ds$ converges, we have, for every $x, \tau \in \mathbb{R}$,

$$0 \leq g_{\tau,l}(x) \leq C J_{\tau,l}(x) + C\beta \int_{-\infty}^{x} e^{-\gamma(x-\xi)} J_{\tau,l}(\xi)d\xi.$$

To show (3.3.7), our aim is to show that, for every $\varepsilon > 0$, there exist $l = l(\varepsilon) > 0$, $k = k(\varepsilon) > 0$ such that, for all $a \in \mathbb{R}$, there exists $\tau \in [a, a+k]$ and that, for every $x \in \mathbb{R}$, we have

$$\begin{aligned} C J_{\tau,l}(x) =& C\left(J^1_{\tau,l}(x) + J^2_{\tau,l}(x)\right) \\ =& C \int_{-\infty}^{x} e^{-\delta_1(x-s)} \frac{1}{l} \int_{s}^{s+l} \left(\mathrm{E}\|F(t+\tau, X(t)) - F(t, X(t))\|^2\right)^{\frac{p}{2}} dtds \\ &+ C \int_{-\infty}^{x} e^{-\delta_2(x-s)} \frac{1}{l} \int_{s}^{s+l} \left(\mathrm{E}\|G(t+\tau, X(t)) - G(t, X(t))\|^2\right)^{\frac{p}{2}} dtds \\ <& \frac{\varepsilon^p}{2} \end{aligned}$$

and

$$\Psi_{\tau,l}(x) := C\beta \int_{-\infty}^{x} e^{-\gamma(x-\xi)} J_{\tau,l}(\xi)d\xi < \frac{\varepsilon^p}{2}. \tag{3.3.9}$$

Let $\varepsilon > 0$. First, we estimate $J^1_{\tau,l}(x)$. For $s, \tau \in \mathbb{R}$, $l > 0$ and $X \in \mathrm{UB}(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2))$, let

$$z_{\tau,l}(s) = \frac{1}{l} \int_{s}^{s+l} \left(\mathrm{E}\|F(t+\tau, X(t)) - F(t, X(t))\|^2\right)^{\frac{p}{2}} dt.$$

Using (H3), we have

$$\begin{aligned} z_{\tau,l}(s) \leq& \frac{2^{\frac{p}{2}}}{l} \int_{s}^{s+l} \left(\mathrm{E}\|F(t+\tau, X(t))\|^2 + \mathrm{E}\|F(t, X(t))\|^2\right)^{\frac{p}{2}} dt \\ \leq& \frac{2^p M^{\frac{p}{2}}}{l} \int_{s}^{s+l} \left(1 + \mathrm{E}\|X(t)\|^2\right)^{\frac{p}{2}} dt \\ \leq& 2^p M^{\frac{p}{2}} \sup_{t\in\mathbb{R}} \left(1 + \mathrm{E}\|X(t)\|^2\right)^{\frac{p}{2}} < \infty. \end{aligned}$$

The solution $X$ has a continuous version. Then, the process $s \mapsto e^{-\delta_1(x-s)} z_{\tau,l}(s)$ defined on $]-\infty, x]$ is uniformly bounded and continuous for every $x \in \mathbb{R}$. So, there exists $x^* = x^*_\varepsilon$, such that, for every $x \in \mathbb{R}$,

$$\begin{aligned} J^1_{\tau,l}(x) =& \int_{-\infty}^{x} e^{-\delta_1(x-s)} z_{\tau,l}(s)ds \leq \int_{-\infty}^{x^*} e^{-\delta_1(x^*-s)} z_{\tau,l}(s)ds + \frac{\varepsilon^p}{8C} \\ \leq& \sum_{n=0}^{+\infty} \int_{x^*-n-1}^{x^*-n} e^{-\delta_1(x^*-s)} z_{\tau,l}(s)ds + \frac{\varepsilon^p}{8C} \leq \sum_{n=0}^{+\infty} e^{-n\delta_1} \left(\max_{s\in[x^*-n-1, x^*-n]} z_{\tau,l}(s)\right) + \frac{\varepsilon^p}{8C}. \end{aligned} \tag{3.3.10}$$

The last term is convergent, thus, there exists $N_1 = N_1(\varepsilon) \geq 1$ such that

$$\sum_{n>N_1} e^{-n\delta_1}\left(\max_{s\in[x^*-n-1,x^*-n]} z_{\tau,l}(s)\right) < \frac{\varepsilon^p}{16C}.$$

