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Résolution du problème de type transport multi-objectifs dans un environnement incertain

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Thema

Solving multi-objective transport problems in an uncertain environment

Résumé

Les problèmes de transport consistent à déterminer la manière de minimiser le coût de transporter des ressources, des marchandises ou des personnes d'un ensemble d'endroits (souvent appelés sources) à un autre ensemble d'endroits (appelée destinations) en utilisant divers types de modes de transport, qui minimise l'ensemble des coûts de cette opération.

Ce sont des problèmes de programmation linéaire d'un type particulier. Hitchock [41] en a donné le modèle et ce modèle a connu de nombreuses extensions et orientations (problème de transport à trois indices, problème de transport à quatre indices, problème de transport multiobjectif, problème de transport à charge fixe, problème de transport dans l'incertitude, ...). Singh et al. [91] ont réalisé une étude sur les extensions floues et stochastiques du problème de transport multi-indice.

Dans cette thèse, nous étudions différents modèles de problèmes de transport multiobjectif dans un environnement incertain et introduisons de nouveaux modèles et méthodes pour leur résolution.

Abstract

Transportation problems involve determining how to minimize the cost of transporting resources, goods, or people from one set of places (often called sources) to another set of places (called destinations) using various types of transport modes that minimize the overall costs.

These are linear programming problems of a particular type. Hitchock [41] has given the model, and this model has had many extensions and directions (three-index transport problem, four-index transport problem, multi-objective transport problem, fixed-load transport problem, transport problem under uncertainty, ...). Singh et al. [91] realized a study on fuzzy and stochastic extensions of the multi-index transport problem.

In this thesis, we study different models of multiobjective transport problems in an uncertain environment and introduce new models and methods for their resolution.

1. Under the supervision of : Méziane AÏDER, Professor, Faculty of Mathematics, USTHB.

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Thiziri Sifaoui

Contents

Abstract

Transportation theory is a specific name given to the field of studying optimal transportation and allocation of resources.

This problem was formalized very early by the French mathematician Gaspard Monge in 1781 and subsequently made important advances, particularly during the Second World War, due to the work of the Soviet investigator Leonid Kantorovich.

The French mathematician Gaspard Monge formalized the problem in 1781 and it was further developed during the Second World War, notably by the Soviet investigator Leonid Kantorovich.

Transportation theory is a specific name given to the field of studying optimal transport and allocation of resources. The French mathematician Gaspard Monge was formalized the problem in 1781 and important advances were known in the field during World War II by the soviet investigator Leonid Kantorovich. In 1941, Hitchock [41] given the special type of linear programming the transportation problem concerns the transport of an amount from a m source to n destination with the objective to minimize the cost of distribution. Then this model known different kind of extension and orientation.(solid transportation problem, multi-item transportation problem, fixed charge transportation problem,...).

Uncertainty refers to situations which involve imperfect or unknown information it applies to predictions of future events, to the physical measurements already made. Uncertainty appear in partially observable and/or stochastic environments as also due to ignorance, indolence or both. to the issues of transportation change with time and space,uncertainty affect the transportation problem.

In the literature, several methods are proposed for solving transportation problems under some specific uncertainty but it is still another kind of uncertainty to express the human thinking.In this thesis, different models of multi-objective transportation problem in an uncertain environment are studied and a new models and methods of resolution are proposed, depending on the different aspects describing the problem.

Introduction

This thesis is headlined "Solving multi-objective transportation problems in an uncertain environment". It represents the research work carried out by me at the Department of Mathematics, laboratory of LAROMAD, University of Mouloud Mammeri Tizi-ouzou Algeria under the direction of Professor Méziane Aïder, from Department of operational research, Faculty of Mathematics, University of Science and Technology Houari Boumediene Bab Ezzouar, Algeria and the co-direction of Professor Carlos Cruze Corona, from the Department of Computer Science and Artificial Intelligence, of University of Granada, Spain.

Operational research, called OR, is the scientific research methods or techniques of mathematics for determining the right decision for a problem. Operational research is used to help people in decision-making that manages large organizations. The main objective of operational research is to improve the performance of existing systems rather than developing new systems. In operational research, a staff of experts in various fields first defines the problem and then expresses it in the form of a set of mathematical equations. After that, the computer analysis is done to solve the problem. Operational research focuses on systems in which human behavior plays an important role. Operational research focuses on the system as a whole rather than on individual parts of the system. Different types of approaches are applied by operational research to address different types of problems. For example, linear programming, Multi-objective and stochastic programming are used to manage complex information. The operational research addresses what information and data are required to make decisions, how to create and implement managerial decisions, etc.

Optimization is an integral part of operational research and an essential tool in decision making and in the analysis of any physical system. In mathematical terms, an optimization problem is the problem of finding a best solution among all possible solutions. The first step in the process of optimization is creating an appropriate model modeling existing situations.

The development of the world imposes that the optimization problems become more complicated. For this reason, Multi-objective optimization problems arise in various fields, such as engineering, economics, and logistics, the optimal decisions need to be taken in the presence of trade-offs between two or more conflicting objectives. For many actual problems, the data cannot be known accurately for a variety of reasons. The first reason is due to simple measurement errors. The second and more fundamental reason is that some data represent information about the future (e. g., product demand or price for a future time period) and simply cannot be known with certainty. In such a case, different theories have been developed (stochastic, fuzzy and uncertainty).

The transportation problem is a very well-known optimization problem in operational research that has been extensively studied in literature. The classical transportation problem (TP) deals with the determination of a plan for transporting a single commodity from m sources to n destinations by one mode of conveyance. The amount of the commodity available at any source is known, the demand for the commodity at each destination is given and the goal is to determine which routes to be considered for shipment and the amount of the shipment so that total transportation cost is minimized. But in the real-world, we always deal with other constraints besides of source constraints and destination constraints, such as product type constraints (when heterogeneous products are to be delivered) or transportation mode constraints (when several modes of transport are available, like trucks, cargo fights, goods trains, ships, etc.). For such cases, the traditional TP turns into the solid transportation problem (STP) in which, three item properties are taken into account in the constraints set instead of two constraints (source and destination) for the classical TP. Most real-world optimization problems in many fields of science, including engineering, economics, and logistics and even daily life need to consider more than one objective to reflect the problem more realistically to satisfy the given set of constraints and the objectives are measured in different scales and conflict at the same time and it is generally an impossible task to find a solution that simultaneously optimizes all the objectives under the same restrictions. Also, modeling our knowledge and observations becomes more challenging as we want to do it in the best manner. The decision-maker likes to find a better way to model his observations and improvisation. To avoid information loss as much as possible, we usually need to attach uncertainty parameters to the models. Because no information is available with certainty, the available information often cannot be understood or properly interpreted, measurement errors are very common, uncontrollable factors are omnipresent, many interpretations can be made, and many authors have provided and applied different approaches to address the problem of uncertainty in transportation problem.

The research contained in this thesis can be summarized as follows:

Chapter 1 gives an introduction to Optimization, Linear and non linear programming, Multi-objective Programming with some methods of resolution such as (Convex combination, Goal Programming (GP), Fuzzy programming, etc.), Programming under uncertainty (Stochastic Programming, Fuzzy Programming and uncertain programming).

Chapter 2 deals with mono and multi-objective transportation problem and presents some methods of resolutions for both cases and gives its extensions and provides state of art of the uncertainty in mono and multi-objective transportation problems: An Annotated Bibiography is given at the end.

Chapter 3 is based on the paper «Uncertain interval programming model for multi-objective multi-item fixed charge solid transportation problem with budget constraint and safety measure ». This chapter presents uncertain interval programming models for multi-objective multi-item fixed charge solid transportation problem with budget constraint and safety measure (MOMIFC-STPBCSM), the parameters of the formulated problem are chosen as uncertain intervals. The unit transportation costs, fixed charges, transportation times, deterioration of items, supplies at origins, demands at destinations, conveyance capacities, budget at each destination, selling prices and purchasing costs, and the safety factor and the desired safety measure are interval uncertain parameters. To formulate the proposed MOMIFCSTPBCSM, we use interval theory and uncertain programming techniques to develop two different models: an Expected Value Model and a Chance-Constrained Model. The equivalent deterministic models are formulated and solved using a linear weighted method, a fuzzy programming method and the goal programming method.

Chapter 4 is based on the paper «Fuzzy multi-objective multi-item fixed charge solid transportation problem with budget constraint and deterioration of item ». In this chapter, we consider a multi-objective multi-item fixed charge solid transportation problem with budget constraints and deterioration of items in a fuzzy environment. We assume that several items are purchased at different sources with various prices, and transported to different destinations by using various kinds of conveyance. In the model we propose, we consider that the unit transportation costs, fixed charges, transportation times, the total deterioration of goods, the selling prices, the purchasing costs, the budget at each destination, are trapezoidal and the inequality constraints are fuzzy. We develop two different models to obtain deterministic equivalent expected value and approximation techniques by the closest interval. Then, we solve the deterministic models by using linear weighted, fuzzy interactive satisfied, and global criteria.

Chapitre 5 addressed Multi-Objective Multi-Item Solid Transportation Problem With Interval-Valued Trapezoidal Fuzzy Numbers. First, the formulation of the problem is presented, then crisp equivalent is formulated and a methodology for its resolution is given. To valid our proposed method, we give a numerical example at the end.

We end our thesis by a conclusion.

Chapter 1

Optimization

1.1 Introduction

This chapter provides an introduction to Optimization, then give the formulation of some mathematical programming (Linear, Nonlinear, Quadratic and Integer programming) then uncertain programming are presented (stochastic, fuzzy and uncertain programming), Additionally, Multi-objective Programming (MOPP) along with some methods to solve them (Interactive, Non-interactive, Goal Programming and Fuzzy Method)are given, at the end, the conclusion of this chapter is pointed.

1.2 Optimization

The word "optimization" has an important role in real world application and the requirement of using the optimization in all aspects of human life and different discipline such as engineering, management, public administration, business etc. imposed this field to be developed more and more. In other word optimization techniques. It is the process of finding the best way of using the existing resources while taking into the account of all the factors that influence decisions in any experiment. The researchers provided continuous and important efforts efforts to develop an efficient algorithms and fast computers.

Our daily life involves to optimize at each step. The decisions which we take in our daily lives do not involve any mathematics tool, but they are based on our experiences from the past, it is not possible to make decisions just on the basis of past experiences. In addition, the decisions made may or may not be optimal because they are simply based on our knowledge and ideas. Consequently, the implications of mathematical theory become important tool for the accuracy. In such situations, optimization theory offers a better alternative to the decision maker on the condition that one can represent the decisions and the system mathematically and use appropriate techniques to obtain the best possible results.

Some developments then took place in the 18th century. The work of Newton, Lagrange and Cauchy, which made it possible to solve certain types of optimization problems in physics and geometry using differential and variation calculation methods, is valuable. The major breakthrough was made by Dantzig (1947) with the development of the Simplex method in linear

programming.
In order to achieve an optimal solution to a problem, several steps must be taken. The first step is to identify the optimization problem itself from the actual decision making situation, keeping in mind the desired benefits, the decision variables and the effort required. Then, the identified problem is formulated as an optimization model and the most appropriate optimization technique is chosen to solve it. In the end, the model is solved and the results are evaluated.

Optimization is used in every area of life. Some typical areas where optimization problems can arise are, industry, economy, commerce, aerodynamics, etc. Optimization has many applications in the study of physical and chemical systems, production planning, location and transportation problems, engineering design, scheduling systems and resource allocation in financial systems.

There is a different kind of optimization, in this thesis we focuses in solving multi-objective transportation problems in an uncertain environment.

1.3 Mathematical programming

Mathematical programming is a branch of operational research Mathematical Programming (MP) is a new field that has appeared in the middle of the 20th century. The Mathematical programming has become very important because of its application in all areas involving decision making. MP is the branch of mathematics that deals with techniques for maximizing or minimizing objective functions/functions subject to linearity/non-linearity/integration constraints on the variables. The MP approach is used to create the mathematical model of the problem we are interested in. As real life models are very complex and can include thousands of decision variables and the number of inequalities (or equations) to represent constraints on decisions, As a result, an appropriate approach is needed for the problem formulation and resolution. The problem is to determine the values of the n vector of components of the decision variables $x_1, ..., x_n$ which optimizes the value of an objective function Z subject to a set of constraints. The standard form of Mathematical programming can be formulated as following:

$$\begin{cases} \min(\max)Z = f(x) \\ g_i(x) \leq\geq= b_i, \\ x \geq 0 \end{cases}$$

Where b_i is the $< i^{th}$ represents the m component vector b called the request vector and only sign between $\leq\geq=$ holds true for each i

Feasible solution :

The solution that satisfies the constraints and non-negativity restriction of a mathematical programming is called a feasible Mathematical programming solution. The set of all feasible solutions to an Mathematical programming can be stated as $F = \{x \mid g_i(x) \leq\geq= b_i, i = 1, ..., m\}$ and $x \geq 0$

Optimum solution:

The feasible solution which optimizes (minimizes or maximizes) the objective of an Mathematical programming is called an optimum solution. Any $x^* \in F$ for which $f(x^*) \leq f(x)$ for all $x \in F$ is called an optimum solution for a minimization Mathematical programming, while any $x^* \in F$ for which $f(x^*) \geq f(x)$ for all $x \in F$ is called an optimum solution for a maximization Mathematical programming.

There exist different mathematical programming problems we can cite some as follows:

- Linear Programming Problem
- Non linear Programming Problem
- Quadratic Programming Problems
- Integer Programming Problems
- Stochastic Programming Problem
- Fuzzy Programming Problem
- uncertain programming
- Multi-objective programming

1.3.1 Linear programming

Linear programming (LP) is also called as the linear optimization. In Mathematical programming, when all the functions are linear, the Mathematical programming is termed as a linear programming problem.It is the optimization (maximization/minimization) of the linear objective function which satisfying a set of linear constraints. It is a mathematical tool for assigning of resources in an optimal way. According to George B. Dantzig, linear programming is a modeling technique useful for the allocation of limited resources such as equipment, machines etc. to several competing activities such as projects, services etc. LP is one of the most powerful and useful techniques for decisions making. It is widely used as a decision tool in areas such as, production, finance, marketing, transportation schedules, assignment problems, determination of the optimal product combination (a combination of products, which gives maximum profit).

Formulating Linear programming

Let us consider three variables c, b and A, $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{n \times m}$.

For maximization, the linear programming can formulate as following.

$$\begin{cases} \max \sum_{j=1}^{n} c_j x_j \\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, ..., m \\ x_j \geq 0 \end{cases}$$

For minimization, the linear programming can formulate as following.

$$\begin{cases} \min \sum_{j=1}^{n} c_j x_j \\ \sum_{j=1}^{n} a_{ij} x_j \geq b_i, i = 1, ..., m \\ x_j \geq 0 \end{cases}$$

Each constraint has a constant part which can be found on the right side of the inequations. When all these conditions are combined, it is said that the linear program is in its standard form.
It is based on a mathematical technique following three methods :

- a graphic solution;
- an algebraic solution;
- the use of the simplex algorithm.

1.3.2 Non linear programming

Non linear programming problem is a mathematical programming problem in which some of the constraints or the objective function or both are non linear.
Non linear models are much more difficult to optimize than linear models. because, for the non linear program, its optimal solution may arise at the boundary or at the interior point of the feasible set. In addition, a local optimum may not always be a global one.

Formulating Non linear programming

A general mathematical model for non linear problem can be formulated as follows:

$$\begin{cases} \min(\max) f(x) \\ g_i(x) \leq \geq = b_i, i = 1, ..., m \\ x_j \geq 0, j = 1, ..., n \end{cases}$$

Where $f(x)$ and $g_i(x)$ are$m + 1$ real valued functions of n decision variables.
The important development in this field was made by Kuhn (2014) and Tucker in 1951.

1.3.3 Quadratic Programming

A special case of the non linear programming arises when the objective functional f is quadratic and the constraints are linear in $x \in \mathbb{R}^n$

Formulating Quadratic programming

A general mathematical model for quadratic programming can be formulated as follows:

$$\begin{cases} \min(\max)\dfrac{1}{2}x^T Bx - x^T b \\ A_i(x) \leq \geq = b_i, i = 1, ..., m \\ x_j \geq 0, j = 1, ..., n \end{cases}$$

where $B \in \mathbb{R}^{n\times n}$ is symmetric, $b \in \mathbb{R}^n$.
Important and relevant techniques for quadratic programming problems are due to Beale (1959) and Wolfe (1959).

1.3.4 Integer Programming

An integer programming problem is a mathematical optimization program in which some or all variables are limited to be integers. In many cases, the term refers to integer linear programming, in which the objective function and constraints (other than integer constraints) are linear.

Formulating Integer programming

A general mathematical model for Integer programming can be formulated as follows:

$$\begin{cases} \max \sum_{j=1}^{n} c_j x_j \\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, ..., m \\ x_j \geq 0 and \quad integer \end{cases}$$

Since rounding the optimal solution is not always a useful technique for obtaining an integer number, Thus, to solve a this kind of problem, a large number of methods have been used developed so far. But among these, two methods are commonly used.

- Branch and Bound Methods
- Cutting Plane Methods

1.4 Uncertain Programming

Modeling our knowledge and observations become more challenging as we want to do it in the best manner. The decision-maker likes to find a better way to model his observations and improvisation. To avoid information loss as much as possible, we usually need to attach uncertainty parameters to the models. To the reason of no information is available with certainty, the available information often cannot be understood or properly interpreted, measurement errors are very common, uncontrollable factors are omnipresent, many interpretations can be made, and many authors have provided and applied different approaches which express different manner and situation

1.4.1 Stochastic Programming

Stochastic programming arise when some or all of the parameters are described as random variables.

Basic Preliminaries

Introducing the concept of random variable, let us define a probability measure by an axiomatic approach.

Definition 1.1. Let Ω be a nonempty set, and a $\sigma-$algebra of subsets (called events) of Ω . The set function Pr is called a probability measure if

- Axiom 1. (Normality) $Pr(\Omega) = 1$;
- Axiom 2. (Nonnegativity) $Pr\{A\} \geq 0$ for any event A;
- Axiom 3. (countable Additivity) For every countable sequence of mutually disjoint events $\{A_i\}$, we have
$$Pr\Big\{\bigcup_{i=1}^{\infty} A_i\Big\} = \sum_{i=1}^{\infty} Pr\Big\{A_i\Big\}$$

Theorem 1.1. *(Liu[63]) Let Ω be a nonempty set, $\mathcal{A}$ a $\sigma-$ algebra over Ω, and pr a probability measure. Then $pr = 0$ and $0 \leq pr\{A\} \leq 1$ for any event A.*

Definition 1.2. Let Ω be a nonempty set, $\mathcal{A}$ a $\sigma-$algebra of subsets of Ω and Pr a probability measure. Then the triplet $(\Omega, \mathcal{A}, pr)$ is called a probability space.

Definition 1.3. A random variable is a measurable function from a probability space $(\Omega, \mathcal{A}, pr)$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\omega \in \Omega | \xi(\omega) \in B\}$ is an event.

Definition 1.4. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a measurable function, and $\xi_1, \xi_2, ..., \xi_n$ random variables defined on the probability space $(\Omega, \mathcal{A}, pr)$.Then $\xi = f(\xi_1, \xi_2, ..., \xi_n)$ is random variable defined by
$$\xi(\omega) = f\xi_1(\omega), \xi_2(\omega), ..., \xi_n(\omega))$$

Definition 1.5. The probability distribution $\phi : \mathbf{R} \to [0, 1]$ of a random variable ξ is defined by
$$\phi(x) = Pr\{\omega \in \Omega | \xi(\omega) \leq x\}$$
That is, $\phi(x)$ is the probability that the random variable ξ takes a value less than or equal to x.

Definition 1.6. The probability density function $\phi : \mathbf{R} \to [0, \infty]$ of a random variable ξ is a function such that
$$\phi(x) = \int_{-\infty}^{x} \phi(y) dy$$
holds for all $x \in \mathbf{R}$,whereϕ is the probability distribution of random variable ξ.

Definition 1.7. [Uniform Distribution] A random variable ξ has a uniform distribution if its probability density function is

$$\begin{cases} \dfrac{1}{b-a} & if \quad a \leq x \leq b \\ 0, & otherwise. \end{cases}$$

denoted by $\mu(a, b)$, where a and b are given real numbers with $a < b$.

Definition 1.8. [Exponential Distribution] A random variable ξ has an exponential distribution if its probability density function is

$$\phi(x) \begin{cases} \dfrac{1}{\beta} \exp(-\dfrac{x}{\beta}) & if \quad x \geq 0 \\ 0, & otherwise. \end{cases}$$

denoted by $\exp(\beta)$, where β is a positive number.

Definition 1.9. [Normal Distribution] A random variable ξ has a normal distribution if its probability density function is

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-u)^2}{2\sigma^2}); \quad x \in \mathbf{R}$$

denoted by $N(\mu, \sigma^2)$, where μ andσ are real numbers.

Theorem 1.2. *([Probability Inversion Theorem]) Let σ be a random variables whose probability density function ϕ exists. Then for any Borel set B of $\mathbf{R}$ we have*

$$Pr\{\xi \in B\} = \int_B \phi(y)dy$$

Definition 1.10. The random variables $\xi_1, \xi_2, ...\xi_m$ are said to be independent if

$$Pr\Big\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\Big\} = \prod_{i=1}^{m} Pr\{\xi_i \in B_i\}$$

For any Borel sets $B_1, B_2, ..., B_m$ of real numbers.

Definition 1.11. Let ξ be a random variables. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} Pr\{\xi \geq r\}dr - \int_{+\infty}^{0} Pr\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite.
Let ξ and η be random variables with finite expected values. For any numbers a and b, it has been proved that

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]$$

That is, the expected value operator has the linearity property.

Example 1.1. Assume that ξ is a discrete random variables taking values x_i with probabilities $p_i, i = 1, 2, ..., m$ respectively.it follows from the definition of expected value operator that

$$E[\xi] = \sum_{i=1}^{m} p_i x_i$$

Theorem 1.3. *(Liu[63]) Let ξ be a random variable whose probability density function ϕ exists. if the lebesgue integral*

$$\int_{-\infty}^{+\infty} x\phi(x)dx$$

is finite, then we have

$$E[\xi] = \int_{-\infty}^{+\infty} x\phi(x)dx$$

Example 1.2. Let ξ be a uniformly distributed random variable on the interval $[a, b]$ then its expected value is

$$E[\xi] = \int_{a}^{b} \frac{x}{b-a}dx = \frac{a+b}{2}$$

Example 1.3. Let ξ be an exponentially distributed random variable $EXP(\beta)$ .Then its expected value is

$$E[\xi] = \int_{0}^{+\infty} \frac{x}{\beta} \exp(-\frac{x}{\beta})dx = \beta$$

Example 1.4. Let ξ be an normally distributed random variable $N(\mu, \sigma^2)$ Then its expected value is

$$E[\xi] = \int_{-\infty}^{+\infty} \frac{x}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)dx = \mu$$

Definition 1.12. A point x is called feasible if and only if the probability measure of the event $\{g_j(x, \xi) \leq 0, j = 1, ..., p\}$ is at least α. In the other words, the constraints will be violated at most $(1-\alpha)$ of time.
The joint chance constraint is separately considered as

$$Pr\{g_j(x, \xi) \leq 0\} \geq \alpha_j \quad , j = 1, ..., p$$

which is referred to as a separate chance constraint.

Definition 1.13. [Deterministic Equivalents]
Let us consider the following form of chance constraint,

$$Pr\{g(x, \xi) \leq 0\} \geq \alpha \quad (****)$$

- The stochastic objective constraint $Pr\{f(x, \xi) \leq \overline{f}\} \geq \beta$ coincides with the form (****) by defining $g(x, \xi) = \overline{f} - f(x, \xi)$;

- The stochastic objective constraint $Pr\{f(x, \xi) \geq \overline{f}\} \geq \beta$ coincides with the form (****) by defining $g(x, \xi) = f(x, \xi) - \overline{f}$;

Theorem 1.4. *(Liu[63]) Assume that the stochastic vector ξ degenerates to a random variable ξ with probability distribution ϕ, and the function $g(x, \xi)$ has the form $g(x, \xi) = h(x) - \xi$. Then $Pr\{g(x, \xi) \leq 0\} \geq \alpha$ if and only if $h(x) \leq K_\alpha$ where K_α is the maximal number such that $Pr\{K_\alpha \leq \xi\} \geq \alpha$*

Remark 1.1. For a continuous random variable ξ the equation $Pr\{K_\alpha \leq \xi\} = 1 - \phi(K_\alpha)$ always holds, and we have $K_\alpha = \phi^{-1}(1-\alpha)$ where ϕ^{-1} is the inverse function of ϕ

1.4.2 Fuzzy Programming

Fuzzy programming arise when some or all of the parameters are known with membership function.

Fuzzy Set Theory

The fuzzy theory was stated by A. Zadeh (1965, It was based on the degree of the membership. He noted that not all sets have clear boundaries and these sets are closer to human thinking.For example, "a set of tall men", "a set of beautiful women", "a set of intelligent people".

Basic Preliminaries

In this section, we review some necessary concepts on fuzzy and their extension.

Definition 1.14. The function $\mu_{\tilde{A}} : X \longrightarrow [0,1]$ is called membership function and the set

$$\tilde{A} = \Big\{(x, \mu_{\tilde{A}}(x)) \mid x \in X\Big\}$$

defined by $\mu_{\tilde{A}}$ for each $x \in X$ is called a fuzzy set.

Definition 1.15. A fuzzy set $\tilde{A}$, defined on the universal set of real numbers $\mathbb{R}$, is said to be a fuzzy number if its membership function has the following characteristics:

- $\tilde{A}$ is convex, i.e.

