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Équations différentielles stochastiques gouvernées par un mouvement brownien fractionnaire

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Abstract

This thesis deals with existence and uniqueness of almost periodic solutions in distribution for stochastic differential equations driven by a fractional Brownian motion with almost periodic coefficients.

More precisely we are interested in the study of existence and uniqueness of almost periodic solutions in distribution to affine stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ of the form :

$$dX_t = (a_0(t) - X_t)dt + (b_0(t) + b(t)X_t)dB_t^H,$$

where a_0, b_0 and b are real valued deterministic functions which assumed to be almost periodic. Here the stochastic integral corresponds to the divergence integral from Malliavin calculus.

Key words : Stochastic differential equations, fractional Brownian motion, almost periodic solution, multiple stochastic integrals, chaos decomposition.

Introduction

Since the introduction of the theory of almost periodic functions by Harald Bohr [9, 10] in the 1920s, it has aroused great interest. In recent years, it has become one of the most attractive topics in the qualitative theory of differential equations because of their significance and applications in physics, mathematical biology, telecommunication, network, finance, control theory and others related fields. For more details on this theory, see for instance [35, 4, 16, 27].

The extension of almost periodicity theory given by Bochner [7] for functions with values in Polish spaces, especially in Banach spaces, allowed to generalize this theory for a wider class of functions, in particular for the class of random functions (stochastic processes): almost periodicity in p-mean, in probability, in distribution (one-distribution, finite distribution and infinite distribution), almost periodicity of moments. Some characterizations of almost periodic functions with values in a Polish space, especially in the space of probability measures, are given in [3, 40, 32].

It is well known that the natural application of the concept of almost periodic random functions is the study of stochastic differential equations, with almost periodic coefficients, driven by different noises (standard Brownian motion, fractional Brownian motion, Lévy, etc.). During the last 30 years, the class of stochastic differential equations driven by standard Brownian motion, $W = \{W_t, t \in \mathbb{R}\}$ is the most studied. The existence and uniqueness of almost periodic solution is carried out by several authors, as L. Arnold, C. Tudor, T. Morozan and G. Da Prato (see e.g. [2, 17, 32]). All these authors deal with almost periodicity in distribution. O. Mellah and P. Raynaud De Fitte [31] show, by counterexamples, that the almost periodicity in p-mean is a strong property for solutions of stochastic differential equations, which implies the nonexistence of almost periodic solution in p-mean in general. However, in the class of stochastic differential equations driven by fractional Brownian motion (fBm for short), until now, to our best knowledge, there are only few works dedicated to the study of existence and uniqueness of almost periodic solutions. We can cite for example, P. Bezandry [5] who attempted to show the existence and uniqueness of almost periodic solutions. Recently, F. Chen and X. Yang [12] showed the existence and uniqueness of almost automorphic (which is a generalization of almost periodicity) solutions in distribution while F. Chen and X. Zhang [13] obtained the same result for another class of differential equations namely the mean-field stochastic differential equations driven by fBm. Note that in all the works cited above, the authors assumed that the coefficients of fractional stochastic part is deterministic. In this thesis, without this condition, we will show the existence and uniqueness of almost periodic solution to affine stochastic differential equations driven by fBm.

A fBm, with Hurst parameter $H \in (0,1)$, introduced by Kolmogorov [26] and studied by Mandelbrot and Van Ness [29], is a centered Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ which has stationary increments and with the covariance function

$$\mathrm{E}(B_t^H B_s^H) = \mathrm{R}_H(t,s) = \frac{1}{2}\big(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\big). \tag{1}$$

Notice that if $H = \frac{1}{2}$, we recover the standard Brownian motion. For $H \neq \frac{1}{2}$ the increments of the fBm are no longer independent (i.e. they are positively correlated if $H > \frac{1}{2}$, and negatively correlated if $H < \frac{1}{2}$). Furthermore, when $H > \frac{1}{2}$, the fBm exhibits long-range dependence. namely, at large time lags, the dependence is so strong that it implies the divergence of the series $\sum_{k=1}^{\infty} R_H(1,k)$. Also the fBm is neither a Markov process nor a semimartingale, as a result, one cannot use the usual Itô stochastic calculus, and alternative methods are required in order to define stochastic integrals with respect to fBm. For this, several different ways of defining stochastic integrals with respect to fBm, $\int_0^t f\,(s)\,dB_s^H$, have been suggested:

- the pathwise approach, using the results of Young [41], Ciesielski [15] and Zähle [42]. In general, in this case, the property $\mathrm{E}(\int_0^t f(s) dB_s^H) = 0$ is not satisfied, see for instance Lin, SJ [28] and Gripenberg and Norros [25].

- The Malliavin calculus approach introduced by Decreusefond and Üstünel [19], Carmona and Coutin [11]. In this approach, the property $\mathrm{E}(\int_0^t f(s) dB_s^H) = 0$ is satisfied. We refer for instance to [23, 36] for equivalent approaches. In some cases, this integral is called Skorohod integral with respect to fBm. It is well known that in the case $H = \frac{1}{2}$, the Skorohod integral may be regarded as an extention of the Itô integral to integrands that are not necessarily adapted, see [20, Theorem 2.9].

Motivated by the previous works, our aim is to study the existence and uniqueness of almost periodic solution in distribution to affine stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, of the form :

$$dX_t = \big(a_0(t) - X_t\big)dt + \big(b_0(t) + b(t)X_t\big)dB_t^H, \tag{2}$$

where a_0, b_0 and b are real valued almost periodic functions.

The main idea in our approach for solving the differential equation (2) consists of applying the chaos decomposition approach. This chaotic expansion method is introduced by Shiota [39] for $H = \frac{1}{2}$ (in the case of standard Brownian motion), and used for $H \in (\frac{1}{2}, 1)$ by Pèrez-Abreu and C. Tudor [36] where multiple fractional integrals were defined by using fractional integrals and derivatives for deterministic functions of several variables and a transfer principle from the multiple Wiener Itô integrals.

The structure of the thesis

The rest of this thesis is organized as follows :

- **Chapter** 1 gathers the needed preliminary results, in particular, it recalls the concept of almost periodicity for deterministic functions and for random functions in different sense (almost periodicity in distribution, in quadratic mean...), as well as their basic properties. It also presents fractional Brownian motion and provides some of its main properties as well as some ways of stochastic integration with respect to it.

- The second chapter of the present PhD project is devoted to study the existence and uniqueness of almost periodic solution in distribution to affine stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ of the form (2).

  The evolution solution of (2) can be expressed as follows

$$X_t = \int_{-\infty}^{t} e^{-(t-s)} a_0(s) ds + \int_{-\infty}^{t} e^{-(t-s)} \left[b_0(s) + b(s) X_s \right] dB_s^H$$

  with the chaos decomposition

$$X_t = \sum_{p=0}^{+\infty} I_p^H(f_p^t(.)),$$

  where I_p^H is the multiple stochastic integral with respect to the fBm and

$$\begin{aligned} f_p^t(t_1, \ldots, t_p) = \operatorname{sym}\{ \mathbb{1}_{-\infty < t_1 < \ldots < t_p < t} e^{-(t-t_1)} b_0(t_1) b(t_2) \ldots b(t_p) \} \\ + \frac{1}{p!} \mathbb{1}_{]-\infty, t]^p}(t_1, \ldots, t_p) b(t_1) \ldots b(t_p) \int_{-\infty}^{t_1 \wedge \ldots \wedge t_p} e^{-(t-s)} a_0(s) ds, \end{aligned}$$

  where $\operatorname{sym}\{.\}$ is the symmetrization function. Under some conditions, we show that the stochastic differential equation (2) has a unique almost periodic solution in distribution.

Chapter 1

Basic tools and definitions

The purpose of this chapter is to describe the framework that will be used in this thesis.

1.1 Almost periodicity

In this section we review some basic definitions and results for the notion of almost periodicity. For additional details, we refer the reader to the books [1, 4, 24].

1.1.1 Almost periodic functions

Definition 1.1.1 (Relatively dense set)**.** A set $A \subset \mathbb{R}$ is relatively dense if there exists a real number $l > 0$, such that

$$A \cap [a, a+l] \neq \emptyset, \text{ for all } a \text{ in } \mathbb{R}.$$

Definition 1.1.2 (Bohr almost periodicity)**.** Let $(\mathbb{E}, d)$ be a complete metric space, we say that a continuous function $f : \mathbb{R} \to \mathbb{E}$ is Bohr almost periodic (or almost periodic) if for all $\varepsilon > 0$, the set

$$T(f, \varepsilon) := \left\{ \tau \in \mathbb{R}, \sup_{t \in \mathbb{R}} d\big(f(t+\tau), f(t)\big) < \varepsilon \right\},$$

is relatively dense (each number $\tau \in T(f, \varepsilon)$ is called ε-almost period or $\varepsilon-$translation). That is for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least an ε-period, that is, a number τ for which

$$\sup_{t \in \mathbb{R}} d\big(f(t+\tau), f(t)\big).$$

Definition 1.1.3 (Bochner almost periodicity)**.** Let $(\mathbb{E}, d)$ be a complete space, let $f : \mathbb{R} \to (\mathbb{E}, d)$ be a continuous function. We say that f is Bochner-almost periodic if the set

$$\left\{ \tilde{f}(t), t \in \mathbb{R} \right\} = \{ f(t + .), t \in \mathbb{R} \},$$

is totally bounded in the space of continuous functions from $\mathbb{R}$ to $\mathbb{E}$ ($\mathcal{C}(\mathbb{R}, \mathbb{E})$) endowed with the topology of uniform convergence on $\mathbb{R}$. That is for every sequence $(h'_n)_{n \in \mathbb{N}}$ of real numbers there corresponds a subsequence $(h_n)_{n \in \mathbb{N}}$ such that the sequence of functions $(f(t + h_n))_{n \in \mathbb{N}}$ is uniformly convergent.

Definition 1.1.4 (Bochner's double sequence criterion)**.** A function f satisfies Bochner's double sequences criterion, that is, for every pair of sequences

$$(\alpha'_n) \subset \mathbb{R} \qquad \text{and} \qquad (\beta'_m) \subset \mathbb{R},$$

there are subsequences

$$(\alpha_n) \subset (\alpha'_n) \qquad \text{and} \qquad (\beta_m) \subset (\beta'_m)$$

respectively with same indexes such that, for every $t \in \mathbb{R}$, the limits

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t+\alpha_n+\beta_m) \qquad \text{and} \qquad \lim_{n\to\infty} f(t+\alpha_n+\beta_n), \tag{1.1}$$

exist and are equal.

Remark 1.1.1. A striking property of Bochner's double sequence criterion is that the limits in (1.1) exist in any of the three modes of convergences: pointwise, uniform on compact intervals and uniform on $\mathbb{R}$. This criterion has thus the advantage that it allows to establish uniform convergence by checking pointwise convergence.

We have the following characterization, due to Bochner.

Theorem 1.1.1 ([8])**.** Let $f : \mathbb{R} \to (\mathbb{E}, d)$ be a continuous function. The following statements are equivalent

1. f is Bohr almost periodic.
2. f is Bochner-almost periodic.
3. f satisfies Bochner's double sequences criterion.

In the following we shall give elementary properties of almost periodic functions.

Proposition 1.1.1. *Every almost periodic function $f : \mathbb{R} \to (\mathbb{E}, d)$ has the following properties :*

1. *f is uniformly bounded.*
2. *f is uniformly continuous.*
3. *If a sequence of almost periodic functions f_n converges uniformly in $\mathbb{R}$ to a function f, then f is also almost periodic.*

Proof. 1. The function f is Bohr almost periodic, that is for all $\varepsilon > 0$ $\exists l_\varepsilon > 0$, such that any interval of length $l(\varepsilon)$ contains $\varepsilon-$ translation. We put $\varepsilon = 1$, $\exists l = l_1 > 0$, such that any interval of length l_1 contains 1 translation. The function f is continuous on $[0, 1]$, then there exists $M > 0$ and $b \in \mathbb{E}$ such that

$$\sup_{x\in[0,l]} d(f(x), b) \leq M.$$

Let $x \in \mathbb{R}$, there exists $\tau \in [-x, -x+l]$ such that $d\left(f(x+\tau), f(x)\right) \leq 1$ and $x+\tau \in [0, l]$. Then

$$d(f(x), b) \leq d(f(x), f(x+\tau)) + d(f(x+\tau), b) \leq 1 + M.$$

2. Let $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that each interval of length l_ε contains $\frac{\varepsilon}{3}-$translation. The function f is continuous on the compact $[-1, 1+l]$ then the function f is uniformly continuous on $[-1, 1+l]$, there exists $0 < \delta < 1$ such that, if $y_1, y_2 \in [-1, 1+l]$ and $|y_1 - y_2| < \delta$ then we have
$$d\left(f(y_1), f(y_2)\right) < \frac{\varepsilon}{3}.$$
Let $x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_2| < \delta$. There exists $\tau \in T(f, \frac{\varepsilon}{3}) \cup [-x_1, -x_1 + l]$, we have $x_1 + \tau \in [0, l] \subset [-1, l] \subset [-1, 1+l]$ and $-1 \leq -\delta \leq x_2 - x_1 \leq x_2 + \tau \leq x_2 - x_1 + l \leq \delta + l \leq 1 + l$, then $x_2 + \tau \in [-1, l]$ and $|(x_2+\tau) - (x_1+\tau)| = |x_2 - x_1| \leq \delta$ then $d(f(x_1+\tau), f(x_2+\tau)) \leq \frac{\varepsilon}{3}$. Since $\tau \in T(f, \frac{\varepsilon}{3})$ then $d(f(x_1), f(x_1+\tau)) \leq \frac{\varepsilon}{3}$ and $d(f(x_2), f(x_2+\tau)) \leq \frac{\varepsilon}{3}$. We get then
$$\begin{aligned} d(f(x_1), f(x_2)) &\leq d(f(x_1), f(x_1+\tau)) + d(f(x_1+\tau), f(x_2+\tau)) + d(f(x_2), f(x_2+\tau)) \\ &\leq \varepsilon. \end{aligned}$$

3. Let $(f_n)_{n\in\mathbb{N}}$ be , given $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $d\left(f_{n_0}(x), f(x)\right) \leq \frac{\varepsilon}{3}$. Let $\tau \in T(f, \frac{\varepsilon}{3})$, then
$$\begin{aligned} d(f(x+\tau), f(x)) &\leq d(f(x+\tau), f_{n_0}(x+\tau)) + d(f_{n_0}(x+\tau), f_{n_0}(x)) + d(f_{n_0}(x), f(x)) \\ &< \varepsilon, \end{aligned}$$
we get then $\tau \in T(f, \varepsilon)$.

□

1.1.2 Almost periodic stochastic processes

The purpose of this subsection is to introduce the concept of almost periodicity for random functions. For these functions, the notion of almost periodicity gives rise to several definitions, mainly: in distribution, in one dimensional distribution, in finite dimensional distribution, in mean square, in probability .... For details we refer the reader to [32, 40, 3] and to the references therein.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We shall denote by $C_b(\mathbb{E})$ the Banach space of continuous and bounded functions $f : \mathbb{E} \to \mathbb{R}$ with
$$\|f\|_\infty = \sup_{x\in\mathbb{E}} |f(x)|,$$
and by $\mathcal{P}(\mathbb{E})$ the set of all probability measures on the Borel σ- field of $\mathbb{E}$.
For $f \in C_b(\mathbb{E})$ we define
$$\|f\|_L = \sup\left\{\frac{f(x) - f(y)}{d_\mathbb{E}(x, y)} : x \neq y\right\},$$
$$|f|_{\mathrm{BL}} = \max\left\{\|f\|_\infty, \|f\|_L\right\},$$

and we define

$$\mathrm{BL}(\mathbb{E}) = \{f \in C_b(\mathbb{E}); \|f\|_{\mathrm{BL}} < \infty\}.$$

For $\mu, \nu \in \mathcal{P}(\mathbb{E})$ we define

$$d_{\mathrm{BL}}(\mu,\nu) = \sup_{\|f\|_{\mathrm{BL}}\leq 1} \left| \int_{\mathbb{E}} f d(\mu-\nu) \right|,$$

which is a complete metric on $\mathcal{P}(\mathbb{E})$ and generates the weak topology [22].

Let $X = (X_t)_{t\in\mathbb{R}}$ be a continuous real-valued stochastic process, we denote by $\mathrm{law}(X_t)$ the distribution of the random variable $X_t : \Omega \to \mathbb{R}$, i.e. $\mathrm{law}(X_t) \in \mathcal{P}(\mathbb{R})$.

We say that X is stochastically continuous at the random variable $X_t : \Omega \to \mathbb{R}$, if for every $s \in \mathbb{R}$,

$$\lim_{t\to s} \mathrm{E}\,|X_t - X_s| = 0.$$

Definition 1.1.5. A stochastically continuous process X is said to be almost periodic in one-dimensional distribution (APOD for short), if the mapping

$$\begin{cases} \mathbb{R} & \to & \mathcal{P}(\mathbb{R}) \\ t & \mapsto & \mathrm{law}(X_t) \end{cases}$$

is almost periodic.

Definition 1.1.6. A stochastically continuous process X is said to be almost periodic in finite dimensional distributions (APFD for short), if for all $n \in \mathbb{N}$ et $t_1 < t_2 < \ldots < t_n$ the mapping

$$\begin{cases} \mathbb{R} & \to & \mathcal{P}(\mathbb{R}^n) \\ t & \mapsto & \mathrm{law}(X_{t_1+t}, \ldots, X_{t_n+t}) \end{cases}$$

is almost periodic.

Definition 1.1.7. A stochastic process X with continuous trajectories is said to be almost periodic in distribution (or in infinite dimensional distributions APD for short) if the mapping

$$\begin{cases} \mathbb{R} & \to & \mathcal{P}(C_k(\mathbb{R},\mathbb{R})) \\ t & \mapsto & \mathrm{law}\big(X_{t+.}\big) \end{cases}$$

is almost periodic (the space of continuous functions $C_k(\mathbb{R},\mathbb{R})$ is endowed with the topology of uniform convergence on compact sets).

A comparative study between different almost periodicity of stochastic processes is given by C. Tudor [40] and completed by F. Bedouhene et al. [3].

1.2 Fractional integrals and derivatives

In this section we recall some important notions and results related to fractional calculus, an exhaustive survey may be found in [38].

1.2.1 Fractional calculus over $[a, b]$

Let $a, b \in \mathbb{R}$, $a < b$. For $p \geq 1$, let $L^p([a, b])$ be the space of real-valued functions on $\mathbb{R}$ such that

$$\|f\|_{L^p} = \left(\int_a^b |f(x)|^p dx\right)^{1/p}.$$

Definition 1.2.1. For $\phi \in L^1([a, b])$ and $\alpha \in (0, 1)$, the left- and right-sided fractional Riemann Liouville integrals of ϕ of order α on $[a, b]$ are given at almost all s by

$$(I_{a+}^{\alpha}\phi)(s) := \frac{1}{\Gamma(\alpha)} \int_a^s \phi(u)\big(s-u\big)^{\alpha-1} du, \quad s \in \mathbb{R},$$

and

$$(I_{b-}^{\alpha}\phi)(s) := \frac{1}{\Gamma(\alpha)} \int_s^b \phi(u)\big(u-s\big)^{\alpha-1} du, \quad s \in \mathbb{R},$$

respectively, where Γ denotes the gamma function (the gamma function is defined via a convergent improper integral $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, $\mathrm{Re}(z) > 0$).

Definition 1.2.2. For a function $\phi(s)$ given in the interval $[a, b]$ and for $\alpha \in (0, 1)$, the left- and right-sided fractional Riemann Liouville derivatives of ϕ of order α are given by

$$D_{a+}^{\alpha}(\phi)(s) := \frac{\alpha}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^s \frac{\phi(u)}{(s-u)^{\alpha}} dr,$$

and

$$D_{b-}^{\alpha}(\phi)(s) := -\frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_s^b \frac{\phi(u)}{(s-u)^{\alpha}} dr,$$

respectively.