So, for every $l > 0$, $\tau \in \mathbb{R}$ and $X \in \mathrm{UB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$, we have

$$J^1_{\tau,l}(x) \leq \sum_{n=0}^{N_1} e^{-n\delta_1}\left(\max_{s\in[x^*-n-1,x^*-n]} z_{\tau,l}(s)\right) + \frac{\varepsilon^p}{16C} + \frac{\varepsilon^p}{8C}.$$

Since $s \mapsto z_{\tau,l}(s)$ is continuous on $[x^*-n-1, x^*-n]$, there exists $s^*_{n,\varepsilon} \in [x^*-n-1, x^*-n]$ such that

$$\max_{s\in[x^*-n-1,x^*-n]} z_{\tau,l}(s) = z_{\tau,l}(s^*_{n,\varepsilon}).$$

Let $s^*_\varepsilon \in \arg\max_{n\in\{0,\cdots N_1\}} z_{\tau,l}(s^*_{n,\varepsilon})$. Then

$$\begin{aligned}
J^1_{\tau,l}(x) &\leq \sum_{n=0}^{N_1} e^{-n\delta_1} z_{\tau,l}(s^*_{n,\varepsilon}) + \frac{\varepsilon^p}{16C} + \frac{\varepsilon^p}{8C} \leq \sum_{n=0}^{N_1} e^{-n\delta_1} z_{\tau,l}(s^*_\varepsilon) + \frac{\varepsilon^p}{16C} + \frac{\varepsilon^p}{8C}\\
&\leq \left(\frac{1-e^{-\delta_1(N_1+1)}}{1-e^{\delta_1}}\right) z_{\tau,l}(s^*_\varepsilon) + \frac{\varepsilon^p}{8C} + \frac{\varepsilon^p}{16C} \leq \frac{1}{1-e^{\delta_1}} z_{\tau,l}(s^*_\varepsilon) + \frac{\varepsilon^p}{16C} + \frac{\varepsilon^p}{8C}.
\end{aligned}$$

Now, by Part 1-a, we have

$$\sup_{t\in\mathbb{R}} \mathrm{E}\left(\|X(t)\|^p\right) < \infty.$$

Thus, $\left(\|X(t)\|^2\right)_{t\in\mathbb{R}}$ is uniformly integrable. Thus, there exists $\eta = \eta_\varepsilon > 0$ such that

$$\sup_{t\in\mathbb{R}} \sup_{\mathrm{P}(B)<\eta} \mathrm{E}\left(\|X(t)\|^2\ \mathbb{1}_B\right) < \frac{\left(1-e^{\delta_1}\right)^{\frac{2}{p}} \varepsilon^2}{2^{\frac{2p+9}{p}} C^{\frac{2}{p}} M^{\frac{1}{2}}}. \tag{3.3.11}$$

Now, the random variable $\omega \mapsto X(\omega) \mid_{[s^*_\varepsilon, s^*_\varepsilon+l]}$ takes its values in $C([s^*_\varepsilon, s^*_\varepsilon + l], \mathbb{H}_2)$, thus, there exist $\eta = \eta_\varepsilon > 0$ and a compact subset $\mathbb{K}_\varepsilon$ of $C([s^*_\varepsilon, s^*_\varepsilon + l], \mathbb{H}_2)$ such that

$$\mathrm{P}\{X(t) \in \mathbb{K}_\varepsilon,\ \forall t \in [s^*_\varepsilon, s^*_\varepsilon + l]\} > 1 - \eta.$$

By consequence of the Arzelà-Ascoli theorem, the set

$$\mathbb{K}^*_\varepsilon = \{X(t)/t \in [s^*_\varepsilon, s^*_\varepsilon + l] \text{ and } X \in \mathbb{K}_\varepsilon\}$$

is a compact subset of $\mathbb{H}_2$, $P(A) < \eta$, where

$$A = \{\exists t \in [s^*_\varepsilon, s^*_\varepsilon + l]/X(t) \notin \mathbb{K}^*_\varepsilon\}.$$