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}, \forall\, x, y \in \mathbb{R}, \forall\, \lambda \in [0,1],$$

- $\tilde{A}$ is normal, i.e.,

$$\exists\, \overline{x} \in \mathbb{R}; \mu_{\tilde{A}}(\overline{x}) = 1$$

- $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Definition 1.16. A fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is said to be trapezoidal fuzzy number if its membership function is given by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & \text{if } x \in [a_1, a_2); \\ 1, & \text{if } x \in [a_2, a_3]; \\ \dfrac{x - a_4}{a_3 - a_4}, & \text{if } x \in (a_3, a_4]; \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

If $a_2 = a_3$ then $\widetilde{A}$ is called a triangular fuzzy number $\widetilde{A} = (a_1, a_2, a_4)$.

In this section, we recall the main notions regarding some representations of uncertain data.

Definition 1.17. For a fuzzy variable $\widetilde{A}$ with membership function $\mu_{\widetilde{A}}(x)$ and for any set B of real numbers, credibility measure of fuzzy event $\{\widetilde{A} \in B\}$ is defined as:

$$Cr\{\widetilde{A} \in B\} = \frac{1}{2}(Pos\{\widetilde{A} \in B\} + Nec\{\widetilde{A} \in B\}), \tag{1.2}$$

where possibility and necessity measures of $\{\widetilde{A} \in B\}$ are respectively defined as:

$$Pos\{\widetilde{A} \in B\} = \sup_{x \in B} \mu_{\widetilde{A}}(x), \tag{1.3}$$

$$Nec\{\widetilde{A} \in B\}) = 1 - \sup_{x \in \overline{B}} \mu_{\widetilde{A}}(x). \tag{1.4}$$

Definition 1.18. Let $\widetilde{A}$ be a fuzzy variable. The expected value of $\widetilde{A}$ is defined as:

$$E\left[\widetilde{A}\right] = \int_0^{+\infty} Cr\{\widetilde{A} \geq r\}\, dr - \int_{-\infty}^{0} Cr\{\widetilde{A} \leq r\}\, dr, \tag{1.5}$$

provided that at least one of the two integrals is finite. For example, expected value of a trapezoidal fuzzy variable $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is:

$$E[\widetilde{A}] = \frac{a_1 + a_2 + a_3 + a_4}{4}. \tag{1.6}$$

Let $\widetilde{A} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number and $[\underline{A}_\alpha, \overline{A}_\alpha]$ its $\alpha-$cut, with:

$$[\underline{A}_\alpha, \overline{A}_\alpha] = [a_1 + (a_2 - a_1)\alpha, a_4 - (a_4 - a_3)\alpha]. \tag{1.7}$$

Using the definition of nearest interval approximation, we get:

$$\underline{C} = \int_0^1 \underline{A}_\alpha \,\mathrm{d}\alpha = \int_0^1 [a_1 + (a_2 - a_1)\alpha]\,\mathrm{d}\alpha = \frac{1}{2}(a_2 + a_1), \quad 0 \leq \alpha \leq 1, \tag{1.8}$$

$$\overline{C} = \int_0^1 \overline{A}_\alpha \,\mathrm{d}\alpha = \int_0^1 [a_4 - (a_4 - a_3)\alpha]\,\mathrm{d}\alpha = \frac{1}{2}(a_4 + a_3), \quad 0 \leq \alpha \leq 1. \tag{1.9}$$

Thus,

$$[\underline{C}, \overline{C}] = \left[\frac{1}{2}(a_2 + a_1), \frac{1}{2}(a_4 + a_3)\right]. \tag{1.10}$$

An interval number is a special case of fuzzy number.

Definition 1.19. An interval number A is defined as

$$A = [a_L, a_R] = \{x : a_L \leq x \leq a_R, x \in R\}. \tag{1.11}$$

Here, $a_L, a_R \in R$ are the lower and upper bounds of the interval A, respectively.

An interval number can also be expressed by its mean and width. In this form, an interval number $A = [a_L, a_R]$ is denoted by $< a_M, a_W >$, where $a_M = \frac{a_L + a_R}{2}$ and $a_W = \frac{a_R - a_L}{2}$ are known as the center and the radius of the interval, respectively.

Definition 1.20. For any two intervals $A = [a_L, a_R] = \langle a_M, a_W \rangle$ and $B = [b_L, b_R] = \langle b_M, b_W \rangle$,

$$A \preceq B \text{ if and only if} \begin{cases} a_M < b_M & \text{for } a_M \neq b_M, \\ a_W \geq b_W & \text{for } a_M = b_M. \end{cases} \tag{1.12}$$

Furthermore $A \prec B$ if and only if $A \preceq B$ and $A \neq B$.

Definition 1.21. A level γ-trapezoidal fuzzy number $\widetilde{A}$ or a generalized trapezoidal fuzzy number $\widetilde{A}$, denoted by $\widetilde{A} = (a_1, a_2, a_3, a_4; \gamma)$, $0 < \gamma \leq 1$, is a fuzzy number with the membership function defined as follows:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \gamma \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2; \\ \gamma, & a_2 \leq x \leq a_3; \\ \gamma \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4; \\ 0, & \text{otherwise.} \end{cases}$$

Let $F_{TN}(\gamma)$ be the family of all level γ-trapezoidal fuzzy numbers, that is:

$$F_{TN}(\gamma) = \{\widetilde{A} = (a_1, a_2, a_3, a_4; \gamma),\ a_1 \leq a_2 \leq a_3 \leq a_4\},\ 0 < \gamma \leq 1\}.$$

Definition 1.22. Let $\widetilde{A}^L \in F_{TN}(\gamma)$ and $\widetilde{A}^U \in F_{TN}(\delta)$. A level (γ, δ)-interval-valued trapezoidal fuzzy number $\widetilde{\widetilde{A}}$, denoted by $\widetilde{\widetilde{A}} = [\widetilde{A}^L, \widetilde{A}^U] = \langle (a_1^L, a_2^L, a_3^L, a_4^L; \gamma),$ $(a_1^U, a_2^U, a_3^U, a_4^U; \delta)\rangle$ is an interval-valued fuzzy set on $\mathbb{R}$ with the lower trapezoidal fuzzy number $\widetilde{A}^L$ expressed by:

$$\mu_{\widetilde{A}^L}(x) = \begin{cases} \gamma \dfrac{x - a_1^L}{a_2^L - a_1^L}, & a_1^L \leq x \leq a_2^L; \\ \gamma, & a_2^L \leq x \leq a_3^L;, \\ \gamma \dfrac{a_4^L - x}{a_4^L - a_3^L}, & a_3^L \leq x \leq a_4^L; \\ 0, & \text{otherwise,} \end{cases}$$

and the upper trapezoidal fuzzy number $\widetilde{A}^U$ expressed by:

$$\mu_{\widetilde{A}^U}(x) = \begin{cases} \delta \dfrac{x - a_1^U}{a_2^U - a_1^U}, & a_1^U \leq x \leq a_2^U; \\ \delta, & a_2^U \leq x \leq a_3^U; \\ \delta \dfrac{a_4^U - x}{a_4^U - a_3^U}, & a_3^U \leq x \leq a_4^U; \\ 0, & \text{otherwise,} \end{cases}$$

where $a_1^L \leq a_2^L \leq a_3^L \leq a_4^L$, $a_1^U \leq a_2^U \leq a_3^U \leq a_4^U$, $0 < \gamma \leq \delta \leq 1, a_1^U \leq a_1^L$, and $a_4^L \leq a_4^U$. Moreover, $\mu_{\widetilde{A}^L}(x) \leq \mu_{\widetilde{A}^U}(x)$.

This means that the least and greatest grades of membership of x in the interval $\widetilde{\widetilde{A}} = [\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x)]$, are $\mu_{\widetilde{A}^L}(x)$ and $\mu_{\widetilde{A}^U}(x)$ respectively.

Let $F_{IVTN}(\gamma, \delta)$ be the family of all level (γ, δ)-interval-valued trapezoidal fuzzy numbers, that is,

$$F_{IVTN}(\gamma, \delta) = \{ \begin{array}{l} \widetilde{\widetilde{A}} = [\widetilde{A}^L, \widetilde{A}^U] = \langle (a_1^L, a_2^L, a_3^L, a_4^L; \gamma), (a_1^U, a_2^U, a_3^U, a_4^U; \delta)\rangle : \\ \widetilde{A}^L \in F_{TN}(\gamma),\ \widetilde{A}^U \in F_{TN}(\delta),\ \ a_1^U \leq a_1^L \leq a_4^L \leq a_4^U \} \end{array}$$

where $0 < \gamma \leq \delta \leq 1$.

Definition 1.23. A triangular interval-valued fuzzy number denoted by:

$$\widetilde{\widetilde{A}} = [\widetilde{A}^L, \widetilde{A}^U] = \langle (a_1^L, a_2^L, a_3^L; \gamma), (a_1^U, a_2^U, a_3^U; \delta)\rangle$$

is an interval-valued fuzzy set on $\mathbb{R}$ with the lower triangular fuzzy number $\widetilde{A}^L$ expressed by :

$$\mu_{\widetilde{A}^L}(x) = \begin{cases} \gamma\dfrac{x-a_1^L}{a_2^L-a_1^L}, & a_1^L \leq x \leq a_2^L; \\ \gamma\dfrac{a_3^L-x}{a_3^L-a_2^L}, & a_2^L \leq x \leq a_3^L; \\ 0, & \text{otherwise,} \end{cases}$$

and the upper triangular fuzzy number $\widetilde{A}^U$ expressed by:

$$\mu_{\widetilde{A}^U}(x) = \begin{cases} \delta\dfrac{x-a_1^U}{a_2^U-a_1^U}, & a_1^L \leq x \leq a_2^U; \\ \delta\dfrac{a_3^U-x}{a_3^U-a_2^L}, & a_2^U \leq x \leq a_3^U; \\ 0, & \text{otherwise,} \end{cases}$$

where $a_1^U \leq a_1^L \leq a_2^U \leq a_2^L \leq a_3^L \leq a_3^U$, $0 < \gamma \leq \delta \leq 1, a_1^U \leq a_1^L$, and $a_4^L \leq a_4^U$. Moreover, $\mu_{\widetilde{A}^L}(x) \leq \mu_{\widetilde{A}^U}(x)$.

Definition 1.24. Let X denote the universe set. An intuitionistic fuzzy set (IFS) $\tilde{A}^I$ in X is defined by a set of ordered triples

$$\tilde{A}^I = \left\{\left\langle x, \mu_{\tilde{A}^I}(x), \rho_{\tilde{A}^I}(x)\right\rangle \mid x \in X\right\}$$

where the functions $\mu_{\tilde{A}^I} : X \to [0,1]$, and $\rho_{\tilde{A}^I} : X \to [0,1]$ respectively represent the membership degree and non membership degree of x in $\tilde{A}$ respectively and verify for each element $x \in X,\ 0 \leq \mu_{\tilde{A}^I}(x) + \rho_{\tilde{A}^I}(x) \leq 1$.

Definition 1.25. An intuitionistic fuzzy set $\tilde{A}^I = \left\{\left\langle x, \mu_{\tilde{A}^I}(x), \rho_{\tilde{A}^I}(x)\right\rangle \mid x \in X\right\}$ is said intuitionistic fuzzy convex if its membership function is fuzzy convex, i.e., $\forall\ x, y \in X$, $\forall\ \lambda \in [0,1]$, $\mu_{\tilde{A}^I}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}^I}(x), \mu_{\tilde{A}^I}(y)\}$ and its non-membership function is concave, i.e. $\rho_{\tilde{A}^I}(\lambda x + (1-\lambda)y) \leq \max\{\rho_{\tilde{A}^I}(x), \rho_{\tilde{A}^I}(y)\}$.

Definition 1.26. An intuitionistic fuzzy set $\tilde{A}^I = \left\{\left\langle x, \mu_{\tilde{A}^I}(x), \rho_{\tilde{A}^I}(x)\right\rangle \mid x \in R\right\}$ of the real number $\mathbb{R}$ is called an intuitionistic fuzzy number if

- $\tilde{A}^I$ is intuitionistic fuzzy normal and intuitionistic fuzzy convex.
- $\mu_{\tilde{A}^I}$ is upper semi continuous and $\rho_{\tilde{A}^I}$ is semi lower continuous.
- Supp $\tilde{A}^I = \left\{x \in R;\ \rho_{\tilde{A}^I} < 1\right\}$ is bounded.

Definition 1.27. A Trapezoidal Intuitionistic Fuzzy Number (TrIFN) $\tilde{A}^I$ is an especial IFN with membership function and non-membership function defined as follows:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2; \\ 1, & a_2 \leq x \leq a_3; \\ \dfrac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\rho_{\widetilde{A}^I}(x) = \begin{cases} \dfrac{x-a_1'}{a_2'-a_1'}, & a_1' \le x \le a_2'; \\ 0, & a_2 \le x \le a_3'; \\ \dfrac{a_4'-x}{a_4'-a_3'}, & a_3' \le x \le a_4'; \\ 1, & \text{otherwise}, \end{cases}$$

where $a_1' < a_1 < a_2' < a_2 < a_3 < a_3' < a_2 < a_4 < a_4'$. This TrIFN is denoted by $\tilde{A}^I = (a_1, a_2, a_3, a_4; a_1', a_2', a_3', a_4')$

Remark 1.2. If $a_2' = a_2 = a_3 = a_3'$, then TrIFN $\tilde{A}^I = (a_1, a_2, a_3, a_4; a_1', a_2', a_3', a_4')$ represents a triangular intuitionistic fuzzy number (TIFN). A TIFN is denoted by $\tilde{A}^I = (a_1, a_2, a_3; a_1', a_2', a_3')$.

Interval-valued intuitionistic fuzzy number (IVIF) expresses more abundant and flexible information than intuitionistic fuzzy numbers. Different authors are defined different type of fuzzy/intuitionistic fuzzy numbers in the literature. We will define a normal IVIF number.

Definition 1.28. An interval-valued intuitionistic fuzzy set $\tilde{A} = \Big\{(b_1^L, b_1^U, a_2, b_3^L, b_3^U), (a_1^L, a_1^U, a_2, a_3^L, a_3^U)\Big\}$ on $\mathbb{R}$ is called an interval-valued intuitionistic fuzzy number if its lower and upper membership and non-membership functions are given as follows:

- Lower and upper membership functions, respectively, are:

$$\mu_{\widetilde{A}^L}(x) = \begin{cases} 1, & x = a_2; \\ f_A(x), & a_1^L < x < a_2; \\ g_A(x), & a_2 \le x \le a_3^L; \\ 0, & x \ge a_3^L \text{ or } x \le a_1^L, \end{cases}$$

and

$$\mu_{\widetilde{A}^U}(x) = \begin{cases} 0, & x > a_3^U \text{ or } x < a_1^U; \\ h_A(x), & a_1^U < x < a_2; \\ I_A(x), & a_2 < x < a_3^U; \\ 1, & x = a_2, \end{cases}$$

- Lower and upper non-membership functions, respectively, are:

$$\rho_{\widetilde{A}^L}(x) = \begin{cases} 0, & x = a_2; \\ J_A(x), & b_1^L < x < a_2; \\ K_A(x), & a_2 < x < b_3^L; \\ 0, & x \ge b_3^L \text{ or } x \le b_1^L, \end{cases}$$

and

$$\rho_{\widetilde{A}^U}(x) = \begin{cases} 1, & x \ge b_3^L \text{ or } x \le b_1^L; \\ L_A(x), & a_2 < x < b_3^L; \\ M_A(x), & b_1^L < x < a_2; \\ 0 & x = a_2. \end{cases}$$

Definition 1.29. An interval-valued triangular Intuitionistic Fuzzy Number is given by $\tilde{A} = \Big\{\Big(b_1^L, b_1^U, a_2, b_3^L, b_3^U\Big), \Big(a_1^L, a_1^U, a_2, a_3^L, a_3^U\Big)\Big\}$ where $b_1^L b_1^U, b_3^L, b_3^U, a_1^L, a_1^U, a_2, a_3^L, a_3^U \in \mathbb{R}$ and its lower, upper membership and non-membership functions are given as :

- lower and upper member membership functions, respectively, are

$$\mu_{\widetilde{A}^L}(x) = \begin{cases} 1, & x = a_2; \\ \dfrac{x - a_1^L}{a_2 - a_1^L}, & a_1^L \le x \le a_2; \\ \dfrac{a_3^L - x}{a_3^L - a_2}, & a_2 \le x \le a_3^L; \\ 0, & x \ge a_3^L \text{ or } x \le a_1^L, \end{cases}$$

$$\mu_{\widetilde{A}^U}(x) = \begin{cases} 0, & x > a_3^U \text{ or } x < a_1^U, \\ \dfrac{x - a_1^U}{a_2 - a_1^U}, & a_1^U < x < a_2; \\ \dfrac{a_3^U - x}{a_3^U - a_2}, & a_2 < x < a_3^U; \\ 1, & x = a_2. \end{cases}$$

- Lower and upper non-membership functions, respectively, are:

$$\rho_{\widetilde{A}^L}(x) = \begin{cases} 0, & x = a_2; \\ \dfrac{a_2 - x}{a_2 - b_1^L}, & b_1^L < x < a_2; \\ \dfrac{a_2 - x}{a_2 - b_1^U}, & b_1^U < x < a_2; \\ 0, & x \le b_3^L \text{ or } x \le a_2, \end{cases}$$

and

$$\rho_{\widetilde{A}^U}(x) = \begin{cases} 1, & x > b_3^L \text{ or } x < b_1^L; \\ \dfrac{x - a_2}{b_3^U - a_2}, & a_2 < x < b_3^U; \\ \dfrac{x - a_2}{b_3^L - a_2}, & a_2 < x < b_3^L, \\ 0, & x = a_2. \end{cases}$$

Definition 1.30. A type-2 fuzzy set(T2 FS) $\tilde{A}$ in X is defined as

$$\tilde{A} = \Big\{((x,u), \mu_{\tilde{A}}(x,u)) : \forall\, x \in X,\ \forall\, u \in J_x \subseteq [0,1]\Big\},$$

where $0 \le \mu_{\tilde{A}}(x,u)) \le 1$ is called the type-2 membership function, J_x is the primary membership of $x \in X$ which is the domain of the secondary membership function $\mu_{\tilde{A}}(x)$. The values $u \in J_x$ for $x \in X$ are called primary membership grades of x. $\tilde{A}$ is also be expressed as

$$\tilde{A} = \int_{x\in X} \int_{u \in J_x} \mu_{\tilde{A}}(x,u))/(x,u), J_x \subseteq [0,1],$$

where $\int\int$ denotes the union over all admissible x and u. For discrete universes of discourse $\int$ is replaced by $\sum$.

Secondary Membership Function: For each values of x, say $x = x^{'}$, the secondary membership function, denoted by $\mu_{\tilde{A}}(x = x^{'}, u), J_{x^{'}} \subseteq [0,1]$ is defined as

$$\mu_{\tilde{A}}(x^{'}, u) \equiv \tilde{\mu}_{\tilde{A}}(x^{'}) = \int_{u \in J_x^{'}} f_{x^{'}}(u)/u,$$

Where

$$0 \le f_x^{'}(u) \le 1.$$

The amplitude of a secondary membership function is called a secondary grade. So far a particular $x = x^{'}$ and $u = u^{'} \in J_{x^{'}}, f_{x^{'}} = \mu_{\tilde{A}}(x^{'}, u^{'})$ is the secondary membership grade.

Definition 1.31. For a type-2 fuzzy set $\tilde{A}$, if all $\mu_{\tilde{A}}(x, u) = 1$ then $\tilde{A}$ is called an interval type-2 fuzzy set, i.e.,

$$\tilde{A} = \int_{x\in X}\int_{u\in J_x} 1/(x, u), u \in J_x \subseteq [0, 1]$$

.

Definition 1.32. Consider the F_{IVTN}

$$\widetilde{\widetilde{A}} = [\widetilde{A}^L, \widetilde{A}^U] = \langle (a_1^L, a_2^L, a_3^L, a_4^L; \gamma), (a_1^U, a_2^U, a_3^U, a_4^U; \delta)\rangle,$$

the Centroid of a trapezoidal into three plane figures namely a triangle, a quadrilateral and a triangle respectively. Let G_1, G_2, G_3 be the Centroids of these three plane figures. The Centroid of these Centroids G_1, G_2, G_3 is considered as the point of reference to define the ranking of generalized Interval valued fuzzy numbers. As the Centroid of these three plane figures is their balancing point, the Centroid of these Centroid points is a much better balancing point.

The Centroids of these plane figures are:

$$G_1 = \Big(\frac{a_1^L + 2a_2^L}{3}, \frac{\gamma}{3}\Big),$$

$$G_2 = \Big(\frac{a_2^L + a_3^L}{2}, \frac{\gamma}{2}\Big),$$

$$G_3 = \Big(\frac{2a_3^L + a_4^L}{3}, \frac{\gamma}{3}\Big),$$

respectively.
It follows that G_1, G_2 and G_3 are not collinear and they form a triangle. Thus, the Centroid of these Centroids is:

$$G(x_0, y_0) = \Big(\frac{2a_1^L + 7a_2^L + 7a_3^L + 2a_4^L}{18}, \frac{7\gamma}{18}\Big).$$

Now we define:

$$S(\mu_{\widetilde{A}^L}) = x_0.y_0 = \frac{2a_1^L + 7a_2^L + 7a_3^L + 2a_4^L}{18}.\frac{7\gamma}{18}.$$

This is the area between the Centroid of the Centroids and the original point.

Similarly, the trapezoid corresponding to the upper membership function is divided into three plane figures. In similar fashion, the Centroid of the three plane figures and the Centroid of these Centroids is evaluated.

The Centroids of these plane figures are:

$$G_1 = \Big(\frac{a_1^U + 2a_2^U}{3}, \frac{\delta}{3}\Big),$$

$$G_2 = \Big(\frac{a_2^U + a_3^U}{2}, \frac{\delta}{2}\Big),$$

$$G_3 = \Big(\frac{2a_3^U + a_4^U}{3}, \frac{\delta}{3}\Big).$$

They are collinear and they form a triangle. Thus the centroid of these Centroids is

$$G(x_0, y_0) = \Big(\frac{2a_1^U + 7a_2^U + 7a_3^U + 2a_4^U}{18}, \frac{7\delta}{18}\Big).$$

Now we define

$$S(\mu_{\widetilde{A}^U}) = x_0.y_0 = \frac{2a_1^U + 7a_2^U + 7a_3^U + 2a_4^U}{18}.\frac{7\delta}{18}.$$

Using the above definitions, the rank of $\widetilde{\widetilde{A}}$ is defined as follows:

$$R(\widetilde{\widetilde{A}}) = \frac{\gamma S(\mu_{\widetilde{A}^L}) + \delta S(\mu_{\widetilde{A}^U})}{\gamma + \delta}.$$

Definition 1.33. Consider the F_{IVTN} $\widetilde{\widetilde{A}} = [\widetilde{A}^L, \widetilde{A}^U] = \langle (a_1^L, a_2^L, a_3^L; \gamma), (a_1^U, a_2^U, a_3^U; \delta) \rangle$. The Centroid of a triangle is considered to be the balancing point of the triangle.
The Centroid of the triangle is: $\Big(\frac{a_1^U + a_2^U + a_3^U}{3}, \frac{\delta}{3}\Big)$.
Now we define

$$S(\mu_{\widetilde{A}^U}) = x_0.y_0 = \frac{a_1^U + a_2^U + a_3^U}{3}.\frac{\delta}{3}.$$

This is the area between the Centroid of the Centroid and the original point.
Similarly,

$$S(\mu_{\widetilde{A}^L}) = x_0.y_0 = \frac{a_1^L + a_2^L + a_3^L}{3}.\frac{\gamma}{3}.$$

Using the above definitions, the rank of $\widetilde{\widetilde{A}}$ is defined as follows:

$$R(\widetilde{\widetilde{A}}) = \frac{\gamma S(\mu_{\widetilde{A}^L}) + \delta S(\mu_{\widetilde{A}^U})}{\gamma + \delta}.$$

Definition 1.34. Let $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B}} \in F_{IVTN}(\gamma, \delta)$. Then the ranking of level (γ, δ)-interval-valued trapezoidal fuzzy numbers in $F_{IVTN}(\gamma, \delta)$ is defined as follows:

$$\begin{aligned} &\widetilde{\widetilde{A}} \prec \widetilde{\widetilde{B}}, \qquad R(\widetilde{\widetilde{A}}) < R(\widetilde{\widetilde{B}}),\\ &\widetilde{\widetilde{A}} \succ \widetilde{\widetilde{B}}, \qquad R(\widetilde{\widetilde{A}}) > R(\widetilde{\widetilde{B}}),\\ &\widetilde{\widetilde{A}} = \widetilde{\widetilde{B}}, \qquad R(\widetilde{\widetilde{A}}) = R(\widetilde{\widetilde{B}}). \end{aligned}$$

1.4.3 Uncertain Programming

Some information and knowledge are usually represented by human language like “about 300km”, “approximately 42 C”, “roughly 70 kg”, “low speed”, “middle age”, and “big size”. How can understand them? some people think that they are subjective probability or they are fuzzy concepts.Although, a lot of surveys showed that those imprecise quantities behave neither like randomness nor like fuzziness.This fact prompts the invention of another mathematical tool, calcified as uncertainty theory, Uncertainty theory was founded by Liu in 2007.