Remark 1.2.1. Let $I_{a+}^{\alpha}(L^p[a, b])$ (resp. $I_{b_-}^{\alpha}(L^p[a, b])$ the image of $L^p([a, b])$ by the operator I_{a+}^{α} (resp. $I_{b_-}^{\alpha}$). The following inversion formulas hold:

$$I_{a+}^{\alpha}\left(D_{a+}^{\alpha} f\right) = f \quad \forall f \in I_{a+}^{\alpha}(L^p[a, b]),$$

and

$$D_{a+}^{\alpha}\left(I_{a+}^{\alpha} f\right) = f \quad \forall f \in L^1([a, b]),$$

and the statements are true for $(I_{b_-}^{\alpha}, D_{b_-}^{\alpha})$.

1.2.2 Fractional calculus over $\mathbb{R}$

Definition 1.2.3. For $\alpha \in (0, 1)$, the Liouville left- and right-sided fractional integral are defined respectively by

$$(I_{+}^{\alpha}\phi)(s) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^s \phi(u)\big(s-u\big)^{\alpha-1} du, \quad s \in \mathbb{R}$$

and

$$(I_{-}^{\alpha}\phi)(s) := \frac{1}{\Gamma(\alpha)} \int_s^{+\infty} \phi(u)\big(s-u\big)^{\alpha-1} du, \quad s \in \mathbb{R}.$$

For $p \geq 1$ we denote by $I_{\pm}^{\alpha}(L^p(\mathbb{R}))$ the class of functions f which can be represented as an $I_{\pm}^{\alpha}$ of some L^p-function ϕ.

In a similar fashion, we recall the left- and right-sided Liouville fractional derivative.

Definition 1.2.4. For $\alpha \in (0,1)$, the left- and right-sided Liouville fractional derivatives are respectively defined by

$$\left(\mathcal{D}_{+}^{\alpha}\phi\right)(s) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\int_{-\infty}^{s}\frac{\phi(u)}{(s-u)^{\alpha}}du, \ \ s \in \mathbb{R},$$

and

$$\left(\mathcal{D}_{-}^{\alpha}\phi\right)(s) := -\frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\int_{-\infty}^{s}\frac{\phi(u)}{(s-u)^{\alpha}}du, \ \ s \in \mathbb{R}.$$

Now let us recall the Marchaaud fractional derivatives (see Samko et al. [38], p.111).

Definition 1.2.5. The Marchaud fractional derivatives $D_{\pm}^{\alpha}$ of order $\alpha \in (0,1)$ for a function ϕ on the whole real line is defined by

$$\left(D_{\pm}^{\alpha}\phi\right)(x) := \lim_{\varepsilon\to 0}\left(D_{\pm,\varepsilon}^{\alpha}\phi\right)(x), \quad x \in \mathbb{R},$$

where

$$\left(D_{\pm,\varepsilon}^{\alpha}\phi\right)(x) := \frac{\alpha}{\Gamma(1-\alpha)}\int_{\varepsilon}^{+\infty}\frac{\phi(x)-\phi(x+r)}{r^{1+\alpha}}dr. \tag{1.2}$$

The next theorem is fundamental for the demonstration of our result, it gives a useful characterization of the space I_{-}^{α} in terms of Marchaud fractional derivatives.

Theorem 1.2.1 ([38])**.** The necessary and sufficient conditions for $f \in I_{-}^{\alpha}(L^p(\mathbb{R}))$, $1 < p < 1/\alpha$ are

1. one of two following conditions is valid

$$\lim_{\varepsilon\to 0} D_{+,\varepsilon}^{\alpha} f \in L^p(\mathbb{R}),$$

$$\sup_{\varepsilon>0}\|D_{+,\varepsilon}^{\alpha} f\|_{L^p(\mathbb{R})} < +\infty.$$

2. $f \in L^r(\mathbb{R})$, where $r = q = p/(1-\alpha p)$ in the necessity part and r is arbitrary $(1 \leq r < +\infty)$ in the sufficiency part.

Moreover there is the following corollary on the equivalence of norms.

Corollary 1.2.1 ([38]). The norm

$$\|\phi\|_{I_-^\alpha(L^p(\mathbb{R}))} = \|f\|_{L^p(\mathbb{R})} \quad \text{with } \phi = I_+^\alpha(f),$$

in the space $I_+^\alpha(L^p(\mathbb{R}))$ is equivalent to the norms

$$\|\phi\|_q + \|\lim_{\varepsilon\to 0} D_{+,\varepsilon}^\alpha f\|_p,$$

$$\|\phi\|_q + \sup_{\varepsilon>0} \|\lim_{\varepsilon\to 0} D_{+,\varepsilon}^\alpha f\|_p,$$

with

$$q = \frac{p}{1-\alpha p}.$$

1.3 Fractional Brownian motion

The fractional Brownian motions are self-similar special process which have several applications in different areas, as : biology, hydrology, finance, telecommunication .... They have the particularity of preserving their distribution by changing scale and time.

In what follows, we will define and give the basic properties of the fBm : properties on the trajectories and two essential properties such as the absence of Markov and semi-martingale properties.

1.3.1 Definition and properties of fBm

Definition 1.3.1. Let $H \in (0,1]$. A fractional Brownian motion (fBm) $B^H = \{B_t^H\}_{t\in\mathbb{R}}$ of Hurst index H is a continuous and centered Gaussian process, with covariance function

$$\mathrm{R}_H(t,s) := \mathrm{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{1.3}$$

Remark 1.3.1. 1. For $H = \frac{1}{2}$, the covariance can be written as

$$\mathrm{R}_{\frac{1}{2}}(t,s) = \frac{1}{2}\left(t + s - |t-s|\right) = t \wedge s,$$

and the process $B^{\frac{1}{2}}$ is a standard Brownian motion.

2. For $H = 1$, $B_t^H = tB_1^H$ almost surely for all $t \geq 0$. Indeed

$$\begin{aligned}
\mathrm{E}[(B_t^H - tB_1^H)^2] =& \mathrm{E}[(B_t^H)^2] + t^2\,\mathrm{E}[(B_1^H)^2] - 2t\,\mathrm{E}[B_t^H B_1^H] \\
=& t^2 + t^2 - t(t^2 + 1 - (t-1)^2) \\
=& 0.
\end{aligned}$$

Existence of the fBm

The existence of the fBm follows from the general existence theorem of centered Gaussian processes with given covariance functions (see [21, page 443]).

Theorem 1.3.1 (Kolmogorov)**.** Let a symmetric function $C : \mathbb{R}^2 \to \mathbb{R}$. Then there is a unique centered Gaussian process $X = (X_t)_{t\in\mathbb{R}}$ having C for covariance function if and only if C is of nonnegative type.

Proposition 1.3.1 ([33])**.** *For any $H \in (0,1]$ the fBm B^H of Hurst parameter H exists (and unique in law).*

Proof. We have to show that R_H is of positive type if and only if $H \leq 1$. First assume that $H > 1$, if we put $t_1 = 1$, $t_2 = 2$ $a_1 = -2$ and $a_2 = 1$, we have

$$a_1^2 R_H(t_1,t_1) + 2a_1a_2R_H(t_1,t_2) + a_2^2R_H(t_2,t_2) = 4 - 2^{2H} < 0,$$

as a result R_H is negative type when $H > 1$.
Now assume that $H = 1$, we observe that

$$R_1(s,t) = \frac{1}{2}(s^2 + t^2 - (t-s)^2) = st,$$

then for all $d \geq 1$, $t_1, \dots, t_d \geq 0$ and $a_1, \dots, a_d \in \mathbb{R}$, we have

$$\begin{aligned}\sum_{i,j=1}^{d} R_H(t_i,t_j)a_ia_j &= \sum_{i,j=1}^{d} t_it_ja_ia_j = \sum_{i=1}^{d}\sum_{j=1}^{d} t_it_ja_ia_j = \sum_{i=1}^{d} t_ia_i \sum_{j=1}^{d} t_ja_j \\ &= \left(\sum_{j=1}^{d} t_ja_j\right)^2 \geq 0.\end{aligned}$$

Now consider the case $H \in (0,1)$. By the change of variable $v = u|x|$ for $x \neq 0$, we have

$$\begin{aligned}\int_0^{+\infty} \frac{1-e^{-u^2x^2}}{u^{1+2H}}du = \int_0^{+\infty} \frac{1-e^{-v^2}}{\left(\frac{v}{|x|}\right)^{1+2H}}\frac{dv}{|x|} &= \int_0^{+\infty} \frac{1-e^{-v^2}}{v^{1+2H}}\frac{|x|^{1+2H}}{|x|}dv \\ &= |x|^{2H}\int_0^{+\infty} \frac{1-e^{-v^2}}{v^{1+2H}}dv.\end{aligned}$$

So we get

$$|x|^{2H} = \frac{1}{\omega(H)}\int_0^{+\infty} \frac{1-e^{-u^2x^2}}{u^{1+2H}}du,$$

where

$$\omega(H) = \int_0^{+\infty} \frac{1-e^{-v^2}}{v^{1+2H}}dv = \frac{\Gamma(1-H)}{2H} \text{ (using integration by parts).}$$

We have that for $s,t \geq 0$,

$$s^{2H} + t^{2H} - |t-s|^{2H}$$

$$
\begin{aligned}
=&\frac{1}{\omega(H)}\int_0^{+\infty}\frac{1-e^{-u^2t^2}}{u^{1+2H}}du+\frac{1}{\omega(H)}\int_0^{+\infty}\frac{1-e^{-u^2s^2}}{u^{1+2H}}du-\frac{1}{\omega(H)}\int_0^{+\infty}\frac{1-e^{-u^2t^2-u^2s^2+2u^2ts}}{u^{1+2H}}du\\
=&\frac{1}{\omega(H)}\int_0^{+\infty}\frac{\left(1-e^{-u^2t^2}\right)\left(1-e^{-u^2s^2}\right)}{u^{1+2H}}du+\frac{1}{\omega(H)}\int_0^{+\infty}\frac{e^{-u^2t^2}\left(e^{2u^2ts}-1\right)e^{-u^2s^2}}{u^{1+2H}}du\\
=&\frac{1}{\omega(H)}\int_0^{+\infty}\frac{\left(1-e^{-u^2t^2}\right)\left(1-e^{-u^2s^2}\right)}{u^{1+2H}}du+\frac{1}{\omega(H)}\int_0^{+\infty}e^{-u^2t^2}e^{-u^2s^2}\frac{\sum_{n\geq 1}(2tsu^2)^n}{n!}du\\
=&\frac{1}{\omega(H)}\int_0^{+\infty}\frac{\left(1-e^{-u^2t^2}\right)\left(1-e^{-u^2s^2}\right)}{u^{1+2H}}du+\frac{1}{\omega(H)}\sum_{n\geq 1}\frac{2^n}{n!}\int_0^{+\infty}\frac{t^ne^{-u^2t^2}s^ne^{-u^2s^2}}{u^{1-2n+2H}},
\end{aligned}
$$

so $\forall d\geq 1$, $t_1,\dots,t_d\geq 0$ and $a_1,\dots,a_d\in\mathbb{R}$,

$$
\begin{aligned}
\sum_{i,j=1}^{d}\frac{1}{2}\left(t_i^{2H}+t_j^{2H}-|t_i-t_j|^{2H}\right)a_ia_j=&\frac{1}{2\omega(H)}\int_0^{+\infty}\frac{\left(\sum_{i=1}^{d}(1-e^{-u^2t_i^2})a_i\right)^2}{u^{1+2H}}du\\
&+\frac{1}{2\omega(H)}\sum_{n\geq 1}\frac{2^n}{n!}\int_0^{+\infty}\frac{\left(\sum_{i=1}^{d}t_i^ne^{-u^2t_i^2}a_i\right)^2}{u^{1-2n+2H}}du\geq 0.
\end{aligned}
$$

That is R_H is non-negative definite for $H\in(0,1)$. □

Proposition 1.3.2. *By Definition 1.3.1 we obtain the following properties:*

1. *For $H\in(0,1)$ the fBm B^H is a Gaussian H selfsimilar process, i,e.*
$$\forall a>0,\left\{a^{-H}B^H_{at}\right\}_{t\in\mathbb{R}}\overset{d}{=}\left\{B^H_t\right\}_{t\in\mathbb{R}}.$$
2. *For $H\in(0,1)$ the fBm B^H has stationary increments, i,e.*
$$\forall h>0,\left\{B^H_{t+h}-B^H_h\right\}_{t\in\mathbb{R}}\overset{d}{=}\left\{B^H_t\right\}_{t\in\mathbb{R}}.$$
3. *For $H\in(0,1)$ and $t\in\mathbb{R}$, we have*
$$E\left(B^H_t\right)^2=|t|^{2H}.$$
4. *For any $s,t\in\mathbb{R}$ we have*
$$E\left(|B^H_t-B^H_s|^2\right)=|t-s|^{2H}.$$

Proof. 1. Let us prove the selfsimilarity property. We have

$$
\begin{aligned}
\mathrm{E}\left(B^H_{at}B^H_{as}\right)=&\frac{1}{2}\left(|as|^{2H}+|at|^{2H}-|at-as|^{2H}\right)\\
=&a^{2H}\frac{1}{2}\left(|s|^{2H}+|t|^{2H}-|t-s|^{2H}\right)\\
=&\mathrm{E}\left(a^HB^H_ta^HB^H_s\right).
\end{aligned}
$$

Thus, since $\left\{B^H_{\alpha t}\right\}_{t\in\mathbb{R}}$ and $\left\{\alpha^HB^H_{\alpha t}\right\}_{t\in\mathbb{R}}$ are centered Gaussian processes, it implies that

$$\left\{a^{-H}B^H_{at}\right\}_{t\in\mathbb{R}}\overset{d}{=}\left\{B^H_t\right\}_{t\in\mathbb{R}}$$

.

2. Let us show that the fBm has stationary increments.

$$\begin{aligned}\mathrm{E}\left((B^H_{t+h}-B^H_h)^2\right) =&\mathrm{E}\left((B^H_{t+h})^2\right)+\mathrm{E}\left((B^H_h)^2\right)-2\,\mathrm{E}\left(B^H_{t+h}B^H_h\right)\\ =&|t+h|^{2H}+|h|^{2H}-2\frac{1}{2}\left(|t+h|^{2H}+|h|^{2H}-|t+h-h|^{2H}\right)\\ =&|t|^{2H}.\end{aligned}$$

3. If we put $s=t$ in the definition 1.3.1, we get

$$\mathrm{E}\left(B^H_t\right)^2=\frac{1}{2}\left(|t|^{2H}+|t|^{2H}-|t-t|^{2H}\right)=|t|^{2H}.$$

4. Fix $s,t\in\mathbb{R}$. Then

$$\begin{aligned}\mathrm{E}\left(|B^H_t-B^H_s|^2\right) =&\mathrm{E}\left((B^H_t)^2+(B^H_s)^2-2(B^H_tB^H_s)\right)\\ =&\mathrm{E}(B^H_t)^2+\mathrm{E}(B^H_s)^2-2\,\mathrm{E}(B^H_tB^H_s)\\ =&|t|^{2H}+|s|^{2H}-2\frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H})\\ =&|t-s|^{2H}.\end{aligned}$$

□

Hölder continuity

Proposition 1.3.3. *Let $H\in(0,1)$, the sample paths of fBm B^H are almost surely Hölder continuous of any order strictly less than H.*

Proof. For any $\alpha>0$ we have

$$\mathrm{E}\left(|B^H_t-B^H_s|^\alpha\right)=\mathrm{E}\left(|B^H_{t-s}|^\alpha\right)=\mathrm{E}(|B^H_1|^\alpha)|t-s|^{\alpha H}.$$

Hence, from the Kolmogorov's continuity criterion we get that the process B^H has sample paths a.s Hölder continuous of order strictly less than H. □

Path differentiability

Proposition 1.3.4 ([29])**.** *For $H\in(0,1)$, the fBm B^H is almost surely not differentiable.*

Proof. We want to show that for all $t_0\in\mathbb{R}$,

$$P\left(\limsup_{t\to t_0}\left|\frac{B^H_t-B^H_{t_0}}{t-t_0}\right|=+\infty\right)=1.$$

Let us denote by

$$\mathcal{N}_{t,t_0}:=\frac{B^H_t-B^H_{t_0}}{t-t_0}.$$

Using the self similarity of B^H, we get

$$\mathcal{N}_{t_0,t}\overset{law}{=}(t-t_0)^{H-1}B^H_1.$$

Define the events

$$A(t) = \left\{ \sup_{0 \le s \le t} \left| \frac{B_s^H}{s} \right| > M \right\},$$

with $M > 0$, then for any sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to zero, we have

$$A(t_n) \supset A(t_{n+1}).$$

Thus

$$P\left(\lim_{n\to+\infty} A(t_n) \right) = \lim_{n\to+\infty} P\left(A(t_n)\right),$$

and

$$P\left(A(t_n)\right) \ge P \left| \left(\frac{B_{t_n}^H}{t_n} \right| > M \right) = P\left(|B_1^H| > t_n^{1-H} M \right)$$

which tends to one as $n \to +\infty$. □

Long-range dependence

Definition 1.3.2. A stationary sequence $(X_n)_{n\in\mathbb{N}}$ exhibits long-range dependence if $\rho : \mathbb{N} \to \mathbb{R}$, defined $\rho(n) = \mathrm{Cov}(X_k, X_{k+n})$ satisfy the condition :

$$\lim_{n\to+\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1,$$

for some constant c and α de $]0;1[$.

Remark 1.3.2. If a stationary sequence $(X_n)_{n\in\mathbb{N}}$ is long-range dependent, then the dependence between X_k and X_{k+n} decays slowly as n tends to infinity and

$$\sum_{n=1}^{+\infty} \rho(n) = \infty.$$

Proposition 1.3.5. *The increments*

$$X_k := B_k^H - B_{k-1}^H \text{ and } X_{k+n} := B_{k+n}^H - B_{k+n-1}^H,$$

of fBm B^H have the Long-range dependence property when $H > \frac{1}{2}$.

Proof. For all $n \in \mathbb{N}^*$, using the selfsimilarity of B^H, we get

$$\begin{aligned} \rho_H(n) :&= \mathrm{Cov}(X_k, X_{k+n}) \\ &= \mathrm{E}(X_k X_{k+n}) \\ &= \mathrm{E}(B_1^H(B_{1+n}^H - B_n^H)) \\ &= \frac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right) \\ &\underset{n\to+\infty}{\sim} H(2H-1)n^{2H-2}. \end{aligned}$$

Note that $\rho_H(n)$ is the general term of a divergent series if and only if $2H - 2 > -1$ i,e. if and only if $H > \frac{1}{2}$. In this case we have $\sum_n |\rho_H(n)| = +\infty$.