By (3.3.11), we obtain

$$\int_A \left(1 + \|X(t)\|^2_{\mathbb{H}_2}\right) d\mathrm{P} < \frac{\left(1-e^{\delta_1}\right)^{\frac{2}{p}} \varepsilon^2}{2^{\frac{2p+8}{p}} C^{\frac{2}{p}} M^{\frac{1}{2}}}.$$

Then, using $(H3)$, we have

$$\begin{aligned}
z_{\tau,l}(s_\varepsilon^*) =& \frac{1}{l}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l} \left(\mathrm{E}\|F(t+\tau,X(t))-F(t,X(t))\|^2\right)^{\frac{p}{2}} dt \\
\leq& \frac{2^{\frac{p}{2}-1}}{l}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l} \left(\int_{\Omega\setminus A} \|F(t+\tau,X(t))-F(t+,X(t))\|^2 \, d\,\mathrm{P}\right)^{\frac{p}{2}} dt \\
&+ \frac{2^{\frac{p}{2}-1}}{l}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l} \left(\int_{A} \|F(t+\tau,X(t))-F(t,X(t))\|^2 \, d\,\mathrm{P}\right)^{\frac{p}{2}} dt \\
\leq& \frac{2^{\frac{p}{2}-1}}{l}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l} \left(\mathrm{E}\|F(t+\tau,X(t))-F(t,X(t))\|^2 \, \mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}} dt \\
&+ \frac{M^p 2^{p-1}}{l}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l} \left(\int_{A} 1+\|X(t)\|_{\mathbb{H}_2}^2 \, d\mathrm{P}\right)^{\frac{p}{2}} dt \\
\leq& \frac{2^{\frac{p}{2}-1}}{l}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l} \left(\mathrm{E}\|F(t+\tau,X(t))-F(t,X(t))\|^2 \, \mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}} dt + \frac{\left(1-e^{\delta_1}\right)\varepsilon^p}{32C}.
\end{aligned}$$

Using again the compactness of $\mathbb{K}_\varepsilon^*$, we can find a finite sequence $u_1,\dots,u_k \in \mathbb{K}_\varepsilon^*$ such that

$$\mathbb{K}_\varepsilon^* \subset \cup_{i=1}^k B\left(u_i, \frac{\left(1-e^{\delta_1}\right)^{\frac{1}{p}}\varepsilon}{\left(2^{\frac{2p+3}{p}}3^{\frac{1}{p}}C^{\frac{1}{p}}\|L\|_{\mathbb{W}^p}\right)}\right).$$

By (H5), we can choose $l_1 = l_1(\varepsilon) > 0$ such that

$$\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1} \|F(t+\tau,u)-F(t,u)\|^p \, dt < \frac{\left(1-e^{\delta_1}\right)\varepsilon^p}{2^{\frac{4p+6}{6}}3kC}, \quad \forall u \in \mathbb{K}_\varepsilon^*, \;\; \tau \in T_{\mathbb{S}_{l_1}^p}(\varepsilon, F, \mathbb{K}_\varepsilon^*). \tag{3.3.12}$$

On $\Omega\setminus A$, since $X(t) \in \mathbb{K}_\varepsilon^*$, it follows that, for every $t \in [s_\varepsilon^*, s_\varepsilon^*+l_1]$ there exists a random variable $i(t) \in \{1,2,\dots,k\}$ such that

$$\left\|X(t)-u_{i(t)}\right\|^2 < \frac{\left(1-e^{\delta_1}\right)^{\frac{2}{p}}\varepsilon^2}{\left(2^{\frac{4p+6}{p}}3^{\frac{2}{p}}C^{\frac{2}{p}}\|L\|_{\mathbb{W}^p}^2\right)}. \tag{3.3.13}$$