Basic Preliminaries

Definition 1.35. Let Γ be a nonempty set and $\mathcal{L}$ be a $\sigma - algebra$ over Γ. The uncertain measure $\mathcal{M}\{\Lambda\}$ meets with the following axioms:

Axiom 1. $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;

Axiom 2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\bar{\Lambda}\} = 1$ for any $\Lambda \in \mathcal{L}$, where $\bar{\Lambda}$ is the complement of Λ;

Axiom 3. For every countable sequence of events $\{\Lambda_i\} \in \mathcal{L}$, we have:

$$\mathcal{M}\Big\{\bigcup_{i=1}^{\infty} \Lambda_i\Big\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

To provide an operational law on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, the product uncertain measure was defined by Liu 2009 as the following product axiom:

Axiom 4. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\Big\{\prod_{k=1}^{\infty} \Lambda_k\Big\} \leq \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k is an arbitrarily chosen event from $\mathcal{L}_k$, for $k = 1, 2, \ldots$.

Definition 1.36. An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 1.37. Suppose that $f : R^n \to R$ is a measurable function, and $\xi_1, ..., \xi_n$ uncertain variables on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then $f(\xi_1, ..., \xi_n)$ is an uncertain variable defined as

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), ..., \xi_n(\gamma)) \ \ \gamma \in \Gamma$$

Example 1.5. Let ξ_1 and ξ_2 be two uncertain variables. Then $\xi = \xi_1 + \xi_2$ is an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) + \xi_2(\gamma) \ \ \gamma \in \Gamma$$

The product $\xi = \xi_1 \xi_2$ is also an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma)\xi_2(\gamma) \ \ \gamma \in \Gamma$$

Definition 1.38. An uncertainty distribution $\phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which

$$\begin{cases} 0 < \phi(x) < 1, \\ \lim\limits_{x \to -\infty} \phi(x) = 0 \\ \lim\limits_{x \to +\infty} \phi(x) = 1. \end{cases}$$

Definition 1.39. Let ξ be an uncertain variable with regular uncertainty distribution $\phi(x)$. Then the inverse function $\phi^{-1}(\beta)$ is called the inverse uncertainty distribution of ξ.

According to inverse uncertainty distribution, Liu 2010 gave the operational law of the strictly monotone function of uncertain variables as follows.

Theorem 1.5. *(Liu[63]) Let $\xi_1, \dots, \xi_n$ be independent uncertain variables with regular uncertainty distribution $\phi_1, \dots, \phi_n$, respectively. If the function $f(x_1, \dots, x_n)$ is strictly increasing with respect to $x_1, \dots, x_m$ and strictly decreasing with respect to $x_{m+1}, \dots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \dots, \xi_n)$$

has the following inverse uncertainty distribution:

$$\psi^{-1}(\beta) = f(\phi_1^{-1}(\beta), \dots, \phi_m^{-1}(\beta), \phi_{m+1}^{-1}(1-\beta), \dots, \phi_n^{-1}(1-\beta)).$$

Liu 2009, introduced the concept of independent uncertain variables in the following way. The uncertain variables $\xi_1, \dots, \xi_n$ are said to be independent if:

$$M\Big\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\Big\}e = \bigwedge_{i=1}^{n} M\{\xi_i \in B_i\}$$

for any Borel sets $B_1, \dots, B_n$. The expected value of uncertain variable ξ was defined by Liu 2007 as

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\}\,\mathrm{d}x - \int_{-\infty}^{0} M\{\xi \leq x\}\,\mathrm{d}x$$

provided that at least one of the two integrals is finite. As a useful representation of expected value, we have:

$$E[\xi] = \int_0^1 \phi^{-1}(\beta)\,\mathrm{d}\beta$$

where ϕ^{-1} is the inverse uncertainty distribution of uncertain variable ξ.

Theorem 1.6. *Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have:*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Definition 1.40. An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\phi(x) = \begin{cases} 0 & \text{if } \; x \geq a \\ \dfrac{x-a}{(b-a)} & \text{if } \; a < x \leq b \\ 1 & \text{if } \; x > b. \end{cases}$$

Denoted by $\mathcal{L}(a, b)$ where a and b are real numbers with $a < b$

Definition 1.41. An uncertain variable ξ is a zigzag uncertain variable, where a, b and c are real numbers with $a < b < c$. and denoted by $\sim Z(a,b,c)$ if its uncertainty distribution distribution is given by:

$$\phi(x) = \begin{cases} 0 & \text{if} \quad x \geq a \\ \dfrac{x-a}{2(b-a)} & \text{if} \quad a < x \leq b \\ \dfrac{x+c-2b}{2(c-b)} & \text{if} \quad b < x \leq c \\ 1 & \text{if} \quad x > c. \end{cases}$$

Definition 1.42. An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\phi(x) = \Big(1 + \exp\Big(\frac{\Pi(e-x)}{\sqrt{3}\sigma}^{-1}\Big), \quad x \in R$$

denoted by $N(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$.

Definition 1.43. An uncertain variable ξ is called lognormal if $ln\xi$ is a normal uncertain variable $N(e,\sigma)$ In other words, a lognormal uncertain variable has an uncertainty distribution

$$\phi(x) = \Big(1 + \exp\Big(\frac{\Pi(e-\ln x)}{\sqrt{3}\sigma}^{-1}\Big), \quad x \in R$$

denoted by $N(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$.

Definition 1.44. An uncertain variable ξ is called discrete if it takes values in $\{x_1, x_2, ..., x_m\}$ and

$$\phi(x_i) = \alpha_i, \quad i = 1, ..., m$$

Where $x_1 < x_2 < ... < x_m = 1$.

The uncertainty distribution ϕ of the discrete uncertain variable is a step function jumping only at $\{x_1, x_2, ..., x_m\}$ i. e.,

Definition 1.45. An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\phi(x) = \begin{cases} \alpha_0, \quad x < x_1 & & \\ \alpha_i & \text{if} & x_i \leq x < x_{i+1} \\ \alpha_m & x \geq x_{i+1} & \end{cases}$$

where $\alpha_0 \equiv 0$ and $\alpha_m \equiv 1$.

Theorem 1.7. *(Liu[63])*

Let ξ be an uncertain variable with continuous uncertainty distribution ϕ. Then for any real number x, we have

$$M\{\xi < x\} = \phi(x)$$

$$M\{\xi > x\} = 1 - \phi(x)$$

Theorem 1.8. *(Liu[63])*

Let ξ be an uncertain variable with continuous uncertainty distribution ϕ. Then for any interval $[a, b]$, we have

$$\phi(b) - \phi(a) \leq M\{a \leq \xi \leq b\} \leq \phi(b) \wedge (1 - \phi(a))$$

Definition 1.46. Let ξ be an uncertain variable with uncertainty distribution ϕ. Then the inverse function ϕ^{-1} is called the inverse uncertainty distribution of ξ. Note that the inverse uncertainty distribution $\phi^{-1}(\alpha)$ is well defined on the open interval $(0, 1)$ If needed, we can extend the domain via

$$\phi^{-1}(0) = \min_{\alpha \to 0} \phi^{-1}(\alpha)$$
$$\phi^{-1}(1) = \min_{\alpha \to 1} \phi^{-1}(\alpha)$$

It is easy to verify that inverse uncertainty distribution is a monotone increasing function on $[0, 1]$.

Example 1.6. The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a, b)$ is

$$\phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b$$

Example 1.7. The inverse uncertainty distribution of zigzag uncertain variable, $\sim Z(a, b, c)$ is

$$\phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b, & \alpha < 0.5 \\ (2 - 2\alpha)b + (2\alpha - 1)c, & \alpha \geq 0.5 \end{cases}$$

Example 1.8. The inverse uncertainty distribution of normal uncertain variable $N(e, \sigma)$ is

$$\phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}$$

Example 1.9. The inverse uncertainty distribution of lognormal uncertain variable $LOGN(e, \sigma)$ is

$$\phi^{-1}(\alpha) = \exp(e)\Big(\frac{\alpha}{1 - \alpha}\Big)^{\sqrt{3}\sigma/\pi}$$

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable

Definition 1.47. The expected value of uncertain variable ξ was defined as

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\} \, \mathrm{d}x - \int_{-\infty}^{0} M\{\xi \leq x\} \, \mathrm{d}x$$

Example 1.10. Let $\xi \sim mathcal L(a, b)$ be a linear uncertain variable. then the expected value expressed as

$$E[\xi] = \frac{a + b}{2}$$

Example 1.11. The zigzag uncertain variable $\xi \sim\sim Z(a, b, c)$ has the expected value expressed as

$$E[\xi] = \frac{a + 2b + c}{4}$$

Example 1.12. The normal uncertain variable $N(e, \sigma)$ has the expected value expressed as is

$$E[\xi] = e$$

Example 1.13. If $\sigma < pi/\sqrt{3}$then the lognormal uncertain variables $\xi \sim LOGN(e, \sigma)$ has the expected value expressed as

$$E[\xi] = \sqrt{3}\sigma \exp(e) CSC(\sqrt{3}\sigma)$$

otherwise

$$E[\xi] = +\infty$$

Theorem 1.9. *Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have:*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

The variance of uncertain variable gives a degree of the spread of the distribution around its expected value.As mall value of variance indicates that the uncertain variable is tightly concentrated around its expected value; and a large value of variance indicates that the uncertain variable has a wide spread around its expected value.

Definition 1.48. Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is defined by

$$V[\xi] = E[(\xi - e)^2]$$

Example 1.14. The variance of linear uncertain variable $\mathcal{L}(a, b)$ is

$$V[\xi] = \frac{(b-a)^2}{12}$$

Example 1.15. The variance of normal uncertain variable $N(e, \sigma)$ is

$$V[\xi] = \sigma^2$$

Definition 1.49. Let ξ be an uncertain variable and $\beta \in (0, 1]$ be the confidence level. Then $\xi_{sup}(\beta)$ and $\xi_{inf}(\beta)$ are respectively the $\beta - optimistic$ and the $\beta - pessimistic$ values of ξ and are defined as follows:

$$\xi_{\text{sup}}(\beta) = \sup\{t | M\{\xi \geq t\} \geq \beta\}$$

$$\xi_{\text{inf}}(\beta) = \inf\{t | M\{\xi \leq t\} \geq \beta\}$$

Theorem 1.10. *(Liu[63]) Let ξ be an uncertain variable with uncertainty distribution ϕ Then its $\beta-$optimistic value and $\beta-$pessimistic value are*

$$\xi_{\text{sup}}(\alpha) = \alpha^{-1}(1 - \alpha)$$

$$\xi_{\text{inf}}(\alpha) = \alpha^{-1}(\alpha)$$

Example 1.16. Let ξ be a linear uncertain variable denoted as $\mathcal{L}(a, b)$. Its $\beta - optimistic$ and $\beta - pessimistic$ values are:

$$\xi_{\text{sup}}(\beta) = (\beta)a + (1 - \beta)b$$

$$\xi_{\text{inf}}(\beta) = (1 - \beta)a + (\beta)b$$

Example 1.17. Let ξ be a zigzag uncertain variable denoted as $Z(a,b,c)$. Its $\beta-optimistic$ and $\beta-pessimistic$ values are:

$$\xi_{inf}(\beta)=\begin{cases}(1-2\beta)a+2\beta b & \text{if} \quad \beta<0.5,\\ (2-2\beta)b+(2\beta-1)c & \text{if} \quad \beta\geq 0.5,\end{cases}$$

$$\xi_{sup}(\beta)=\begin{cases}2\beta b+(1-2\beta)c & \text{if} \quad \beta<0.5,\\ (2\beta-1)a+(2-2\beta)b & \text{if} \quad \beta\geq 0.5.\end{cases}$$

Example 1.18. Let ξ be a normal uncertain variable denoted as $N(e,\sigma)$. Its $\beta-optimistic$ and $\beta-pessimistic$ values are:

$$\xi_{\sup}(\beta)=e-\frac{\sigma\sqrt{3}}{\pi}\ln\frac{\beta}{1-\beta}$$

$$\xi_{\inf}(\beta)=e+\frac{\sigma\sqrt{3}}{\pi}\ln\frac{\beta}{1-\beta}$$

Example 1.19. Let ξ be a Lognormal uncertain variable denoted as $LOGN(e,\sigma)$. Its $\beta-optimistic$ and $\beta-pessimistic$ values are:

$$\xi_{\sup}(\beta)=\exp(e)\Big(\frac{1-\beta}{\beta}\Big)^{\sqrt{3}\sigma/\pi}$$

$$\xi_{\inf}(\beta)=\exp(e)\Big(\frac{\beta}{1-\beta}\Big)^{\sqrt{3}\sigma/\pi}$$

Suppose that ξ is an uncertain variable with uncertainty distribution ϕ Then its entropy is defined by

$$H[\xi]=\int_{-\infty}^{+\infty}S(\phi(x))dx$$

Where $S(t)=-t\ln t-(1-t)\ln(1-t)$

Example 1.20. Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$ Then its entropy is

$$H[\xi]=-\int_a^b\Big(\frac{x-a}{b-a}\ln\frac{x-a}{b-a}+\frac{b-x}{b-a}\ln\frac{b-x}{b-a}\Big)dx=\frac{b-a}{2}$$

Example 1.21. Let ξ be a Zigzag uncertain variable $Z(a,b,c)$ Then its entropy is

$$H[\xi]=\frac{c-a}{2}$$

Example 1.22. Let ξ be a normal uncertain variable $N(e,\sigma)$ Then its entropy is

$$H[\xi]=\frac{\pi\sigma}{\sqrt{3}}$$

1.5 Multi-objective Programming

Multi-objective optimization (also called multi-objective programming, vector optimization, multi-criteria optimization, multi-attribute optimization or Pareto optimization) is a field of multi-criteria decision making, concerning mathematical optimization problems involving several objective functions to be optimized simultaneously. Multi objective optimization has been applied to many fields of science, including engineering, where optimal decisions must be made in the presence of trade-offs between two or more objectives that may be in conflict.

Formulation of Multi objective programming :

The standard form of Multi objective programming can be formulated as following:

$$\begin{cases} \min(\max) f^1(x), ..., f^n(x) \\ g_i(x) \leq\geq= b_i, \\ x \geq 0 \end{cases}$$

Where $f^k(x), k = 1, ..., n$ is the n objecives

Existing Solution Procedures

We can cite some methods for solving this kind of problem as following

Interactive Method

In this method the decision maker DM Play an important role in the process of the decision, once the set of the efficient solutions determined the DM choose the best compromise solution. The main steps of interactive method summarized as follows:

1: Determine the optimal linear compromise solution

2: present the solution obtained for the DM

3: If the DM satisfied with the solution obtained, stop else

4: Generate other solutions, go to 2.

Non-interactive Method

In this method, The DM enumerates all the efficient solutions and select an appropriate solution at the end.

1: Determine an initial efficient basic solution

2: all efficient basic solutions are enumerated and constructed

Goal Programming

This method largely applied to solve multi objective problem introduced by Charnes and Cooper (1961), in this method the DM involve to find, specify and analyze the goals levels for each objectives.

We can formulate the goal Programming as follows:

$$\begin{cases} \min \sum_{k=1}^{K} (d_k^+ + d_k^-) \\ Z_k(x) - d_k^+ + d_k^- = G_k, \quad \forall\ k \\ g_i(x) \leq \geq = b_i, \\ \\ d_k^+, d_k^- \geq 0, \quad \forall k \end{cases}$$

Where G_k are the goals specified by the decision maker (DM).
d_k^+, d_k^- are the positives and negatives deviations from the goals (under-achievement and over-achievement) .

Fuzzy Method

Fuzzy set theory is important tool to treat and analyze the optimization problems, proposed by Zadeh in 1965, it considered as a powerful tool to treat the incomplete and imprecise information.

First, let us consider the lower bound and upper bound for each k objectives

$$L_k = \min f^k, k = 1, ..., n$$

$$U_k = \max f^k, k = 1, ..., n$$

Then construct the membership function for each objective

$$\mu_k(f^k) \begin{cases} 1 & for f^k \geq U_k \\ \dfrac{f_k - L_k}{U_k - L_k} & for U_k \geq f^k \geq L_k \\ 0 & for f^k \geq L_k \end{cases}$$

Using the membership function defined above and following the fuzzy decision of Bellman and Zadeh (1970), the model can written as

$$\begin{cases} \min \max \mu_k(f^k) \\ g_i(x) \leq \geq = b_i, \end{cases}$$

Introducing an auxiliary variable β , it can be reduced to the following conventional model.

$$\begin{cases} \max \beta \\ \beta \leq \mu_k(f^k) g_i(x) \leq \geq = b_i, \end{cases}$$

1.6 Conclusion

The work of this chapter concerned the presentation of optimization problem then the formulation of some mathematical programming (Linear, Nonlinear, Quadratic and Integer programming) are given then uncertain programming like (stochastic, fuzzy and uncertain programming)are presented, Additionally, Multi-objective Programming (MOPP) along with exact methods (Interactive, Non-interactive, Goal Programming and Fuzzy Method)are given.

Chapter 2

Mono and multi objective transportation problems

2.1 Introduction

This chapter deals with mono and multi objective transportation problem and presents some methods of resolutions for both cases and gives its extensions and provides state of the art of the uncertainty in mono and multi-objective transportation problems: An Annotated Bibliography is given at the end.

2.2 The classical Transportation Problem

The goal of transportation problem is to determine the optimal strategy for The distribution of a commodity from a set of supply such as factories, called sources, to various receiving centers, such as warehouses, called destinations, in order to minimize total distribution costs. Each source is able to provide a fixed number of units of the product, usually called capacity or availability, and each destination has a fixed demand, often referred to as a requirement.

2.2.1 The Mathematical formulation of Transportation problem

Following, the formulation and feasibility conditions of transportation problem, two main models can be considered in the classical sense (balanced and non balanced).
First, let us use the following notation

- m : number of sources of the transportation problem;
- n : number of destinations;
- a_i**:** The amount of product available at i^{th} origin;
- b_j **:** The demand of product at j^{th} destination;
- C_{ij} **:** The unit cost for the transportation problem from i^{th} origin to j^{th} destination.

- TP with equality constraints (balanced problem), which consists of

$$(1)\begin{cases} \min \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij}, & & & \\ \sum_{j=1}^{n} x_{ij} & = & a_i & i=1,\dots,m, \\ \sum_{i=1}^{m} x_{ij} & = & b_j & j=1,\dots,n, \\ x_{ij} & \geq & 0 & i=1,\dots,m, j=1,\dots,n. \end{cases}$$

(1) is feasible if and only if the condition holds
Total supply = Total demand.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

This is called a balanced transportation problem

- TP with inequality constraints (Non balanced problem), which consists of

$$(2)\begin{cases} \min \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij}, & & & \\ \sum_{j=1}^{n} x_{ij} & \leq & a_i & i=1,\dots,m, \\ \sum_{i=1}^{m} x_{ij} & \geq & b_j & j=1,\dots,n, \\ x_{ij} & \geq & 0 & i=1,\dots,m, j=1,\dots,n. \end{cases}$$

(2) is feasible if and only if
Total supply $\geq$ Total demand.

$$\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$$

This is called a balanced transportation problem.
The problem (2) can be transformed as problem (1) by adding a dummy destination $j = n + 1$ characterized by the level of demand

$$b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$$

and corresponding costs $C_{in+1} = 0, i \in I$

2.2.2 Methods for solving classical Transportation Problem (TP):

The Methods of solving transportation problem are

1: Formulate the problem. Formulate the given problem and set up in a matrix form. Check whether the problem is a balanced or unbalanced transportation problem. If unbalanced, add dummy source (row) or dummy destination (column) as required.

2: Obtain the initial feasible solution. The initial feasible solution can be obtained by any of the following three methods:

1: Northwest Corner Method (NWC)

2: Least Cost Method (LCM)

3: Vogels Approximation Method (VAM)

3: Find the optimal solution using

1: Modified distribution method (MODI)

2: Stepping Stone Method

2.3 Extensions of classical Transportation Problem

2.3.1 Solid Transportation Problem(STP):

The Solid Transportation Problem (STP) is a generalization of the classical TP. The need to consider this type of TP occurs when there are different types of products are to be transported using heterogeneous modes of transport called conveyances. Therefore, three item properties are taken into account in the constraints set of STP instead of two constraints (source and destination). The STP was stated by Shell who discussed four different cases based on the given data on the item properties such as three planar sums, two planar sums, one planar and one axial sum, and three axial sums. An STP is defined as follows. Suppose that a homogeneous product is to be transported from each of m sources to n destinations. The sources are production facilities, warehouses, or supply points that are characterized by the available capacities The sources are production facilities, warehouses, or supply points that are characterized by the available capacities a_i,$i = 1, ..., m$. The destinations are expending points, warehouses, or demand points that are characterized by required levels of demands b_j, $j = 1, ..., n$. Let e_k, be the number of units transported by k conveyance $k = 1, ..., K$ from sources to destinations. The conveyances may be trucks, air freights, freight trains, and ships.
The mathematical formulation of STP is represented as following:

$$(3)\left\{\begin{array}{llll} \min Z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} C_{ijk}x_{ijk} & & & \\ \qquad\qquad \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk} & (\leq, =) & a_i, & \forall\, i = 1, ..., m \\ \qquad\qquad \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk} & (\geq, =) & b_j, & \forall\, j = 1, ..., n \\ \qquad\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} & (\leq, =) & e_k, & \forall\, k = 1, ..., K \\ \qquad\qquad x_{ijk} & \geq & 0, & \forall\, i, j, k \end{array}\right.$$

Where, c_{ijk} represent the unit transportation cost from i origin to j destination through k conveyance, x_{ijk} is the decision variable represent quantity of goods to be transported from i origin to j destination through k conveyance

(3) is feasible if and only if the condition

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{k=1}^{K} e_k$$

holds. if not the same idea of TP holds with STP.

2.3.2 Multi-item Transportation Problem(MITP):

In multi-item transportation problem(MITP), more than one type of good is transported instead of one type of good. If p items are to be transported and c_{ij}^k be the unit transportation cost from i source to j destination for $p(p = 1, ..., P)$ item.
The mathematical formulation of MITP is represented as following:

$$
(4)\begin{cases}
\min Z = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}^{p} x_{ij}^{p} & & & \\
\qquad\qquad \sum_{j=1}^{n} x_{ij}^{p} & (\leq, =) & a_i^p, & \forall\, i = 1, ..., m;\ p = 1, ..., P \\
\qquad\qquad \sum_{i=1}^{m} x_{ij}^{p} & (\geq, =) & b_j^p, & \forall\, j = 1, ..., n;\ p = 1, ..., P \\
\qquad\qquad x_{ij}^{p} & \geq & 0, & \forall\, i, j, p
\end{cases}
$$

2.3.3 Fixed Charge Transportation Problem (FCTP):

A fixed charge is any type of expense that recurs on a regular basis, regardless of the volume of business. Fixed charges mainly include loan (principal and interest) and lease payments, but the definition of "fixed charges" may broaden out to include insurance, utilities, and taxes for the purposes of drawing up loan covenants by lenders. Before a business sets up, it lists all the necessary upfront and ongoing expenses. The expenses are then separated into two buckets: fixed and variable. The variable expenses depend on the volume of business. For example, a salesperson's commission is determined by how much of the company's products or services are sold. Fixed expenses, on the other hand, exist regardless of the volume of business. Suppose d_{ij} be the fixed cost associated with route (i, j) The mathematical formulation of FCTP is given as:

$$
(5)\begin{cases}
\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} + d_{ij}y_{ij} & & & \\
\qquad\qquad \sum_{j=1}^{n} x_{ij} & (\leq, =) & a_i, & \forall\, i = 1, ..., m \\
\qquad\qquad \sum_{i=1}^{m} x_{ij} & (\geq, =) & b_j, & \forall\, j = 1, ..., n \\
\qquad\qquad x_{ij} & \geq & 0, & \forall\, i, j
\end{cases}
$$

y_{ij} is defined such that if $x_{ij} > 0$ then $y_{ij} = 1$, otherwise it will be 0.

2.3.4 Transportation Problem with budget constraint TPBC:

A transportation problem is often associated with additional constraints which consists in budget constraint, the activity of transportation are usually limited (limitation of the budget at each destination). The mathematical formulation of TPBC is represented as following

$$(6)\begin{cases} \min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} \\ \qquad \sum_{j=1}^{n} x_{ij} \quad (\leq, =) \quad a_i, \quad \forall\, i = 1, ..., m \\ \qquad \sum_{i=1}^{m} x_{ij} \quad (\geq, =) \quad b_j, \quad \forall\, j = 1, ..., n \\ \qquad \sum_{i=1}^{m} x_{ij} \quad (\leq=) \quad B_j, \quad \forall\, j = 1, ..., n \\ \qquad x_{ij} \quad \geq \quad 0, \quad \forall\, i, j \end{cases}$$

2.4 Multi objective Transportation Problem (MOTP):

In most of the real-life application, it is required to take into account more than one objective to make the problem more realistic. The objectives can be transportation cost, reliability of transportation, time of transportation and deterioration of products. If o objective are to be optimized and c_{ij}^{o} represents the unit transportation penalty(transportation cost, profit, time etc.) for o objective $o = 1, ..., O$, then mathematical formulation is given as:

$$(7)\begin{cases} \min / \max Z_o = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}^{o}x_{ij}, o = 1, ..., O \\ \qquad \sum_{j=1}^{n} x_{ij} \quad (\leq, =) \quad a_i, \quad \forall\, i = 1, ..., m \\ \qquad \sum_{i=1}^{m} x_{ij} \quad (\geq, =) \quad b_j, \quad \forall\, j = 1, ..., n \\ \qquad x_{ij} \quad \geq \quad 0, \quad \forall\, i, j \end{cases}$$

2.4.1 Methods for solving Multi objective Transportation Problem (MOTP):

2.4.2 Interactive Method

In interactive method, an interactive solution procedure is made, its steps are repeated and the DM specifies preference information progressively during the solution process. In other terms, the preference request phase and the generation of solutions alternate until the DM has found the most favorable solution. After each step, some information is given to the DM, and she/he/they is/are asked to answer some questions concerning a critical evaluation of the proposed solution or to provide some kind of information to express her/his preference. In this manner DM leads the solution process and only a part of the optimal Pareto solutions needs to be generated and evaluated. The main steps of interactive method are given as following

1: Initialize

2: Generate a Pareto optimal starting point

3: Ask the preference information from the DM

4: Generate a new Pareto optimal solution according to the preference and present it to the DM.If several solutions were generated, ask the DM to select the best solution.