□

Semimartingale property

Proposition 1.3.6 ([34]). *Let B^H be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, then B^H is not a semimartingale.*

Proof. Let us consider for fixed $p > 0$,

$$Y_{n,p} = n^{pH-1} \sum_{j=1}^{n} \left| B^H_{\frac{j}{n}} - B^H_{\frac{j-1}{n}} \right|^p .$$

We consider also

$$\tilde{Y}_{n,p} = n^{-1} \sum_{j=1}^{n} \left| B^H_j - B^H_{j-1} \right|^p ,$$

By the selfsimilar property of B^H, the sequence $\{Y_{n,p}; n \geq 1\}$ has the same distribution as $\left\{\tilde{Y}_{n,p}; n \geq 1\right\}$. On the other hand we have the sequence $(B^H_j - B^H_{j-1})_{j\in\mathbb{Z}}$ is stationary (due to the stationarity of fBm) and ergodic. Thus by the ergodic theorem we get

$$\tilde{Y}_{n,p} \rightarrow \mathrm{E}\left(|B^H_1 - B^H_0|^p\right) = (|B_1|^p) ,$$

almost surely in $L^1(\Omega)$ as $n \rightarrow +\infty$. So that , $Y_{n,p}$ converges in probability to $\mathrm{E}\left(|B_1|^p\right)$ when $n \rightarrow +\infty$. Therefore,

$$V_{n,p} := \sum_{j=1}^{n} \left| B^H_{\frac{j}{n}} - B^H_{\frac{j-1}{n}} \right|^p = \frac{Y_{n,p}}{n^{pH-1}} \xrightarrow{P} \begin{cases} 0 & \text{if } pH > 1 \\ \infty & \text{if } pH < 1 \end{cases}$$

Then the index of $p-$variation of B^H is $\frac{1}{H}$. Since for every semimartingale the index must belong to $[0, 1]$ or 2, the fBm B^H cannot be a semimartingale except when $H = \frac{1}{2}$. □

Markov property

Proposition 1.3.7. *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, the fractional Brownian motion B^H is not a Markov process.*

Proof. We proceed by contradiction. Assume that B^H is a Markov process. Since it is a Gaussian process, we must have for all $0 \leq s \leq t \leq u$

$$\mathrm{E}\left[B^H_s B^H_u\right] \mathrm{E}[(B_t)^2] = \mathrm{E}\left[B^H_s B^H_t\right] \mathrm{E}\left[B^H_t B^H_u\right] . \tag{1.4}$$

If we put $s = 1, t = 2, u = 3$ in (1.4), we obtain

$$\begin{aligned} R_H(1, 3)R_H(2, 2) &= R_H(1, 2)R_H(2, 3) \\ \frac{1}{2}\left(1 + 3^{2H} - 2^{2H}\right).2^{2H} &= \frac{1}{2}\left(1 + 2^{2H} - 1\right)\frac{1}{2}\left(2^{2H} + 3^{2H} - 1\right) \\ \frac{3}{4}2^{2H} + \frac{1}{4}.2^{2H}.3^{2H} - \frac{3}{4}2^{2H}.2^{2H} &= 0 \\ \frac{2^H}{4}\left(3 + 3^{2H} - 3.2^{2H}\right) &= 0. \end{aligned}$$

It follows that the function

$$H \mapsto 3 + 3^{2H} - 3.2^{2H} = 0,$$

for $H = \frac{1}{2}$ or for $H = 1$. The only possible case ($H = \frac{1}{2}$) corresponds to standard Brownian motion.

The proof of the proposition is done.

□

1.3.2 Stochastic integral representations

In this section, we will present three different integral representations of the fBm with respect to classical Brownian motion. These representations establish a relation between the classical Brownian motion with rich properties and the fBm (without Markov and semi-martingale properties).

The first representation of fBm in terms of classical Brownian motion on the whole real line is due to Mandelbrot and Van Ness [30].

Proposition 1.3.8. *Let B^H be a fBm with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $\{W_t\}_{t\in\mathbb{R}}$ be a classical Brownian motion. Then B^H has the moving average representation*

$$B_t^H = \frac{1}{C_1(H)} \left(\int_{-\infty}^{0} \left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right] dW_u + \int_0^t (t-u)^{H-\frac{1}{2}} dW_u \right), \tag{1.5}$$

with

$$C_1(H) = \left(\int_0^{+\infty} \left[(1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right]^2 ds + \frac{1}{2H} \right)^{1/2}.$$

Proof. Let us set

$$f_t(u) = (t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}}, \quad u \in \mathbb{R}, \quad t \geq 0.$$

Notice that $f_t \in L^2(\mathbb{R})$, i.e. $\int_{\mathbb{R}} f_t(u)^2 du < +\infty$. Now for $H \geq \frac{1}{2}$, $f_t(s) \sim (H - \frac{1}{2})(-s)^{H-\frac{3}{2}}$ as $u \to -\infty$. Hence, the Wiener integral is well defined. For $t \geq 0$, we set

$$X_t = \int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dW_s,$$

we have then

$$\begin{aligned} \mathrm{E}\,(X_t)^2 &= \int_{\mathbb{R}} \left[(t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right]^2 du \\ &= \int_{\mathbb{R}} \left[t^{\frac{1}{2}-H} \left(1 - \frac{u}{t} \right)_+^{H-\frac{1}{2}} - t^{\frac{1}{2}-H} \left(-\frac{u}{t} \right)_+^{H-\frac{1}{2}} \right]^2 du. \end{aligned}$$

Making the change of variable $s = \frac{u}{t}$, we get

$$\mathrm{E}\,(X_t)^2 = \int_{\mathbb{R}} \left[t^{\frac{1}{2}-H} (1-s)_+^{H-\frac{1}{2}} - t^{H-\frac{1}{2}} (-s)_+^{H-\frac{1}{2}} \right]^2 t\; ds$$

$$
\begin{aligned}
&= t^{2H} \int_{\mathbb{R}} \left[(1-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right]^2 ds \\
&= t^{2H} \int_{-\infty}^{0} \left[(1-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right]^2 ds + t^{2H} \int_0^1 (1-s)^{2H-1} ds \\
&= t^{2H} \int_0^{+\infty} \left[(1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right]^2 ds + \frac{t^{2H}}{2H} \\
&= [C_1(H)]^2 \, t^{2H}.
\end{aligned}
$$

For all $s < t$, we have

$$
\begin{aligned}
X_t - X_s &= \int_{\mathbb{R}} \left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right) dW_u - \int_{\mathbb{R}} \left((s-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right) dW_u \\
&= \int_{\mathbb{R}} \left((t-u)_+^{H-\frac{1}{2}} - (s-u)_+^{H-\frac{1}{2}} \right) dW_u,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{E}\,|X_t - X_s|^2 &= \mathrm{E}\,| \int_{\mathbb{R}} \left((t-u)_+^{H-\frac{1}{2}} - (s-u)_+^{H-\frac{1}{2}} \right) dW_u|^2 \\
&= \int_{\mathbb{R}} \left((t-u)_+^{H-\frac{1}{2}} - (s-u)_+^{H-\frac{1}{2}} \right)^2 du,
\end{aligned}
$$

by the change of variable $v = u - s$, we get

$$
\begin{aligned}
\mathrm{E}\,|X_t - X_s|^2 &= \int_{\mathbb{R}} \left((t-s-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right)^2 du \\
&= \int_{\mathbb{R}} (t-s)^{2H-1} \left((1 - \frac{u}{t-s})_+^{H-\frac{1}{2}} - (\frac{-u}{t-s})_+^{H-\frac{1}{2}} \right)^2 du,
\end{aligned}
$$

Making the change of variable $v = \frac{u}{t-s}$, we get

$$
\begin{aligned}
\mathrm{E}\,|X_t - X_s|^2 &= (t-s)^{2H-1} \int_{\mathbb{R}} \left((1-v)_+^{H-\frac{1}{2}} - (v)_+^{H-\frac{1}{2}} \right)^2 (t-s) dv \\
&= (t-s)^{2H} \int_{-\infty}^{0} \left[(1-v)^{H-\frac{1}{2}} - (-v)^{H-\frac{1}{2}} \right]^2 dv + (t-s)^{2H} \int_0^1 (1-v)^{2H-1} dv \\
&= (t-s)^{2H} \int_0^{+\infty} \left[(1+v)^{H-\frac{1}{2}} - v^{H-\frac{1}{2}} \right]^2 dv + \frac{(t-s)^{2H}}{2H} \\
&= (C_1(H))^2 \, (t-s)^{2H}.
\end{aligned}
$$

We have

$$
\mathrm{E}\,|X_t - X_s|^2 = \mathrm{E}\,(X_t)^2 + \mathrm{E}\,(X_s)^2 - 2\,\mathrm{E}\,|X_t X_s|,
$$

we get then

$$
\mathrm{E}\,|X_t X_s| = \frac{[C_1(H)]^2}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right).
$$

We deduce that the Gaussian centered $\{X_t, t \geq 0\}$ has the same covariance function as the fBm. We obtain then the following equality in distribution

$$
B_t^H \overset{d}{=} \frac{1}{C_1(H)} \int_{-\infty}^{0} \left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right] dW_u + \frac{1}{C_1(H)} \int_0^t (t-u)^{H-\frac{1}{2}} dW_u
$$

$$=\frac{1}{C_1(H)}X_t.$$

□

The equality (1.5) is a representation of the fBm on the whole real line. For a second representation of the fBm, we propose its representation on a finite interval $[0, T]$. We will see that the fBm is a special case of the Volterra processes, that is to say defined as a stochastic integral of a time-dependent deterministic kernel K with respect to a standard Brownian motion W

$$B_t^H = \int_0^t K_H(t,s)dW_s.$$

Case $H \in (\frac{1}{2}, 1)$

Proposition 1.3.9. *Let $H \in (\frac{1}{2}, 1)$, the kernel K_H is given by*

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}},$$

where

$$c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-1/2)} \right)^{1/2},$$

with β is the Bêta function [1].

Proof. It is readily checked that the covariance function $\mathrm{R}_H(t,s)$ can be written as

$$\mathrm{R}_H(t,s) = H(2H-1)\int_0^t \int_0^s |r-u|^{2H-2}\, dudr.$$

If we suppose that $r \wedge u = u$ making the change of variable $z = \frac{r-v}{u-v}$, we get

$$\begin{aligned}
&\int_0^u v^{1-2H}(r-v)^{H-\frac{3}{2}}(u-v)^{H-\frac{3}{2}}dv \\
&\qquad =\int_{\frac{r}{u}}^{\infty} \left(\frac{zu-r}{z-1}\right)^{1-2H} (z(u-v))^{H-\frac{3}{2}} \left(\frac{r-v}{z}\right)^{H-\frac{3}{2}} \frac{(u-v)^2}{r-u}dz \\
&\qquad =\int_{\frac{r}{u}}^{\infty} (zu-r)^{1-2H} z^{H-\frac{3}{2}} (u-v)^{H-\frac{3}{2}} \left(\frac{r-u}{u-v}\right)^{2H-1} (u-v)^{H-\frac{3}{2}} \frac{(u-v)^2}{r-u}dz \\
&\qquad =\int_{\frac{r}{u}}^{\infty} (zu-r)^{1-2H} z^{H-\frac{3}{2}} (r-u)^{2H-2}dz \\
&\qquad =(r-u)^{2H-2}\int_{\frac{r}{u}}^{\infty} (zu-r)^{1-2H} z^{H-\frac{3}{2}}dz.
\end{aligned}$$

Then by making the change of variable again $x = \frac{r}{z}$, $dx = \frac{-ru}{(uz)^2}dz$, we obtain

$$(r-u)^{2H-2}\int_{\frac{r}{u}}^{\infty} (zu-r)^{1-2H} z^{H-\frac{3}{2}}dz$$

[1] $\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt.$

$$
\begin{aligned}
&=(r-u)^{2H-2}\int_0^1 (zu-xzu)^{1-2H} z^{h-\frac{3}{2}} \frac{(uz)^2}{ru} dx\\
&=(r-u)^{2H-2}\int_0^1 (zu)^{1-2H}(1-x)^{1-2H} z^{H-\frac{3}{2}} \frac{(uz)^2}{ru} dx\\
&=(r-u)^{2H-2}\int_0^1\int_0^1 \left(\frac{r}{xu}\right)^{-H+\frac{3}{2}} u^{2-2H}(1-x)^{1-2H} r^{-1} dx\\
&=(ru)^{\frac{1}{2}-H}(r-u)^{2H-2}\int_0^1 (1-x)x^{H-\frac{3}{2}} dx\\
&=(ru)^{\frac{1}{2}-H}(r-u)^{2H-2}\beta\left(2-2H, H-\frac{1}{2}\right).
\end{aligned}
$$

We then have

$$
|r-u|^{2H-2} = \frac{(ru)^{H-\frac{1}{2}}}{\beta\left(2-2H, H-\frac{1}{2}\right)}\int_0^{r\wedge u} v^{1-2H}(r-v)^{H-\frac{3}{2}}(u-v)^{H-\frac{3}{2}} dv. \tag{1.6}
$$

If we consider

$$
K_H(t,s) = C_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,
$$

where

$$
C_H = \left(\frac{(H(2H-1))}{\beta(2-2H, H-1/2)}\right)^{\frac{1}{2}}, \text{ and } t > s.
$$

By (1.6) we have thus

$$
\begin{aligned}
&\int_0^{t\wedge s} K_H(t,u)K_H(s,u)du\\
&=C_H^2\int_0^{t\wedge s}\left(\int_u^t (y-u)^{H-\frac{3}{2}} y^{H-\frac{1}{2}} dy\right)\left(\int_u^s (z-u)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} dz\right) u^{1-2H} du\\
&=C_H^2\int_0^t\int_0^s (yz)^{H-\frac{1}{2}}\left(\int_0^{y\wedge z} u^{1-2H}(z-u)^{H-\frac{3}{2}}(y-u)^{H-\frac{3}{2}} du\right) dz dy\\
&=C_H^2\beta(2-2H, H-\frac{1}{2})\int_0^t\int_0^s |y-z|^{2H-2} dz dy\\
&=R_H(t,s).
\end{aligned}
$$

□

Remark 1.3.3. Using the fractional calculus, the covariance function R_H can be rewritten as

$$
R_H(t,s) = c\int_0^T \left(r^{\frac{1}{2}-H}(I_{T^-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \mathbb{1}_{[0,t)}(u))(r)\right)\left(r^{\frac{1}{2}-H}(I_{T^-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \mathbb{1}_{[0,s)}(u))(r)\right) dr,
$$

where

$$
c = \left(\frac{\Gamma(H-\frac{1}{2})^2 H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}.
$$

Case $H \in (0, \frac{1}{2})$

Proposition 1.3.10 ([34]). *Let $H \in (0, \frac{1}{2})$. The kernel K_H is given by*

$$K_H(t,s) = b_H \Big[\Big(\frac{t}{s}\Big)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du \Big],$$

where

$$b_H = \Big[\frac{2H}{(1-2H)\beta(1-2H, H+1/2)} \Big]^{\frac{1}{2}},$$

and satisfies

$$R_H = \int_0^t K_H(t,u) K_H(s,u) du. \tag{1.7}$$

Proof. First we consider the diagonal case $t = s$. Let us set

$$\phi(s) = \int_0^s K_H(s,u)^2 du,$$

then we have

$$\begin{aligned}\phi(s) =& b_H^2 \Big[\int_0^s \Big(\frac{s}{u}\Big)^{2H-1} (s-u)^{2H-1} du \\ &- (2H-1) \int_0^s s^{H-\frac{1}{2}} u^{1-2H} (s-u)^{H-\frac{1}{2}} \Big(\int_u^s v^{H-\frac{3}{2}} (v-u)^{H-\frac{1}{2}} dv \Big) du \\ &+ (H - \frac{1}{2})^2 \int_0^s u^{1-2H} \Big(\int_u^s v^{H-\frac{3}{2}} (v-u)^{H-\frac{1}{2}} dv \Big)^2 du \Big].\end{aligned}$$

By making the change of variable $u = sx$ in the first integral, we get

$$\begin{aligned}\phi(s) =& b_H^2 \Big[s^{2H} \int_0^1 x^{1-2H} (1-x)^{2H-1} dx \\ &- (2H-1) \int_0^s v^{H-\frac{3}{2}} \Big(\int_0^v u^{1-2H} (s-u)^{H-\frac{1}{2}} (v-u)^{H-\frac{1}{2}} du \Big) dv \\ &+ 2(H - \frac{1}{2})^2 \int_0^s \int_0^v \int_0^w u^{1-2H} (v-u)^{H-\frac{1}{2}} (w-u)^{H-\frac{1}{2}} w^{H-\frac{3}{2}} v^{H-\frac{3}{2}} du dw dv \Big].\end{aligned}$$

Now we make the change of variable $u = vx$ and $v = sy$ for the second term, we get

$$\begin{aligned}\phi(s) =& b_H^2 \Big[s^{2H} \beta(2-2H, 2H) \\ &- (2H-1) s^{H-\frac{1}{2}} s^{H+\frac{1}{2}} \int_0^1 \int_0^1 x^{1-2H} (1-yx)^{H-\frac{1}{2}} (1-x)^{H-\frac{1}{2}} dx dy \\ &+ 2(H - \frac{1}{2})^2 \int_0^s \int_0^v \int_0^w u^{1-2H} (v-u)^{H-\frac{1}{2}} (w-u)^{H-\frac{1}{2}} w^{H-\frac{3}{2}} v^{H-\frac{3}{2}} du dw dv \Big].\end{aligned}$$

We make the change of variable $u = wx$, $w = vy$ for the third term, we get

$$\phi(s) = b_H^2 \Big[s^{2H} \beta(2-2H, 2H)$$

$$
\begin{aligned}
&-(2H-1)s^{2H}\int_0^1\int_0^1 x^{1-2H}(1-yx)^{H-\frac{1}{2}}(1-x)^{H-\frac{1}{2}}dxdy\\
&+2(H-\frac{1}{2})^2\frac{1}{2H}s^{2H}\int_0^1\int_0^1 x^{1-2H}(1-yx)^{H-\frac{1}{2}}(1-x)^{H-\frac{1}{2}}dxdy\Big]\\
=&b_H^2 s^{2H}\Big[\beta(2-2H,2H)-(2H-1)\left(\frac{1}{4H}+\frac{1}{2}\right)\\
&\qquad\int_0^1\int_0^1 x^{1-2H}(1-yx)^{H-\frac{1}{2}}(1-x)^{H-\frac{1}{2}}dxdy\Big]\\
=&s^{2H}.
\end{aligned}
$$

Notice that

$$
\frac{\partial K_H}{\partial t}(t,s)=b_H\left(H-\frac{1}{2}\right)\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}. \tag{1.8}
$$

We suppose that $s<t$. Differentiating equality (1.7) with respect to t, we would like to show

$$
H(t^{2H-1}-(t-s)^{2H-1})=\int_0^s\frac{\partial K_H}{\partial t}(t,u)K_H(s,u)du. \tag{1.9}
$$

Let us set

$$
\phi(t,s)=\int_0^s\frac{\partial K_H}{\partial t}(t,u)K_H(s,u)du.
$$

Using equality (1.8), we get

$$
\begin{aligned}
\phi(t,s)=&b_H^2\left(H-\frac{1}{2}\right)\int_0^s\left(\frac{t}{u}\right)^{H-\frac{1}{2}}(t-u)^{H-\frac{3}{2}}\left(\frac{s}{u}\right)^{H-\frac{1}{2}}(s-u)^{H-\frac{3}{2}}du\\
&-b_H^2\left(H-\frac{1}{2}\right)^2\int_0^s\left(\frac{t}{u}\right)^{H-\frac{1}{2}}(t-u)^{H-\frac{3}{2}}u^{\frac{1}{2}-H}\\
&\qquad\left(\int_u^s v^{H-\frac{3}{2}}(v-u)^{H-\frac{1}{2}}dv\right)du.
\end{aligned}
$$

Now we make the change of variable $u=sx$ for the first integral and $u=vx$ for the second, we get

$$
\begin{aligned}
\phi(t,s)=&b_H^2\left(H-\frac{1}{2}\right)(ts)^{H-\frac{1}{2}}\int_0^1 x^{1-2H}\left(\frac{t}{s}-x\right)^{H-\frac{3}{2}}(1-x)^{H-\frac{1}{2}}dx\\
&-b_H^2\left(H-\frac{1}{2}\right)^2 t^{H-\frac{1}{2}}\int_0^1 x^{1-2H}(1-x)^{H-\frac{1}{2}}\left(\frac{t}{v}-x\right)^{H-\frac{3}{2}}dx\\
=&b_H^2\left(H-\frac{1}{2}\right)(ts)^{H-\frac{1}{2}}\gamma\left(\frac{t}{s}\right)\\
&-b_H^2\left(H-\frac{1}{2}\right)^2 t^{H-\frac{1}{2}}\int_0^1(v)^{H-\frac{3}{2}}\gamma\left(\frac{t}{v}\right)dv,
\end{aligned}
$$

where

$$
\gamma(y)=\int_0^1 x^{1-2H}(y-x)^{H-\frac{3}{2}}(1-x)^{H-\frac{1}{2}},
$$

for $y > 1$. Therefore equality (1.9) is equivalent to

$$
\begin{aligned}
&b_H^2\left(H-\frac{1}{2}\right)\left[s^{H-\frac{1}{2}}\gamma\left(\frac{t}{s}\right)-\left(H-\frac{1}{2}\right)\int_0^1 v^{H-\frac{3}{2}}\gamma\left(\frac{t}{v}\right)dv\right]\\
&\quad= H(t^{H-\frac{1}{2}}-(t-s)^{2H-1}t^{H-\frac{1}{2}}).
\end{aligned}
\tag{1.10}
$$