We have then

$$
\begin{aligned}
&\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}\left(\mathrm{E}\|F(t+\tau,X(t))-F(t,X(t))\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}dt\\
&\qquad\le 2^{2p-2}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}\left(\mathrm{E}\|F(t+\tau,X(t))-F(t,u_{i(t)})\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}dt\\
&\qquad\quad+2^{2p-2}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}\left(\mathrm{E}\|F(t+\tau,u_{i(t)})-F(t,u_{i(t)})\|^2\right)^{\frac{p}{2}}dt\\
&\qquad\quad+2^{2p-2}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}\left(\mathrm{E}\|F(t,u_{i(t)})-F(t,X(t))\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}dt\\
&\qquad\le 2^{2p-2}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}L^p(t+\tau)\left(\mathrm{E}\|X(t)-u_{i(t)})\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}dt\\
&\qquad\quad+2^{2p-2}\sum_{j=1}^{k}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}\left(\mathrm{E}\|F(t+\tau,u_j)-F(t,u_j)\|^2\right)^{\frac{p}{2}}dt\\
&\qquad\quad+2^{2p-2}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}L^p(t)\left(\mathrm{E}\|X(t)-u_{i(t)})\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}dt\\
&\qquad:=S_1(s_\varepsilon^*)+S_2(s_\varepsilon^*)+S_3(s_\varepsilon^*).
\end{aligned}
$$

Using (3.3.13), we have

$$
\begin{aligned}
S_1(s_\varepsilon^*)&=2^{2p-2}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}L^p(t+\tau)\left(\mathrm{E}\|X(t)-u_{i(t)})\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}dt\\
&\le 2^{2p-2}\sup_{t\in[s_\varepsilon^*,s_\varepsilon^*+l_1]}\left(\mathrm{E}\|X(t)-u_{i(t)})\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}L^p(t+\tau)dt\\
&\le 2^{2p-2}\|L\|_{W^p}^p\left(\mathrm{E}\|X(t)-u_{i(t)})\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}\\
&<\frac{\left(1-e^{\delta_1}\right)\varepsilon^p}{2^5 3C}.
\end{aligned}
$$

Following the same steps for $S_3(s_\varepsilon^*)$, by (3.3.13), we obtain

$$
S_3(s_\varepsilon^*)=2^p\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}L^p(t)\left(\mathrm{E}\|X(t)-u_{i(t)})\|^2\,\mathbb{1}_{\Omega\setminus A}\right)^{\frac{p}{2}}dt<\frac{\left(1-e^{\delta_1}\right)\varepsilon^p}{2^5 3C}.
$$

For $S_2(s_\varepsilon^*)$, using Jensen's inequality (since $u\mapsto\|u\|^{\frac{p}{2}}$ is a convex function) and (3.3.12), we have

$$
\begin{aligned}
S_2(s_\varepsilon^*)&=2^p\sum_{j=1}^{k}\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}\left(\mathrm{E}\|F(t+\tau,u_j)-F(t,u_j)\|^2\right)^{\frac{p}{2}}dt\\
&\le 2^p\sum_{j=1}^{k}\mathrm{E}\left(\frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1}\|F(t+\tau,u_j)-F(t,u_j)\|^p\,dt\right)\\
&<\frac{\left(1-e^{\delta_1}\right)\varepsilon^p}{2^5 3C}.
\end{aligned}
$$

This implies that

$$z_{\tau,l_1}(s_\varepsilon^*) = \frac{1}{l_1}\int_{s_\varepsilon^*}^{s_\varepsilon^*+l_1} \left(\mathrm{E}\|F(t+\tau, X(t)) - F(t, X(t))\|^2\right)^{\frac{p}{2}} dt < \frac{\left(1-e^{\delta_1}\right)\varepsilon^p}{16C}.$$

Thus

$$J^1_{\tau,l_1}(x) < \frac{\varepsilon^p}{4C} \qquad \forall \tau \in T_{\mathbb{S}^p_{l_1}}(\varepsilon, F, \mathbb{K}^*_\varepsilon). \tag{3.3.14}$$

For the same reason as for $J_{\tau,l_1}(x)$, we can choose $l_2 = l_2(\varepsilon) > 0$ and we can find a compact subset $\mathbb{K}^{**}_\varepsilon$ of $\mathbb{H}_2$ (the same procedure as for $\mathbb{K}^*_\varepsilon$) such that

$$J^2_{\tau,l}(x) < \frac{\varepsilon^p}{4C} \qquad \forall \tau \in T_{\mathbb{S}^p_{l_2}}(\varepsilon, G, \mathbb{K}^{**}_\varepsilon). \tag{3.3.15}$$

Let $L_1 = \max\{l_1, l_2\}$. From (3.3.14) and (3.3.15), we have, for every $\tau \in T_{\mathbb{S}^p_{L_1}}(\varepsilon, F, \mathbb{K}^*_\varepsilon) \cap T_{\mathbb{S}^p_{L_1}}(\varepsilon, G, \mathbb{K}^{**}_\varepsilon)$,

$$CJ_{\tau,L_1}(x) = C\left(J^1_{\tau,L_1}(x) + J^2_{\tau,L_1}(x)\right) < \frac{\varepsilon^p}{2}. \tag{3.3.16}$$