2.4.3 Non-interactive Methods

The process of solving of these methods depends on the determination of the set of efficient solutions and, finally, the DM is responsible for the choice of the preferred solution from this set. In this situation, the solution procedure may take a long time to search the possible region for the efficient solution(s). Another point to note here is that the DM in difficulty in making tradeoffs between the alternatives because of his/her inexperience and/or incomplete information about the decision environment.

The main steps of interactive methods are summarized as follows:

1: Determine an initial efficient basic solution,

2: All efficient basic solutions are enumerated and constructed.

2.4.4 Goal Programming

Goal programming (GP) is one of the most efficient techniques for solving multi-objective optimization problems. It is used as a powerful tool that builds on the highly developed and tested techniques of linear programming and at the same time offers a simultaneous solution to the complex system of competing objectives and a widely accepted and applied technique, mainly due to the underlying philosophy of satisfiable. Goal programming has been applied in various fields and is an excellent model for a wide variety of real world problems with multiple goals. In general terms, GP minimizes undesirable deviations from target values. It is a special multi-criteria trade off method that assumes that the decision maker knows the values of the goals and their relative importance. It can be designed to take into account several objectives simultaneously in the search for a compromise solution and can be based on mathematical programming. The basic approach of goal programming is to set up a specific numeric goal G_o for each objective function $F_o(x)$ then the total deviation from the specified goals $\sum_{o=1}^{O} d_o$is minimized, where d_o is deviation from the goal G_o for o objective function. We can formulate the typical goal Programming as follows:

$$
\begin{cases}
\min & \displaystyle\sum_{o=1}^{O}(d_o^+ + d_o^-) & \\
& Z_o(x) - d_o^+ + d_o^- = G_o, & \forall\, o \\
& \displaystyle\sum_{j=1}^{n} x_{ij} = a_i, & \forall\, i \\
& \displaystyle\sum_{i=1}^{m} x_{ij} = b_j, & \forall\, j \\
& x_{ij} \geq 0 & \forall\, i, j, \\
& d_o^+, d_o^- \geq 0, & \forall\, o
\end{cases}
$$

2.4.5 Fuzzy Optimization Method

Let a multi objective transportation problem (MOTP) problem (7) with o objective functions $F_o(x)$ In problem (7), it is improbable that all objective functions will simultaneously achieve their optimal value. Thus, in practice, the decision-maker (DM) chooses a satisfactory solution, depending on the aspiration level fixed for each objective Suppose that the DM provides imprecise aspiration levels such as, the objective function $F_o(x)$ should be essentially less than or equal to some value G_o MOLP can be expressed as following:

$$
\begin{cases}
Find \quad X(x_1,...,x_n) \\
such\ that \\
F_o \lessapprox G_o, \\
or \\
F_o \gtrapprox G_o, \\
\sum_{j=1}^{n} x_{ij} = a_i, \qquad \forall\, i \\
\sum_{i=1}^{m} x_{ij} = b_j, \qquad \forall\, j \\
x_{ij} \geq 0 \quad \forall\, i,j,
\end{cases}
$$

Each expression $F_o \lessapprox G_o$ is represented by a fuzzy set called fuzzy goal, whose membership function $\mu_o : R \to [0,1]$, provides a membership degree(satisfaction degree) λ_o to which the o fuzzy inequality is satisfied. In order to define the membership function μ_o, the DM provides the L_o and U_o as lower and upper bounds of the objective functions $F_o(x)$. For each objective function $F_o(x)$, find the lower bound(minimum value) L_o and upper bound (maximum value) U_o. there are different kind of membership function .

Linear membership function:

$$
\mu_k(Z_k(x)) \begin{cases}
1, & if \quad Z_k(x) \leq L_k, \\
1 - \dfrac{Z_k - L_k}{U_k - L_k}, & if \quad L_k \leq Z_R(x) \leq U_k, \\
0, & if \quad Z_k(x) \geq U_k
\end{cases}
$$

Exponential membership function:

$$
\mu_k(Z_k(x)) \begin{cases}
1, & if \quad Z_k(x) \leq L_k, \\
\dfrac{e^{-a\phi_k(x)} - e^{-a}}{1 - e^{-a}}, & if \quad L_k \leq Z_R(x) \leq U_k, \\
0, & if \quad Z_k(x) \geq U_k
\end{cases}
$$

Where

$$
\phi_k(x) = \frac{Z_k - L_k}{U_k - L_k}
$$

and a is a nonn zero parameter perscribed by the DM.

Hyperbolic membership function:

$$
\mu_k(Z^k(x)) \begin{cases}
1, & if \quad Z_k(x) \leq L_k, \\
\alpha_k Z_k(x) - \alpha_k \dfrac{(L_k + U_k)}{2} & if \quad L_k < Z_k(x) < U_k \\
0, & if \quad Z_k(x) \geq U_k
\end{cases}
$$

With

$$
\alpha_k = \frac{6}{(U_k - L_k)}
$$

Quadratic membership function:

$$\mu_k(Z^k(x)) \begin{cases} 1, & if \quad Z_k(x) \leq L_k, \\ \dfrac{U_k - Z_k}{U_k - L_k} + a_{k1} Z_k^2 - & \\ a_{k1}(L_k + U_k) Z_k + a_{k1} & \\ L_k U_k & if \quad L_k < Z_k(x) < U_k \\ 0, & if \quad Z_k(x) \geq U_k \end{cases}$$

Where a_{k1} is a nonn zero parameter perscribed by the DM.

Using the fuzzy decision $(\min - \max)$of Bellman and Zadeh and introducing the auxiliary variable λ we get the formulation of fuzzy programming as follows

$$\begin{cases} \max \lambda \\ \lambda \leq \mu_o(F_o(x)), \\ \displaystyle\sum_{j=1}^{n} x_{ij} = a_i, & \forall\ i \\ \displaystyle\sum_{i=1}^{m} x_{ij} = b_j, & \forall\ j \\ x_{ij} \geq 0 \quad \forall\ i, j, \end{cases}$$

In the very recent past, most optimization practitioners and researchers have been looking for new approaches that combine efficiency and ability to find the global optimum some authors combine goal programming and fuzzy programming and other researcher combine more than two.

2.5 Modeling approaches of uncertainty in transportation problems

In many real-world problems, the available data are neither accurate nor precise. There may be several reasons for this, such as insufficient information, imprecision of information, incompleteness of information, uncertainty of information, unreliability of information, doubt about information, ...

In order to describe and extract useful information hidden in uncertain data and to use it correctly and wisely in practical problems, researchers have developed many theories and proposed many approaches to deal with this problem.

The most frequently used approaches to address the uncertainty or unknown information in transportation problem are considered below:

2.5.1 Fuzzy Approach

A fuzzy transport problem occurs when the supplies, demands, capacities and costs of transportation of a transport problem are unknown and characterized by fuzzy sets (/variables). A Fuzzy set, has imprecise boundaries, was introduced by Zadeh [103], as an extension of the classical (crisp) set. A Fuzzy number is described by a membership function which has a value in the real unit interval $[0, 1]$, while a crisp number is described with membership value that is either 0 or 1. In a fuzzy set, data is represented by different grades of membership function such as triangular, trapezoidal, and LR Fuzzy.

Fuzzy Sets have known many types of extensions. Chiang [20] pointed out that it is better to represent the constraints as interval-valued triangular fuzzy numbers instead of normal fuzzy numbers. Intuitionistic fuzzy sets were introduced by Atanassov (1986) as generalization of fuzzy set where membership and non-membership degree were used. Atanassov and Gargov [4] generalized intuitionistic fuzzy sets as interval-valued intuitionistic fuzzy sets. In which each membership and non-membership degrees are intervals rather than exact numbers. Type-2 fuzzy sets are proposed by Zadeh [104], defined by both primary and secondary membership to provide more degrees of freedom and flexibility.Interval type-2 fuzzy sets can be viewed as a special case of general type-2 fuzzy sets that all the values of secondary membership are equal to 1.

2.5.2 Interval Approach

Interval programming is one of the approaches to tackle uncertainty. It possesses some interesting characteristics because it does not require the specification or the assumption of probabilistic distributions (as in stochastic programming) or possibilistic distributions (as in fuzzy programming). Interval programming just assumes that information about the range of variation of some (or all) of the parameters is available, which allows to specify a model with interval coefficients.

Interval transportation problems arise when supplies, demands, capacities and cost of transportation in transportation problem are expressed as Interval sets (/variables).

2.5.3 Stochastic Approach

When the parameter values in the objective function and in the constraints of the STP can not be known in advance, one may treat them as random variables according to the statistical experience (we consider that the sample data are enough). Then the STP becomes a stochastic STP, usually denoted by SSTP. Thus, for a solution, the corresponding objective functions become random variables and the constraints become relations implying random variables. For these conditions, it is impossible to handle the problem by classical deterministic methods, and probability theory should be employed to interpret these conditions. The main idea is to choose some criteria to rank random variables, since it is not possible to rank them directly.

Following these ideas, different models for the SSTP can be constructed, according to different ranking criteria.

2.5.4 Uncertain Approach

Considering the various complexity in real world, several authors were aware of a fact that it was usually inappropriate to consider the various parameters of the transport problem as crisp numbers. They should be considered as variables. But in many cases, no investigated data are available to estimate a probability distribution of these variables. Since it is well known that probability theory has no effect in the case of shortage of sufficient observed data, there is no choice but to invite some domain experts to evaluate the above parameters. Therefore, for dealing with such situations, uncertainty theory was founded by Liu [61] based on normality, duality, subadditivity and product axioms.

Nowadays, uncertainty theory has become a branch of axiomatic mathematics for modeling belief degrees.

2.5.5 Rough Approach

To deal with uncertainty, rough set theory as developed by Pawlak [79] can be successfully used. Rough set theory has been proved to be a very powerful mathematical tool dealing with vague description of objects. A fundamental assumption in rough set theory is that any object from a universe is perceived through available information, and such information may not be sufficient to characterize the object exactly. One way is the approximation of a set by other sets. Thus a rough set may be defined by a pair of crisp sets, called the lower and the upper approximations, that are originally produced by an equivalence relation.

Rough transportation problems arise when the parameters in transportation problem are initially taken as rough variables based on subjective estimation of experts. Further, these rough estimates are suitably approximated as uncertain normal variables and the conceptual uncertain programming model has been developed.

2.6 Classification of the papers Reviewed

Table 2.1: Papers addressing uncertain transportation problem

Author(s) (year)	Fuzzy Approaches					
	Fuzzy set	Type-2 Set	Interval Type 2 Set	Intuitionistic Set	Interval-valued Set	Interval-valued Intuitionistic set
Chanas et al. [16] (1984)	Yes					
Chanas et al. [15] (1993)	Yes					
Gen et al. [35] (1995)	Yes					
Li et al. [60] (1997)	Yes					
Jiménez and Verdegay [45] (1998)	Yes					
Jiménez and Verdegay [46] (1999)	Yes					
Chiang [20] (2005)					Yes	
Liu [66] (2006)	Yes					
Pramanik and Roy [80] (2008)	Yes					
Dutta and Murthy [27] (2010)	Yes					
Pandian and Nagarajan [78] (2010).	Yes					
Samuel and Venkatachalapathy [85] (2011)	Yes					
Gupta et al. [38](2012)	Yes					
Gupta and Kumar [38] (2012)					Yes	
Shanmugasundari and Ganesan [87] (2013)	Yes					
Malini and Kennedy [71] (2013)	Yes					
Kumar and Kaur [53] (2014)	Yes					
Antony et al. [2] (2014)				Yes		
Ebrahimnejad [29] (2014)	Yes					

Ebrahimnejad [30] (2015)	Yes				
Singh and Yadav [96] (2015)				Yes	
Singh and Yadav [97] (2016)				Yes	
Sinha et al. [98] (2016)			Yes		
Ebrahimnejad [31] (2016)				Yes	
Kour et al. [51](2017)				Yes	
Dutta and Jana [26] (2017)		Yes			
Ebrahimnejad and Verdegay [32] (2018),				Yes	
Roy et al. [82] (2018				Yes	
Bharati and Singh [9] (2018)					Yes
Mahmoodirad et al. [69] (2019),				Yes	

Table 2.2: Contributions of papers addressing uncertain transportation problem

Author(s) [Reference] (year)	type	Contribution
Chanas et al. [16] (1984)	SO	Using α-cut with fuzzy linear programming
Chanas et al. [15] (1993)	SO	Provides the links between the interval TP and fuzzy ones
Gen et al. [35] (1995)	MO	Applied genetic algorithm
Li et al. [60] (1997)	MO	Applied genetic algorithm
Jimenez and Verdegay [45] (1998)	SO	Propose a new approach based on the idea of Chanas et al. [16] (1984)
Jiménez and Verdegay [46] (1999)	MO	Applied evolutionary algorithm
Chiang [20] (2005)	SO	Express the constraints as $(1-\alpha, 1-\beta)$interval-valued fuzzy numbers
Liu [66] (2006)	SO	Proposed a new method for solving fuzzy solid transportation problem
Pramanik and Roy [80] (2008)	MO	Presents a priority based fuzzy goal programming approach
Dutta and Murthy [27] (2010)	MO	Applying multi-choice goal programming
Mahapatra et al. [68] (2010)	MO	Presents a solution procedure for solving multi-objective stochastic transportation problem
Pandian and Nagarajan [78] (2010)	SO	Proposed a fuzzy zero point method
Samuel and Venkatachalapathy [85] (2011)	SO	Proposed Modified Vogels approximation
Gupta and Kumar [38] (2012)	MO	Proposed Mehar's method
Sheng and Yao [89] (2012)	SO	Present an uncertain transportation model
Nagarajan and Jeyaraman [75] (2010)	MO	Present multi-objective interval solid transportation problem under stochastic environment
Hussain and Kumar [43] (2012)	SO	Propose intuitionistic fuzzy zero point Method
Shanmugasundari and Ganesan [87] (2013)	SO	Develop the fuzzy version of vogel and MODI algorithm
Malini and Kennedy [71] (2013)	SO	Propose a new model based on octagonal fuzzy numbers

Biswal and Samal [10] (2013)	MO	studies the multi-objective stochastic transportation problem with cauchy random variables
Kumar and Kaur [53] (2014)	SO	Presents a new method based on tabular representation
Antony et al. [2] (2014)	SO	Using vogel's approximation for solving intuitionistic fuzzy transportation problem
Ebrahimnejad [29] (2014)	SO	Reduce the computational complexity of the existing method
Guo et al. [37] (2015)	SO	Present the uncertain random transportation problem
Ebrahimnejad [30] (2015)	SO	Present two step method for solving fuzzy transportation problem
Singh and Yadav [96] (2015)	SO	Adopt the classical method to the intuitionistic fuzzy environment
Sinha et al. [98] (2016)	MO	fuzzy number
Ebrahimnejad [31] (2016) A.	SO	introduces a new formulation of transportation problem with all parameters expressed as (α, β) interval-valued trapezoidal fuzzy numbers.
Dalman and Sivri [24] (2017)	MO	studies a multi-objective solid transportation problem with interval parameters
Kour et al. [51] (2017)	SO	Presents two methods for solving intuitionistic fuzzy TP
Chen et al. [18](2017)	MO	Propose expected value goal programming model and chance-constrained goal programming model
Ebrahimnejad and Verdegay [32] (2018)	SO	Propose a new approach for solving fully intuitionistic fuzzy transportation problems
Roy et al. [82] (2018)	MO	Propose a new approach for solving intuitionistic fuzzy multi-objective transportation problem
Bharati and Singh [9] (2018)	MO	Solve an Interval-Valued Intuitionistic Fuzzy transportation problem
Dash and Mohanty [25] (2018)	SO	Present the transportation problem with rough parameters
Dutta and Jana [26] (2017)	MO	Presents the application of expectations of the reductions for type-2 trapezoidal fuzzy variables and its application to a multi-objective solid transportation problem
Singh et al. [92] (2019)	SO	propose multi-objective solid transportation problem under stochastic environment with gamma or Erlang distribution.
Mahmoodirad et al. [69] (2019)	SO	Propose a new solution approach for solving fully intuitionistic fuzzy transportation problem.

Table 2.3: Papers addressing uncertain transportation problem

Author(s)(year)			
	Objective	Constraints	Decision variable
Chanas et al. [16] (1984)		Yes	
Chanas et al. [15] (1993)		Yes	
Gen et al. [35] (1995)		Yes	
Li et al. [60] (1997)	Yes		
Jimenez and Verdegay [45] (1998)		Yes	
Jiménez and Verdegay [46] (1999)		Yes	
Chiang [20] (2005)		Yes	
Liu [66] (2006)	Yes	Yes	Yes
Jimenez and Verdegay [47] (1996)		Yes	
Jimenez and Verdegay [45] (1998)		Yes	
Jimenez and Verdegay [45] (1998)		Yes	
Pramanik and Roy [80] (2008)	Yes	Yes	
Dutta and Murthy [27] (2010)		Yes	
Mahapatra et al. [68] (2010)		Yes	
Mahapatra et al. [67] (2010)		Yes	
Nagarajan and Jeyaraman [75] (2010)	Yes	Yes	
Pandian and Natarajan [78] (2010)	Yes	Yes	Yes
Samuel and Venkatachalapathy [85] (2011)	Yes	Yes	Yes
Gupta et al. [39] (2012)	Yes	Yes	Yes
Gupta and Kumar [38] (2012)	Yes	Yes	
Sheng and Yao [89] (2012)	Yes	Yes	
Hussain and Kumar [43] (2012)		Yes	
Shanmugasundari and Ganesan [87] (2013)	Yes	Yes	Yes
Malini and Kennedy [71] (2013)	Yes	Yes	
Biswal and Samal [10] (2013)		Yes	
Kumar and Kaur [53] (2014)	Yes	Yes	Yes
Antony et al. [2] (2014)	Yes	Yes	Yes
Ebrahimnejad [29] (2014)	Yes		
Guo et al. [37](2015)	Yes	Yes	
Ebrahimnejad [30] (2015)	Yes	Yes	Yes
Singh and Yadav [96] (2015)		Yes	

Sinha et al. [98] (2016)	Yes	Yes	
Yadav [97] (2016)	Yes		
Ebrahimnejad [31] (2016)	Yes	Yes	Yes
Dalman and Sivri [24] (2017)	Yes	Yes	
Dutta and Jana [26] (2017)	Yes	Yes	
Kour et al. [51] (2017)	Yes		
Chen et al. [18] (2017)	Yes	Yes	
Ebrahimnejad and Verdegay [32] (2018)	Yes	Yes	Yes
Roy et al. [82] (2018)	Yes	Yes	
Bharati and Singh [9] (2018)	Yes	Yes	
Dash and Mohanty [25] (2018)	Yes	Yes	
Dash and Mohanty [25] (2018)	Yes	Yes	
Singh et al. [92] (2019)		Yes	
Mahmoodirad et al. [69] (2019)	Yes	Yes	Yes

Table 2.4: Papers addressing uncertain transportation problem

Author(s)(year)	Approaches			
	Rough	Interval	Stochastic	Uncertain
Jimenez and Verdegay [47] (1996)		Yes		
Jimenez and Verdegay [45] (1998)		Yes		
Mahapatra et al. [68] (2010)			Yes	
Mahapatra et al. [67] (2010)			Yes	
Nagarajan and Jeyaraman [75] (2010)		Yes	Yes	
Sheng and Yao [89] (2012),				Yes
Biswal and Samal[10] (2013),			Yes	
Guo et al. [37] (2015)			Yes	Yes
Dalman and Sivri [24] (2017),		Yes		
Chen et al. [18](2017)				Yes
Dash and Mohanty [25] (2018)	Yes			
Singh et al. [92] (2019)			Yes	

Tables 1-3 and table 11 shows the papers examined, categorized according different approaches used for modeling the uncertainties in transportation problem. Analyzing the results, we can note that the Fuzzy approach with their extension is the most used comparing with other approaches.

Tables 4-7 shows the papers examined, categorized according number of objectives and the contribution

of each other, we can note that the exact method are the most used when solving the deterministic equivalent and authors try to adopt the classical method to uncertain environment when the problem are with single objective.
Tables 8-10 shows the papers examined, categorized according the uncertain parameters, we can note than there are a few papers dealt with uncertain decision variable.

2.7 Conclusion

The work of this chapter deals with mono and multi-objective transportation problem and presents some methods of resolutions for both cases and gives its extensions and provides state of the art of the uncertainty in mono and multi-objective transportation problems: An Annotated Bibliography is given at the end.

Chapter 3

An uncertain interval programming model for a multi-objective multi-item fixed charge solid transportation problem with budget constraint and safety measure

3.1 Introduction

This chapter proposes a new model, an uncertain interval programming model for multi-objective multi-item fixed charge solid transportation problem with budget constraint and safety measure in which unit transportation costs, fixed charges, transportation times, deterioration of items, supplies at origins, demands at destinations, conveyance capacities, budget at destinations, selling prices, purchasing costs, safety factors and desired safety measures are expressed as interval uncertain parameters.First, To formulate the problem, two different models: an Expected Value Model and chance-constrained model, by using theories of interval and uncertain programming techniques are given. Then, using the linear weighted method, the fuzzy programming method, and the goal programming method. The equivalent deterministic models are formulated and solved The subsequent sections of this chapter are organized as follows. Section 2 proposes a literature study of different variants of transportation problems. Section 3 gives the preliminary concepts for our contribution. Section 4 describes the proposed models. In section 5, we give the formulation of uncertain programming expected value and chance-constrained methods of MOMIFCSTPBCSM. In Section 6, we present the crisp equivalent of the proposed models and some related theorems. In section 7, three compromise multi-objective solving methodologies are presented. In section 8 we present numerical examples. Section 9 contains the main conclusion.

3.2 Literature study of transportation problems under uncertainty

In literature, there are a lot of works made in the field of TP. In Table 1, we cite some recent reviews of different variants of TP under uncertainty.

Table 3.1: Some recent studies under different variants of transportation problem.

Author(s)(year)	Uncertain environment	Type	Uncertain programming model(s)	Model variants						
				TP	STP	Items	FTP	Budget constraint	Deterioration of items	Safety measure
Nagarajan and Jeyaraman [76] (2014)	Interval-valued under Stochastic environment	MO	Expected value	Yes	Yes	No	No	No	No	No
Nagarajan and Jeyaraman [75] (2010)	Interval-valued under Stochastic environment	MO	Chance constrained	Yes	Yes	No	No	No	No	No
Baidya and Bera [5] (2014)	Interval-valued	SO	Theory of interval	Yes	Yes	No	No	Yes	No	No
Baidya and Bera [6] (2014)	Interval-valued	SO	Theory of interval	Yes	Yes	No	No	No	No	Yes
Dalman and Sivri [24] (2017)	Interval-valued	MO	Theory of interval	Yes	Yes	No	No	No	No	No
Dalman et al. [23] (2016)	Interval-valued	MO	Theory of interval	Yes	Yes	Yes	No	No	No	No
Dalman et al. [22] (2016)	Uncertainty theory	MO	Expected value with chance constrained	Yes	Yes	Yes	No	No	No	No
Majumder et al. [70] (2018)	Uncertainty theory	MO	Expected value chance constrained Dependent chance constrained	Yes	Yes	Yes	Yes	Yes	No	No

Chen et al. [18] (2017)	Uncertainty Theory	MO	Expected value Chance constrained	Yes	Yes	No	No	No	No	No
Ebrahimnejad [31] (2016)	Interval-valued Trapezoidal fuzzy numbers	Single	Fuzzy linear programming	Yes	No	No	No	No	No	No
Midya and Roy [73] (2017)	Interval-valued	SO	Theory of interval	Yes	No	No	Yes	No	No	No
Kar et al. [48] (2018)	Fuzzy	MO	Chance constrained	Yes	Yes	Yes	No	No	No	No
Kundu et al. [55] (2017)	Rough	SO	Chance constrained	Yes	Yes	No	Yes	No	No	No
Gao and Kar [34] (2017)	Uncertainty	SO	Expected value Chance constrained	Yes	Yes	No	Yes	No	No	No
Kundu et al. [59] (2015)	Type-2 fuzzy	SO	Nearest interval Chance constrained	Yes	Yes	Yes	No	No	No	No
Kundu et al. [58] (2014)	Type-2 fuzzy	SOe	Critical value+centroid Chance constrained	Yes	No	No	Yes	No	No	No
Roy and Midya [83] (2019)	intuitionistic fuzzy	MO	Centroid method	Yes	Yes	No	Yes	No	No	No
The proposed model	Interval-valued under uncertainty theory	MO	Theory of interval Expected value Chance constrained	Yes	Yes	Yes	Yes	Yes	Yes	Yes

Despite all the developments that have been made, several gaps in the literature can be taken into account in TP. A far as we know, none has already considered an uncertain interval programming model for multi-objective multi-item fixed charge solid transportation problem with budget constraint and safety measure. Here we presents another kind of transportation problem that has not yet been studied.

3.3 Preliminary

Definition 3.1 (Baidya and Bera [5]). An interval number A is defined as

$$A = [a_L, a_R] = \{x : a_L \leq x \leq a_R, x \in \mathbb{R}\}.$$

Here $a_L, a_R \in \mathbb{R}$ are the lower and upper bounds of the interval A, respectively.