Differentiating the left-hand of equality (1.10) with respect to t, we get

$$
\begin{aligned}
&b_H^2\left(H-\frac{1}{2}\right)\left(H-\frac{3}{2}\right)\left[s^{H-\frac{3}{2}}\delta\left(\frac{t}{s}\right)-\left(H-\frac{1}{2}\right)\int_0^1 v^{H-\frac{5}{2}}\delta\left(\frac{t}{v}\right)dv\right]\\
&:=\mu(t,s),
\end{aligned}
\tag{1.11}
$$

where

$$\delta(y)=\int_0^1 x^{1-2H}(y-x)^{H-\frac{5}{2}}(1-x)^{H-\frac{1}{2}},$$

for $y > 1$. Using the change of variable $z=\frac{y(1-x)}{y-x}$, we obtain

$$
\begin{aligned}
\delta(y)=&\int_0^1\left(\frac{y(1-z)}{y-z}\right)^{1-2H}(y-z)^{H-\frac{5}{2}}\left(\frac{z(y-x)}{y}\right)^{H-\frac{1}{2}}\frac{(y-x)^2}{y(y-1)}dz\\
=&y^{-3H+\frac{1}{2}}(y-1)^{-1}\int_0^1(1-z)^{1-2H}z^{H-\frac{1}{2}}(y-x)^{2H-1}(y-z)^{2H-1}dz\\
=&y^{-3H+\frac{1}{2}}(y-1)^{-1}\int_0^1(1-z)^{1-2H}z^{H-\frac{1}{2}}\left(\frac{y(y-1)}{y-z}\right)^{2H-1}(y-z)^{2H-1}dz\\
=&y^{-H-\frac{1}{2}}(y-1)^{2H-2}\beta\left(2-2H,H+\frac{1}{2}\right).
\end{aligned}
\tag{1.12}
$$

And finally, replacing (1.12) in (1.11), we get

$$
\begin{aligned}
\mu(t,s)=\;& b_H^2\left(H-\frac{1}{2}\right)\left(H-\frac{3}{2}\right)\beta\left(2-2H,H+\frac{1}{2}\right)t^{-H-\frac{1}{2}}s(t-s)^{2H-2}\\
&-b_H^2\left(H-\frac{1}{2}\right)^2\left(H-\frac{3}{2}\right)\beta\left(2-2H,H+\frac{1}{2}\right)t^{-H-\frac{1}{2}}\int_0^s(t-v)^{2H-2}dv\\
=&b_H^2\left(H-\frac{1}{2}\right)\left(H-\frac{3}{2}\right)\beta\left(2-2H,H+\frac{1}{2}\right)t^{-H-\frac{1}{2}}s(t-s)^{2H-2}\\
&+b_H^2\left(H-\frac{1}{2}\right)\left(H-\frac{3}{2}\right)\beta\left(2-2H,H+\frac{1}{2}\right)\frac{1}{2}t^{-H-\frac{1}{2}}\left((t-s)^{2H-1}-t^{2H-1}\right)\\
=&b_H^2\left(H-\frac{1}{2}\right)\left(H-\frac{3}{2}\right)\beta\left(2-2H,H+\frac{1}{2}\right)\\
&\qquad\left[t^{-H-\frac{1}{2}}s(t-s)^{2H-2}+\frac{1}{2}t^{-H-\frac{1}{2}}\left((t-s)^{2H-1}-t^{2H-1}\right)\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
\beta\left(2-2H,H+\frac{1}{2}\right)=&\beta\left(H+\frac{1}{2},2-2H\right)=\beta\left(H+\frac{1}{2},1-2H+1\right)\\
&=\frac{(1-2H)}{(H+\frac{1}{2})+(1-2H)}\beta\left(H+\frac{1}{2},1-2H\right)
\end{aligned}
$$

$$=\frac{(1-2H)}{(\frac{3}{2}-H)}\beta\left(1-2H, H+\frac{1}{2}\right).$$

We deduce that

$$\mu(t,s) = H(1-2H)\left(t^{-H-\frac{1}{2}}s(t-s)^{2H-2} + \frac{1}{2}t^{-H-\frac{1}{2}}(t-s)^{2H-1} - \frac{1}{2}t^{H-\frac{3}{2}}\right).$$

$\mu(t,s)$ corresponds to derivative with respect to t of the right-hand side of (1.10). □

Remark 1.3.4. We have by using the fractional calculus, the kernel K_H can be expressed as

$$K_H(t,s) = cs^{\frac{1}{2}-H}\left(D_{T^-}^{\frac{1}{2}-H}u^{H-\frac{1}{2}}\mathbb{1}_{[0,t)}(u)\right)(s).$$

Case $H = \frac{1}{2}$

It is obvious that

$$K_{\frac{1}{2}}(t,s) = \mathbb{1}_{[0,t]}(s).$$

Indeed

$$B_t^{\frac{1}{2}} = \int_0^t K_{\frac{1}{2}}(t,s)dW_s = \int_0^t \mathbb{1}_{[0,t]}(s)dW_s = W_t.$$

And finally for a third representation of the fBm, we propose "the spectral representation" also called "the harmonizable representation" (see [33]).

Proposition 1.3.11. *For $H \in (0,\frac{1}{2})\cup(\frac{1}{2},1)$, the fractional Brownian motion $B^H = \{B_t^H\}_{t\geq 0}$, has the following representation*

$$B_t^H = \frac{1}{C_2(H)}\left(\int_{-\infty}^0 \frac{1-\cos(ut)}{|u|^{H+\frac{1}{2}}}dW_u + \int_0^{+\infty}\frac{\sin(ut)}{|u|^{H+\frac{1}{2}}}dW_u\right), \tag{1.13}$$

where

$$C_2(H)^2 = 2\int_0^{+\infty}\frac{1-\cos u}{u^{2H+1}}du < +\infty.$$

Proof. The Wiener integral (1.13) is well defined. For $t > s$ set

$$Y_t = \frac{1}{C_2(H)}\left(\int_{-\infty}^0 \frac{1-\cos(ut)}{|u|^{H+\frac{1}{2}}}dW_u + \int_0^{+\infty}\frac{\sin(ut)}{|u|^{H+\frac{1}{2}}}dW_u\right).$$

We have

$$\begin{aligned} Y_t - Y_s =& \frac{1}{C_2(H)}\left(\int_{-\infty}^0 \frac{1-\cos(ut)}{|u|^{H+\frac{1}{2}}}dW(u) + \int_0^{+\infty}\frac{\sin(ut)}{|u|^{H+\frac{1}{2}}}dW(u)\right) \\ &- \frac{1}{C_2(H)}\left(\int_{-\infty}^0 \frac{1-\cos(us)}{|u|^{H+\frac{1}{2}}}dW(u) + \int_0^{+\infty}\frac{\sin(us)}{|u|^{H+\frac{1}{2}}}dW(u)\right) \end{aligned}$$

$$=\frac{1}{C_2(H)}\left(\int_{-\infty}^{0}\frac{\cos(us)-\cos(ut)}{|u|^{H+\frac{1}{2}}}dW_u+\int_{0}^{+\infty}\frac{\sin(ut)-\sin(us)}{|u|^{H+\frac{1}{2}}}dW_u\right),$$

then

$$\mathrm{E}\left((Y_t-Y_s)^2\right)=\frac{1}{[C_2(H)]^2}\left[\int_{-\infty}^{0}\frac{(\cos(ut)-\cos(us))^2}{|u|^{2H+1}}du+\int_{0}^{+\infty}\frac{(\sin(ut)-\sin(us))^2}{|u|^{2H+1}}du\right]$$
$$=\frac{1}{[C_2(H)]^2}\int_{0}^{+\infty}\frac{(\cos(ut)-\cos(us))^2}{u^{2H+1}}du+\frac{1}{[C_2(H)]^2}\int_{0}^{+\infty}\frac{(\sin(ut)-\sin(us))^2}{u^{2H+1}}du$$
$$=\frac{1}{[C_2(H)]^2}\int_{0}^{+\infty}\frac{\cos^2(ut)+\cos^2(us)-2\cos(ut)\cos(us)+\sin^2(ut)+\sin^2(us)-2\sin(ut)\sin(us)}{u^{2H+1}}du$$
$$=\frac{1}{[C_2(H)]^2}\int_{0}^{+\infty}\frac{2-2\cos(ut)\cos(us)-2\sin(ut)\sin(us)}{u^{2H+1}}du$$
$$=\frac{1}{[C_2(H)]^2}\int_{0}^{+\infty}\frac{2-2\cos(u(t-s))}{u^{2H+1}}du$$
$$=\frac{2(t-s)^{2H}}{[C_2(H)]^2}\int_{0}^{+\infty}\frac{1-\cos v}{v^{2H+1}}dv$$
$$=(t-s)^{2H}.$$

Now using the fact that $Y_0=0$ we get that for any $t\geq s>0$,

$$\begin{aligned}\mathrm{E}(Y_sY_t)=&\frac{1}{2}\left(\mathrm{E}\left((Y_s-Y_0)^2\right)\right)+\mathrm{E}\left((Y_t-Y_0)^2\right)-\mathrm{E}\left((Y_t-Y_s)^2\right)\\=&\frac{1}{2}\left(s^{2H}+t^{2H}-(t-s)^{2H}\right),\end{aligned}$$

we deduce that the centered Gaussian process $\{Y_t,t\geq 0\}$ has the covariance R_H of a fBm with index H. □

1.3.3 The integrand space

We fix a fBm $\{B_t^H\}_{t\in\mathbb{R}}$ with Hurst parameter $H\in(0,1)$ defined on the probability space $(\Omega,\mathcal{F},P)$ such that $\mathcal{F}=\sigma\left(B_t^H,t\in\mathbb{R}\right)$. Let $L^2(\Omega)$ be the space of real-valued random variables with a finite quadratic-mean.

We denote by $\mathcal{E}$ the space of elementary (or step) functions defined on real line by all functions of the form

$$f(u)=\sum_{k=1}^{n}f_k\mathbb{1}_{[u_k,u_{k+1})}(u),\ \ \forall u\in\mathbb{R},$$

where $f_k\in\mathbb{R}$. The integral on this space with respect to the fBm is defined by

$$I^H(f)=\sum_{k=1}^{n}f_k\left(B^H(u_{k+1})-B^H(u_k)\right):=\int_{\mathbb{R}}f(u)dB_u^H,$$

which is a centred Gaussian random variable. The linear Gaussian space

$$\left\{I^H(f);\ f\in\mathcal{E}\right\},$$

is a subset of

$$\mathfrak{I}(B^H) := \{X \in L^2(\Omega);\ \text{ s.t. } \exists (f_n)_{n\in\mathbb{N}} \subset \mathcal{E},\ \ I^H(f_n) \xrightarrow{L^2(\Omega)} X\},$$

which is also a linear space of centred Gaussian random variables with variance

$$\operatorname{Var}(X) = E(X^2) = \lim_{n\to\infty} E(I^H(f_n)^2) = \lim_{n\to\infty} \operatorname{Var}(I^H(f_n)),$$

for every $X \in \mathfrak{I}(B^H)$.
If f is an element of equivalence class of sequences of elementary functions $(f_n)_{n\in\mathbb{N}}$ such that $I^H(f_n) \xrightarrow{L^2(\Omega)} X$ ($X \in \mathfrak{I}(B^H)$), then X can be defined as the integral of f with respect to the fBm

$$X := \int_{\mathbb{R}} f(u)dB_u^H.$$

For every $X \in \mathfrak{I}(B^H)$, the existence of function (or integrand) f such that $X = \int_{\mathbb{R}} f(u)dB_u^H$ is insured according to the values of H :

1. In the case $H = \frac{1}{2}$ (which means that $B^{\frac{1}{2}} = W$) the set of all functions f forms the whole Hilbert space $L^2(\mathbb{R})$ which is isometric to $\mathfrak{I}(W)$.

2. In the case $H \neq \frac{1}{2}$, the problem is dealt by V. Pipiras and M.S. Taqqu [37, Proposition 2.1], where for $H \in (0, \frac{1}{2})$, the authors show the existence of a functional space which is a Hilbert space isometric to $\mathfrak{I}(B^H)$. However, for $H \in (\frac{1}{2}, 1)$, they show the existence of a functional space which is not complete and isometric to a proper subspace of $\mathfrak{I}(B^H)$.

In the following, we will give the construction of the integrand spaces for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ (for more details see [37]). The representation of fBm on the whole real line will allow us to define the integrand space. But before, let us recall the definition of fractional integrals and fractional derivatives.

Let W be a red two-sided standard Brownian motion. Recall that a fBm with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ has the moving average representation :

$$B_t^H = d_H \int_{\mathbb{R}} \Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \Big) dW_s, \tag{1.14}$$

where

$$d_H = \frac{1}{C_1(H)} = \frac{1}{\Big(\int_0^\infty \big((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\big)^2 ds + \frac{1}{2H}\Big)^{\frac{1}{2}}}.$$

Remark 1.3.5. Some remarks are in order.

- For $H \in (\frac{1}{2}, 1)$ the fractional kernel K can be expressed as a function of $I_-^{H-\frac{1}{2}}$. Precisely, we have

$$K(t,s) = d_H \Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \Big)$$

$$= d_H(H - \frac{1}{2}) \int_{\mathbb{R}} 1\!\!1_{[0,t)}(u)(u-s)_+^{H-\frac{3}{2}} du$$
$$= d_H \Gamma(H + \frac{1}{2}) I_-^{H-\frac{1}{2}} \big(1\!\!1_{[0,t)}\big)(s),$$

where $1\!\!1_{[0,t)}$ is interpreted as $-1\!\!1_{[t,0)}$, if $t < 0$.

- If $H \in (0, \frac{1}{2})$ the fractional Kernel $K(t,s)$ is given as function of $D_-^{\frac{1}{2}-H}$ (for the proof, see [37, Lemma 3.1])

$$K(t,s) = d_H\Big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\Big) = d_H\Gamma(H+\frac{1}{2})\big(D_-^{\frac{1}{2}-H} 1\!\!1_{[0,t)}\big)(s).$$

- Using the representation (1.14), we define the fractional integral, with respect to B^H, of every step function $f \in \mathcal{E}$, by the Wiener-Itô integral, with respect to the standard Brownian motion W as follows.

  1. If $\frac{1}{2} < H < 1$

  $$I^H(f) = \int_{\mathbb{R}} f(s) dB_s^H = d_H \Gamma(H + \frac{1}{2}) \int_{\mathbb{R}} \big(I_-^{H-\frac{1}{2}}(f)\big)(s) dW_s.$$

  2. If $0 < H < \frac{1}{2}$

  $$I^H(f) = \int_{\mathbb{R}} f(u) dB_u^H = d_H \Gamma(H + \frac{1}{2}) \int_{\mathbb{R}} \big(D_-^{\frac{1}{2}-H} f\big)(s) dW_s.$$

  We say that the definition of fractional integral is obtained by using the transfer principle from the Wiener-Itô integral.

- For every $f, g \in \mathcal{E}$, we have

$$E\big(I^H(f)I^H(g)\big) = (d_H)^2\Gamma(H+\frac{1}{2})^2 \int_{\mathbb{R}} \big(I_-^{H-\frac{1}{2}}(f)\big)(s)\big(I_-^{H-\frac{1}{2}}(g)\big)(s) ds$$

  for $H \in (\frac{1}{2}, 1)$ and

$$E\big(I^H(f)I^H(g)\big) = (d_H)^2\Gamma(H+\frac{1}{2})^2 \int_{\mathbb{R}} \big(D_-^{\frac{1}{2}-H} f\big)(s)\big(D_-^{\frac{1}{2}-H} g\big)(s) ds.$$

  for $H \in (0, \frac{1}{2})$.

We start by giving the class of integrands when $H \in (0, \frac{1}{2})$. It follows from Theorem 6.1 in [38] that, for every $f = I_-^{\frac{1}{2}-H}\phi$ for some $\phi \in L^2(\mathbb{R})$,

$$D_-^{\frac{1}{2}-H} f = D_-^{\frac{1}{2}-H}(I_-^{\frac{1}{2}-H}\phi) = \phi,$$

and hence

$$\int_{\mathbb{R}} [(D_-^{\frac{1}{2}-H} f)(s)]^2 ds = \int_{\mathbb{R}} \phi^2(u) du.$$

It is natural to introduce the following class of functions(see [37])

$$L^2_H(\mathbb{R}) = \left\{ f : \exists\, \phi \in L^2(\mathbb{R}) \text{ such that } f = I_-^{\frac{1}{2}-H}\phi \right\},$$

which is a linear space with the inner product

$$\langle f, g\rangle_{L^2_H(\mathbb{R})} = \left(d_H \Gamma(H+\frac{1}{2}) \right)^2 \int_{\mathbb{R}} \big(D_-^{\frac{1}{2}-H} f\big)(s) \big(D_-^{\frac{1}{2}-H} g\big)(s) ds,$$

and with the associated norm

$$\|f\|^2_{L^2_H(\mathbb{R})} = \langle f, f\rangle_{L^2_H(\mathbb{R})}.$$

The set of elementary functions $\mathcal{E}$ is dense in $L^2_H(\mathbb{R})$. The space $L^2_H(\mathbb{R})$ is complete and hence it is isometric to $\mathfrak{I}(B^H)$ (see Theorem 3.3 of [37]).

We now consider the case when the Hurst parameter H belongs to the interval $H \in (\frac{1}{2}, 1)$. The space of integrands with respect to the fBm B^H, denoted by $\mathcal{L}^2_H(\mathbb{R})$, is defined as follows

$$\mathcal{L}^2_H(\mathbb{R}) = \left\{ f ; I_-^{H-\frac{1}{2}}(f) \in L^2(\mathbb{R}) \right\} = \left\{ f ; \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u)(u-s)_+^{H-\frac{3}{2}} du \right)^2 ds < \infty \right\}.$$

This space is linear and endowed with the inner product

$$\langle f, g\rangle_H = (d_H)^2 \Gamma(H+\frac{1}{2})^2 \int_{\mathbb{R}} \big(I_-^{H-\frac{1}{2}}(f)\big)(s) \big(I_-^{H-\frac{1}{2}}(g)\big)(s) ds, \;\; f, g \in L^2_H(\mathbb{R}). \tag{1.15}$$

The set of elementary functions $\mathcal{E}$ is dense in $\mathcal{L}^2_H(\mathbb{R})$. The space $\mathcal{L}^2_H(\mathbb{R})$ is not complete ([37, Proposition 3.2]).

Remark 1.3.6. From the definition of the inner product (1.15), the following isometry holds

$$\langle f, g\rangle_H = (d_H)^2 \Gamma(H+\frac{1}{2})^2 \langle I_-^{H-\frac{1}{2}}(f), I_-^{H-\frac{1}{2}}(g)\rangle_{L^2(\mathbb{R})}.$$

We observe that the elements of the integrand space $\mathcal{L}^2_H(\mathbb{R})$ are not explicitly defined. However, Proposition 1.3.12 and the following identity (given in [25, page 404])

$$\big(H-\frac{1}{2}\big)^2 d_H^2 \int_{-\infty}^{t\wedge s} (s-u)^{H-\frac{3}{2}} (t-u)^{H-\frac{3}{2}} du = \varphi(s,t), \tag{1.16}$$

where

$$\varphi(s,t) = H(2H-1)|s-t|^{2H-2}; \;\; s, t \in \mathbb{R}$$

give the explicit form of the elements of $\mathcal{L}^2_H(\mathbb{R})$.