Next, let us consider $\Psi_{\tau,l}(x)$ defined in (3.3.9). Using (H3), we have, for any $\xi, \tau \in \mathbb{R}$, $l > 0$, $X \in \mathrm{UB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$ and for $s \le \xi$,

$$\begin{aligned}
J_{\tau,l}(\xi) =& \int_{-\infty}^{\xi} e^{-\delta_1(\xi-s)}\frac{1}{l}\int_s^{s+l} \left(\mathrm{E}\|F(t+\tau, X(t)) - F(t, X(t))\|^2\right)^{\frac{p}{2}} dtds \\
&+ \int_{-\infty}^{\xi} e^{-\delta_1(\xi-s)}\frac{1}{l}\int_s^{s+l} \left(\mathrm{E}\|G(t+\tau, X(t)) - G(t, X(t))\|^2\right)^{\frac{p}{2}} dtds \\
\le& 2\int_{-\infty}^{\xi} e^{-\delta_1(\xi-s)}\frac{2^p M^{\frac{p}{2}}}{l}\int_s^{s+l} \left(1+\mathrm{E}\|X(t)\|^2\right)^{\frac{p}{2}} dt \\
\le& 2^{p+1}M^{\frac{p}{2}}\sup_{t\in\mathbb{R}}\left(1+\mathrm{E}\|X(t)\|^2\right)^{\frac{p}{2}}\int_{-\infty}^{\xi} e^{-\delta_1(\xi-s)} < \infty.
\end{aligned}$$

Then, for any $x \in \mathbb{R}$, the process $\xi \mapsto e^{\gamma\xi}J_{\tau,l}(\xi)$ is uniformly bounded and continuous on $]-\infty, x]$. We can find a real number $x' = x'_\varepsilon$ such that

$$\begin{aligned}
\Psi_{\tau,l}(x) \le& C\beta\int_{-\infty}^{x'} e^{-\gamma(x'-\xi)}J_{\tau,l}(\xi)d\xi + \frac{\varepsilon^p}{4} \\
=& C\beta\sum_{n=0}^{\infty}\int_{x'-n-1}^{x'-n} e^{-\gamma(x'-\xi)}J_{\tau,l}(\xi)d\xi + \frac{\varepsilon^p}{4} \le C\beta\sum_{n=0}^{\infty}e^{-n\gamma}\int_{x'-n-1}^{x'-n} J_{\tau,l}(\xi)d\xi + \frac{\varepsilon^p}{4}.
\end{aligned}$$

As in (3.3.10), we see that $J_{\tau,l}$ is bounded, and we have

$$J_{\tau,l}(\xi) \le C\beta\sum_{n=0}^{\infty}e^{-n\gamma}J_{\tau,l}(\xi^*_{n,\varepsilon}) + \frac{\varepsilon^p}{4}, \tag{3.3.17}$$

where $\xi^*_{n,\varepsilon} = \arg\max_{\xi\in[x'-n-1,x'-n]} J_{\tau,l}(\xi)$. Since the series in (3.3.17) converges, we can find an integer $N_2 = N_2(\varepsilon) > 1$ such that

$$\sum_{n>N_2} e^{-n\gamma}J_{\tau,l}(\xi^*_{n,\varepsilon}) < \frac{\varepsilon^p}{8}.$$

Then

$$\Psi_{\tau,l}(x) \leq \sum_{n=0}^{N_2} e^{-n\gamma} J_{\tau,l}(\xi^*_{n,\varepsilon}) + \frac{\varepsilon^p}{8} + \frac{\varepsilon^p}{4}$$
$$\leq \sum_{n=0}^{N_2} e^{-n\gamma} J_{\tau,l}(\xi^*_{\varepsilon}) + \frac{\varepsilon^p}{8} + \frac{\varepsilon^p}{4} \leq \frac{1}{1-e^{-\gamma}} J_{\tau,l}(\xi^*_{\varepsilon}) + \frac{\varepsilon^p}{8} + \frac{\varepsilon^p}{4},$$