An interval number can also be expressed by its mean and width. In this form, an interval number $A = [a_L, a_R]$ is denoted by $< a_M, a_W >$, where $a_M = \frac{1}{2}(a_L + a_R)$ and $a_W = \frac{1}{2}(a_R - a_L)$ are known respectively as the center and the radius of the interval.

Definition 3.2 (Liu [61]). An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 3.3. To any uncertain variable ξ (cf. Definition 3.), we associate an uncertain interval $[\xi_L, \xi_R]$ that we also denote by $< \xi_M, \xi_W >$, where $\xi_M = \frac{1}{2}(\xi_L + \xi_R)$ and $\xi_W = \frac{1}{2}(\xi_R - \xi_L)$ are known as the uncertain center and the uncertain radius of the uncertain interval, respectively.

Definition 3.4. Let $\xi_A = [\xi_{a_L}, \xi_{a_R}]$ and $\xi_B = [\xi_{b_L}, \xi_{b_R}]$ be two intervals and $k \in \mathbb{R}$, a scalar. By means of the set-theoretic definitions and from the fact that interval numbers are ordered sets of real numbers, the following four formulas are derivable:

. **Addition:** $\xi_A + \xi_B = [\xi_{a_L}, \xi_{a_R}] + [\xi_{b_L}, \xi_{b_R}] = [\xi_{a_L} + \xi_{b_L}, \xi_{a_R} + \xi_{b_R}]$.

. **Substraction:** $\xi_A - \xi_B = [\xi_{a_L}, \xi_{a_R}] - [\xi_{b_L}, \xi_{b_R}] = [\xi_{a_L} - \xi_{b_R}, \xi_{a_R} - \xi_{b_L}]$.

. **Scalar Multiplication:**

$$k\xi_A = k[\xi_{a_L}, \xi_{a_R}] = \begin{cases} [k\xi_{a_L}, k\xi_{a_R}] & \text{if } k \geq 0 \\ [k\xi_{a_R}, k\xi_{a_L}] & \text{if } k < 0 \end{cases}$$

. **Multiplication:**

$$\begin{aligned} \xi_A \times \xi_B &= [\xi_{a_L}, \xi_{a_R}] \times [\xi_{b_L}, \xi_{b_R}] \\ &= [\min\{\xi_{a_L}\xi_{b_L}, \xi_{a_L}\xi_{b_R}, \xi_{a_R}\xi_{b_L}, \xi_{a_R}\xi_{b_R}\}, \\ &\qquad \max\{\xi_{a_L}\xi_{b_L}, \xi_{a_L}\xi_{b_R}, \xi_{a_R}\xi_{b_L}, \xi_{a_R}\xi_{b_R}\}] \,. \end{aligned}$$

. **Division:** $\dfrac{\xi_A}{\xi_B} = \xi_A \times \dfrac{1}{\xi_B} = [\xi_{a_L}, \xi_{a_R}] \times \left[\dfrac{1}{\xi_{b_R}}, \dfrac{1}{\xi_{b_L}}\right]$ provided that $0 \notin [\xi_{b_L}, \xi_{b_R}]$.

Definition 3.5. For any two intervals $\xi_A = [\xi_{a_L}, \xi_{a_R}] = \langle \xi_{a_M}, \xi_{a_W} \rangle$ and $\xi_B = [\xi_{b_L}, \xi_{b_R}] = \langle \xi_{b_M}, \xi_{b_W} \rangle$,

$$\xi_A \preceq \xi_B \text{ if and only if} \begin{cases} \xi_{a_M} < \xi_{b_M} & \text{for } \xi_{a_M} \neq \xi_{b_M}, \\ \xi_{a_W} \geq \xi_{b_W} & \text{for } \xi_{a_M} = \xi_{b_M}. \end{cases}$$

Furthermore $\xi_A \prec \xi_B$ if and only if $\xi_A \preceq \xi_B$ and $\xi_A \neq \xi_B$.

3.4 Problem description

In a single-objective optimization problem, the goal of the decision-maker is to optimize a single objective (e.g. maximization of the profit). Although, the complexity of real-world applications can not be represented only by one objective. The decision-maker is interested in several objectives such as maximizing the profit of his production while minimizing the production time and the item deterioration. Therefore, several objective functions have to be considered simultaneously to model more precisely the expectation of the decision-maker. This is the scope of multi-objective optimization. The transportation problem is a special type of linear programming where the objective is to minimize the cost of distributing a product from several sources or origins to many destinations. Here we talk about a single-objective transportation optimization problem. A solid transportation problem is a general case of transportation problems in which a homogenous product will be transported from several sources or origins to many destinations via k different conveyances, $k \geq 2$, (trucks, cargos vans, good trains, etc.). We search to find a transportation plan under the framework of the multi-objective optimization problem defined above. A balanced condition in STP assumes that the total supply at sources equals both the total demands at the destinations and the total conveyance capacities. But with today's highly competitive market, how and when to send products to customers in better conditions becomes a major challenge for actors. In our study, we suppose the following situations:

- Multiple heterogeneous items are considered for shipment from sources to destinations via different kinds of conveyances.
- Different kinds of items and services can be purchased by a consumer with his profit at their purchase prices. The indicated prices are generally limited (limited by the budget for each destination).
- When a transportation activity is initiated from the source i to the destination j, in addition to a transportation cost, a fixed charge should be considered.
- Several objectives, that are conflicting in nature, are to be simultaneously optimized under the same restrictions, such as minimization of time, minimization of deterioration of the items, maximization of profit (we always try to maximize profits by avoiding deterioration of items in a short period of time and it is often not interesting to make profits after a long period or with a deterioration of the items).
- All parameters are expressed as intervals-valued under uncertainty.
- The safety factor is estimated and taken into account in the constraints.
- The sum of availabilities at each source is greater than or equal to the sum of demands at destinations for each item in the right and left limits.
- The sum of conveyance capacities is greater than or equal to the total demand for all items in the right and left limits.

Here, we will use the following notations.

- m : number of sources of the transportation problem;
- n : number of destinations;
- K : number of conveyances (modes of transportation);
- P : number of items;
- $[\xi_{a^p_{L_i}}, \xi_{a^p_{R_i}}]$: uncertain interval amount of product available at i^{th} origin for p^{th} item;
- $[\xi_{b^p_{L_j}}, \xi_{b^p_{R_j}}]$: uncertain interval demand of product at j^{th} destination for p^{th} item;

- $[\xi_{e_{L_k}}, \xi_{e_{R_k}}]$: uncertain interval amount of product which can be carried by the k^{th} conveyance;
- $[\xi_{C^p_{L_{ijk}}}, \xi_{C^p_{R_{ijk}}}]$: uncertain interval cost for the transportation problem from i^{th} origin to j^{th} destination by k^{th} conveyance of p^{th} item;
- $[\xi_{F^p_{L_{ijk}}}, \xi_{F^p_{R_{ijk}}}]$: uncertain interval fixed charge for the transportation from i^{th} origin to j^{th} destination by k^{th} conveyance of p^{th} item;
- $[\xi_{T^p_{L_{ijk}}}, \xi_{T^p_{R_{ijk}}}]$: uncertain interval time required for the transportation from i^{th} origin to j^{th} destination by k^{th} conveyance of p^{th} item;
- $[\xi_{D^p_{L_{ijk}}}, \xi_{D^p_{R_{ijk}}}]$: uncertain interval deterioration of goods for the transportation from i^{th} origin to j^{th} destination by k^{th} conveyance of p^{th} item;
- $[\xi_{S^p_{L_j}}, \xi_{S^p_{R_j}}]$: uncertain interval selling price per unit for p^{th} item at j^{th} destination;
- $[\xi_{V^p_{L_i}}, \xi_{V^p_{R_i}}]$: uncertain interval purchasing cost per unit for p^{th} item at i^{th} source;
- $[\xi_{B_{L_j}}, \xi_{B_{R_j}}]$: budget at j^{th} destination;
- $[\xi_{m^p_{L_{ijk}}}, \xi_{m^p_{R_{ijk}}}]$: The uncertain interval safety value for the transportation of p^{th} item from i^{th} source to j^{th} destination by k^{th} conveyance,
- $[\xi_{M_L}, \xi_{M_R}]$: uncertain interval of desired safety measure for the whole transportation problem;
- x^p_{ijk}: unknown quantity to be transported from i^{th} origin to j^{th} destination by k^{th} conveyance of p^{th} item (decision variable);
- y^p_{ijk} : represents the binary decision variable

$$y^p_{ijk} = \begin{cases} 1 \text{ if } x^p_{ijk} > 0 \forall\, i, j, k, p, \\ 0 \text{ otherwise;} \end{cases}$$

- $\forall\, i$: means "for $i = 1, \ldots, m$";
- $\forall\, j$: means "for $j = 1, \ldots, n$";
- $\forall\, k$: means "for $k = 1, \ldots, K$";
- $\forall\, p$: means "for $p = 1, \ldots, P$".

The mathematical model of MOMIFCSTPBCSM is formulated as follows:

$$
(P_1)\begin{cases}
\max Z^1 = \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\Big\{\Big([\xi_{S^p_{L_j}},\xi_{S^p_{R_j}}]-[\xi_{V^p_{L_i}},\xi_{V^p_{R_i}}]-[\xi_{C^p_{L_{ijk}}},\xi_{C^p_{R_{ijk}}}]\Big)x^p_{ijk} \\
\qquad\qquad -[\xi_{F^p_{L_{ijk}}},\xi_{F^p_{R_{ijk}}}]y^p_{ijk}\Big\} \\
\min Z^2 = \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}[\xi_{T^p_{L_{ijk}}},\xi_{T^p_{R_{ijk}}}]y^p_{ijk} \\
\min Z^3 = \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}[\xi_{D^p_{L_{ijk}}},\xi_{D^p_{R_{ijk}}}]y^p_{ijk} \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K}x^p_{ijk}=[\xi_{a^p_{L_i}},\xi_{a^p_{R_i}}] \qquad \forall\, i,p \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K}x^p_{ijk}=[\xi_{b^p_{L_j}},\xi_{b^p_{R_j}}] \qquad \forall\, j,p \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x^p_{ijk}=[\xi_{e_{L_k}},\xi_{e_{R_k}}] \qquad \forall\, k \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Big\{\Big([\xi_{V^p_{L_i}},\xi_{V^p_{R_i}}]+[\xi_{C_{L^p_{ijk}}},\xi_{C_{R^p_{ijk}}}]\Big)x^p_{ijk} \\
\qquad +[\xi_{F^p_{L_{ijk}}},\xi_{F^p_{R_{ijk}}}]y^p_{ijk}\Big\}=[\xi_{B_{L_j}},\xi_{B_{R_j}}] \qquad \forall\, j, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}[\xi_{m^p_{L_{ijk}}},\xi_{m^p_{R_{ijk}}}]y^p_{ijk}=[\xi_{M_L},\xi_{M_R}] \\
x^p_{ijk}\ge 0,\quad y^p_{ijk}=\begin{cases}1 & \text{if } x^p_{ijk}>0,\\ 0 & \text{otherwise}\end{cases} \qquad \forall\, i,j,k,p
\end{cases}
$$

In our model, we have three uncertain interval objectives and each parameter is lying between a right and a left limits. The first objective maximizes the total profit of transportation of all items from each source to every destination. The second objective minimizes the total time of transportation of all items from each source to every destination. The third objective minimizes the deterioration of all items from each source to every destination.

All the parameters in the constraints are lying between uncertain right and left limits. The first constraint means that the total quantity p of items that have to be transported from source i lies between a left limit $\xi_{a^p_{L_i}}$ and a right limit $\xi_{a^p_{R_i}}$. The second constraint means that the destination parameter lies between a left limit $\xi_{b_{L_i}}$ and a right limit $\xi_{b_{R_i}}$. The third constraint means that the amount p of items transported from source i to destination j via conveyance k lies between a left limit $\xi_{e_{L_i}}$ and a right limit $\xi_{e_{R_i}}$. The fourth constraint means that the total cost and purchase price of item p at source i and fixed charge for transporting item p from source i to destination j via conveyance k at destination j lies between a left limit $\xi_{B_{L_j}}$ and a right limit $\xi_{B_{R_j}}$. The fifth constraint means that the safety factors for different routes, modes, and items lie between left limits ξ_{M_L} and right limits ξ_{M_R}.

It is a very hard task to solve the problem defined above. It contains uncertain interval parameters. We first use the theories of interval, then uncertain programming techniques to obtain a crisp equivalent.

3.5 Formulation of uncertain programming model

In this section we formulate the uncertain programming model of the proposed MOMIFCSTPBCSM, by adopting the concepts of Alefed [1] and Moore [74] to express the equivalent of the upper bounds

Z_R^1, Z_R^2, Z_R^3 and the lower bounds Z_L^1, Z_L^2, Z_L^3 and the centers Z_C^1, Z_C^2, Z_C^3 of the objective functions of the original problem P_1 as follows:

$$Z_L^1 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left\{\left(\xi_{S^p_{L_j}} - \xi_{V^p_{R_i}} - \xi_{C^p_{R_{ijk}}}\right)x^p_{ijk} - \xi_{F^p_{R_{ijk}}} y^p_{ijk}\right\}$$
$$Z_R^1 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left\{\left(\xi_{S^p_{R_j}} - \xi_{V^p_{L_i}} - \xi_{C^p_{L_{ijk}}}\right)x^p_{ijk} - \xi_{F^p_{L_{ijk}}} y^p_{ijk}\right\}$$
$$Z_C^1 = \frac{Z_L^1 + Z_R^1}{2}$$
$$Z_L^2 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{T^p_{L_{ijk}}} y^p_{ijk}$$
$$Z_R^2 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{T^p_{R_{ijk}}} y^p_{ijk}$$
$$Z_C^2 = \frac{Z_L^2 + Z_R^2}{2}$$
$$Z_L^3 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{D^p_{L_{ijk}}} y^p_{ijk}$$
$$Z_R^3 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{D^p_{R_{ijk}}} y^p_{ijk}$$
$$Z_C^3 = \frac{Z_L^3 + Z_R^3}{2}$$
$$Z^1 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left\{\left(\xi_{S^p_j} - \xi_{V^p_i} - \xi^p_{C_{ijk}}\right)x^p_{ijk} - \xi_{F^p_{ijk}} y^p_{ijk}\right\}$$
$$Z^1 \in \left[Z_L^1, Z_R^1\right]$$
$$Z^2 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{T^p_{ijk}} y^p_{ijk}$$
$$Z^2 \in \left[Z_L^2, Z_R^2\right]$$
$$Z^3 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{D^p_{ijk}} y^p_{ijk}$$
$$Z^3 \in \left[Z_L^3, Z_R^3\right]$$

By using Hu and Wang's approach [42], we obtain the following equivalent of the constraints:

$$\begin{cases}
\xi_{a^p_{L_i}} \le \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} \le \xi_{a^p_{R_i}} & \forall\, i, p \\
\xi_{b^p_{L_j}} \le \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} \le \xi_{b^p_{R_j}} & \forall\, j, p \\
\xi_{e_{L_k}} \le \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} \le \xi_{e_{R_k}} & \forall\, k \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\left\{\frac{\xi_{V^p_{L_i}} + \xi_{C^p_{L_{ijk}}} + \xi_{V^p_{R_i}} + \xi_{C^p_{R_{ijk}}}}{2} x^p_{ijk} + \left(\frac{\xi_{F^p_{R_{ijk}}} + \xi_{F^p_{L_{ijk}}}}{2}\right) y^p_{ijk}\right\} \le \frac{\xi_{B_{R_j}} + \xi_{B_{L_j}}}{2} & \forall\, j \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\xi_{m^p_{L_{ijk}}} + \xi_{m^p_{R_{ijk}}}}{2} y^p_{ijk} \le \frac{\xi_{M_L} + \xi_{M_R}}{2} &
\end{cases}$$

Finally the uncertain programming model of MOMIFCSTPBCSM is presented as:

$$
(P_2)\left\{
\begin{array}{llll}
\max Z_R^1 = \sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K}\left\{\left(\xi_{S^p_{R_j}} - \xi_{V^p_{L_i}} - \xi_{C^p_{L_{ijk}}}\right)x^p_{ijk} - \xi_{F^p_{L_{ijk}}} y^p_{ijk}\right\} & & & \\
\max Z_C^1 = \sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K}\left\{\left(\xi_{S^p_{C_j}} - \xi_{V^p_{C_i}} - \xi_{C^p_{C_{ijk}}}\right)x^p_{ijk} - \xi_{F^p_{C_{ijk}}} y^p_{ijk}\right\} & & & \\
\min Z_R^2 = \sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K}\xi_{T^p_{R_{ijk}}} y^p_{ijk} & & & \\
\min Z_C^2 = \sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K}\xi_{T^p_{C_{ijk}}} y^p_{ijk} & & & \\
\min Z_R^3 = \sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K}\xi_{D^p_{R_{ijk}}} y^p_{ijk} & & & \\
\min Z_C^3 = \sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K}\xi_{D^p_{C_{ijk}}} y^p_{ijk} & & & \\
\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K} x^p_{ijk} - \xi_{a^p_{L_i}} & \geq & 0 & \forall\, i,p \\
\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K} x^p_{ijk} - \xi_{a^p_{R_i}} & \leq & 0 & \forall\, i,p \\
\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K} x^p_{ijk} - \xi_{b^p_{L_j}} & \geq & 0 & \forall\, j,p \\
\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K} x^p_{ijk} - \xi_{b^p_{R_j}} & \leq & 0 & \forall\, j,p \\
\sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} x^p_{ijk} - \xi_{e_{L_k}} & \geq & 0 & \forall\, k \\
\sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} x^p_{ijk} - \xi_{e_{R_k}} & \leq & 0 & \forall\, k \\
\sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{K}\left\{\dfrac{\xi_{V^p_{L_i}} + \xi_{C^p_{L_{ijk}}} + \xi_{V^p_{R_i}} + \xi_{C^p_{R_{ijk}}}}{2} x^p_{ijk}\right. & & & \\
\left. + \dfrac{\xi_{F^p_{R_{ijk}}} + \xi_{F^p_{L_{ijk}}}}{2} y^p_{ijk}\right\} - \dfrac{\xi_{B_{R_j}} + \xi_{B_{L_j}}}{2} & \leq & 0, & \forall\, j \\
\sum\limits_{p=1}^{P}\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{K} \dfrac{\xi_{m^p_{L_{ijk}}} + \xi_{m^p_{R_{ijk}}}}{2}) y^p_{ijk} - \dfrac{\xi_{M_L} + \xi_{M_R}}{2} & \leq & 0 & \\
x^p_{ijk} \geq 0, \quad y^p_{ijk} = \begin{cases} 1 & \text{if } x^p_{ijk} > 0, \\ 0 & \text{otherwise} \end{cases} & & & \forall\, i,j,k,p
\end{array}
\right.
$$

There are several models for classifying uncertain variables, but the most commonly used are the EVM and the CCM, which consist of optimizing the EVM and the CCM that are the most appropriate objective functions in the multi-objective domain.

3.5.1 Expected Value Model

Liu [63] presented the Expected Value Model of uncertain programming. This kind of model optimizes some expected objective functions subject to some expected constraints. In what follows, we give the formulation of our problem as an Expected Value Model.

$$
(P_3)\begin{cases}
\max E\Big[Z_R^1\Big] = E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\Big\{\Big(\xi_{S_{R_j}^p} - \xi_{V_{L_i}^p} - \xi_{C_{L_{ijk}}^p}\Big)x_{ijk}^p - \xi_{F_{L_{ijk}}^p}y_{ijk}^p\Big\}\Big] \\
\max E\Big[Z_C^1\Big] = E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\Big\{\Big(\xi_{S_{C_j}^p} - \xi_{V_{C_i}^p} - \xi_{C_{C_{ijk}}^p}\Big)x_{ijk}^p - \xi_{F_{C_{ijk}}^p}y_{ijk}^p\Big\}\Big] \\
\min E\Big[Z_R^2\Big] = E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{T_{R_{ijk}}^p}y_{ijk}^p\Big] \\
\min E\Big[Z_C^2\Big] = E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{T_{C_{ijk}}^p}y_{ijk}^p\Big] \\
\min E\Big[Z_R^3\Big] = E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{D_{R_{ijk}}^p}y_{ijk}^p\Big] \\
\min E\Big[Z_C^3\Big] = E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{D_{C_{ijk}}^p}y_{ijk}^p\Big] \\
E\Big[\sum_{j=1}^{n}\sum_{k=1}^{K}x_{ijk}^p - \xi_{a_{L_i}^p}\Big] \quad \geq \quad 0 \qquad \forall\, i,p, \\
E\Big[\sum_{j=1}^{n}\sum_{k=1}^{K}x_{ijk}^p - \xi_{a_{R_i}^p}\Big] \quad \leq \quad 0 \qquad \forall\, i,p, \\
E\Big[\sum_{i=1}^{m}\sum_{k=1}^{K}x_{ijk}^p - \xi_{b_{L_j}^p}\Big] \quad \geq \quad 0 \qquad \forall\, j,p \\
E\Big[\sum_{i=1}^{m}\sum_{k=1}^{K}x_{ijk}^p - \xi_{b_{R_j}^p}\Big] \quad \leq \quad 0 \qquad \forall\, j,p \\
E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk}^p - \xi_{e_{L_k}}\Big] \quad \geq \quad 0 \qquad \forall\, k \\
E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk}^p - \xi_{e_{R_k}}\Big] \quad \leq \quad 0 \qquad \forall\, k, \\
E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Big\{\frac{\xi_{V_{L_i}^p} + \xi_{C_{L_{ijk}}^p} + \xi_{V_{R_i}^p} + \xi_{C_{R_{ijk}}^p}}{2}x_{ijk}^p + \frac{\xi_{F_{R_{ijk}}^p} + \xi_{F_{L_{ijk}}^p}}{2}y_{ijk}^p\Big\} \\
\qquad -\frac{\xi_{B_{R_j}} + \xi_{B_{L_j}}}{2}\Big] \quad \leq \quad 0, \qquad \forall\, j \\
E\Big[\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\xi_{m_{L_{ijk}}^p} + \xi_{m_{R_{ijk}}^p}}{2}y_{ijk}^p - \frac{\xi_{M_L} + \xi_{M_R}}{2}\Big] \quad \leq \quad 0 \\
x_{ijk}^p \geq 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, i,j,k,p
\end{cases}
$$

In model (P_3), the first objective is to maximize the overall center and right expected profit, the second objective consists to minimize the center and right expected total transportation times and the third objective aims to minimize the center and right total expected of deterioration of goods during the transportation, under some expected constraints.

3.5.2 Chance Constrained Model

Charnes and Cooper proposed in 1959 the basic idea of the chance-constrained model. Then Liu [63] presented a chance-constrained model for uncertain programming. This kind of model offers a powerful means of modeling uncertain decision systems with the assumption that the uncertain constraints will hold at least α time, where α is referred to as the confidence level provided as an appropriate safety margin by the decision-maker.

We formulate the chance-constrained model for our problem as follows:

$$
(P_4)\left\{\begin{array}{lll}
\max \overline{Z^1}^U & & \\
\min \overline{Z^2}^L & & \\
\min \overline{Z^3}^L & & \\
M\Big\{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\Big[\Big(\xi_{S_j^p}-\xi_{V_i^p}-\xi_{C_{ijk}}^p\Big)x_{ijk}^p-\xi_{F_{ijk}^p}y_{ijk}^p\Big]\geq\overline{Z^1}^U\Big\} & \geq & \alpha_1 \\
M\Big\{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{T_{ijk}^p}y_{ijk}^p\leq\overline{Z^2}^L\Big\} & \geq & \alpha_2 \\
M\Big\{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\xi_{D_{ijk}^p}y_{ijk}^p\leq\overline{Z^3}^L\Big\} & \geq & \alpha_3 \\
M\Big\{\sum_{j=1}^{n}\sum_{k=1}^{K}x_{ijk}^p-\xi_{a_{L_i}^p}\geq 0\Big\} & \geq & \beta_{a_{L_i}^p} \quad \forall\, j,p \\
M\Big\{\sum_{j=1}^{n}\sum_{k=1}^{K}x_{ijk}^p-\xi_{a_{R_i}^p}\leq 0\Big\} & \geq & \beta_{a_{R_i}^p} \quad \forall\, j,p \\
M\Big\{\sum_{i=1}^{m}\sum_{k=1}^{K}x_{ijk}^p-\xi_{b_{L_j}^p}\geq 0\Big\} & \geq & \beta_{b_{L_j}^p} \quad \forall\, i,p \\
M\Big\{\sum_{i=1}^{m}\sum_{k=1}^{K}x_{ijk}^p-\xi_{b_{R_j}^p}\leq 0\Big\} & \geq & \beta_{b_{R_j}^p} \quad \forall\, i,p \\
M\Big\{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk}^p-\xi_{e_{L_k}}\geq 0\Big\} & \geq & \beta_{e_{L_k}} \quad \forall\, k \\
M\Big\{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk}^p-\xi_{e_{R_k}}\leq 0\Big\} & \geq & \beta_{e_{R_k}}, \quad \forall\, k \\
M\Bigg\{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Big\{\frac{\xi_{V_{L_i}^p}+\xi_{C_{L_{ijk}}^p}+\xi_{V_{R_i}^p}+\xi_{C_{R_{ijk}}^p}}{2}x_{ijk}^p+\frac{\xi_{F_{R_{ijk}}^p}+\xi_{F_{L_{ijk}}^p}}{2}y_{ijk}^p\Big\} & & \\
\qquad -\frac{\xi_{B_{R_j}}+\xi_{B_{L_j}}}{2}\leq 0\Bigg\} & \geq & \rho_j^1 \quad \forall\, j \\
M\Bigg\{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\xi_{m_{L_{ijk}}^p}+\xi_{m_{R_{ijk}}^p}}{2}y_{ijk}^p-\frac{\xi_{M_L}+\xi_{M_R}}{2}\leq 0\Bigg\} & \geq & \rho^2 \\
x_{ijk}^p\geq 0, \qquad y_{ijk}^p=\begin{cases}1 & \text{if } x_{ijk}^p>0\\ 0 & \text{otherwise}\end{cases} & & \forall\, i,j,k,p
\end{array}\right.
$$

In CCM $\alpha_1,\alpha_2,\alpha_3,\beta_{a_{L_i}^p},\beta_{a_{R_i}^p},\beta_{b_{L_j}^p},\beta_{b_{R_j}^p},\beta_{e_{L_k}},\beta_{e_{R_k}},\rho_j^1$ and ρ^2 respectively, are predetermined confidence levels.