Proposition 1.3.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Then $f \in \mathcal{L}^2_H(\mathbb{R})$ if and only if*

$$\int_{\mathbb{R}^2} f(t) f(s) \varphi(t,s) dt ds < \infty.$$

Proof. Let $f \in \mathcal{L}_H^2(\mathbb{R})$. We have by Fubini Theorem and identity (1.16)

$$\begin{aligned}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(u)(u-s)_+^{H-\frac{3}{2}}du\Big)^2 ds &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} f(r)(r-s)_+^{H-\frac{3}{2}}dr\int_{\mathbb{R}} f(t)(t-s)_+^{H-\frac{3}{2}}dt\Big)ds \\ &= \int_{\mathbb{R}}\Big(\iint_{\mathbb{R}^2} f(r)(r-s)_+^{H-\frac{3}{2}} f(t)(t-s)_+^{H-\frac{3}{2}}drdt\Big)ds \\ &= \iint_{\mathbb{R}^2} f(r)f(t)\Big(\int_{-\infty}^{r\wedge t}(r-s)^{H-\frac{3}{2}}(t-s)^{H-\frac{3}{2}}ds\Big)drdt \\ &= \frac{1}{(H-\frac{1}{2})^2 d_H^2}\iint_{\mathbb{R}^2} f(r)f(t)\varphi(r,t)drdt.\end{aligned}$$

□

Remark 1.3.7. 1. The space $\mathcal{L}_H^2(\mathbb{R})$ can be endowed with another scalar product

$$\langle f,g\rangle_{H,\mathbb{R}} = \iint_{\mathbb{R}^2} f(r)g(t)\varphi(r,t)drdt$$

with the associated norm

$$|f|_{H,\mathbb{R}}^2 = \langle f,f\rangle_{H,\mathbb{R}}.$$

2. The operator Γ_H defined from $\mathcal{L}_H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ by

$$\Gamma_H(f)(u) = d_H\big(H-\frac{1}{2}\big)\int_{\mathbb{R}} f(t)(t-u)_+^{H-\frac{3}{2}}dt = d_H\Gamma(H+\frac{1}{2})\big(I_-^{H-\frac{1}{2}}f\big)(u),$$

for every $f \in \mathcal{L}_H^2(\mathbb{R})$, is an isometry.

3. The stochastic integral with respect to the fBm B^H is defined by

$$I^H(f) = \int_{\mathbb{R}} f(s)dB_s^H = \int_{\mathbb{R}}\Gamma_H(f)(s)dW_s = I^{1/2}(\Gamma_H(f)). \tag{1.17}$$

4. The fractional kernel, which fits to the definition (1.17), still denoted by K, is given by

$$K_H(t,s) = d_H\Gamma(H+\frac{1}{2})\Big(I_-^{H-\frac{1}{2}}\mathbb{1}_{[0,t)}\Big)(s).$$

For more details, see [25].

5. The set

$$L_H^2(\mathbb{R}) := \Big\{f:\mathbb{R}\to\mathbb{R};\ \iint_{\mathbb{R}^2}\varphi(r,t)|f(r)f(t)|drdt<\infty\Big\},$$

is a strict subspace (dense) of $\mathcal{L}_H^2(\mathbb{R})$. Endowed with the norm

$$||f||_{H,\mathbb{R}}^2 :=< |f|,|f| >_{H,\mathbb{R}},$$

it is Banach space. For more details on this space, see for instance [37].

Proposition 1.3.13 ([37]). *We have the following space embeddings:*

$$L^1(\mathbb{R})\cap L^2(\mathbb{R}) \subset L^{\frac{1}{H}}(\mathbb{R}) \subset L_H^2(\mathbb{R}) \subset \mathcal{L}_H^2(\mathbb{R}).$$

1.3.4 The multiple integrand space

All the results and definitions seen in the subsection below are naturally generalized to the case of multiple integrals with respect to the fBm, but with some different properties, as the Gaussian property (the multiple integral is not a Gaussian random variable in general). In this subsection, we just recall the most important definitions and results for our work.

We fix H such that $\frac{1}{2} < H < 1$ and denote by $L^2_H(\mathbb{R}^p)$ the class of all functions $f \in L^2(\mathbb{R}^p)$ such that

$$\int_{\mathbb{R}^{2p}} \prod_{j=1}^{p} \varphi(x_j, y_j) \, |f(x_1, \ldots, x_p) f(y_1, \ldots, y_p)| \, dx_1 \ldots dx_p dy_1 \ldots dy_p < \infty,$$

which is a Banach space with the norm

$$\|f\|^2_{H,\mathbb{R}^p} := \langle |f|, |f| \rangle_{H,\mathbb{R}^p},$$

for $f \in L^2_H(\mathbb{R}^p)$, where $\langle .,. \rangle_{H,\mathbb{R}^p}$ is an inner product on $L^2_H(\mathbb{R}^p)$ defined, for $f, g \in L^2_H(\mathbb{R}^p)$, by

$$\langle f, g \rangle_{H,\mathbb{R}^p} = \int_{\mathbb{R}^{2p}} \prod_{j=1}^{p} \varphi(x_j, y_j) f(x_1, \ldots, x_p) g(y_1, \ldots, y_p) dx_1 \ldots dx_p dy_1 \ldots dy_p.$$

The multiple fractional integral denoted by $I_-^{H-\frac{1}{2},p}$ of a function $f \in L^2_H(\mathbb{R}^p)$ is given by

$$\left(I_-^{H-\frac{1}{2},p} f\right)(x_1, \ldots, x_p) = \frac{1}{\Gamma(H-\frac{1}{2})^p} \int_{x_1}^{+\infty} \cdots \int_{x_p}^{+\infty} \frac{f(t_1, \ldots, t_p)}{\prod_{j=1}^{p}(t_j - x_j)^{\frac{3}{2}-H}} dt_1 \ldots dt_p.$$

The operator

$$\Gamma_H^{(p)} : L^2_H(\mathbb{R}^p) \to L^2(\mathbb{R}^p),$$

defined, for every $f \in L^2_H(\mathbb{R}^p)$, by

$$\begin{aligned}
\Gamma_H^{(p)}(f)(x_1, \ldots, x_p) &= \left[d_H\left(H - \frac{1}{2}\right)\right]^p \int_{x_1}^{+\infty} \cdots \int_{x_p}^{+\infty} \frac{f(t_1, \ldots, t_p)}{\prod_{j=1}^{p}(t_j - x_j)^{\frac{3}{2}-H}} dt_1 \ldots dt_p \\
&= \left[d_H \Gamma\left(H + \frac{1}{2}\right)\right]^p \frac{1}{\Gamma(H-\frac{1}{2})^p} \int_{x_1}^{+\infty} \cdots \int_{x_p}^{+\infty} \frac{f(t_1, \ldots, t_p)}{\prod_{j=1}^{p}(t_j - x_j)^{\frac{3}{2}-H}} dt_1 \ldots dt_p \\
&= \left[d_H \Gamma\left(H + \frac{1}{2}\right)\right]^p \left(I_-^{H-\frac{1}{2},p} f\right)(x_1, \ldots, x_p),
\end{aligned}$$

is an isometry.

Remark 1.3.8. Observe that $I_-^{\frac{1}{2}-H,p}$ is the product operator $\left(I_-^{\frac{1}{2}-H}\right)^{\otimes p}$.

If $0 < H < \frac{1}{2}$, we denote by $D_-^{\frac{1}{2}-H,p}$ the right-sided multiple fractional derivatives of order $\frac{1}{2} - H$, which is the inverse of the multiple fractional integrals $I_-^{\frac{1}{2}-H,p}$. In this case

$$L^2_H(\mathbb{R}^p) = \left\{ f : \exists \phi \in L^2(\mathbb{R}^p) \text{ such that } f = I_-^{\frac{1}{2}-H,p} \phi \right\}.$$

Thus

$$L^2_H(\mathbb{R}^p) = I_-^{\frac{1}{2}-H,p}\left(L^2(\mathbb{R}^p)\right).$$

We define the operator

$$\Gamma_H^{(p)}(f) : L^2_H(\mathbb{R}^p) \to L^2(\mathbb{R}^p)$$

by

$$\Gamma_H^{(p)}(f)(x_1,\dots,x_p) = \left[\frac{\Gamma\left(H+\frac{1}{2}\right)}{c_H}\right]^p \left(D_-^{\frac{1}{2}-H,p} f\right)(x_1,\dots,x_p); f \in L^2_H(\mathbb{R}^p),$$

where $D_-^{\frac{1}{2}-H,p}$ is the product operator $\left(D_-^{\frac{1}{2}-H}\right)^{\otimes p}$.

Proposition 1.3.14. *([36, Proposition 3.7])*

1. *The space $L^2_H(\mathbb{R}^p)$ is a Hilbert space with respect to the inner product*

$$\langle f, g\rangle_{L^2_H(\mathbb{R}^p)} = \left\langle \Gamma_H^{(p)} f, \Gamma_H^{(p)} g\right\rangle_{L^2(\mathbb{R}^p)}.$$

2. *The operator $\Gamma_H^{(p)}$ is one to one and onto.*

For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, we denote by $I_p^{1/2}(f)$ the multiple Wiener Itô integral of $f \in L^2(\mathbb{R}^p)$ with respect to the standard Brownian motion W (for more details on multiple Wiener Itô integral see [34] and [20]), that is,

$$I_p^{1/2}(f) = \int_{\mathbb{R}^p} f(t_1,\dots,t_p)dW_{t_1}\dots dW_{t_p}.$$

The following definition of the multiple integral with respect to the fBm B^H is one of the most important tools in this work.

Definition 1.3.3. For an element $f \in L^2_H(\mathbb{R}^p)$, we define its multiple integral by

$$I_p^H(f) := \int_{\mathbb{R}^p} f(t_1,\dots,t_p)dB^H_{t_1}\dots dB^H_{t_p},$$

and we have

$$I_p^H(f) = I_p^{1/2}(\Gamma_H^{(p)} f), \tag{1.18}$$

that is,

$$\int_{\mathbb{R}^p} f(t_1,\dots,t_p)dB^H_{t_1}\dots dB^H_{t_p} := \int_{\mathbb{R}^p} (\Gamma_H^{(p)} f)(t_1,\dots,t_p)dW_{t_1}\dots dW_{t_p}.$$

This definition is the same as that given in [23] and [18], for the proof see [36, Theorem 3.14]. Notice that for $p = 0$ and $f = f_0$ (constant), we set $I_0^H(f_0) = f_0$. For more details on multiple integrals with respect to fBm, see [6].

We denote by $L^2_{s,H}(\mathbb{R}^p)$ the subspace of all symmetric functions $f \in L^2_H(\mathbb{R}^p)$. Every function $f \in L^2_H(\mathbb{R}^p)$ is symmetrizable, we denote $\mathrm{sym}(f)$ its symmetrization. In [36, Lemma 3.4], the following inequality is proved

$$|\mathrm{sym}(f)|_{H,\mathbb{R}^p} \le |f|_{H,\mathbb{R}^p}. \tag{1.19}$$

1.3.5 Chaos Expansion

We denote by $L^2_H(\Omega,\mathcal{F},P)$ the space of all random variables F which have an orthogonal fractional chaos decomposition of the form

$$F = \mathrm{E}(F) + \sum_{p=1}^{+\infty} I^H_p(f_p), \tag{1.20}$$

where $f_p \in L^2_{s,H}(\mathbb{R}^p)$ and

$$\sum_{p=1}^{+\infty} p!|f_p|^2_{H,\mathbb{R}^p} < +\infty.$$

The chaos decomposition (1.20) is unique, i.e.

$$L^2_H(\Omega,\mathcal{F},P) = \oplus_{p=0}^{+\infty} I^H_p(L^2_{s,H}(\mathbb{R}^p)).$$

The following two results point out the difference between the two cases, $H \in (0,\frac{1}{2})$ and $H \in (\frac{1}{2},1)$.

Proposition 1.3.15 ([36]). *If $H \in (\frac{1}{2},1)$, then the subspace $L^2_H(\Omega,\mathcal{F},P)$ is total in $L^2(\Omega,\mathcal{F},P)$. Moreover, for every $p \geq 1$, the fractional chaos of order p, $I^H_p(L^2_{s,H}(\mathbb{R}^p))$ is not closed. In particular $L^2_H(\Omega,\mathcal{F},P)$ is not closed and it is strictly included in $L^2(\Omega,\mathcal{F},P)$.*

Proposition 1.3.16 ([36]). *If $H \in (0,\frac{1}{2})$, the fractional chaos of order $p \geq 1$, $I^H_p(L^2_{s,H}(\mathbb{R}^p))$, is closed and agrees with the Wiener chaos $I^{1/2}_p(L^2_s(\mathbb{R}^p))$. We have the equality*

$$L^2_H(\Omega,\mathcal{F},P) = L^2(\Omega,\mathcal{F},P).$$

1.3.5.1 Fractional anticipating integral

We denote by $L^2\left(\Omega,\mathcal{F},P,L^2_H(\mathbb{R})\right)$ the space of all measurable processes with trajectories in the Banach space $\left(L^2_H(\mathbb{R}),||.||_{H,\mathbb{R}}\right)$. For every $X \in L^2\left(\Omega,\mathcal{F},P,L^2_H(\mathbb{R})\right)$, we have, for a.e $t \in \mathbb{R}$, $X_t \in L^2_H(\Omega,\mathcal{F},P)$, i.e. for each $p \in \mathbb{N}$, there exists $f_p(.,t) \in L^2_{s,H}(\mathbb{R}^p)$ such that

$$X_t = \sum_{p=0}^{+\infty} I^H_p(f_p(.,t)),$$

with $f_p \in L^2_H(\mathbb{R}^{p+1})$ (t is considered here as variable) , for all $p \geq 1$ and

$$\sum_{p=1}^{+\infty} p!\|f_p\|^2_{H,\mathbb{R}^{p+1}} < +\infty.$$

We consider the operator

$$\delta^{\mathrm{ch}}_H : \left\{ \begin{array}{lll} L^2_H\left(\Omega,\mathcal{F},\mathbf{P},L^2_H(\mathbb{R})\right) & \to & L^2(\Omega,\mathcal{F},\mathbf{P}) \\ X & \mapsto & \delta^{\mathrm{ch}}_H(X) = \sum_{p=0}^{+\infty} I^H_{p+1}(\mathrm{sym}(f_p)) \end{array} \right. .$$

The domain of definition of this operator, denoted by D_H^{ch}, is the set of all processes $X \in L^2(\Omega, \mathcal{F}, P, L_H^2(\mathbb{R}))$ such that the series

$$\sum_{p=0}^{+\infty} I_{p+1}^H(\mathrm{sym}(f_p))$$

converges in $L^2(\Omega, \mathcal{F}, P)$.

Proposition 1.3.17. *([36, Proposition 3.35])*

1. *Assume that the space $L_H^2(\mathbb{R})$ is endowed with the norm $|.|_{H,\mathbb{R}}$ and $X \in D_H^{ch}$, then*

$$E\left(|X|_{H,\mathbb{R}}^2\right) = \sum_1^{\infty} p! \, |f|_{H,\mathbb{R}^{p+1}}^2 .$$

2. *The operator $\delta_H^{ch} : D_H^{ch} \to L^2(\Omega, \mathcal{F}, \mathbf{P})$ is closable.*

The closure of the operator defined above is denoted again by δ_H^{ch}. If the process $X 1\!\!1_{]-\infty,a]} \in D_H^{ch}$ then we define

$$\int_{-\infty}^a X_s dB_s^H = \delta_H^{\mathrm{ch}}(X 1\!\!1_{]-\infty,a]}).$$

Note that in the case $H \in (0, \frac{1}{2})$, the operator δ_H^{ch} corresponds to the extended divergence operator (see the work of Cheridito and Nualart [14]).

Chapter 2

Almost periodic solutions in distribution to affine stochastic differential equations driven by a fractional Brownian motion

The aim of this chapter is to use the chaos expansion approach to show an existence and uniqueness of almost periodic solution for linear stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Here the coefficients are real valued almost periodic functions. We provide an explicit expression for the chaos decomposition of the solution and show the almost periodicity in infinite dimensional distribution of the solution.

In the sequel, we fix a normalized fBm $(B_t^H)_{t\in\mathbb{R}}$ with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $(W_t)_{t\in\mathbb{R}}$ a standard Brownian motion defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbf{P})$ (i.e $(B_t^H)_{t\in\mathbb{R}}$ and $(W_t)_{t\in\mathbb{R}}$ generate the same filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$), we recall the well known representation formula

$$B^H(t) = \int_{\mathbb{R}} K_H(t,s) dW_s,$$

where

- $K_H(t,s) = d_H \Gamma(H + \frac{1}{2}) \Big(I_-^{H-\frac{1}{2}} \mathbb{1}_{[0,t[} \Big)(s)$, for $\frac{1}{2} < H < 1$.
- $K_H(t,s) = d_H \Gamma(H + \frac{1}{2}) \Big(D_-^{\frac{1}{2}-H} \mathbb{1}_{[0,t[} \Big)(s)$, for $0 < H < \frac{1}{2}$.

We consider the fractional affine equation

$$dX_t = [a_0(t) - X_t]dt + [b_0(t) + b(t)X_t]dB_t^H, \tag{2.1}$$

where, $a_0,\ b_0,\ b : \mathbb{R} \to \mathbb{R}$ are continuous functions. In the sequel, we suppose that

1. The mappings $a_0,\ b_0,\ b$ are almost periodic.

2. The functions
$$(t,s) \in \mathbb{R}^2 \mapsto G_0(t,s) := 1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} b_0(s),$$
and
$$(t,s) \in \mathbb{R}^2 \mapsto G(t,s) := 1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} b(s),$$
are elements of $L^2_H(\mathbb{R}^2)$.

3. Moreover, in the case $0 < H < \frac{1}{2}$ we add the condition
$$\left\| \left(\int_0^{+\infty} \frac{G(x_1,x_2) - G(x_1, x_2 + r_2)}{r_2^{\bar{q}(\frac{3}{2}-H)}} dr_2 \right)^{\frac{1}{\bar{q}}} \right\|_{L^2(\mathbb{R}^2)} < \infty,$$
where $\bar{q} = \frac{1}{1-H}$.

Theorem 2.1. *Under the conditions (1)-(3), the equation (2.1) has a unique evolution solution $X \in D^{ch}_H$, which can be expressed as follows :*
$$X_t = \int_{-\infty}^t e^{-(t-s)} a_0(s) ds + \int_{-\infty}^t e^{-(t-s)} \left[b_0(s) + b(s) X_s \right] dB^H_s$$
and with the chaos decomposition
$$X_t = \sum_{p=0}^{+\infty} I^H_p(f^t_p(.)), \tag{2.2}$$
where
$$f^t_0 = \int_{-\infty}^t e^{-(t-s)} a_0(s) ds; \tag{2.3}$$
$$f^t_1(s) = 1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} [b_0(s) + b(s) f^s_0]; \tag{2.4}$$
$$f^t_p(t_1,\dots,t_p) = \frac{1}{p} \sum_{j=1}^p 1\!\!1_{]-\infty,t]}(t_j) e^{-(t-t_j)} b(t_j) f^{t_j}_{p-1}(., \hat{t}_j); \;\; p \geq 2. \tag{2.5}$$

and $f^{t_j}_{p-1}(., \hat{t}_j)$ denote the function of $(p-1)$ variables without the variable t_j. More precisely
$$\begin{aligned} f^t_p(t_1,\dots,t_p) = \mathrm{sym}\{ 1\!\!1_{-\infty<t_1<\ldots<t_p<t} e^{-(t-t_1)} b_0(t_1) b(t_2) \dots b(t_p) \} \\ + \frac{1}{p!} 1\!\!1_{]-\infty,t]^p}(t_1,\dots,t_p) b(t_1) \dots b(t_p) \int_{-\infty}^{t_1\wedge\ldots\wedge t_p} e^{-(t-s)} a_0(s) ds. \end{aligned} \tag{2.6}$$

Furthermore, X is almost periodic in distribution.