where $\xi^*_\varepsilon = \arg\max_{n\in\{0,\cdots,N_2\}} J_{\tau,l}(\xi^*_{n,\varepsilon})$. To estimate $J_{\tau,l}(\xi^*_\varepsilon)$, we follow the same procedure as for $z_{\tau,l}(s^*_\varepsilon)$. We get that, there exist $L_2 = L_2(\varepsilon)$ and a compact subset $\mathbb{K}^{3*}_\varepsilon$ of $\mathbb{H}_2$, and that we can choose a set relatively dense $T_{\mathbb{S}^p_{L_2}}(\varepsilon, F, \mathbb{K}^{3*}_\varepsilon) \cap T_{\mathbb{S}^p_{L_2}}(\varepsilon, G, \mathbb{K}^{3*}_\varepsilon)$, such that,

$$J_{\tau,L_2}(\xi^*_\varepsilon) < \frac{\varepsilon^p}{8}\left(1-e^{-\gamma}\right) \quad \forall \tau \in T_{\mathbb{S}^p_{L_2}}(\varepsilon, F, \mathbb{K}^{3*}_\varepsilon) \cap T_{\mathbb{S}^p_{L_2}}(\varepsilon, G, \mathbb{K}^{3*}_\varepsilon).$$

This implies that, for every $\tau \in T_{\mathbb{S}^p_{L_2}}(\varepsilon, F, \mathbb{K}^{3*}_\varepsilon) \cap T_{\mathbb{S}^p_{L_2}}(\varepsilon, G, \mathbb{K}^{3*}_\varepsilon)$ and $X \in \mathrm{UB}\big(\mathbb{R}, \mathrm{L}^2(\mathrm{P}, \mathbb{H}_2)\big)$, we have

$$\Psi_{\tau,L_2}(x) < \frac{\varepsilon^p}{2}. \tag{3.3.18}$$

We have proved that, for $l = \max\{L_1, L_2\}$, there exists a set relatively dense $T_{\mathbb{S}^p_l}(\varepsilon, F, G, \mathbb{K})$ such that (3.3.16) and (3.3.18) are verified and

$$g_{\tau,l}(x) < \varepsilon^p.$$

Therefore, for every $\varepsilon > 0$, there exist $l = l(\varepsilon) > 0$ such that for every $\tau \in T_{\mathbb{S}^p_l}(\varepsilon, F, G, \mathbb{K})$, we have

$$\sup_{x\in\mathbb{R}} \left( \frac{1}{l} \int_x^{x+l} \left( \mathrm{E}\|X^\tau(t) - X(t)\|^2 \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} < \varepsilon.$$

Hence, the solution $X$ to equation (3.3.2) is Weyl almost periodic in one-dimensional distribution.

## Third part

In this last part, we show that the solution $X$ is Weyl almost periodic in finite-dimensional distribution.

To do it, we just show it for $n = 2$, the reasoning is similar for any $n > 2$.

Let $d_{\mathbb{H}_2^2}$ be the distance on $\mathbb{H}_2^2$ defined by

$$d_{\mathbb{H}_2^2}((x_1, x_2), (y_1, y_2)) = \max_{i\in\{1,2\}} d_{\mathbb{H}_2}(x_i, y_i),$$

where $d_{\mathbb{H}_2}$ be the distance on $\mathbb{H}_2$ such that

$$d_{\mathbb{H}_2}(x, y) = \|x - y\|_{\mathbb{H}_2},$$

et

$$\|(x_1, x_2), (y_1, y_2)\|_{\mathbb{H}_2} = \max_{i\in\{1,2\}} \|x_i - y_i\|_{\mathbb{H}_2}.$$

Let $t_1, t_2 \in \mathbb{R}$. Using Tudor's notation [Tud95], we define, for every $t \in \mathbb{R}$,

$$\mu_t := \text{law}(X(t)) \text{ and } \mu_t^{t_1,t_2} := \text{law}(X(t_1+t), X(t_2+t))$$