The objectives $\overline{Z^1}^U,\overline{Z^2}^L,\overline{Z^3}^L$ determine the critical values corresponding to the first and second and third constraints respectively. The first constraint determines the α_1-optimistic value of the overall profit corresponding to α_1-transportation plan, and the second constraint determines the α_2-pessimistic value of the shipping/transportation time with respect to α_2-transportation plan and the third constraint determines the α_3-pessimistic value of the deterioration of good with respect to α_3-transportation plan. The remaining constraints also will hold at their corresponding confidence levels $\beta_{a_{R_i}^p},\beta_{b_{L_j}^p},\beta_{b_{R_j}^p},\beta_{e_{L_k}},\beta_{e_{R_k}},\rho_j^1$ and ρ^2 respectively.

3.6 Crisp equivalents of the models

We adopt the theorem provided by Majumder et al. [70] for obtaining the crisp equivalent of the proposed model as follows:

Theorem 3.1. *(Majumder et al. [70]) Let* $\xi_{S_j^p}$,$\xi_{V_i^p}$,$\xi_{C_{ijk}^p}$, $\xi_{F_{ijk}^p}$,$\xi_{T_{ijk}^p}$, ,$\xi_{D_{ijk}^p}$,$\xi_{a_{L_i}^p}$, $\xi_{a_{R_i}^p}$, $\xi_{b_{L_i}^p}$, $\xi_{b_{R_i}^p}$,$\xi_{e_{L_k}}$,$\xi_{e_{R_k}}$, $\xi_{B_{R_j}}$, $\xi_{B_{L_j}}$,$\xi_{m_{L_{ijk}}^p}$,$\xi_{m_{R_{ijk}}^p}$, ξ_{M_L} *and* ξ_{M_R} *be independent uncertain variables associated with the regular uncertainty distributions*
$\phi_{\xi_{S_j^p}}$,$\phi_{\xi_{V_i^p}}$,$\phi_{\xi_{C_{ijk}^p}}$, $\phi_{\xi_{F_{ijk}^p}}$, $\phi_{\xi^p_{T_{ijk}}}$, $\phi_{\xi_{D_{ijk}^p}}$, $\phi_{\xi_{a_{L_i}^p}}$, $\phi_{\xi_{a_{R_i}^p}}$, $\phi_{\xi_{b_{L_i}^p}}$, $\phi_{\xi_{b_{R_i}^p}}$, $\phi_{\xi_{e_{L_k}}}$,$\phi_{\xi_{e_{R_k}}}$, $\phi_{\xi_{B_{R_j}}}$, $\phi_{\xi_{B_{L_j}}}$, $\phi_{\xi_{m_{L_{ijk}}^p}}$,$\phi_{\xi_{m_{R_{ijk}}^p}}$, $\phi_{\xi_{M_L}}$ *and* $\phi_{\xi_{M_R}}$ *respectively.*

Then the crisp equivalents of Expected Value Model and Chance Constrained Model are presented by respectively the models (P_5) *and* (P_6) *presented below.*

3.6.1 Expected Value Model

The model P_3 is equivalent to the following model:

$$
(P_5)\left\{\begin{array}{lll}
\max E\Big[Z_R^1\Big] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\Big\{\Big(\int_0^1 \phi^{-1}_{\xi_{S^p_{R_j}}}(\beta)\,\mathrm{d}\beta - \int_0^1 \phi^{-1}_{\xi_{V^p_{L_i}}}(\beta)\,\mathrm{d}\beta & & \\
\qquad - \int_0^1 \phi^{-1}_{\xi^p_{C_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta\Big)x^p_{ijk} - \int_0^1 \phi^{-1}_{\xi_{F^p_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta\, y^p_{ijk}\Big\} & & \\
\max E\Big[Z_C^1\Big] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\Big\{\Big(\int_0^1 \phi^{-1}_{\xi_{S^p_{C_j}}}(\beta)\,\mathrm{d}\beta - \int_0^1 \phi^{-1}_{\xi_{V^p_{C_i}}}(\beta)\,\mathrm{d}\beta & & \\
\qquad - \int_0^1 \phi^{-1}_{\xi^p_{C_{C_{ijk}}}}(\beta)\,\mathrm{d}\beta\Big)x^p_{ijk} - \int_0^1 \phi^{-1}_{\xi_{F^p_{C_{ijk}}}}(\beta)\,\mathrm{d}\beta\, y^p_{ijk}\Big\} & & \\
\min E\Big[Z_R^2\Big] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\int_0^1 \phi^{-1}_{\xi_{T^p_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta\, y^p_{ijk} & & \\
\min E\Big[Z_C^2\Big] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\int_0^1 \phi^{-1}_{\xi_{T^p_{C_{ijk}}}}(\beta)\,\mathrm{d}\beta\, y^p_{ijk} & & \\
\min E\Big[Z_R^3\Big] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\int_0^1 \phi^{-1}_{\xi_{D^p_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta\, y^p_{ijk} & & \\
\min E\Big[Z_C^3\Big] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\int_0^1 \phi^{-1}_{\xi_{T^p_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta\, y^p_{ijk} & & \\
\qquad\qquad \sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} \quad \geq \quad \int_0^1 \phi^{-1}_{\xi_{a^p_{L_i}}}(\beta)\,\mathrm{d}\beta & & \forall\, i,p, \\
\qquad\qquad \sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} \quad \leq \quad \int_0^1 \phi^{-1}_{\xi_{a^p_{R_i}}}(\beta)\,\mathrm{d}\beta, & & \forall\, i,p \\
\qquad\qquad \sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} \quad \geq \quad \int_0^1 \phi^{-1}_{\xi_{b^p_{L_j}}}(\beta)\,\mathrm{d}\beta & & \forall\, j,p \\
\qquad\qquad \sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} \quad \leq \quad \int_0^1 \phi^{-1}_{\xi_{b^p_{R_j}}}(\beta)\,\mathrm{d}\beta & & \forall\, j,p \\
\qquad\qquad \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} \quad \geq \quad \int_0^1 \phi^{-1}_{\xi_{e_{L_k}}}(\beta)\,\mathrm{d}\beta & & \forall\, k \\
\qquad\qquad \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} \quad \leq \quad \int_0^1 \phi^{-1}_{\xi_{e_{R_k}}}(\beta)\,\mathrm{d}\beta & & \forall\, k \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Bigg\{\Bigg(\dfrac{\int_0^1 \phi^{-1}_{\xi_{V^p_{L_i}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi_{V^p_{R_i}}}(\beta)\,\mathrm{d}\beta}{2} & & \forall\, j \\
\quad + \dfrac{\int_0^1 \phi^{-1}_{\xi^p_{C_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{C_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2}\Bigg)x^p_{ijk} & & \\
\quad + \dfrac{\int_0^1 \phi^{-1}_{\xi^p_{F_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{F_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2}\, y^p_{ijk}\Bigg\} \quad \leq \quad \dfrac{\int_0^1 \phi^{-1}_{\xi_{B_{R_j}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi_{B_{L_j}}}(\beta)\,\mathrm{d}\beta}{2} & & \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\dfrac{\int_0^1 \phi^{-1}_{\xi^p_{m_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{m_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2}\, y^p_{ijk} \leq \dfrac{\xi_{M_L} + \xi_{M_R}}{2} & & \\
\quad x^p_{ijk} \geq 0, \qquad y^p_{ijk} = \begin{cases} 1 & \text{if } x^p_{ijk} > 0 \\ 0 & \text{otherwise} \end{cases} & & \forall\, i,j,k,p
\end{array}\right.
$$

3.6.2 Chance Constrained Model

The model P_4 is equivalent to the following model

$$
(P_6)\left\{\begin{array}{llll}
\multicolumn{4}{c}{\max \overline{Z^1}^U} \\
\multicolumn{4}{c}{\min \overline{Z^2}^L} \\
\multicolumn{4}{c}{\min \overline{Z^3}^L} \\
\multicolumn{4}{l}{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left\{\left(\phi^{-1}_{\xi_{S^p_j}}(\alpha_1)-\phi^{-1}_{\xi_{V^p_i}}(\alpha_1)-\phi^{-1}_{\xi^p_{C_{ijk}}}(\alpha_1)\right)x^p_{ijk}-\phi^{-1}_{\xi_{F^p_{ijk}}}(\alpha_1)y^p_{ijk}\right\} \geq \overline{Z^1}^U} \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\phi^{-1}_{\xi_{T^p_{ijk}}}(\alpha_2)y^p_{ijk} & \leq & \overline{Z^2}^L & \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\phi^{-1}_{\xi_{D^p_{ijk}}}(\alpha_3)y^p_{ijk} & \leq & \overline{Z^3}^L & \\
\sum_{j=1}^{n}\sum_{k=1}^{K}x^p_{ijk}-\phi^{-1}_{\xi_{a^p_{L_i}}}(\beta_{a^p_{L_i}}) & \geq & 0 & \forall\, j,p \\
\sum_{j=1}^{n}\sum_{k=1}^{K}x^p_{ijk}-\phi^{-1}_{\xi_{a^p_{R_i}}}(1-\beta_{a^p_{R_i}}) & \leq & 0 & \forall\, j,p \\
\sum_{i=1}^{m}\sum_{k=1}^{K}x^p_{ijk}-\phi^{-1}_{\xi_{b^p_{L_j}}}(\beta_{b^p_{L_j}}) & \geq & 0 & \forall\, i,p \\
\sum_{i=1}^{m}\sum_{k=1}^{K}x^p_{ijk}-\phi^{-1}_{\xi_{b^p_{R_j}}}(1-\beta_{b^p_{R_j}}) & \leq & 0 & \forall\, i,p \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x^p_{ijk}-\phi^{-1}_{\xi_{e_{L_k}}}(\beta_{e_{L_k}}) & \geq & 0 & \forall\, k \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x^p_{ijk}-\phi^{-1}_{\xi_{e_{R_k}}}(1-\beta_{e_{R_k}}) & \leq & 0 & \forall\, k \\
\multicolumn{4}{l}{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\left\{\frac{\phi_{\xi_{V^p_{L_i}}}{}^{-1}(\rho^1_j)+\phi_{\xi_{C^p_{L_{ijk}}}}{}^{-1}(\rho^1_j)+\phi^{-1}_{\xi_{V^p_{R_i}}}(\rho^1_j)+\phi^{-1}_{\xi_{C^p_{R_{ijk}}}}(\rho^1_j)}{2}x^p_{ijk}\right.} \\
\left.+\frac{\phi^{-1}_{\xi_{F^p_{R_{ijk}}}}(\rho^1_j)+\phi^{-1}_{\xi_{F^p_{L_{ijk}}}}(\rho^1_j)}{2}y^p_{ijk}\right\}-\frac{\phi^{-1}_{\xi_{B_{R_j}}}(1-\rho^1_j)+\phi^{-1}_{\xi_{B_{L_j}}}(1-\rho^1_j)}{2} & \leq & 0 & \forall\, j \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\phi^{-1}_{\xi_{m^p_{L_{ijk}}}}(\rho^2)+\phi^{-1}_{\xi_{m^p_{R_{ijk}}}}(\rho^2)}{2}y^p_{ijk} & & & \\
\qquad -\frac{\phi^{-1}_{\xi_{M_L}}(1-\rho^2)+\phi^{-1}_{\xi_{M_R}}(1-\rho^2)}{2} & \leq & 0 & \\
x^p_{ijk}\geq 0,\qquad y^p_{ijk}=\begin{cases}1 & \text{if } x^p_{ijk}>0\\ 0 & \text{otherwise}\end{cases} & & & \forall\, i,j,k,p
\end{array}\right.
$$

3.7 Methodologies for crisp equivalence

In this section, we discuss three different multi-objective programming resolution techniques:

- Linear weighted method;
- Fuzzy programming method;
- Goal programming method.

We will use these methods to generate compromise solutions of EVM and CCM.

3.7.1 Linear weighted method

We use the linear weighted method to convert an interval multi-objective transportation problem into its equivalent single objective transportation problem (SOTP) by using a weighted sum of the objective functions to reflect the importance of each objective determined by the decision maker (DM).

- For the Expected Value Model, the linear weighted method can be presented as follows

$$
(P_7)\left\{\begin{array}{lll}
\min -\lambda_R^1 E\left[Z_R^1\right] - \lambda_C^1 E\left[Z_C^1\right] + \lambda_R^2 E\left[Z_R^2\right] + \lambda_C^2 E\left[Z_C^2\right] + \lambda_R^3 E\left[Z_R^3\right] + \lambda_C^3 E\left[Z_C^3\right] & & \\
\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p \geq \int_0^1 \phi^{-1}_{\xi_{a^p_{L_i}}}(\beta)\,\mathrm{d}\beta & & \forall\, i,p \\
\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p \leq \int_0^1 \phi^{-1}_{\xi_{a^p_{R_i}}}(\beta)\,\mathrm{d}\beta & & \forall\, i,p \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p \geq \int_0^1 \phi^{-1}_{\xi_{b^p_{L_j}}}(\beta)\,\mathrm{d}\beta & & \forall\, j,p \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p \leq \int_0^1 \phi^{-1}_{\xi_{b^p_{R_j}}}(\beta)\,\mathrm{d}\beta & & \forall\, j,p \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p \geq \int_0^1 \phi^{-1}_{\xi_{e_{L_k}}}(\beta)\,\mathrm{d}\beta & & \forall\, k \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p \leq \int_0^1 \phi^{-1}_{\xi_{e_{R_k}}}(\beta)\,\mathrm{d}\beta & & \forall\, k \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Bigg\{\Bigg(\frac{\int_0^1 \phi^{-1}_{\xi_{V^p_{L_i}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi_{V^p_{R_i}}}(\beta)\,\mathrm{d}\beta}{2} & & \forall\, j \\
+\frac{\int_0^1 \phi^{-1}_{\xi^p_{C_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{C_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2}\Bigg) x_{ijk}^p & & \\
+\frac{\int_0^1 \phi^{-1}_{\xi^p_{F_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{F_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2} y_{ijk}^p\Bigg\} \leq \frac{\int_0^1 \phi^{-1}_{\xi_{B_{R_j}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi_{B_{L_j}}}(\beta)\,\mathrm{d}\beta}{2} & & \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\int_0^1 \phi^{-1}_{\xi^p_{m_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{m_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2} y_{ijk}^p \leq \frac{\xi_{M_L}+\xi_{M_R}}{2} & & \\
\lambda_R^1+\lambda_C^1+\lambda_R^2+\lambda_C^2+\lambda_R^3+\lambda_C^3 = 1 & & \\
\lambda_C^1, \lambda_R^2, \lambda_C^2, \lambda_R^3, \lambda_C^3 \in [0,1]. & & \\
x_{ijk}^p \geq 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0 \\ 0 & \text{otherwise} \end{cases} & & \forall\, i,j,k,p
\end{array}\right.
$$

- For the Chance Constrained Model, the linear weighted method can be presented as follows:

$$
P_8 \left\{
\begin{array}{lll}
\min -\lambda^1\overline{Z^1} + \lambda^2\overline{Z^2} + \lambda^3\overline{Z^3} & & \\
\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} - \phi^{-1}_{\xi_{a^p_{L_i}}}(\beta_{a^p_{L_i}}) & \geq 0 & \forall\, j,p \\
\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} - \phi^{-1}_{\xi_{a^p_{R_i}}}(1-\beta_{a^p_{R_i}}) & \leq 0 & \forall\, j,p \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \left\{ \left(\phi^{-1}_{\xi_{S^p_j}}(\alpha_1) - \phi^{-1}_{\xi_{V^p_i}}(\alpha_1) - \phi^{-1}_{\xi_{C^p_{ijk}}}(\alpha_1) \right) x^p_{ijk} - \phi^{-1}_{\xi_{F^p_{ijk}}}(\alpha_1) y^p_{ijk} \right\} \geq \overline{Z^1}^U & & \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \phi^{-1}_{\xi_{T^p_{ijk}}}(\alpha_2) y^p_{ijk} & \leq \overline{Z^2}^L & \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \phi^{-1}_{\xi_{D^p_{ijk}}}(\alpha_3) y^p_{ijk} & \leq \overline{Z^3}^L & \\
\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} - \phi^{-1}_{\xi_{a^p_{L_i}}}(\beta_{a^p_{L_i}}) & \geq 0 & \forall\, j,p \\
\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} - \phi^{-1}_{\xi_{a^p_{R_i}}}(1-\beta_{a^p_{R_i}}) & \leq 0 & \forall\, j,p \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} - \phi^{-1}_{\xi_{b^p_{L_j}}}(\beta_{b^p_{L_j}}) & \geq 0 & \forall\, i,p \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} - \phi^{-1}_{\xi_{b^p_{R_j}}}(1-\beta_{b^p_{R_j}}) & \leq 0 & \forall\, i,p \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} - \phi^{-1}_{\xi_{e_{L_k}}}(\beta_{e_{L_k}}) & \geq 0 & \forall\, k \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} - \phi^{-1}_{\xi_{e_{R_k}}}(1-\beta_{e_{R_k}}) & \leq 0 & \forall\, k \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \left\{ \frac{\phi_{\xi_{V^p_{L_i}}}{}^{-1}(\rho^1_j) + \phi^{-1}_{\xi_{C^p_{L_{ijk}}}}(\rho^1_j) + \phi^{-1}_{\xi_{V^p_{R_i}}}(\rho^1_j) + \phi^{-1}_{\xi_{C^p_{R_{ijk}}}}(\rho^1_j)}{2} x^p_{ijk} \right. & & \\
\left. + \frac{\phi^{-1}_{\xi_{F^p_{R_{ijk}}}}(\rho^1_j) + \phi^{-1}_{\xi_{F^p_{L_{ijk}}}}(\rho^1_j)}{2} y^p_{ijk} \right\} & & \\
-(\frac{\phi^{-1}_{\xi_{B_{R_j}}}(1-\rho^1_j) + \phi^{-1}_{\xi_{B_{L_j}}}(1-\rho^1_j)}{2} & \leq 0 & \forall\, j \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \frac{\phi^{-1}_{\xi_{m^p_{L_{ijk}}}}(\rho^2) + \phi^{-1}_{\xi_{m^p_{R_{ijk}}}}(\rho^2)}{2} y^p_{ijk} & & \\
-(\frac{\phi^{-1}_{\xi_{M_L}}(1-\rho^2) + \phi^{-1}_{\xi_{M_R}}(1-\rho^2)}{2} & \leq 0 & \\
\lambda^1 + \lambda^2 + \lambda^3 & = 1 & \\
\lambda^1, \lambda^2, \lambda^3 & \in [0,1] & \\
x^p_{ijk} \geq 0, \qquad y^p_{ijk} = \begin{cases} 1 & \text{if } x^p_{ijk} > 0 \\ 0 & \text{otherwise} \end{cases} & & \forall\, i,j,k,p
\end{array}
\right.
$$

Theorem 3.2. *A feasible solution of the crisp equivalent of EVM in* (P_5) *is:*

- *an optimal solution of the compromise model* (P_7) *if it is Pareto optimal to the multi-objective model* (P_5)*.*
- *a Pareto optimal solution of the multi-objective model* (P_5) *if it is an optimal solution of the compromise model* (P_7)*.*

Proof. Let h^* be an optimal solution of the compromise model P_7, which is not Pareto optimal to the multi-objective model P_5. Then there exists a Pareto optimal solution h, which dominates h^* or $h \prec h^*$. It follows that:

$$-\lambda_R^1 E\big[(Z_R^1)^h\big] - \lambda_C^1 E\big[(Z_C^1)^h\big] + \lambda_R^2 E\big[(Z_R^2)^h\big] + \lambda_C^2 E\big[(Z_C^2)^h\big] + \lambda_R^3 E\big[(Z_R^3)^h\big] + \lambda_C^3 E\big[(Z_C^3)^h\big]$$

$$< \quad -\lambda_R^1 E\big[(Z_R^1)^{h_*}\big] - \lambda_C^1 E\big[(Z_C^1)^{h_*}\big] + \lambda_R^2 E\big[(Z_R^2)^{h_*}\big] + \lambda_C^2 E\big[(Z_C^2)^{h_*}\big] + \lambda_R^3 E\big[(Z_R^3)^{h_*}\big] + \lambda_C^3 E\big[(Z_C^3)^{h^*}\big]$$

where $\lambda_R^1 + \lambda_C^1 + \lambda_R^2 + \lambda_C^2 + \lambda_R^3 + \lambda_C^3 = 1$, $\lambda_R^1, \lambda_C^1, \lambda_R^2, \lambda_C^2, \lambda_R^3, \lambda_C^3 \in [0,1]$.
This may imply that h^* is not an optimal solution of model P_7 which contradicts the hypothesis that h^* is the optimal solution of model P_7.
Let h^* be a Pareto optimal solution of the model (P_5), which is not an optimal solution of the model P_7. Then, there exists an optimal solution $h^{'}$ of the model (P_7) such that

$$\begin{aligned}&-\lambda_R^1 E\Big[(Z_R^1)^{h\prime}\Big] - \lambda_C^1 E\Big[(Z_C^1)^{h\prime}\Big] + \lambda_R^2 E\Big[(Z_R^2)^{h\prime}\Big] + \lambda_C^2 E\Big[(Z_C^2)^{h\prime}\Big] + \lambda_R^3 E\Big[(Z_R^3)^{h\prime}\Big] \\ &+\lambda_C^3 E\Big[(Z_C^3)^{h^{'}}\Big] \quad < \quad -\lambda_R^1 E\Big[(Z_R^1)^{h_*}\Big] - \lambda_C^1 E\Big[(Z_C^1)^{h_*}\Big] + \lambda_R^2 E\Big[(Z_R^2)^{h_*}\Big] + \lambda_C^2 E\Big[(Z_C^2)^{h_*}\Big] \\ &\qquad + \lambda_R^3 E\Big[(Z_R^3)^{h_*}\Big] + \lambda_C^3 E\Big[(Z_C^3)^{h^*}\Big]\end{aligned}$$

where $\lambda_R^1 + \lambda_C^1 + \lambda_R^2 + \lambda_C^2 + \lambda_R^3 + \lambda_C^3 = 1$, and $\lambda_R^1, \lambda_C^1, \lambda_R^2, \lambda_C^2, \lambda_R^3, \lambda_C^3 \in [0,1]$.

This implies that $h^{'}$ is a Pareto optimal to model P_5. It contradicts the initial hypothesis that h^* is a Pareto optimal solution of model P_5.

Similar proofs can be done for the multi-objective model P_6 and the compromise model P_8.

□ □

3.7.2 Fuzzy programming method

Zadeh [103] proposed in 1965 the fuzzy set theory and Zimmermann developed in 1978 the fuzzy programming technique to solve multi-objective linear programs.
The fuzzy set theory is considered as an important tool to treat and analyze optimization problems and a powerful mathematical tool for dealing with incomplete and imprecise information.
We use a fuzzy programming method to solve our problem.

- For the Expected Value Model, the fuzzy programming method can be presented as follows:

$$
P_9 \left\{
\begin{array}{lll}
& \max \lambda & \\
\dfrac{E[Z^1] - E[Z^1]^L}{E[Z^1]^U - E[Z^1]^L} & \geq & \lambda \\
\dfrac{E[Z^2]^U - E[Z^2]}{E[Z^2]^U - E[Z^2]^L} & \geq & \lambda \\
\dfrac{E[Z^3]^U - E[Z^3]}{E[Z^3]^U - E[Z^3]^L} & \geq & \lambda \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p & \geq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{a^p_{L_i}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, i,p \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p & \leq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{a^p_{R_i}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, i,p \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p & \geq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{b^p_{L_j}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, j,p \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p & \leq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{b^p_{R_j}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, j,p \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p & \geq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{e_{L_k}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, k \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p & \leq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{e_{R_k}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, k \\
\multicolumn{3}{l}{\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Bigg\{\Bigg(\frac{\int_0^1 \phi^{-1}_{\xi_{V^p_{L_i}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi_{V^p_{R_i}}}(\beta)\,\mathrm{d}\beta}{2} \qquad \forall\, j} \\
\multicolumn{3}{l}{\displaystyle\quad + \frac{\int_0^1 \phi^{-1}_{\xi^p_{C_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{C_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2}\Bigg) x_{ijk}^p} \\
\multicolumn{3}{l}{\displaystyle + \frac{\int_0^1 \phi^{-1}_{\xi^p_{F_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{F_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2} y_{ijk}^p\Bigg\} \leq \frac{\int_0^1 \phi^{-1}_{\xi_{B_{R_j}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi_{B_{L_j}}}(\beta)\,\mathrm{d}\beta}{2}} \\
\multicolumn{3}{l}{\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \frac{\int_0^1 \phi^{-1}_{\xi^p_{m_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta + \int_0^1 \phi^{-1}_{\xi^p_{m_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2} y_{ijk}^p \leq \frac{\xi_{M_L} + \xi_{M_R}}{2}} \\
\multicolumn{3}{l}{x_{ijk}^p \geq 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, i,j,k,p}
\end{array}
\right.
$$

where:

$E[Z^1]^L, E[Z^2]^L, E[Z^3]^L$ are the lower bounds and $E[Z^1]^U, E[Z^2]^U, E[Z^3]^U$ are the upper bounds of the objectives $E[Z^1], E[Z^2], E[Z^3]$.