Remark 2.1. It is straightforward to see that the solution $X = (X_t : t \in \mathbb{R})$ (which is a two-sided process) satisfies the condition
$$\lim_{t\to-\infty} X_t = 0,$$
and the evolution solution X is actually a strong solution (see [36]).

To prove this result, we use the following two lemmas, but before stating them, let us denote by

$$M = \max\bigl(\sup_{\mathbb{R}}(b(t), \sup_{\mathbb{R}}(b_0(t))\bigr), \quad m = \sup_{\mathbb{R}}(a_0(t)), L = \sup_{\mathbb{R}^2}(F(t,s)) \text{ and } C_H = \frac{\frac{1}{2}-H}{\Gamma(\frac{1}{2}+H)},$$

where

$$F(t,s) = \int_{-\infty}^{t}\int_{-\infty}^{s} \varphi(r,u)e^{-(t+s-(r+u))}drdu.$$

Lemma 2.1. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a real sequence, $l : \mathbb{R} \to \mathbb{R}$ a continuous bounded function such that

$$1\!\!1_{]-\infty,t]}(s)e^{-(t-s)}l(s) \in L^2_H(\mathbb{R}^2).$$

If the sequence $(l(.+\alpha_n))_{n\in\mathbb{N}}$ converges uniformly to $l^*(.)$, then

$$1\!\!1_{]-\infty,t]}(s)e^{-(t-s)}l^*(s) \in L^2_H(\mathbb{R}^2).$$

Proof. Case $\frac{1}{2} < H < 1$.
By dominated convergence theorem and Fatou lemma we get:

$$\iint_{\mathbb{R}^2} \varphi(t_2,s_2)\int_{-\infty}^{t_2}\int_{-\infty}^{s_2} \varphi(t_1,s_1)e^{-(t_2-t_1)}e^{-(s_2-s_1)}|l^*(t_1)l^*(s_1)|dt_1ds_1dt_2ds_2 =$$
$$\iint_{\mathbb{R}^2} \varphi(t_2,s_2)\lim_{n\to\infty}\int_{-\infty}^{t_2}\int_{-\infty}^{s_2} \varphi(t_1,s_1)e^{-(t_2-t_1)}e^{-(s_2-s_1)}|l(t_1+\alpha_n)l(s_1+\alpha_n)|dt_1ds_1dt_2ds_2 \leq$$
$$\underline{\lim}\iint_{\mathbb{R}^2} \varphi(t_2,s_2)\int_{-\infty}^{t_2+\alpha_n}\int_{-\infty}^{s_2+\alpha_n} \varphi(t_1,s_1)e^{-(t_2+\alpha_n-t_1)}e^{-(s_2+\alpha_n-s_1)}|l(t_1)l(s_1)|dt_1ds_1dt_2ds_2 < \infty,$$

thus

$$1\!\!1_{]-\infty,t]}(s)e^{-(t-s)}l^*(s) \in L^2_H(\mathbb{R}^2).$$

Case $0 < H < \frac{1}{2}$.
We denote by

$$g_n(t,s) := 1\!\!1_{]-\infty,t]}(s)e^{-(t-s)}l(s+\alpha_n),$$

and

$$g^*(t,s) = 1\!\!1_{]-\infty,t]}(s)e^{-(t-s)}l(s+\alpha_n).$$

By the definition of $D_-^{\frac{1}{2}-H}$, given in (1.2), and using the fact that

$$D_-^{\alpha}(e^{-t})(x) = e^{-x},$$

the calculation of the first fractional derivative of g_n, with respect to the first variable t, gives

$$\begin{aligned} D_-^{\frac{1}{2}-H}(g_n)(x_1) =& C_H l(s+\alpha_n)\int_0^{+\infty} \frac{1\!\!1_{[s,+\infty[}(x_1)e^{-(x_1-s)} - 1\!\!1_{[s,+\infty[}(x_1+r_1)e^{-(x_1+r_1-s)}}{r_1^{\frac{3}{2}-H}}dr_1 \\ =& C_H l(s+\alpha_n)\int_0^{+\infty} \frac{e^{-(x_1-s)} - e^{-(x_1+r_1-s)}}{r_1^{\frac{3}{2}-H}}dr_1 1\!\!1_{[s,+\infty[}(x_1) \end{aligned}$$

$$
\begin{aligned}
&- C_H l(s+\alpha_n) \int_{s-x_1}^{+\infty} \frac{e^{-(x_1+r_1-s)}}{r_1^{\frac{3}{2}-H}} dr_1 1\!\!1_{]-\infty,s[}(x_1) \\
=& l(s+\alpha_n) e^{-(x_1-s)} 1\!\!1_{[s,+\infty[}(x_1) - C_H l(s+\alpha_n) \int_{s-x_1}^{+\infty} \frac{e^{-(x_1+r_1-s)}}{r_1^{\frac{3}{2}-H}} dr_1 1\!\!1_{]-\infty,s[}(x_1).
\end{aligned}
$$

Thus, for every $n \in \mathbb{N}$,

$$
\begin{aligned}
D_-^{\frac{1}{2}-H,2}(g_n)(x_1,x_2) = D_-^{\frac{1}{2}-H}\left(D_-^{\frac{1}{2}-H}(g_n)(x_1)\right)(x_2) = C_H \left(\int_0^{\infty} \frac{(A_n(x_1,x_2) - A_n(x_1,x_2+r_2))}{r_2^{\frac{3}{2}-H}} dr_2 \right) \\
- (C_H)^2 \left(\int_0^{\infty} \frac{(B_n(x_1,x_2) - B_n(x_1,x_2+r_2))}{r_2^{\frac{3}{2}-H}} dr_2 \right),
\end{aligned}
$$

where

$$
A_n(x_1,x_2) = 1\!\!1_{]-\infty,x_1]}(x_2) e^{-(x_1-x_2)} l(x_2+\alpha_n),
$$

and

$$
B_n(x_1,x_2) = l(x_2+\alpha_n) \int_{x_2-x_1}^{+\infty} \frac{e^{-(x_1+r_1-x_2)}}{r_1^{\frac{3}{2}-H}} dr_1 1\!\!1_{]x_1,\infty[}(x_2).
$$

Since $g_n \in L_H^2(\mathbb{R}^2)$ for every $n \in \mathbb{N}$, then

$$
\left\| D_-^{\frac{1}{2}-H,2} g_n \right\|_{L^2(\mathbb{R}^2)} < \infty.
$$

By using the dominated convergence theorem, the uniform convergence of $(l(.+\alpha_n))_{n\in\mathbb{N}}$ to $l^*(.)$ and Fatou lemma, with the same techniques those used in the case $\frac{1}{2} < H < 1$(inversion of limits and inversion between limits and integrals), we obtain:

$$
\begin{aligned}
\left\| D_-^{\frac{1}{2}-H,2} g^* \right\|_{L^2(\mathbb{R}^2)} &= \left\| \lim_{n\to\infty} D_-^{\frac{1}{2}-H,2} g_n \right\|_{L^2(\mathbb{R}^2)} \\
&\leq \underline{\lim} \left\| D_-^{\frac{1}{2}-H,2} g_n \right\|_{L^2(\mathbb{R}^2)} = \left\| D_-^{\frac{1}{2}-H,2} g \right\|_{L^2(\mathbb{R}^2)} \\
&< +\infty.
\end{aligned}
$$

Thus

$$
1\!\!1_{]-\infty,t]}(s) e^{-(t-s)} l^*(s) \in L_H^2(\mathbb{R}^2).
$$

□

Lemma 2.2. The operator $\Im_H : D_H^{ch} \to L^2(\Omega, \mathcal{F}, P)$, defined by

$$
\Im_H(X)(t) = \int_{-\infty}^{t} e^{-(t-s)} a_0(s) ds + \int_{-\infty}^{t} e^{-(t-s)} \left[b_0(s) + b(s) X_s \right] dB_s^H; \quad t \in \mathbb{R},
$$

for every $X \in D_H^{ch}$, is well-defined. Moreover, if $X \in D_H^{ch}$ with the chaos decomposition

$$
X_t = \sum_{p=0}^{+\infty} I_p^H(f_p^t(.)),
$$

then X is a fixed point of $\Im_H$ if and only if $f_p^t(.)$ satisfies (2.3), (2.4) and (2.5).

Proof. First, let us show that the operator $\Im_H$ is well-defined. Let $X \in D_H^{ch}$, therefore, for each $p \in \mathbb{N}$, there exist $f_p(.,t) \in L^2_{s,H}(\mathbb{R}^p)$ such that

$$X_t = \sum_{p=0}^{+\infty} I_p^H(f_p(.,t)), \tag{2.7}$$

with $f_p \in L^2_H(\mathbb{R}^{p+1})$, for all $p \geq 1$ and

$$\sum_{p=1}^{+\infty} p! \|f_p\|^2_{H,\mathbb{R}^{p+1}} < +\infty.$$

We denote by (Y_t) the image of (X_t) by $\Im_H$, that is

$$Y_t = \int_{-\infty}^t e^{-(t-s)} a_0(s) ds + \int_{-\infty}^t e^{-(t-s)} \left[b_0(s) + b(s)X_s\right] dB_s^H, \ \forall t \in \mathbb{R}. \tag{2.8}$$

By replacing (2.7) in (2.8) we get

$$Y_t = \int_{-\infty}^t e^{-(t-s)} a_0(s) ds + \delta_H^{ch}(Z^t), \ \forall t \in \mathbb{R},$$

where

$$Z^t(s) = \sum_{p=0}^{\infty} I_p^H(g_p^t(.,s))$$

and

$$\begin{aligned} g_0^t(s) &:= \mathbb{1}_{]-\infty,t]}(s) e^{-(t-s)} [b_0(s) + b(s) f_0(s)]; \\ g_p^t(.,s) &:= \mathbb{1}_{]-\infty,t]}(s) e^{-(t-s)} b(s) f_p(.,s), \ \ p \geq 1. \end{aligned}$$

Case $\frac{1}{2} < H < 1$. As $X \in D_H^{ch}$, b_0 and b are bounded (almost periodic), it is straightforward to show that, for every $t \in \mathbb{R}$, $Z^t \in D_H^{ch}$ and thus

$$\delta_H^{ch}(Z^t) = \sum_{p=0}^{\infty} I_{p+1}^H\big(\text{sym}(g_p^t)\big),$$

converges in $L^2(\Omega, \mathcal{F}, P)$.
Case $0 < H < \frac{1}{2}$. Let us check that, for every $t \in \mathbb{R}$, $g_p^t \in L^2_H(\mathbb{R}^p)$, for every $p \in \mathbb{N}$.
We start with the case $p = 0$, then

$$\begin{aligned} g_0^t(s) =& \mathbb{1}_{]-\infty,t]}(s) e^{-(t-s)} b_0(s) + \mathbb{1}_{]-\infty,t]}(s) e^{-(t-s)} b(s) f_0(s) \\ :=& G_0^t(s) + G^t(s) f_0(s). \end{aligned}$$

Since $G_0^t \in L^2_H(\mathbb{R})$, then

$$\left(D_-^{\frac{1}{2}-H} G_0^t\right)(x) = C_H \int_0^{+\infty} \frac{G_0^t(x) - G_0^t(x+r)}{r^{\frac{3}{2}-H}} dr$$

is in $L^2(\mathbb{R})$.

The fractional derivative of $G^t f_0$ gives:

$$\begin{aligned}\left(D_-^{\frac{1}{2}-H}G^t f_0\right)(x) =& C_H\left\{ 1\!\!1_{]-\infty,t]}(x)e^{-(t-x)}b(x)\int_0^{+\infty}\frac{\left[f_0(x)-f_0(x+r)\right]}{r^{\frac{3}{2}-H}}dr \right.\\ &\left. + \int_0^{+\infty}\frac{\left[G^t(x)-G^t(x+r)\right]f_0(x+r)}{r^{\frac{3}{2}-H}}dr\right\} \\ =& e^{-(t-x)}1\!\!1_{]-\infty,t]}(x)b(x)\left(D_-^{\frac{1}{2}-H}f_0\right)(x) \\ &+ C_H\int_0^{+\infty}\frac{\left[G^t(x)-G^t(x+r)\right]f_0(x+r)}{r^{\frac{3}{2}-H}}dr.\end{aligned} \tag{2.9}$$

Therefore

$$\begin{aligned}\left\|D_-^{\frac{1}{2}-H}g_0^t\right\|_{L^2(\mathbb{R})} \leq \left\|D_-^{\frac{1}{2}-H}G_0^t\right\|_{L^2(\mathbb{R})} + M\left\|D_-^{\frac{1}{2}-H}f_0\right\|_{L^2(\mathbb{R})} \\ + \left\|C_H\int_0^{+\infty}\frac{\left[G^t(x)-G^t(x+r)\right]f_0(x+r)}{r^{\frac{3}{2}-H}}dr\right\|_{L^2(\mathbb{R})}.\end{aligned}$$

The first two terms of the previous sum are finite. It remains to show that the last term is finite too.

The Hölder's inequality (with $q=\frac{1}{H}=\frac{2}{1-2(\frac{1}{2}-H)}$ and its conjugate $\bar{q}=\frac{1}{1-H}$) and the assumption (3) imply

$$\begin{aligned}&\left\|C_H\int_0^{+\infty}\frac{\left[G^t(x)-G^t(x+r)\right]f_0(x+r)}{r^{\frac{3}{2}-H}}dr\right\|_{L^2(\mathbb{R})} \\ \leq& C_H\left\|\left(\int_0^{+\infty}\frac{\left|G^t(x)-G^t(x+r)\right|^{\bar{q}}}{r^{\bar{q}(\frac{3}{2}-H)}}dr\right)^{1/\bar{q}}\left(\int_0^{+\infty}|f_0(x+r)|^q dr\right)^{1/q}\right\|_{L^2(\mathbb{R})} \\ \leq& C_H\left\|\left(\int_0^{+\infty}\frac{\left|G^t(x)-G^t(x+r)\right|^{\bar{q}}}{r^{\bar{q}(\frac{3}{2}-H)}}dr\right)^{1/\bar{q}}\left(\int_{\mathbb{R}}|f_0(x+r)|^q dr\right)^{1/q}\right\|_{L^2(\mathbb{R})} \\ \leq& C_H\left\|f_0\right\|_{L^q(\mathbb{R})}\left\|\left(\int_0^{+\infty}\frac{\left|G^t(x)-G^t(x+r)\right|^{\bar{q}}}{r^{\bar{q}(\frac{3}{2}-H)}}dr\right)^{1/\bar{q}}\right\|_{L^2(\mathbb{R})}.\end{aligned}$$

From the equivalent norms given in [38, Corollary 1 page 129], we deduce that there exists a real constant $K>0$ such that

$$\left\|f_0\right\|_{L^q(\mathbb{R})} \leq K\left\|f_0\right\|_{H,\mathbb{R}}.$$

Thus, we can conclude that $g_0^t \in L_H^2(\mathbb{R})$, for every $t\in\mathbb{R}$. Furthermore,

$$\left\|g_0^t\right\|_{H,\mathbb{R}} \leq \left\|G_0^t\right\|_{H,\mathbb{R}} + M\left\|f_0\right\|_{H,\mathbb{R}} + C_H K N_t\left\|f_0\right\|_{H,\mathbb{R}},$$

where

$$N_t = \left\|\left(\int_0^{+\infty}\frac{\left|G^t(x)-G^t(x+r)\right|^{\bar{q}}}{r^{\bar{q}(\frac{3}{2}-H)}}dr\right)^{1/\bar{q}}\right\|_{L^2(\mathbb{R})}.$$

Now, let us show that $g_p^t \in L_H^2(\mathbb{R}^{p+1})$, for $p \geq 1$. We have

$$\begin{aligned}
&\big(D_-^{\frac{1}{2}-H,p+1} g_p^t\big)(x_1,\ldots,x_p,x_{p+1}) = \big(D_-^{\frac{1}{2}-H}\ \big(D_-^{\frac{1}{2}-H,p} g_p^t\big)(x_1,\ldots,x_p)\big)(x_{p+1})\\
=&C_H\Big\{G^t(x_{p+1})\int_0^{+\infty}\frac{\big(D_-^{\frac{1}{2}-H,p}f_p\big)(x_{p+1})-\big(D_-^{\frac{1}{2}-H,p}f_p\big)(x_{p+1}+r_{p+1})}{r_{p+1}^{\frac{3}{2}-H}}dr_{p+1}\\
&+\int_0^{+\infty}\frac{[G^t(x_{p+1})-G^t(x_{p+1}+r_{p+1})]\big(D_-^{\frac{1}{2}-H,p}f_p\big)(x_{p+1}+r_{p+1})}{r_{p+1}^{\frac{3}{2}-H}}dr_{p+1}\Big\}.
\end{aligned} \tag{2.10}$$

With the same manner as in the case $p=0$, we get $g_p^t \in L_H^2(\mathbb{R}^{p+1})$ for all $p \geq 1$ and

$$\left\|g_p^t\right\|_{H,\mathbb{R}^{p+1}} \leq M\left\|f_p\right\|_{H,\mathbb{R}^{p+1}} + C_H K N_t \left\|f_p\right\|_{H,\mathbb{R}^{p+1}}.$$

Thus,

$$\sum_{p\geq 1} p!\|g_p^t\|^2_{H,\mathbb{R}^{p+1}} < +\infty.$$

We deduce that,

$$\delta_H^{ch}(Z^t) = \sum_{p=0}^{\infty} I_{p+1}^H\big(\mathrm{sym}(g_p^t)\big),$$

converges in $L^2(\Omega,\mathcal{F},P)$.
Then, in both cases, for every $t\in\mathbb{R}$, $Y_t \in L^2(\Omega,\mathcal{F},P)$ with the chaos decomposition:

$$Y_t = \sum_{p=0}^{\infty} I_p^H(h_p^t(.)),$$

where

$$\begin{aligned}
h_0^t &:= \int_{-\infty}^t e^{-(t-s)}a_0(s)ds;\\
h_1^t(s) &:= 1\!\!1_{]-\infty,t]}(s)e^{-(t-s)}[b_0(s)+b(s)f_0(s)];\\
h_p^t(t_1,\ldots,t_p) &:= \frac{1}{p}\sum_{j=1}^{p} 1\!\!1_{]-\infty,t]}(t_j)e^{-(t-t_j)}b(t_j)f_{p-1}^{t_j}(\hat{\mathrm{t}}_j).
\end{aligned}$$

With the same arguments as used to prove that $Z^t \in D_H^{ch}$, we can also show that for every $t\in\mathbb{R}$, the kernels $h_p^t(.) \in L_{s,H}^2(\mathbb{R}^p)$.
Second, let $X \in D_H^{ch}$ such that $\Im_H(X)(t) = X_t$ for every $t\in\mathbb{R}$, we have then

$$\sum_{p=0}^{+\infty} I_p^H(f_p^t) = \int_{-\infty}^t e^{-(t-s)}a_0(s)ds \quad + \int_{-\infty}^t e^{-(t-s)}\left[b_0(s)+b(s)\sum_{p=0}^{+\infty}I_p^H(f_p^s)\right]dB^H(s),$$

$$f_0^t + I_1^H(f_1^t) + I_2^H(f_2^t) + \sum_{p\geq 3} I_p^H(f_p^t) = \int_{-\infty}^t e^{-(t-s)}a_0(s)ds$$

$$
\begin{aligned}
&+\int_{-\infty}^{t} e^{-(t-s)}b_0(s)dB^H(s)+\int_{-\infty}^{t} e^{-(t-s)}b(s)f_0^s dB^H(s)\\
&\quad+\int_{-\infty}^{t} e^{-(t-s)}b(s)I_1^H(f_1^s)dB^H(s)\\
&\quad+\int_{-\infty}^{t} e^{-(t-s)}b(s)I_2^H(f_2^s)dB^H(s)+\int_{-\infty}^{t} e^{-(t-s)}b(s)\sum_{p\geq 3} I_p^H(f_p^t)dB^H(s),
\end{aligned}
$$