Then, we have, for every $t, \tau \in \mathbb{R}$,

$$\begin{aligned}
d_{\text{BL}}(\mu_{t+\tau}^{t_1,t_2}, \mu_t^{t_1,t_2}) &\leq \left\{ \left(\mathrm{E}\|((X(t_1+t+\tau), (X(t_2+t+\tau)) - ((X(t_1+t), (X(t_2+t))\|^2\right)^{\frac{1}{2}} \right\} \\
&\leq \left\{ \left( \mathrm{E}\left( \sum_{i=1}^{2} \|X(t_i+t+\tau) - X(t_i+t)\|^2 \right) \right)^{\frac{1}{2}} \right\} \\
&\leq \left\{ \left(\mathrm{E}\|X(t_1+t+\tau) - X(t_1+t)\|^2\right)^{\frac{1}{2}} \right\} + \left\{ \left(\mathrm{E}\|X(t_2+t+\tau) - X(t_2+t)\|^2\right)^{\frac{1}{2}} \right\} \\
&\leq \mathbf{W}^2(\mu_{t_1+t+\tau}, \mu_{t_1+t}) + \mathbf{W}^2(\mu_{t_2+t+\tau}, \mu_{t_2+t}) \\
&= \mathbf{W}^2(\tilde{\mu}_{t_1+t+\tau}, \mu_{t_1+t}) + \mathbf{W}^2(\tilde{\mu}_{t_2+t+\tau}, \mu_{t_2+t}),
\end{aligned}$$

where $\tilde{\mu}_{t_1+t+\tau} = \text{law}(X^\tau(t+t_1))$ and $\tilde{\mu}_{t_2+t+\tau} = \text{law}(X^\tau(t+t_2))$. It implies that

$$\begin{aligned}
d_{\text{BL}}(\mu_{t+\tau}^{t_1,t_2}, \mu_t^{t_1,t_2}) &\leq \mathbf{W}^2(\tilde{\mu}_{t_1+t+\tau}, \mu_{t_1+t}) + \mathbf{W}^2(\tilde{\mu}_{t_2+t+\tau}, \mu_{t_2+t}) \\
&\leq \left(\mathrm{E}\|X^\tau(t_1+t) - X(t_1+t)\|^2\right)^{\frac{1}{2}} + \left(\mathrm{E}\|X^\tau(t_2+t) - X(t_2+t)\|^2\right)^{\frac{1}{2}},
\end{aligned}$$

where $X^\tau$ the mild solution to (3.3.5). Then, for every $x \in \mathbb{R}$ and $l > 0$, we have

$$\begin{aligned}
&\frac{1}{l}\int_x^{x+l} d_{\text{BL}}^p(\mu_{t+\tau}^{t_1,t_2}, \mu_t^{t_1,t_2})dt \\
&\quad \leq \frac{1}{l}\int_x^{x+l} \left( \left(\mathrm{E}\,\|X^\tau(t_1+t), X(t+t_1)\|^2\right)^{\frac{1}{2}} + \left(\mathrm{E}\,\|X^\tau(t_2+t), X(t+t_2)\|^2\right)^{\frac{1}{2}} \right)^p dt \\
&\quad \leq 2^{p-1}\frac{1}{l}\int_x^{x+l} \left(\mathrm{E}\,\|X^\tau(t_1+t), X(t+t_1)\|^2\right)^{\frac{p}{2}} + \left(\mathrm{E}\,\|X^\tau(t_2+t), X(t+t_2)\|^2\right)^{\frac{p}{2}} dt.
\end{aligned}$$

We follow the same steps as in the second part. We find, for every $\varepsilon > 0$, that there exist $l = l(\varepsilon)$ and $k = k(\varepsilon) > 0$ such that any interval of length $k$ contains at least a number $\tau$ for which, for every $i \in \{1, 2\}$,

$$\sup_{x\in\mathbb{R}} \frac{1}{l}\int_x^{x+l} \left(\mathrm{E}\,\|X^\tau(t+t_i), X(t+t_i)\|^2\right)^{\frac{p}{2}} dt < \frac{\varepsilon^p}{2^p}.$$

So, for every $\varepsilon > 0$ there exist $l = l(\varepsilon) > 0$ and a relatively dense set $T_{\mathbb{S}_l^p}(\varepsilon, f) \subset \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} \left( \frac{1}{l}\int_x^{x+l} d_{\text{BL}}^p(\mu_{t+\tau}^{t_1,t_2}, \mu_t^{t_1,t_2})dt \right)^{\frac{1}{p}} < \varepsilon.$$

Thus $X$ is also Weyl almost periodic in finite-dimensional distribution.

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