- For Chance Constrained Model, the fuzzy method can be presented as follows:

$$
(P_{10})\left\{
\begin{array}{lcll}
\max \lambda & & & \\
\dfrac{\overline{Z^1} - \overline{Z^1}^L}{\overline{Z^1}^U - \overline{Z^1}^L} & \geq & \lambda & \\
\dfrac{\overline{Z^2}^U - \overline{Z^2}}{\overline{Z^2}^U - \overline{Z^2}^L} & \geq & \lambda & \\
\dfrac{\overline{Z^3}^U - \overline{Z^3}}{\overline{Z^3}^U - \overline{Z^3}^L} & \geq & \lambda & \\
\multicolumn{4}{l}{\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left\{\left(\phi^{-1}_{\xi_{S^p_j}}(\alpha_1) - \phi^{-1}_{\xi_{V^p_i}}(\alpha_1) - \phi^{-1}_{\xi^p_{C_{ijk}}}(\alpha_1)\right)x^p_{ijk} - \phi^{-1}_{\xi_{F^p_{ijk}}}(\alpha_1)y^p_{ijk}\right\} \geq \overline{Z^1}^U} \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\phi^{-1}_{\xi_{T^p_{ijk}}}(\alpha_2)y^p_{ijk} & \leq & \overline{Z^2}^L & \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\phi^{-1}_{\xi_{D^p_{ijk}}}(\alpha_3)y^p_{ijk} & \leq & \overline{Z^3}^L & \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K}x^p_{ijk} - \phi^{-1}_{\xi_{a^p_{L_i}}}(\beta_{a^p_{L_i}}) & \geq & 0 & \forall\, j,p \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K}x^p_{ijk} - \phi^{-1}_{\xi_{a^p_{R_i}}}(1-\beta_{a^p_{R_i}}) & \leq & 0 & \forall\, j,p \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K}x^p_{ijk} - \phi^{-1}_{\xi_{b^p_{L_j}}}(\beta_{b^p_{L_j}}) & \geq & 0, & \forall\, i,p \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K}x^p_{ijk} - \phi^{-1}_{\xi_{b^p_{R_j}}}(1-\beta_{b^p_{R_j}}) & \leq & 0 & \forall\, i,p \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x^p_{ijk} - \phi^{-1}_{\xi_{e_{L_k}}}(\beta_{e_{L_k}}) & \geq & 0 & \forall\, k \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x^p_{ijk} - \phi^{-1}_{\xi_{e_{R_k}}}(1-\beta_{e_{R_k}}) & \leq & 0 & k \\
\multicolumn{4}{l}{\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\left\{\frac{\phi_{\xi_{V^p_{L_i}}}{}^{-1}(\rho^1_j) + \phi_{\xi_{C^p_{L_{ijk}}}}{}^{-1}(\rho^1_j) + \phi^{-1}_{\xi_{V^p_{R_i}}}(\rho^1_j) + \phi^{-1}_{\xi_{C^p_{R_{ijk}}}}(\rho^1_j)}{2}x^p_{ijk}\right.} \\
\left. + \frac{\phi^{-1}_{\xi_{F^p_{R_{ijk}}}}(\rho^1_j) + \phi^{-1}_{\xi_{F^p_{L_{ijk}}}}(\rho^1_j)}{2}y^p_{ijk}\right\} - \frac{\phi^{-1}_{\xi_{B_{R_j}}}(1-\rho^1_j) + \phi^{-1}_{\xi_{B_{L_j}}}(1-\rho^1_j)}{2} & \leq & 0 & \forall\, j \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\phi^{-1}_{\xi_{m^p_{L_{ijk}}}}(\rho^2) + \phi^{-1}_{\xi_{m^p_{R_{ijk}}}}(\rho^2)}{2}y^p_{ijk} & & & \\
-\frac{\phi^{-1}_{\xi_{M_L}}(1-\rho^2) + \phi^{-1}_{\xi_{M_R}}(1-\rho^2)}{2} & \leq & 0 & \\
\lambda^1 + \lambda^2 + \lambda^3 & = & 1 & \\
\lambda^1, \lambda^2, \lambda^3 & \in & [0,1] & \\
x^p_{ijk} \geq 0, \qquad y^p_{ijk} = \begin{cases} 1 & \text{if } x^p_{ijk} > 0 \\ 0 & \text{otherwise} \end{cases} & & & \forall\, i,j,k,p
\end{array}
\right.
$$

where:
$\overline{Z^1}^L, \overline{Z^2}^L, \overline{Z^3}^L$ are the lower bounds and $\overline{Z^1}^U, \overline{Z^2}^U, \overline{Z^3}^U$ are the upper bounds of the objectives $\overline{Z^1}, \overline{Z^2}, \overline{Z^3}$.

Theorem 3.3. *A feasible solution of the crisp equivalent of EVM in* (P_5) *is:*

- *an optimal solution of the compromise model* (P_9) *if it is Pareto optimal to the multi-objective model* (P_5)*;*
- *a Pareto optimal solution of the multi-objective model* (P_5) *if it is an optimal solution of the compromise model* (P_9)*.*

Proof.
Let h^* be an optimal solution of the compromise model (P_9), which is not Pareto optimal to the multi-objective model (P_5). Then, there exists a Pareto optimal solution h, which dominates h^* or $h \prec h^*$. This implies

$$\frac{E[(Z^1)^h] - E[Z^1]^L}{E[Z^1]^U - E[Z^1]^L} > \frac{E[(Z^1)^{h_*}] - E[Z^1]^L}{E[Z^1]^U - E[Z^1]^L},$$

where

$$Z^1 = \lambda_R^1 Z_R^1 + \lambda_C^1 Z_C^1, \quad \lambda_R^1 + \lambda_C^1 = 1, \quad \lambda_R^1, \lambda_C^1 \in [0,1]$$

and

$$\frac{E[Z^2]^U - E[Z^2]^h}{E[Z^2]^U - E[Z^2]^L} < \frac{E[Z^2]^U - E[(Z^2)^{h_*}]}{E[Z^2]^U - E[Z^2]^L},$$

where

$$Z^2 = \lambda_R^2 Z_R^2 + \lambda_C^2 Z_C^2, \quad \lambda_R^2 + \lambda_C^2 = 1, \quad \lambda_R^2, \lambda_C^2 \in [0,1]$$

and

$$\frac{E[Z^3]^U - E[Z^3]^h}{E[Z^3]^U - E[Z^3]^L} < \frac{E[Z^3]^U - E[(Z^3)^{h_*}]}{E[Z^3]^U - E[Z^3]^L},$$

where

$$Z^3 = \lambda_R^3 Z_R^3 + \lambda_C^3 Z_C^3, \quad \lambda_R^3 + \lambda_C^3 = 1, \quad \lambda_R^3, \lambda_C^3 \in [0,1].$$

Let us denote

$$\mu_1 E[(Z^1)^H] = \begin{cases} 1, & \text{if} \quad E[Z^1]^U \le E[(Z^1)^H], \\ \dfrac{E[(Z^1)^H] - E[Z^1]^L}{E[Z^1]^U - E[Z^1]^L}, & \text{if} \quad E[Z^1]^L \le E[(Z^1)^H] \le E[Z^1]^U, \\ 0, & \text{if} \quad E[(Z^1)^H] \le E[Z^1]^L \end{cases}$$

and

$$\mu_2 E[(Z^2)^H] = \begin{cases} 1, & \text{if} \quad E[(Z^2)^H] \le E[Z^2]^L, \\ \dfrac{E[Z^2]^U - E[(Z^2)^H]}{E[Z^2]^U - E[Z^2]^L}, & \text{if} \quad E[Z^2]^L \le E[(Z^2)^H] \le E[Z^2]^U, \\ 0, & \text{if} \quad E[Z^2]^U \le E[(Z^2)^H] \end{cases}$$

and

$$\mu_3 E[(Z^3)^H] \begin{cases} 1, & \text{if} \quad E[(Z^3)^H] \le E[Z^3]^L, \\ \dfrac{E[Z^3]^U - E[(Z^3)^H]}{E[Z^3]^U - E[Z^3]^L}, & \text{if} \quad E[Z^3]^L \le E[(Z^3)^H] \le E[Z^3]^U, \\ 0, & \text{if} \quad E[Z^3]^U \le E[(Z^3)^H] \end{cases}$$

for $H \in \{h, h^*\}$.

This means that there exists a λ, such that $\lambda > \lambda^*$. Therefore, it follows that h^* is not an optimal solution of the model (P_9) and this is a contradiction with the initial assumption (that h^* is an optimal solution of model (P_9)).
Let h^* be a Pareto optimal solution of the model (P_5), which is not an optimal solution of the model (P_9). Then, there exists an optimal solution $h^{'}$ of the model P_9 such that $\mu_1(E[(Z^1)^{h^{'}}]) > \mu_1(E[(Z^1)^{h^*}])$ and $\mu_2(E[(Z^2)^{h^{'}}]) < \mu_2(E[(Z^2)^{h^*}])$ and $\mu_3(E[(Z^3)^{h^{'}}]) < \mu_3(E[(Z^3)^{h^*}])$.

Therefore,

$$\frac{E[(Z^1)^{h'}] - E[Z^1]^L}{E[Z^1]^U - E[Z^1]^L} > \frac{E[(Z^1)^{h^*}] - E[Z^1]^L}{E[Z^1]^U - E[Z^1]^L}$$

and

$$\frac{E[Z^2]^U - E[(Z^2)^{h'}]}{E[Z^2]^U - E[Z^2]^L} < \frac{E[Z^2]^U - E[(Z^2)^{h^*}]}{E[Z^2]^U - E[Z^2]^L}$$

and

$$\frac{E[Z^3]^U - E[(Z^3)^{h'}]}{E[Z^3]^U - E[Z^3]^L} < \frac{E[Z^3]^U - E[(Z^3)^{h^*}]}{E[Z^3]^U - E[Z^3]^L}.$$

Hence, h^* is not a Pareto optimal solution of the model (P_5), which contradicts our initial hypothesis (that h^* is a Pareto optimal solution of (P_5)).

Similar proofs can be easily done for the multi-objective model (P_6) and the compromise model (P_9).

□ □

3.7.3 Goal programming method

The initial, precise and detailed development of the concept of goal programming is due to Charnes and Cooper in 1961 [17]. In their paper, they proposed a model and an approach for dealing with multi-objective linear programming problems in which conflicting objectives were included as constraints. Since it might be impossible to satisfy exactly all such goals, one, attempts to minimize the sum of the absolute values of the deviations from such goals.

- For the Expected Value Model, the goal programming method can be presented as follows:

$$
(P_{11})\left\{
\begin{array}{lll}
\min d_{01}+d_{11}+d_{02}+d_{12}+d_{03}+d_{13} & & \\
-E[Z^1]-d01+d11 & = & -E[Z^1]^U \\
E[Z^2])+d02-d12 & = & E[Z^2]^L \\
E[Z^3])+d03-d13 & = & E[Z^3]^L \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p & \geq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{a^p_{L_i}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, i,p \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p & \leq & \int_0^1 \phi^{-1}_{\xi_{a^p_{R_i}}}(\beta)\,\mathrm{d}\beta, \qquad \forall\, i,p \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p & \geq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{b^p_{L_j}}}(\beta)\,\mathrm{d}\beta, \qquad \forall\, j,p \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p & \leq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{b^p_{R_j}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, j,p \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p & \geq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{e_{L_k}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, k \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p & \leq & \displaystyle\int_0^1 \phi^{-1}_{\xi_{e_{R_k}}}(\beta)\,\mathrm{d}\beta \qquad \forall\, k \\
\multicolumn{3}{l}{\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Bigg\{\Bigg(\frac{\int_0^1 \phi^{-1}_{\xi_{V^p_{L_i}}}(\beta)\,\mathrm{d}\beta+\int_0^1 \phi^{-1}_{\xi_{V^p_{R_i}}}(\beta)\,\mathrm{d}\beta}{2} \qquad\qquad \forall\, j} \\
\multicolumn{3}{l}{\displaystyle\quad+\frac{\int_0^1 \phi^{-1}_{\xi^p_{C_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta+\int_0^1 \phi^{-1}_{\xi^p_{C_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2}\Bigg)x_{ijk}^p} \\
\multicolumn{3}{l}{\displaystyle\quad+\frac{\int_0^1 \phi^{-1}_{\xi^p_{F_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta+\int_0^1 \phi^{-1}_{\xi^p_{F_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2}y_{ijk}^p\Bigg\} \leq \frac{\int_0^1 \phi^{-1}_{\xi_{B_{R_j}}}(\beta)\,\mathrm{d}\beta+\int_0^1 \phi^{-1}_{\xi_{B_{L_j}}}(\beta)\,\mathrm{d}\beta}{2}} \\
\multicolumn{3}{l}{\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\int_0^1 \phi^{-1}_{\xi^p_{m_{L_{ijk}}}}(\beta)\,\mathrm{d}\beta+\int_0^1 \phi^{-1}_{\xi^p_{m_{R_{ijk}}}}(\beta)\,\mathrm{d}\beta}{2}y_{ijk}^p \leq \frac{\xi_{M_L}+\xi_{M_R}}{2}} \\
\multicolumn{3}{l}{x_{ijk}^p \geq 0, \qquad y_{ijk}^p=\begin{cases}1 & \text{if } x_{ijk}^p>0\\ 0 & \text{otherwise}\end{cases} \qquad \forall\, i,j,k,p}
\end{array}\right.
$$

where:

$-E[Z^1]^U$, $E[Z^2]^L$, $E[Z^3]^L$ are the lower bounds of the objectives $-E[Z^1]$, $E[Z^2]$, $E[Z^3]$, respectively.

- For the Chance Constrained Model, the goal programming method can be presented as follows:

$$
(P_{12})\left\{
\begin{array}{llll}
\min d_{01}+d_{11}+d_{02}+d_{12}+d_{03}+d_{13} & & & \\
-\overline{Z^1}-d_{01}+d_{11} & = & -\overline{Z^1}^U & \\
\overline{Z^2}-d_{02}+d_{12} & = & \overline{Z^2}^L & \\
\overline{Z^3}^U-d_{03}+d_{13} & = & \overline{Z^3}^L & \\
\multicolumn{4}{l}{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left\{\left(\phi^{-1}_{\xi_{S^p_j}}(\alpha_1)-\phi^{-1}_{\xi_{V^p_i}}(\alpha_1)-\phi^{-1}_{\xi^p_{C_{ijk}}}(\alpha_1)\right)x^p_{ijk}-\phi^{-1}_{\xi_{F^p_{ijk}}}(\alpha_1)y^p_{ijk}\right\}\geq\overline{Z^1}^U} \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\phi^{-1}_{\xi_{T^p_{ijk}}}(\alpha_2)y^p_{ijk} & \leq & \overline{Z^2}^L & \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\phi^{-1}_{\xi_{D^p_{ijk}}}(\alpha_3)y^p_{ijk} & \leq & \overline{Z^3}^L & \\
\sum_{j=1}^{n}\sum_{k=1}^{K}x^p_{ijk}-\phi^{-1}_{\xi_{a^p_{L_i}}}(\beta_{a^p_{L_i}}) & \geq & 0 & \forall\, j,p, \\
\sum_{j=1}^{n}\sum_{k=1}^{K}x^p_{ijk}-\phi^{-1}_{\xi_{a^p_{R_i}}}(1-\beta_{a^p_{R_i}}) & \leq & 0 & \forall\, j,p \\
\sum_{i=1}^{m}\sum_{k=1}^{K}x^p_{ijk}-\phi^{-1}_{\xi_{b^p_{L_j}}}(\beta_{b^p_{L_j}}) & \geq & 0 & \forall\, i,p \\
\sum_{i=1}^{m}\sum_{k=1}^{K}x^p_{ijk}-\phi^{-1}_{\xi_{b^p_{R_j}}}(1-\beta_{b^p_{R_j}}) & \leq & 0 & \forall\, i,p \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x^p_{ijk}-\phi^{-1}_{\xi_{e_{L_k}}}(\beta_{e_{L_k}}) & \geq & 0 & \forall\, k, \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x^p_{ijk}-\phi^{-1}_{\xi_{e_{R_k}}}(1-\beta_{e_{R_k}}) & \leq & 0 & \forall\, k \\
\multicolumn{4}{l}{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\left\{\frac{\phi^{-1}_{\xi_{V^p_{L_i}}}(\rho^1_j)+\phi^{-1}_{\xi_{C^p_{L_{ijk}}}}(\rho^1_j)+\phi^{-1}_{\xi_{V^p_{R_i}}}(\rho^1_j)+\phi^{-1}_{\xi_{C^p_{R_{ijk}}}}(\rho^1_j)}{2}x^p_{ijk}\right.} \\
\multicolumn{4}{l}{\left.+\frac{\phi^{-1}_{\xi_{F^p_{R_{ijk}}}}(\rho^1_j)+\phi^{-1}_{\xi_{F^p_{L_{ijk}}}}(\rho^1_j)}{2}y^p_{ijk}\right\}-\frac{\phi^{-1}_{\xi_{B_{R_j}}}(1-\rho^1_j)+\phi^{-1}_{\xi_{B_{L_j}}}(1-\rho^1_j)}{2}\leq 0 \qquad \forall\, j} \\
\multicolumn{4}{l}{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\phi^{-1}_{\xi_{m^p_{L_{ijk}}}}(\rho^2)+\phi^{-1}_{\xi_{m^p_{R_{ijk}}}}(\rho^2)}{2}y^p_{ijk}-\frac{\phi^{-1}_{\xi_{M_L}}(1-\rho^2)+\phi^{-1}_{\xi_{M_R}}(1-\rho^2)}{2}\leq 0} \\
x^p_{ijk}\geq 0, \qquad y^p_{ijk}=\begin{cases}1 & \text{if } x^p_{ijk}>0\\ 0 & \text{otherwise}\end{cases} & & & \forall\, i,j,k,p
\end{array}
\right.
$$

where:

$-\overline{Z^1}^U, \overline{Z^2}^L, \overline{Z^3}^L$ are the lower bounds of the objectives $-\overline{Z^1}, \overline{Z^2}, \overline{Z^3}$, respectively.

Theorem 3.4. *A feasible solution of the crisp equivalent of EVM in* (P_5) *is:*

- *an optimal solution of the compromise model* (P_{11}) *if it is Pareto optimal to the multi-objective model* (P_5)*;*
- *a Pareto optimal solution of the multi-objective model* (P_5) *if it is an optimal solution of the compromise model* (P_{11})*.*

Proof.
Let h^* is an optimal solution of the compromise model (P_{11}), which is not Pareto optimal to the multi-objective model (P_5). Then, there exists a Pareto optimal solution h, which dominates h^* or $h \prec h^*$.

This implies:

$$-E[(Z^1)^h] - d01 + d11 + E[Z^1]^U < -E[(Z^1)^{h_*}] - d01 + d11 + E[Z^1]^U$$
$$Z^1 = \lambda_R^1 Z_R^1 + \lambda_C^1 Z_C^1, \quad \lambda_R^1 + \lambda_C^1 = 1, \quad \lambda_R^1, \lambda_C^1 \in [0,1]$$

and:

$$E[(Z^2)^h] + d02 - d12 - E[Z^2]^L < E[(Z^2)^{h_*}] + d02 - d12 - E[Z^2]^L$$
$$Z^2 = \lambda_R^2 Z_R^2 + \lambda_C^2 Z_C^2, \quad \lambda_R^2 + \lambda_C^2 = 1, \quad \lambda_R^2, \lambda_C^2 \in [0,1]$$

and:

$$E[(Z^3)^h] + d03 - d13 - E[Z^3]^L < E[(Z^3)^{h_*}] + d03 - d13 - E[Z^3]^L$$
$$Z^3 = \lambda_R^3 Z_R^3 + \lambda_C^3 Z_C^3, \quad \lambda_R^3 + \lambda_C^3 = 1, \quad \lambda_R^3, \lambda_C^3 \in [0,1]$$

This eventually implies that h^* is not an optimal solution of model (P_{11}) which contradicts the choice of h^*.

Let h^* be the Pareto optimal solution of model (P_5) which is not an optimal solution of the model (P_{11}). Then there exists an optimal solution $h^{'}$ of the model P_{11} such that:

$$-E[(Z^1)^{h\prime}] - d01 + d11 + E[Z^1]^U < -E[(Z^1)^{h_*}] - d01 + d11 + E[Z^1]^U,$$
$$Z^1 = \lambda_R^1 Z_R^1 + \lambda_C^1 Z_C^1, \quad \lambda_R^1 + \lambda_C^1 = 1, \quad \lambda_R^1, \lambda_C^1 \in [0,1],$$

and

$$E[(Z^2)^{h\prime}] + d02 - d12 - E[Z^2]^L < E[(Z^2)^{h_*}] + d02 - d12 - E[Z^2]^L,$$
$$Z^2 = \lambda_R^2 Z_R^2 + \lambda_C^2 Z_C^2, \quad \lambda_R^2 + \lambda_C^2 = 1, \quad \lambda_R^2, \lambda_C^2 \in [0,1],$$

and

$$E[(Z^3)^{h\prime}] + d03 - d13 - E[Z^3]^L < E[(Z^3)^{h_*}] + d03 - d13 - E[Z^3]^L,$$
$$Z^3 = \lambda_R^3 Z_R^3 + \lambda_C^3 Z_C^3, \quad \lambda_R^3 + \lambda_C^3 = 1, \quad \lambda_R^3, \lambda_C^3 \in [0,1].$$

This eventually implies that $h^{'}$ is a Pareto optimal to model P_5. This contradicts our initial hypothesis that h^* is Pareto optimal solution of model (P_5).

Similar proofs can be done for the multi-objective model (P_5) and the compromise model (P_{12}). $\square$ $\square$

3.8 Numerical Examples

Let us consider two items $p = 1, 2$, two supplies $i = 1, 2$, three demands $j = 1, 2, 3$ and two kinds of conveyances $k = 1, 2$.

Unit transportation costs, fixed charges, transportation times, deterioration of items, supplies at origins, demands at destinations, conveyance capacities, budget at destinations, selling prices, purchasing cost, the safety factor and desired safety measure are expressed as interval zigzag variables.

We search to maximize the total profit and to minimize both the time and the deterioration of the items.

We use two different soft-computing tools (MATLAB and LINGO-17.0) to solve the models, and we suppose that the total safeties are:

$$[z(600, 687, 776), z(500, 737, 776)]$$

Table 3.2: The supplies $[\xi_{a^1_{L_i}}, \text{ß}_{a^1_{R_i}}]$.

i	1	2
$[\xi_{a^1_{L_i}}, \xi_{a^1_{R_i}}]$	$[z(48, 50, 52), z(82, 95, 120)]$	$[z(39, 40, 41), z(60, 100, 120)]$
$[\xi_{a^2_{L_i}}, \xi_{a^2_{R_i}}]$	$[z(14, 40, 50), z(82, 95, 120)]$	$[z(48, 50, 52), z(78, 80, 82)]$

Table 3.3: The unit purchase costs of items 1 and 2 at two different sources $[\xi_{V^1_{L_i}}, \xi_{V^1_{R_i}}]$.

i	1	2
$[\xi_{V^1_{L_i}}, \xi_{V^1_{R_i}}]$	$[z(10, 13, 16), z(12, 14, 20)]$	$[z(10, 13, 14), z(12, 18, 20)]$
$[\xi_{V^2_{L_i}}, \xi_{V^2_{R_i}}]$	$[z(2, 14, 15), z(8, 19, 20)]$	$[z(8, 12, 15), z(11, 20, 25)]$

Table 3.4: Budget availability at destinations.

B_1	[z(1980,2000,2020),z(1990,2000,2010)]
B_2	[z(1400,1500,1600),z(1300,1500,1700)]
B_3	[z(2000,2500,3000),z(2400,2500,2600)]

Table 3.5: The demands $[\xi_{b^1_{L_j}}, \xi_{b^1_{R_j}}]$

j	1	2	3
$[\xi_{b^1_{L_j}}, \xi_{b^1_{R_j}}]$	$[z(18,20,23), z(20,22,25)]$	$[z(12,13,14), z(20,30,60)]$	$[z(20,50,80), z(40,60,80)]$
$[\xi_{b^2_{L_j}}, \xi_{b^2_{R_j}}]$	$[z(16,40,45), z(25,50,65)]$	$[z(10,15,60), z(35,45,70)]$	$[z(10,15,60), z(20,40,80)]$

Table 3.6: Unit selling prices of items 1 and 2 at three different destinations $[\xi_{S^1_{L_j}}, \xi_{S^1_{R_j}}]$

j	1	2	3
$[\xi_{S^1_{L_j}}, \xi_{S^1_{R_j}}]$	$[z(22,24,28), z(28,29,32)]$	$[z(22,23,24), z(22,32,36)]$	$[z(22,25,28), z(24,26,28)]$
$[\xi_{S^2_{L_j}}, \xi_{S^2_{R_j}}]$	$[z(21,23,24), z(22,24,25)]$	$[z(20,25,30), z(33,35,37)]$	$[z(20,25,30), z(24,26,30)]$

Table 3.7: The capacities of conveyances $[\xi_{e_{L_k}}, \xi_{e_{R_k}}]$

k	1	2
$[\xi_{e_{L_k}}, \xi_{e_{R_k}}]$	$[z(65,68,71), z(98,100,102)]$	$[z(100,110,120), z(150,160,170)]$

Table 3.8: Transportation costs for the items.