By identifying the corresponding Wiener chaos, we get

$$
f_0^t=\int_{-\infty}^{t} e^{-(t-s)}a_0(s)ds,
$$

$$
f_1^t(s)=1\!\!1_{]-\infty,t]}(s)e^{(t-s)}\left[b_0(s)+b(s)f_0^s\right].
$$

$$
\int_{-\infty}^{t} e^{-(t-s)}b(s)I_1^H(f_1^s)dB^H(s)=I_2^H\left(\widetilde{f_1^*(.)1\!\!1_{]-\infty,t]}(*)e^{-(t-*)}b(*)}\right),
$$

where $\widetilde{f_1^*(.)1\!\!1_{]-\infty,t]}(*)e^{-(t-*)}b(*)}$ denotes the symetrization of the function $f_1^*(.)1\!\!1_{]-\infty,t]}(*)e^{-(t-*)}b(*)$ in 2 variables. We get then

$$
f_2^t(t_1,t_2)=\tfrac{1}{2}\left[1\!\!1_{]-\infty,t]}(t_1)e^{-(t-t_1)}b(t_1)f_1^{t_1}(t_2)+1\!\!1_{]-\infty,t]}(t_2)e^{-(t-t_2)}b(t_2)f_1^{t_2}(t_1)\right].
$$

By induction we will get identity (2.5) for every p .
Conversely, assume that f_p^t satisfy (2.3), (2.4) and (2.5), it's straightforward to check that X is a fixed point of $\Im_H$. □

Proof of Theorem 2.1. ***Step 1: existence and uniqueness of evolution solution in*** D_H^{ch}**.**

For the existence, by using Lemma 2.2, it suffices to show that the process X defined in (2.2)-(2.6) is an element of D_H^{ch}.
Case $H\in(\frac{1}{2},1)$: Clearly by using (1.19), for every $p\in\mathbb{N}$ and $t\in\mathbb{R}$, we have $f_p^t(.)\in L^2_{s,H}(\mathbb{R}^p)$ and

$$
\sum_{p=1}^{+\infty} p!\|f_p^t(.)\|^2_{H,\mathbb{R}^p}<+\infty,
$$

which means that for every $t\in\mathbb{R}$, $X(t)\in L^2_H(\Omega,\mathcal{F},\mathbf{P})$. Let us show that $f_p\in L^2_H(\mathbb{R}^{p+1})$ for every $p\geq 1$. We denote by g_p the multiple function

$$
g_p(t_1,\dots,t_{p+1})=1\!\!1_{-\infty<t_1<\dots<t_p<t_{p+1}}e^{-(t_{p+1}-t_1)}b_0(t_1)b(t_2)\dots b(t_p)
$$

and by h_p the multiple function

$$
h_p(t_1,\dots,t_{p+1})=1\!\!1_{]-\infty,t_{p+1}]^p}(t_1,\dots,t_p)b(t_1)\dots b(t_p)\int_{-\infty}^{t_1\wedge\dots\wedge t_p} e^{-(t_{p+1}-s)}a_0(s)ds.
$$

Thanks to the condition (2), we can show that $g_p,h_p\in L^2_H(\mathbb{R}^{p+1})$, therefore $f_p\in L^2_H(\mathbb{R}^{p+1})$ and

$$
\|f_p\|^2_{H,\mathbb{R}^{p+1}}\leq\frac{2}{(p!)^2}(M^2L)^{p-1}\big(\|g\|^2_{H,\mathbb{R}^2}+m\|h\|^2_{H,\mathbb{R}^2}\big),\ \ \forall p\geq 1,
$$

thus

$$\sum_{p=1}^{+\infty} p! \|f_p\|^2_{H,\mathbb{R}^{p+1}} < +\infty.$$

It remains to show that

$$\sum_{p=0}^{\infty} I^H_{p+1}\big(\text{sym}(f_p)\big)$$

converges in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. For this purpose, it is enough to show that

$$\sum_{p=0}^{+\infty} (p+1)! \|f_p\|^2_{H,\mathbb{R}^{p+1}} < +\infty.$$

We have

$$(p+1)! \|f_p\|^2_{H,\mathbb{R}^{p+1}} \leq \frac{4}{(p-1)!} (M^2 L)^{p-1} \big(\|g\|^2_{H,\mathbb{R}^2} + m \|h\|^2_{H,\mathbb{R}^2}\big), \;\; \forall p \geq 1,$$

which gives the convergence in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ of the series

$$\sum_{p=0}^{\infty} I^H_{p+1}\big(\text{sym}(f_p)\big).$$

Case $H \in (0, \frac{1}{2})$:

With the same manner as the calculations made in the proof of Lemma 2.1 (case $0 < H < \frac{1}{2}$), the assumption $G(t,s), G_0(t,s) \in L^2_H(\mathbb{R}^2)$ gives us

$$D_-^{\frac{1}{2}-H,2}(G)(x_1,x_2) =$$
$$C_H \int_0^{+\infty} \frac{G(x_1,x_2) - G(x_1,x_2+r_2)}{r_2^{\frac{3}{2}-H}} dr_2 - (C_H)^2 \int_0^{+\infty} \frac{B(x_1,x_2) - B(x_1,x_2+r_2)}{r_2^{\frac{3}{2}-H}} dr_2$$

is in $L^2(\mathbb{R}^2)$, where

$$B(x_1,x_2) = b(x_2) \int_{x_2-x_1}^{+\infty} \frac{e^{-(x_1+r_1-x_2)}}{r_1^{\frac{3}{2}-H}} dr_1 \mathbb{1}_{]x_1,\infty]}(x_2).$$

In the same way, we get the calculation for $G_0(t,s)$.
Now, let us check that $f_1 \in L^2_H(\mathbb{R}^2)$, where

$$f_1(s,t) = \mathbb{1}_{]-\infty,t]}(s) e^{-(t-s)} b_0(s) + \mathbb{1}_{]-\infty,t]}(s) e^{-(t-s)} b(s) f_0(s) := G_0(t,s) + F_1(s,t).$$

By assumption, $G_0(t,s) \in L^2_H(\mathbb{R}^2)$. It remains to show that $F_1(t,s) \in L^2_H(\mathbb{R}^2)$. Using the fact that

$$D_-^{\frac{1}{2}-H}(e^{-t})(x) = e^{-x},$$

the calculation of the first fractional derivative of $F_1(t,s)$ with respect to the first variable t, gives

$$D_-^{\frac{1}{2}-H}(F_1)(x_1) = b(s) f_0(s) e^{-(x_1-s)} \mathbb{1}_{[s,+\infty[}(x_1) - C_H b(s) f_0(s) \int_{s-x_1}^{+\infty} \frac{e^{-(x_1+r_1-s)}}{r_1^{\frac{3}{2}-H}} dr_1 \mathbb{1}_{]-\infty,s[}(x_1)$$

$$
\begin{aligned}
&= b(s)f_0(s)e^{-(x_1-s)}\mathbb{1}_{]-\infty,x_1]}(s) - C_H b(s)f_0(s)\int_{s-x_1}^{+\infty}\frac{e^{-(x_1+r_1-s)}}{r_1^{\frac{3}{2}-H}}dr_1\mathbb{1}_{]x_1,\infty[}(s)\\
&:= \left(G(x_1,s) - C_H B(x_1,s)\right)f_0(s).
\end{aligned}
$$

We have

$$
\begin{aligned}
&D_-^{\frac{1}{2}-H,2}\left(F_1\right)(x_1,x_2) = D_-^{\frac{1}{2}-H}\left(D_-^{\frac{1}{2}-H}\left(F_1\right)(x_1)\right)(x_2)\\
&= C_H\int_0^{+\infty}\frac{\left(G(x_1,x_2)-C_HB(x_1,x_2)\right)f_0(x_2) - \left(G(x_1,x_2+r_2)-C_HB(x_1,x_2+r_2)\right)f_0(x_2+r_2)}{r_2^{\frac{3}{2}-H}}dr_2\\
&= C_H\left(\left(G(x_1,x_2)-C_HB(x_1,x_2)\right)\right)\int_0^{+\infty}\frac{f_0(x_2)-f_0(x_2+r_2)}{r_2^{\frac{3}{2}-H}}dr_2\\
&+ C_H\int_0^{+\infty}\frac{\left(\left(G(x_1,x_2)-C_HB(x_1,x_2)\right)-\left(G(x_1,x_2+r_2)-C_HB(x_1,x_2+r_2)\right)\right)f_0(x_2+r_2)}{r_2^{\frac{3}{2}-H}}dr_2.
\end{aligned}
$$

By using the Hölder's inequality and the assumption (3) as in the proof of Lemma 2.2, the norm of the above quantity in $L^2(\mathbb{R}^2)$ gives:

$$
\begin{aligned}
&\left\|D_-^{\frac{1}{2}-H,2}\left(F_1\right)\right\|_{L^2(\mathbb{R}^2)} \leq \left\|\sup_{x_2\in\mathbb{R}}\left|G(.,x_2)-C_HB(.,x_2)\right|\right\|_{L^2(\mathbb{R})}\left\|D_-^{\frac{1}{2}-H}f_0\right\|_{L^2(\mathbb{R})}\\
&+\left\|C_H\int_0^{+\infty}\frac{\left(\left(G(x_1,x_2)-C_HB(x_1,x_2)\right)-\left(G(x_1,x_2+r_2)-C_HB(x_1,x_2+r_2)\right)\right)f_0(x_2+r_2)}{r_2^{\frac{3}{2}-H}}dr_2\right\|_{L^2(\mathbb{R}^2)}\\
&\leq K_2\left\|D_-^{\frac{1}{2}-H}f_0\right\|_{L^2(\mathbb{R})} + C_H\left\|\left(\int_0^{+\infty}\frac{\left|G(x_1,x_2)-G(x_1,x_2+r_2)\right|^{\bar{q}}}{r_2^{\bar{q}(\frac{3}{2}-H)}}dr_2\right)^{\frac{1}{\bar{q}}}\left(\int_0^{+\infty}\left|f_0(x_2+r_2)\right|^q dr_2\right)^{\frac{1}{q}}\right\|_{L^2(\mathbb{R}^2)}\\
&+(C_H)^2\left\|\left(\int_0^{+\infty}\frac{\left|B(x_1,x_2)-B(x_1,x_2+r_2)\right|^{\bar{q}}}{r_2^{\bar{q}(\frac{3}{2}-H)}}dr_2\right)^{\frac{1}{\bar{q}}}\left(\int_0^{+\infty}\left|f_0(x_2+r_2)\right|^q dr_2\right)^{\frac{1}{q}}\right\|_{L^2(\mathbb{R}^2)}\\
&\leq K_2\left\|D_-^{\frac{1}{2}-H}f_0\right\|_{L^2(\mathbb{R})} + C_H\left\|\left(\int_0^{+\infty}\frac{\left|G(x_1,x_2)-G(x_1,x_2+r_2)\right|^{\bar{q}}}{r_2^{\bar{q}(\frac{3}{2}-H)}}dr_2\right)^{\frac{1}{\bar{q}}}\right\|_{L^2(\mathbb{R}^2)}\left(\int_{\mathbb{R}}\left|f_0(r_2)\right|^q dr_2\right)^{\frac{1}{q}}\\
&+(C_H)^2\left\|\left(\int_0^{+\infty}\frac{\left|B(x_1,x_2)-B(x_1,x_2+r_2)\right|^{\bar{q}}}{r_2^{\bar{q}(\frac{3}{2}-H)}}dr_2\right)^{\frac{1}{\bar{q}}}\right\|_{L^2(\mathbb{R}^2)}\left(\int_{\mathbb{R}}\left|f_0(r_2)\right|^q dr_2\right)^{\frac{1}{q}}\\
&\leq K_2\left\|D_-^{\frac{1}{2}-H}f_0\right\|_{L^2(\mathbb{R})} + Q\left\|f_0\right\|_{L^q(\mathbb{R})} \leq K_2\left\|f_0\right\|_{H,\mathbb{R}} + QK\left\|f_0\right\|_{H,\mathbb{R}},
\end{aligned}
$$

where K is the same constant as that given in the proof of lemma 2.2, which is obtained from the equivalent norms given in [38, Corollary 1 page 129],

$$
\begin{aligned}
Q :=& \left((C_H)^2\left\|\left(\int_0^{+\infty}\frac{\left|B(x_1,x_2)-B(x_1,x_2+r_2)\right|^{\bar{q}}}{r_2^{\bar{q}(\frac{3}{2}-H)}}dr_2\right)^{\frac{1}{\bar{q}}}\right\|_{L^2(\mathbb{R}^2)}\right)\\
&+\left(C_H\left\|\left(\int_0^{+\infty}\frac{\left|G(x_1,x_2)-G(x_1,x_2+r_2)\right|^{\bar{q}}}{r_2^{\bar{q}(\frac{3}{2}-H)}}dr_2\right)^{\frac{1}{\bar{q}}}\right\|_{L^2(\mathbb{R}^2)}\right)
\end{aligned}
$$

and

$$K_2 = \left\| \sup_{x_2 \in \mathbb{R}} |(G(.,x_2) - C_H B(.,x_2)| \right\|_{L^2(\mathbb{R})}.$$

Thus $f_1 \in L^2_H(\mathbb{R}^2)$ and

$$\|f_1\|_{H,\mathbb{R}^2} \leq \|G_0\|_{H,\mathbb{R}^2} + K_2 \|f_0\|_{H,\mathbb{R}} + QK \|f_0\|_{H,\mathbb{R}}.$$

For $p \geq 2$, we use the recurrence form (2.5) and with the same manner we obtain:

$$\|f_{p+1}\|_{H,\mathbb{R}^{p+1}} \leq K_2 \|f_p\|_{H,\mathbb{R}^p} + QK \|f_p\|_{H,\mathbb{R}^p}.$$

Thus $X \in D_H^{ch}$ in both cases . The uniqueness is deduced from the explicit form (2.6).

Step 2: almost periodicity. To prove that X is almost periodic in distribution, we use Bochner's double sequences criterion. Let (α'_n) and (β'_n) be two sequences in $\mathbb{R}$, we show that there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indices such that, for every $t \in \mathbb{R}$, the limits

$$\lim_{n\to+\infty} \lim_{m\to+\infty} \mu(t + \alpha_n + \beta_m) \qquad \text{and} \lim_{n\to+\infty} \mu(t + \alpha_n + \beta_n),$$

exist and are equal, where $\mu(t) := \text{law}(X_t)$ is the distribution of X_t.

We have assumed that the functions a_0, b_0 and b are almost periodic, then there are subsequences $(\alpha_n) \subset (\alpha'_n)$ and $(\beta_n) \subset (\beta'_n)$ with same indices such that

$$\lim_{n\to\infty} \lim_{m\to\infty} a_0(t + \alpha_n + \beta_m) = \lim_{n\to\infty} a_0(t + \alpha_n + \beta_n) =: a_0^*(t), \tag{2.11}$$

$$\lim_{n\to\infty} \lim_{m\to\infty} b_0(t + \alpha_n + \beta_m) = \lim_{n\to\infty} b_0(t + \alpha_n + \beta_n) =: b_0^*(t), \tag{2.12}$$

and

$$\lim_{n\to\infty} \lim_{m\to\infty} b(t + \alpha_n + \beta_m) = \lim_{n\to\infty} b(t + \alpha_n + \beta_n) =: b^*(t). \tag{2.13}$$

These limits exist in any of the three modes of convergences: pointwise, uniform on compact intervals and uniform on $\mathbb{R}$. We now denote by $(\gamma_n)_{n\in\mathbb{N}}$ the sequence $(\alpha_n + \beta_n)_{n\in\mathbb{N}}$. From step 1, we can deduce that, for each fixed integer n,

$$X_t^n = \int_{-\infty}^t e^{-(t-s)} a_0(s + \gamma_n) ds + \int_{-\infty}^t e^{-(t-s)} \left[b_0(s + \gamma_n) + b(s + \gamma_n) X_s^n \right] dB_s^H$$

is the solution of

$$dX_t^n = [a_0(t + \gamma_n) - X_t^n]dt + [b_0(t + \gamma_n) + b(t + \gamma_n)X_t^n]dB_t^H, \tag{2.14}$$

with chaos decomposition

$$X_t^n = \sum_{p=0}^{+\infty} I_p^H(f_p^{t,n}(.)) = \sum_{p=0}^{+\infty} I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,n}(.))\Big),$$

where

$$f_p^{t,n}(t_1,\dots,t_p) = \mathrm{sym}\{\mathbb{1}_{-\infty<t_1<\dots<t_p<t} e^{-(t-t_1)} b_0(t_1+\gamma_n) b(t_2+\gamma_n)\dots b(t_p+\gamma_n)\}$$
$$+\frac{1}{p!}\mathbb{1}_{]-\infty,t]^p}(t_1,\dots,t_p) b(t_1+\gamma_n)\dots b(t_p+\gamma_n)\int_{-\infty}^{t_1\wedge\dots\wedge t_p} e^{-(t-s)} a_0(s+\gamma_n)ds. \quad (2.15)$$

Also from step 1 and Lemma 2.1, we deduce that

$$X_t^* = \int_{-\infty}^t e^{-(t-s)} a_0^*(s)ds + \int_{-\infty}^t e^{-(t-s)}\left[b_0^*(s)+b^*(s)X_s^*\right]dB_s^H,$$

is the solution of

$$dX_t^* = [a_0^*(t) - X_t^*]dt + [b_0^*(t)+b^*(t)X_t^*]dB_t^H, \quad (2.16)$$

with chaos decomposition

$$X_t^* = \sum_{p=0}^{+\infty} I_p^H(f_p^{t,*}(.)) = \sum_{p=0}^{+\infty} I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,*}(.))\Big),$$

where

$$f_p^{t,*}(t_1,\dots,t_p) = \mathrm{sym}\{\mathbb{1}_{-\infty<t_1<\dots<t_p<t} e^{-(t-t_1)} b_0^*(t_1) b^*(t_2)\dots b^*(t_p)\}$$
$$+\frac{1}{p!}\mathbb{1}_{]-\infty,t]^p}(t_1,\dots,t_p) b^*(t_1)\dots b^*(t_p)\int_{-\infty}^{t_1\wedge\dots\wedge t_p} e^{-(t-s)} a_0^*(s)ds. \quad (2.17)$$

We have

$$X_{t+\gamma_n} = \sum_{p=0}^{+\infty} I_p^H(f_p^{t+\gamma_n}(.)) = \sum_{p=0}^{+\infty} I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t+\gamma_n}(.))\Big).$$

The changes of variables, $\sigma = t+\gamma_n$ and $\sigma_j = t_j+\gamma_n$ for all $j=1,p$, transform

$$\sum_{p=0}^{+\infty} I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t+\gamma_n}(.))\Big)$$

to

$$\sum_{p=0}^{+\infty} \hat{I}_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,n}(.)\Big),$$

where

$$\hat{I}_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,n})\Big) = \int_{\mathbb{R}^p}\Gamma_H^{(p)}(f_p^{t,n})(t_1,\dots,t_p)d\tilde{W}_{n,t_1}\dots d\tilde{W}_{n,t_p},$$

and $\tilde{W}_{n,t_j} = W_{t_j+\gamma_n} - W_{\gamma_n}$ is a two-sided standard Brownian motion with the same distribution as W_{t_j}.
Let us show that, for each fixed $t\in\mathbb{R}$,

$$I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,n}(.))\Big) \text{ converges in quadratic mean to } I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,*}(.))\Big).$$

Note that for $p=0$ the process is deterministic and we get the convergence easily. Let

$$M^* = \max\big(M, \sup(b_0^*), \sup(b^*)\big).$$

For $p > 0$, using the Itô isometry and the fact that the operator $\Gamma_H^{(p)}$ is an isometry, we obtain

$$\begin{aligned}\mathrm{E}\left|I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,n}(.))\Big) - I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,*}(.))\Big)\right|^2 &= \left\|\Gamma_H^{(p)}\Big(f_p^{t,n}(.) - f_p^{t,*}(.)\Big)\right\|_{L^2(\mathbb{R}^p)}^2\\ &= \left|f_p^{t,n}(.) - f_p^{t,*}(.)\right|_{H,\mathbb{R}^p}^2.\end{aligned}$$

For $\frac{1}{2} < H < 1$, it is adequate to replace $f_p^{t,n}$ and $f_p^{t,*}$ with their explicit forms (2.15) and (2.17) respectively. We get

$$\begin{aligned}\left|f_p^{t,n}(.) - f_p^{t,*}(.)\right|_{H,\mathbb{R}^p}^2 \leq & 2|\mathrm{sym}\{\mathbb{1}_{-\infty<t_1<\ldots<t_p<t} e^{-(t-\cdot_1)} b_0(\cdot_1+\gamma_n) b(\cdot_2+\gamma_n)\ldots b(\cdot_p+\gamma_n)\}\\ & - \mathrm{sym}\{\mathbb{1}_{-\infty<t_1<\ldots<t_p<t} e^{-(t-\cdot_1)} b_0^*(\cdot_1) b^*(\cdot_2)\ldots b^*(\cdot_p)\}|_{H,\mathbb{R}^p}^2\\ & + 2|\frac{1}{p!}\mathbb{1}_{]-\infty,t]^p}(\cdot_1,\ldots,\cdot_p) b(\cdot_1+\gamma_n)\ldots b(\cdot_p+\gamma_n)\int_{-\infty}^{\cdot_1\wedge\ldots\wedge\cdot_p} e^{-(t-s)} a_0(s+\gamma_n)ds\\ & - \frac{1}{p!}\mathbb{1}_{]-\infty,t]^p}(\cdot_1,\ldots,\cdot_p) b^*(\cdot_1)\ldots b^*(\cdot_p)\int_{-\infty}^{\cdot_1\wedge\ldots\wedge\cdot_p} e^{-(t-s)} a_0^*(s)ds|_{H,\mathbb{R}^p}^2\\ \leq & 2p\big(I_1 + I_2 + \ldots I_k + \ldots + I_p\big)\\ & + 2(p+1)\big(J_1 + J_2 + \ldots + J_k + \ldots + J_{p+1}\big),\end{aligned}$$

where I_k and J_k are such that

$$\begin{aligned}I_k =& \\ & \left|\mathrm{sym}\{\mathbb{1}_{-\infty<t_1<\ldots<t_p<t} e^{-(t-\cdot_1)} b_0^*(\cdot_1) b^*(\cdot_2)\ldots b^*(\cdot_{k-1})\big(b(\cdot_k+\gamma_n) - b^*(\cdot_k)\big) b(\cdot_{k+1}+\gamma_n)\ldots b(\cdot_p+\gamma_n)\}\right|_{H,\mathbb{R}^p}^2\\ & \leq \sup_{\mathbb{R}} |b(t_k+\gamma_n) - b^*(t_k)|^2 \frac{L}{(p!)^2}\big((M^*)^2 L\big)^{p-1},\end{aligned}$$

$$\begin{aligned}J_k =& \\ & \left|\frac{1}{p!}\mathbb{1}_{]-\infty,t]^p} b^*(\cdot_1)\ldots b^*(\cdot_{k-1})\big(b(\cdot_k+\gamma_n) - b^*(\cdot_k)\big) b(\cdot_{k+1}+\gamma_n)\ldots b(\cdot_p+\gamma_n)\int_{-\infty}^{\cdot_1\wedge\ldots\wedge\cdot_p} e^{-(t-s)} a_0(s+\gamma_n)ds\right|_{H,\mathbb{R}^p}^2\\ & \leq \sup_{\mathbb{R}} |b(t_k+\gamma_n) - b^*(t_k)|^2 \frac{Lm^2}{(p!)^2}\big((M^*)^2 L\big)^{p-1}.\end{aligned}$$

and

$$\begin{aligned}J_{p+1} &= \left|\frac{1}{p!}\mathbb{1}_{]-\infty,t]^p} b^*(\cdot_1)\ldots b^*(\cdot_p)\int_{-\infty}^{\cdot_1\wedge\ldots\wedge\cdot_p} e^{-(t-s)}\big(a_0(s+\gamma_n) - a_0^*(s)\big)ds\right|_{H,\mathbb{R}^p}^2\\ &\leq \sup_{\mathbb{R}} |a_0(s+\gamma_n) - a_0^*(s)|^2 \frac{L(M^*)^2}{(p!)^2}\big((M^*)^2 L\big)^{p-1}.\end{aligned}$$

From (2.11)-(2.13) and the above inequalities, we deduce that

$$\lim_{n\to\infty} \mathrm{E}\left|I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,n}(.))\Big) - I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,*}(.))\Big)\right|^2 = 0.$$

Hence

$$I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,n}(.))\Big),$$

converges in quadratic mean (uniformly with respect to t) to

$$I_p^{1/2}\Big(\Gamma_H^{(p)}(f_p^{t,*}(.))\Big).$$

Using the previous inequalities again, we get the convergence of X_t^n in quadratic mean to X_t^* uniformly with respect to $t \in \mathbb{R}$, which gives the convergence in distribution of X_t^n to X_t^*.
For $0 < H < \frac{1}{2}$, we use the recurrence form (2.5). First, let us show that, for $p = 1$,

$$\left\| f_1^{t,n}(.) - f_1^{t,*}(.) \right\|_{H,\mathbb{R}^1} \to 0 \text{ as } n \to \infty.$$

We have

$$f_1^{t,n}(s) = \mathbb{1}_{]-\infty,t]}(s)e^{-(t-s)}b_0(s+\gamma_n) + \mathbb{1}_{]-\infty,t]}(s)e^{-(t-s)}b(s+\gamma_n)f_0^n(s) := g_1^{t,n}(s) + h_1^{t,n}(s)$$

and

$$f_1^{t,*}(s) = \mathbb{1}_{]-\infty,t]}(s)e^{-(t-s)}b_0^*(s) + \mathbb{1}_{]-\infty,t]}(s)e^{-(t-s)}b^*(s)f_0^*(s) := g_1^{t,*}(s) + h_1^{t,*}(s),$$

The fractional derivative of $g_1^{t,n} - g_1^{t,*}$, with respect to the variable s, gives

$$\begin{aligned}
D_-^{\frac{1}{2}-H}\left(g_1^{t,n} - g_1^{t,*}\right)(x_1) &= C_H \int_0^{+\infty} \frac{\left(g_1^{t,n}(x_1) - g_1^{t,*}(x_1)\right) - \left(g_1^{t,n}(x_1+r) - g_1^{t,*}(x_1+r)\right)}{r^{\frac{3}{2}-H}} dr \\
=& C_H e^{-(t-x_1)}\left(\int_0^{t-x_1} \frac{[b_0(x_1+\gamma_n) - b_0^*(x_1)] - e^r[b_0(x_1+r+\gamma_n) - b_0^*(x_1+r)]}{r^{\frac{3}{2}-H}} dr\right)\mathbb{1}_{]-\infty,t]}(x_1) \\
&+ C_H e^{-(t-x_1)}\left(\frac{[b_0(x_1+\gamma_n) - b_0^*(x_1)]}{(\frac{1}{2}-H)(t-x_1)^{\frac{1}{2}-H}}\right)\mathbb{1}_{]-\infty,t]}(x_1).
\end{aligned}$$

By using the limits inversion theorems, we get the convergence of this sequence to zero uniformly with respect to t and x_1 (here we use the uniform norm) and by the dominated convergence theorem, we obtain the convergence in $L^2(\mathbb{R})$ uniformly with respect to t. Thus

$$\left\| g_1^{t,n}(.) - g_1^{t,*}(.) \right\|_{H,\mathbb{R}} \to 0 \text{ as } n \to \infty.$$

It remains to show that

$$\left\| h_1^{t,n}(.) - h_1^{t,*}(.) \right\|_{H,\mathbb{R}} \to 0 \text{ as } n \to \infty.$$

We have

$$\begin{aligned}
D_-^{\frac{1}{2}-H}\left(h_1^{t,n} - h_1^{t,*}\right)(x_1) = C_H e^{-(t-x_1)}\Bigg(\int_0^{+\infty} & \frac{\mathbb{1}_{]-\infty,t]}(x_1)[b(x_1+\gamma_n)f_0^n(x_1) - b^*(x_1)f_0^*(x_1)]}{r^{\frac{3}{2}-H}} \\
&- \frac{\mathbb{1}_{]-\infty,t]}(x_1+r)e^r[b(x_1+r+\gamma_n)f_0^n(x_1+r) - b^*(x_1+r)f_0^*(x_1+r)]}{r^{\frac{3}{2}-H}} dr\Bigg).
\end{aligned}$$

With the same manner as below, we can show that this sequence converges to zero in $L^2(\mathbb{R})$ uniformly with respect to t . Thus, we conclude that

$$\left\| f_1^{t,n}(.) - f_1^{t,*}(.) \right\|_{H,\mathbb{R}} \to 0 \text{ as } n \to \infty.$$

We have also

$$D_-^{\frac{1}{2}-H}\left(h_1^{t,n}-h_1^{t,*}\right)(x_1)=e^{-(t-x_1)}1\!\!1_{]-\infty,t]}(x_1)b(x_1+\gamma_n)D^{\frac{1}{2}-H}\left(f_0^n\right)(x_1)$$
$$+C_He^{-(t-x_1)}\int_0^{+\infty}\frac{\left[1\!\!1_{]-\infty,t]}(x_1)b(x_1+\gamma_n)-1\!\!1_{]-\infty,t]}(x_1+r)b(x_1+r+\gamma_n)\right]f_0^n(x_1+r)}{r^{\frac{3}{2}-H}}dr$$
$$-e^{-(t-x_1)}1\!\!1_{]-\infty,t]}(x_1)b^*(x_1)D^{\frac{1}{2}-H}\left(f_0^*\right)(x_1)$$
$$-C_He^{-(t-x_1)}\int_0^{+\infty}\frac{\left[1\!\!1_{]-\infty,t]}(x_1)b^*(x_1)-1\!\!1_{]-\infty,t]}(x_1+r)b^*(x_1+r)\right]f_0^*(x_1+r)}{r^{\frac{3}{2}-H}}dr,$$

by using the same development as used for the formula (2.9), we obtain

$$\|h_1^{t,n}-h_1^{t,*}\|_{H,\mathbb{R}}\leq M\|f_0^n\|_{H,\mathbb{R}}+M\|f_0^*\|_{H,\mathbb{R}}+C_HKN_t\|f_0^n\|_{H,\mathbb{R}}+C_HKN_t\|f_0^*\|_{H,\mathbb{R}}.$$

Thus,

$$\|f_1^{t,n}-f_1^{t,*}\|_{H,\mathbb{R}}\leq\|h_1^{t,n}-h_1^{t,*}\|_{H,\mathbb{R}}+(M+C_HKN_t)\|f_0^n\|_{H,\mathbb{R}}+(M+C_HKN_t)\|f_0^*\|_{H,\mathbb{R}}.$$

Now, let us show that for $p\geq 2$,

$$\|f_p^{t,n}-f_p^{t,*}\|_{H,\mathbb{R}^p}\to 0\text{ as }n\to\infty.$$

By formula (2.5), we have

$$f_p^{t,n}(t_1,\dots,t_p)-f_p^{t,n}(t_1,\dots,t_p)=\frac{1}{p}\sum_{p=1}^{p}1\!\!1_{]-\infty,t]}(t_j)e^{-(t-t_j)}[b(t_j+\gamma_n)f_{p-1}^{t_j,n}(\hat{t_j})-b^*(t_j)f_{p-1}^{t_j,*}(\hat{t_j})].$$

By using the same technique as in formula (2.10) and with the same manner as the case $p=1$ (the second step, for h_1^t), we obtain

$$\|f_p^{t,n}-f_p^{t,*}\|_{H,\mathbb{R}^p}\to 0\text{ as }n\to\infty.$$

Furthermore,

$$\|f_p^{t,n}-f_p^{t,*}\|_{H,\mathbb{R}^p}\leq(M+C_HKN_t)\|f_{p-1}^n\|_{H,\mathbb{R}^p}+(M+C_HKN_t)\|f_{p-1}^*\|_{H,\mathbb{R}^p}.$$

Using the last inequality, we deduce that the series

$$\sum_{p=1}^{\infty}p!\|f_p^{t,n}-f_p^{t,*}\|_{H,\mathbb{R}^p}\leq\sum_{p=1}^{+\infty}(M+C_HKN_t)\,p!\|f_{p-1}^n\|_{H,\mathbb{R}^p}+\sum_{p=1}^{+\infty}(M+C_HKN_t)\,p!\|f_{p-1}^*\|_{H,\mathbb{R}^p},$$

converges uniformly with respect to n, thus by the inversion limits we get the convergence of X_t^* in quadratic mean to X_t^*, which gives the convergence in distribution of X_t^n to X_t^*. It remains to show that X_t^n and $X_{t+\gamma_n}$ have the same distribution for every $t\in\mathbb{R}$. From the transfer principle given by (1.18), we are able to define the Skorohod integral with respect to fBm (B_t^H) by Skorohod integral with respect to the two-sided standard Brownian motion (W_t) as follows :

$$X_t=\int_{-\infty}^t e^{-(t-s)}a_0(s)ds+\int_{-\infty}^t e^{-(t-s)}\left[b_0(s)+b(s)X_s\right]dB_s^H=\sum_{p=0}^{+\infty}I_p^H(f_p^t(.))$$

$$= \sum_{p=0}^{+\infty} I_p^{1/2}\left(\Gamma_H^{(p)}(f_p^t(.))\right) = f_0^t + \sum_{p=1}^{+\infty} I_p^{1/2}\left(\Gamma_H^{(p)}(f_p^t(.))\right) = f_0^t + \sum_{p=0}^{+\infty} I_{p+1}^{1/2}\left(\Gamma_H^{(p+1)}(f_{p+1}^t(.))\right)$$
$$= \int_{-\infty}^{t} e^{-(t-s)} a_0(s)ds + \int_{-\infty}^{t} U_s^t dW_s,$$

where $(U_s^t)_{s\in\mathbb{R}}$, for each $t \in \mathbb{R}$, is a process with chaos decomposition

$$U_s^t = \sum_{p=0}^{+\infty} I_p^{1/2}\left(\Gamma_H^{(p)}(f_p^t(., s))\right).$$

If the integrand process $(U_s^t)_{s\in\mathbb{R}}$ is predictable, we get the classical Itô integral. The change of variable $\sigma = s+\gamma_n$ gives another Itô integral with respect to $\tilde{W}_{n,t}$, where $\tilde{W}_{n,t} = W_{t+\gamma_n} - W_{\gamma_n}$ is standard Brownian motion with the same distribution as W_t and from the independence of the increments of standard Brownian motion, we deduce that the process $X_{t+\gamma_n}$ has the same distribution as X_t^n. Thus it suffices to show that the solution X is predictable process. From the definition of iterated Wiener integrals, the explicit definition of the kernels f_p and the convergence in square mean of the series we deduce that the process $(U_s^t)_{s\in\mathbb{R}}$ is predictable. Thus for $t \in \mathbb{R}$, the sequence $X_{t+\gamma_n}$ converges in distribution to X_t^*, i.e.

$$\lim_{n\to+\infty} \mu(t + \alpha_n + \beta_n) = \mu^*(t) := \text{law}\, X_t^*,$$

By analogy and using (2.11)-(2.13) we can prove that

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \mu(t + \alpha_m + \beta_n) = \mu^*(t).$$

We have thus proved that X has almost periodic one-dimensional distributions.

Now, let us show that the solution X is almost periodic in distribution (infinite dimensional distributions). For fixed $\tau \in \mathbb{R}$ and for each $n \in \mathbb{N}$, we consider

$$X_t^n = e^{(t-\tau)} X_\tau^n + \int_\tau^t e^{-(t-s)} a_0(s + \alpha_n)ds + \int_\tau^t U_s^{t,n} dW_s,$$

the evolution solution to the equation (2.14) and

$$X_t^* = e^{(t-\tau)} X_\tau^* + \int_\tau^t e^{-(t-s)} a_0^*(s)ds + \int_\tau^t U_s^{t,*} dW_s,$$

the evolution solution to the equation (2.16). By the forgoing :

the sequence $(X_\tau^n)_{n\in\mathbb{N}}$ converges in quadratic mean to X_τ^*,

and since $(U_s^{t,n})_{s\in\mathbb{R}}$ and $(U_s^{t,*})_{s\in\mathbb{R}}$ are predictable, the processes

$$\left(\int_\tau^t U_s^{t,n} dW_s\right)_{t\in[\tau,T]} \text{ and, for every} n \in \mathbb{N}, \left(\int_\tau^t U_s^{t,*} dW_s\right)_{t\in[\tau,T]},$$

are martingales for every $T > \tau$. Thus, by using the Doob's inequality, we obtain

$$\sup_{[\tau,T]} |X_t^n - X_t^*| \to 0,$$

in quadratic mean for all $T > \tau$, then

$$d_{\mathrm{BL}}(X^n, X^*) \longrightarrow 0 \text{ in } \mathcal{P}(C([\tau, T], \mathbb{R})) \text{ for all } T > \tau.$$

The uniqueness in law of the distribution of the evolution solution to (2.14) and (2.16) and the convergence in distribution of $X_{t+\gamma_n}$ to X_t^* imply that

$$d_{\mathrm{BL}}(X_{.+\gamma_n}, X^*) \longrightarrow 0 \text{ in } \mathcal{P}(C([\tau, T], \mathbb{R})) \text{ for all } T > \tau.$$

Therefore

$$d_{\mathrm{BL}}(X_{.+\gamma_n}, X^*) \longrightarrow 0 \text{ in } \mathcal{P}(C_k(\mathbb{R}, \mathbb{R})).$$

Now we can easily conclude the almost periodicity in distribution of the solution. $\square$

Conclusion

The purpose of this thesis was to study the existence and uniqueness of almost periodic solution to stochastic differential equations driven by a fractional Brownian motion with almost periodic coefficients. We are interested in a stochastic affine differential equation generated by a fractional Brownian motion of Hurst index $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$, the coefficient were deterministic functions. In this perspective the stochastic integral considered was in the sense of divergence operator for $H \in (\frac{1}{2}, 1)$, and the extended divergence operator for the case $H \in (0, \frac{1}{2})$ from Malliavin calculus. We have used the chaos decomposition approach to establish our result Theorem 2.1. The solution is given in terms of explicit expressions for the kernels of the fractional multiple integrals in its chaos expansion. We have shown that this solution is almost periodic in distribution (infinite dimensional distribution).

As a research prospect, our work can be pursued in different directions :

- Studying a more general class of stochastic differential equations driven by a fractional Brownian motion.
- Consider another type of almost periodicity : in the sense of Stepanov, Weyl, ...

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Résumé

Cette thèse porte sur l'étude d'éxistence et unicité de solution presque périodique en distribution pour des équations différentielles stochastiques gouvernées par un mouvement brownien fractionnaire.

En particulier, nous nous intéressons à l'étude d'existence et unicité d'une solution presque périodique en loi infinidimensionnelle pour une équation différentielle stochastique affine régie par un mouvement brownien fractionnaire d'indice de Hurst $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, sous la forme :

$$dX_t = (a_0(t) - X_t)dt + (b_0(t) + b(t)X_t)dB_t^H,$$

où a_0, b_0 et b sont des fonctions définies de $\mathbb{R}$ à valeurs dans $\mathbb{R}$, presque périodiques et l'intégrale stochastique considérée est l'intégrale divergence du calcul de Malliavin.

Mots clefs : Équations différentielles stochastiques, mouvement brownien fractionnaire, solution presque périodique, intégrales stochastiques multiples, décomposition chaotique.