$p = 1$	i	j			j		
		1	2	3	1	2	3
	1	[z(8,9,16),z(10,20,25)]	[z(2,12,13),z(20,22,25)]	[z(12,14,26),z(20,24,28)]	[z(5,8,14),z(11,13,24)]	[z(8,20,28),z(8,20,28)]	[z(3,15,16),z(9,12,21)]
	2	[z(1,13,18),z(10,13,23)]	[z(18,19,20),z(19,20,21)]	[z(10,15,18),z(13,15,28)]	[z(8,10,15) ,z(11,13,25)]	[z(8,10,12),z(8,10,18)]	[z(8,15,20),z(16,18,34)]
	k	1			2		
$p = 2$	i	j			j		
		1	2	3	1	2	3
	1	[z(8,12,21),z(10,12,22)]	[z(8,25,27),z(8,25,27)]	[z(10,15,17),z(11,12,23)]	[z(10,11,21),z(13,14,26)]	[z(24,26,28),z(27,29,30)]	[z(8,20,22),z(10,20,25)]
	2	[z(10,11,21),z(13,14,27)]	[z(5,25,28),z(9,25,30)]	[z(2,17,18),z(14,16,30)]	[z(4,14,21), z(16,18,34)]	[z(10,11,21),z(8,16,20)]	[z(1,17,18),z(14,15,29)]
	k	1			2		

Table 3.9: Transportation times for the items.

$p = 1$	i	j			j		
		1	2	3	1	2	3
	1	[z(5,6,10),z(5,7,12)]	[z(4,5,10),z(5,6,11)]	[z(6,7,10),z(9,10,19)]	[z(6,7,10),z(7,8,15)]	[z(5,6,10),z(6,7,13)]	[z(7,8,10),z(7,9,16)]
	2	[z(4,8,14) ,z(8,9,17)]	[z(4,5,10),z(6,7,13)]	[z(6,8,10),z(7,9,16)]	[z (4,6,8),z (6,8,14)]	[z(9,10,12),z(9,11,20)]	[z(8,10,19),z(10,11,21)]
	k	1			2		
$p = 2$	i	j			j		
		1	2	3	1	2	3
	1	[z(4,7,8),z(9,10,11)]	[z(4,5,12),z(6,7,13)]	[z(9,11,15),z(11,12,22)]	[z(4,6,16),z(8,9,17)]	[z(4,6,10)z(5,7,12)]	[z(8,10,15),z(10,11,20)]
	2	[z(7,10,16),z(11,18,19)]	[z(2,7,14),z(6,8,16)]	[z(6,10,13),z(9,11,19)]	[z(6,8,14) z(8,10,18)]	[z(5,8,14),z(7,9,16)]	[z(10,11,20),z(10,12,32)]
k	1				2		

Table 3.10: Fixed charge costs for the items.

$p=1$	i	j			j		
		1	2	3	1	2	3
	1	[z(11,13,14),z(22,25,28)]	[z(11,13,14),z(13,14,16)]	[z(12,13,16),z(22,26,31)]	[z(11,13,14),z(15,17,18)]	[z(8,10,14),z(18,21,23)]	[z(10,11,12),z(11,13,14)]
	2	[z(11,13,14),z(18,22,24)]	[z(11,13,14)z(15,18,19)]	[z(11,13,14),z(20,22,25)]	[z(1,6,10),z(10,11,14)]	[z(11,13,14),z(14,15,20)]	[z(2,9,10),z(10,11,14)]
	k	1			2		
$p=2$	i	j			j		
		1	2	3	1	2	3
	1	[z(11,13,14),(20,23,24)]	[z(11,13,14),z(14,15,17)]	[z(11,13,14),z(24,26,29)]	[z(11,13,14),z(14,17,19)]	[z(11,13,14),z(19,20,24)]	[z(11,13,14),z(12,14,17)]
	2	[z(11,13,14),z(19,22,25)]	[z(11,13,14),(21,24,26)]	[z(11,13,14),(18,20,23)]	[z(6,7,10),z(11,12,14)]	[z(7,10,12),z(12,13,16)]	[z(11,13,14),z(15,18,24)]
	k	1			2		

Table 3.11: Deterioration of the items.

$p=1$	i	j			j		
		1	2	3	1	2	3
	1	[z(7,10,11),z(10,14,20)]	[z(6,8,10),z(7,10,11)]	[z(7,10,14),z(9,11,15)]	[z(7,9,10),z(9,10,11)]	[z(7,10,16),z(10,11,18)]	[z(7,10,16),z(9,10,18)]
	2	[z(7,9,10),z(9,10,12)]	[z(10,12,18),z(12,13,20)]	[z(7,10,18),z(12,13,20)]	[z(6,8,9),z(7,9,10)]	[z(10,12,14),z(12,14,15)]	[z(10,12,13),z(11,14,15)]
	k	1			2		
$p=2$	i	j			j		
		1	2	3	1	2	3
	1	[z(8,9,10),z(13,15,20)]	[z(8,9,10),z(9,14,15)]	[z(8,9,10),z(10,11,16)]	[z(8,9,10),z(11,13,20)]	[z(8,9,10),z(8,10,17)]	[z(8,9,10),z(10,11,19)]
	2	[z(5,10,12),z(7,17,20)]	[z(7,9,10),z(9,10,20)]	[z(2,10,12),z(2,19,20)]	[z(6,10,12) ,z(6,14,16)]	[z(7,8,12),z(9,10,14)]	[z(10,12,18),z(12,14,20)]
	k	1			2		

Table 3.12: Safety factors

$p=1$	i	j			j		
		1	2	3	1	2	3
	1	[z(6,10,20),z(10,11,23)]	[z(5,6,10),z(5,10,13)]	[z(6,10,15),z(6,10,19)]	[z(6,8,10) ,z(6,10,14)]	[z(6,9,20),z(6,10,23)]	[z(7,10,12),z(7,10,20)]
	2	[z(5,8,20),z(5,10,21)]	[z(10,11,13),z(10,12,23)]	[z(7,10,12),z(10,13,22)]	[z(5,7,10) ,z(5,9,12)]	[z(2,8,10),z(10,14,15)]	[z(9,10,14),z(9,14,17)]
	k	1			2		
$p=2$	i	j			j		
		1	2	3	1	2	3
	1	[z(10,15,23),z(12,17,26)]	[z(3,10,15),z(5,15,19)]	[z(6,15,16),z(6,15,18)]	[z(7,13,17),z(7,13,27)]	[z(5,10,15),z(5,10,20)]	[z(6,10,20),z(6,10,24)]
	2	[z(7,10,20),z(7,12,25)]	[z(5,7,20),z(7,8,24)]	[z(8,10,12),z(10,11,20)]	[z(6,10,12),z(6,11,22)]	[z(7,8,10),z(7,10,16)]	[z(10,12,20),z(10,12,24)]
	k	1			2		

Table 3.13: Optimum results of EVM and CCM for uncertain variables using the linear weighted method

Weights			EVM			CCM					
							cl_1			cl_2	
λ_1	λ_2	λ_3	$E[Z_1]$	$E[Z_2]$	$E[Z_3]$	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3
0.4	0.4	0.2	[494.485, 1360.1]	[83.75, 108.5]	[92.5, 123.25]	[500, 1390]	[81.75, 120]	[90, 120]	[320, 1250]	[80, 90]	[90, 100]
0.2	0.4	0.4	[494.485, 1360.1]	[83.75, 108.5]	[92.5, 123.25]	[495, 1385]	[80, 140]	[95, 100]	[320, 1250]	[80, 90]	[90, 100]
0.4	0.2	0.4	[494.485, 1360.1]	[83.75, 108.5]	[92.5, 123.25]	[495, 1385]	[80, 140]	[95, 100]	[310, 1230]	[75, 80]	[90, 95]
0.5	0.25	0.25	[494.485, 1360.1]	[83.75, 108.5]	[92.5, 123.25]	[495, 1385]	[80, 140]	[95, 100]	[320, 1250]	[80, 90]	[90, 100]
0.25	0.5	0.25	[494.485, 1360.1]	[83.75, 108.5]	[92.5, 123.25]	[495, 1385]	[80, 140]	[95, 100]	[320, 1250]	[80, 90]	[90, 100]
0.25	0.25	0.5	[494.485, 1360.1]	[83.75, 108.5]	[92.5, 123.25]	[495, 1385]	[80, 140]	[95, 100]	[320, 1250]	[80, 90]	[90, 100]
0.1	0.1	0.8	[450.9, 1400]	[79.8, 110.8]	[95.3, 120]	[495, 1385]	[80, 140]	[95, 100]	$[32f0, 1250]$	[80, 90]	[90, 100]
0.1	0.8	0.1	[450.9, 1400]	[79.8, 110.8]	[95.3, 120]	[480, 1400]	[79, 108]	[90, 130]	[310, 1230]	[75, 80]	[90, 95]
0.8	0.1	0.1	[450.9, 1400]	[79.8, 110.8]	[95.3, 120]	[480, 1400]	[79, 108]	[90, 130]	[310, 1230]	[75, 80]	[90, 95]

Table 3.15: Results obtained by the Expected Value Model

Transportation plan using fuzzy method	Z^1, Z^2, Z^3
$x_{111}^1 = 22.25, x_{132}^1 = 39.03, x_{132}^1 = 39.03$	[494.485, 1360.1]
$x_{231}^1 = 10.96, x_{222}^1 = 29.03, x_{111}^2 = 27.18$	[83.75, 108.5]
$x_{131}^2 = 8.81, x_{211}^2 = 8.06$	
$x_{222}^2 = 25.75, x_{232}^2 = 16.18$	[92.5, 123.25]
Transportation plan using linear weighted method	Z^1, Z^2, Z^3
$x_{111}^1 = 22.25, x_{132}^1 = 39.03, x_{132}^1 = 39.03$	[494.485, 1360.1]
$x_{231}^1 = 10.96, x_{222}^1 = 29.03, x_{111}^2 = 27.18$	[83.75, 108.5]
$x_{131}^2 = 8.81, x_{211}^2 = 8.06$	
$x_{222}^2 = 25.75, x_{232}^2 = 16.18$	[92.5, 123.25]
Transportation plan using goal programming method	Z^1, Z^2, Z^3
$x_{211}^1 = 21.47, x_{231}^1 = 8.52, x_{122}^1 = 3$	[448.2, 1195.8]
$x_{132}^1 = 47, x_{222}^1 = 10, x_{111}^2 = 35.25$	[81.5, 104.5]
$x_{131}^2 = 2.75, x_{222}^2 = 27.75$	
$x_{232}^2 = 22.25$	[93.25, 126.25]

Table 3.14: Results for CCM for zigzag uncertain variables

Chance levels			Zigzag uncertain variables		
			Objective values of model		
α_1	α_2	α_3	Z_1	Z_2	Z_3
0.1	0.1	0.1	$[524.84, 1390.1]$	$[90, 120]$	$[95, 128.25]$
0.2	0.2	0.2	$[500.9, 1400]$	$[95, 110]$	$[94, 125]$
0.3	0.3	0.3	$[420, 1450]$	$[88, 113]$	$[98, 129]$
0.4	0.4	0.4	$[450, 1420]$	$[92, 115]$	$[91, 128]$
0.5	0.5	0.5	$[415, 1445]$	$[91, 119]$	$[92.5, 126]$
0.6	0.6	0.6	$[410, 1440]$	$[109, 115]$	$[100, 136]$
0.7	0.7	0.7	$[405, 1438]$	$[90, 100]$	$[93, 110]$
0.8	0.8	0.8	$[340, 1290]$	$[75, 87]$	$[90, 95]$
0.9	0.9	0.9	$[300, 1158]$	$[70, 80]$	$[88, 94]$

3.9 Results and discussion

In this section, we present the results we obtained by the two models:

- Expected Value Model (EVM);

Table 3.16: Results obtained by Chance Constrained Model class 1

Transportation plan using fuzzy method	Z^1, Z^2, Z^3
$x^1_{211} = 22.3, x^1_{132} = 39.8, x^1_{222} = 12.9$	[510, 1257]
$x^1_{232} = 6.7, x^2_{111} = 8.35, x^2_{131} = 33.04$	[86, 88.6]
$x^2_{212} = 35.24, x^2_{222} = 14.55$	[91.8, 94.2]
Transportation plan using linear weighted method	Z^1, Z^2, Z^3
$x^1_{221} = 25, x^1_{132} = 36.8, x^1_{222} = 12.9$	[449, 1007]
$x^1_{232} = 6.7, x^2_{111} = 8.35, x^2_{131} = 33.04$	[83, 85]
$x^2_{212} = 35.24, x^2_{222} = 14.55$	[91.8, 94.2]
Transportation plan using goal programming method	Z^1, Z^2, Z^3
$x^1_{211} = 22.3, x^1_{132} = 39.8, x^1_{222} = 12.9$	[510, 1257]
$x^1_{232} = 6.7, x^2_{111} = 8.35, x^2_{131} = 33.04$	[86, 88.6]
$x^2_{212} = 35.24, x^2_{222} = 14.55$	[91.8, 94.2]

Table 3.17: Results obtained by Chance Constrained Model for class 2

Transportation plan using fuzzy method	Z^1, Z^2, Z^3
$x^1_{231} = 26.2, x^1_{112} = 20.6, x^1_{132} = 29.8$	[310, 1230]
$x^1_{222} = 14, x^2_{111} = 38.2, x^2_{131} = 3.8$	[75, 80]
$x^2_{211} = 2.8, x^2_{222} = 24.19, x^2_{232} = 23.4$	[90, 95]
Transportation plan using linear weighted method	Z^1, Z^2, Z^3
$x^1_{231} = 26.2, x^1_{112} = 20.6, x^1_{132} = 29.8$	[310, 1230]
$x^1_{222} = 14, x^2_{111} = 38.2, x^2_{131} = 3.8$	[75, 80]
$x^2_{211} = 2.8, x^2_{222} = 24.19, x^2_{232} = 23.4$	[90, 95]
Transportation plan using goal programming method	Z^1, Z^2, Z^3
$x^1_{231} = 26.2, x^1_{112} = 20.6, x^1_{132} = 29.8$	[310, 1230]
$x^1_{222} = 14, x^2_{111} = 38.2, x^2_{131} = 3.8$	[75, 80]
$x^2_{211} = 2.8, x^2_{222} = 24.19, x^2_{232} = 23.4$	[90, 95]

- Chance Constrained Model (CCM);

of the proposed (MOMIFCSTPBCSM), using three compromise programming methods:

- Linear weighted method;
- Fuzzy programming method;
- Goal programming method.

The numerical examples are shown in Tables 3-13. Here we have used two kinds of soft-computing tools: MATLAB and LINGO-17.0. We present the results for crisp equivalents of both EVM and CCM along with their transportation plans. For CCM, cl_1 represents all the chance levels having the values within the interval $[0, 0.5)$ and cl_2 represents all the chance levels having the values within the interval $(0.5, 1]$.

For cl_1, the values of chance levels are set as:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.4,$$

$$\beta_{a^p_{L_i}} = \beta_{a^p_{R_i}} = \beta_{b^p_{L_j}} = \beta_{b^p_{R_j}} = \beta_{e_{L_k}} = \beta_{e_{R_k}} = \rho^1_j = \rho^2 = 0.45,\ \forall i, j, k, p.$$

For cl_2, the values of chance levels are set as:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.8,$$

$$\beta_{a^p_{L_i}} = \beta_{a^p_{R_i}} = \beta_{b^p_{L_j}} = \beta_{b^p_{R_j}} = \beta_{e_{L_k}} = \beta_{e_{R_k}} = \rho^1_j = \rho^2 = 0.85,\ \forall i, j, k, p.$$

We combined the results obtained for the weighted linear method of EVM and CCM in Table 14 with different weight values λ_1, λ_2 and λ_3.It can be seen that the optimal results of the EVM and CCM for uncertain variables vary with the values of λ_1, λ_2 and λ_3. The results of the EVM and CCM are non-dominated by each other at different values λ_1, λ_2 and λ_3.

The results obtained using the three methods to solve the crispy equivalent models of EVM and CCM are presented in Tables 16 to 18. Here, the values of the weights λ_1, λ_2 and λ_3 for the linear weighted method are considered equal.

In Table 16, we have presented the results obtained by EVM. The solutions generated by the fuzzy method and the linear weighted method are the same and these solutions are non dominated by those of the goal programming method.

Table 17 presents the results obtained by CCM for cl_1. The solutions generated by the fuzzy method and the goal programming are the same and these solutions are non dominated by those of the linear weighted method.

Table 18 presents the results obtained by CCM for cl_2. The solutions generated by the fuzzy method and the goal programming and the linear weighted method are the same.

From Tables 14-18, we observe that the linear weighted method is computationally more efficient than both the fuzzy programming method and the goal programming method, for both EVM and CCM. It generates different solutions according to the decision-maker's preferences.

3.10 Sensitivity analysis

The corresponding results of the CCM are reported in Table 15. The models are solved by using the linear weighted method and we gave the same value to λ_1, λ_2 and λ_3, by changing the values of the

chance levels α_1, α_2 and α_3.
When α_1, α_2 and α_3 are in $[0, 0.5)$ the other parameters are fixed as follows:

$$\beta_{a^p_{L_i}} = \beta_{a^p_{R_i}} = \beta_{b^p_{L_j}} = \beta_{b^p_{R_j}} = \beta_{e_{L_k}} = \beta_{e_{R_k}} = \rho^1_j = \rho^2 = 0.4, \ \forall i, j, k, p.$$

and when α_1, α_2 and α_3 are in $(0.5, 1]$, the other parameters are fixed as follows:

$$\beta_{a^p_{L_i}} = \beta_{a^p_{R_i}} = \beta_{b^p_{L_j}} = \beta_{b^p_{R_j}} = \beta_{e_{L_k}} = \beta_{e_{R_k}} = \rho^1_j = \rho^2 = 0.8, \ \forall i, j, k, p.$$

Table 15 shows that the optimal solutions are different and are not dominated by each other. We also observe that the parameter values have a great influence on the quality of the model solutions. Thus, sensitivity analysis can help the decision-maker to choose and make decisions.

3.11 Conclusion

This study presents an uncertain interval multi-objective multi-item fixed charge solid transportation problem with budget constraint and safety measure. We consider the parameters as interval zigzag uncertain variables. We have used the interval theory to transform MOMIFCSTPBCSM into an uncertain program. Then we have formulated two models EVM and CCM to transform the uncertain program into deterministic equivalents. The equivalent multi-objective deterministic obtained problems are solved by using the linear weighted method, the fuzzy programming method and the goal programming method. The related theorems are proved and numerical examples are illustrated and their results are analyzed.

Chapter 4

A multi-objective multi-item fixed charge solid transportation problem with budget constraint and deterioration of item under the framework of fuzziness theory

4.1 Introduction

In this chapter, we consider a new model consisting of a multi-objective multi-item fixed charge solid transportation problem with budget constraint and deterioration of item under the framework of fuzziness theory, in which we consider the unit transportation costs, fixed charges, transportation times, total deterioration of goods, selling prices, purchasing costs, budget at each destination as trapezoidal and the equality constraints as fuzzy. The problem is formulated in three different models: the expected value, the concept minimum of fuzzy numbers and the nearest interval approximation. These models are solved by new methods which consist of linear weighted fuzzy interactive satisfied global criteria methods combined with a flexible index. To prove the performance of our methods, we made a comparison with similar methods.
This chapter is organized as follows. In section 2, we present our proposed model and give its mathematical formulation. In Section 3, we present the expected value and the interval nearest of the proposed model and we formulate and describe the crisp equivalent for each model. For each model, we give three compromise multi-objective solving methodologies. In section 4, we give numerical examples and discuss the obtained results in section 5. Section 6 contains the main conclusion and some future work.

4.2 The proposed model

In our model, we suppose the following situations:

- ◇ Multiple heterogeneous items are considered for shipment from source i to destination j via kind of conveyance k.

- ◇ Different kinds of items and services that can be purchased by a consumer with his or her profit at their given prices are usually limited (budget limitation at each destination).
- ◇ When a transportation activity is initiated from source i to destination j, there is a fixed charge to be considered in addition to the cost of transportation (which represents income before interest and income taxes).
- ◇ Many objectives are to be optimized under the same restrictions which are of contradictory nature such as minimizing time, minimizing deterioration of items, maximizing profit (we always try to maximize profit by avoiding deterioration of items in a short period of time and it is not interesting to make profit neither after a long period of time nor with deterioration of items).
- ◇ The inequality constraints are considered in a fuzzy environment (in reality, inequality constraints are usually not strict).
- ◇ The parameters are expressed as trapezoidal fuzzy numbers (in financial markets, one usually encounters lack of data, uncertainty about the future, measurement error, etc.).
- ◇ The total availability from all sources is greater than or equal to the total demand from all destinations, for each item.
- ◇ The sum of conveyance capacities is greater than or equal to the total demand for all items.

In our study, we formulated a new model which consists of a fuzzy multi-objective multi-item fixed charge solid transportation problem with budget constraint and deterioration of the items with fuzzy inequality.

In the case of the fixed cost problem, the transportation activity involves two types of costs: a fixed cost that must be incurred to start the activity and a typical variable cost to ship the product from source i to destination j. This means that a fixed charge will be added to the cost of the direct transport, if $x_{ijk}^p > 0$.

In our problem, the fixed charge is included in the model as follows:

$$y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \quad \forall\, i, j, k, p, \\ 0 & \text{otherwise,} \end{cases}$$

The multi-objective multi-item fixed charge solid transportation problem with budget constraint under fuzziness is formulated as follows:

$$
(P_1)\begin{cases}
\max Z^1 = \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\{(\widetilde{S}_j^p - \widetilde{V}_i^p - \widetilde{C}_{ijk}^p)x_{ijk}^p - \widetilde{F}_{ijk}^p y_{ijk}^p\} \\
\min Z^2 = \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\widetilde{T}_{ijk}^p y_{ijk}^p \\
\min Z^3 = \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\widetilde{D}_{ijk}^p y_{ijk}^p \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p \ \tilde{\leq}\ \widetilde{a}_i^p, \quad \forall\, i,p, \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p \ \tilde{\geq}\ \widetilde{b}_j^p, \quad \forall\, j,p, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p \ \tilde{\leq}\ \widetilde{e}_k, \quad \forall\, k, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\{(\widetilde{V}_i^p + \widetilde{C}_{ijk}^p)x_{ijk}^p + \widetilde{F}_{ijk}^p y_{ijk}^p\} \ \tilde{\geq}\ \widetilde{B}_j, \quad \forall\, j, \\
x_{ijk}^p \geq 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \\ 0 & \text{otherwise,} \end{cases} \quad \forall\, i,j,k,p.
\end{cases}
$$

In model (P_1):

- The first objective is to maximize total profit after transporting all the different items from each source to each destination by different conveyances.
- The second objective is to minimize the total transportation time for shipping all the different items from each source to each destination by different conveyances.
- The third objective is to minimize the total deterioration of the items intended for shipment of all the different items from each source to each destination by different conveyances.
- The first constraint means that the total quantity of the item p to be transported from the source i does not exceed the availability $\widetilde{a_i^p}$.
- The second constraint means that the total quantity of item p that has arrived at destination j satisfies at least the demand $\widetilde{b_j^p}$.
- The third constraint means that the quantity of item p transported from source i to destination j by means of conveyance k does not exceed quantity of product $\widetilde{e_k}$ that can be carried by conveyance k.
- The fourth constraint means that the total expenses to destination j, including the purchase price of the item p at source i, the direct cost and the fixed costs for shipping p from source i to destination j via conveyance k does not exceed the budget $\widetilde{B_j}$.

Human languages are usually involved with imperfect or unknown information and are in uncertainty and it is impossible to describe exactly the existing state or a future outcome.

In our model, we assume that unit transportation costs, fixed charges, transportation time, total deterioration of goods, selling price, purchase cost, budget at each destination are expressed in fuzzy terms and all inequality constraints are fuzzy.

The corresponding formulations of Nearest interval approximations and expected value models are discussed in the next section.

4.3 Formulation of crisp equivalent

4.3.1 Nearest interval approximations approach

Using the nearest interval approximation of fuzzy numbers as defined in § 2.2, the (P_1) model will be converted to its equivalent interval model as follows:

$$
(P_2)\left\{
\begin{array}{l}
\max Z^1 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\Big\{\big([S^p_{L_j}, S^p_{R_j}] - [V^p_{L_i}, V^p_{R_i}] - [C^p_{L_{ijk}}, C^p_{R_{ijk}}]\big)x^p_{ijk} \\
\qquad\qquad - [F^p_{L_{ijk}}, F^p_{R_{ijk}}]y^p_{ijk}\Big\} \\
\min Z^2 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}[T^p_{L_{ijk}}, T^p_{R_{ijk}}]y^p_{ijk} \\
\min Z^3 = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}[D^p_{L_{ijk}}, D^p_{R_{ijk}}]y^p_{ijk} \\
\qquad \sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} \ \tilde{\leq}\ [a^p_{L_i}, a^p_{R_i}], \qquad \forall\, i, p, \\
\qquad \sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} \ \tilde{\geq}\ [b^p_{L_j}, b^p_{R_j}], \qquad \forall\, j, p, \\
\qquad \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} \ \tilde{\leq}\ [e^p_{L_k}, e^p_{R_k}], \qquad \forall\, k, \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Big\{\big([V^p_{L_i}, V^p_{R_i}] + [C_{L^p_{ijk}}, C_{R^p_{ijk}}]\big)x^p_{ijk} + [F^p_{L_{ijk}}, F^p_{R_{ijk}}]y^p_{ijk}\Big\} \\
\qquad \tilde{\geq}\ [B_{L_j}, B_{R_j}], \qquad \forall\, j, \\
x^p_{ijk} \geq 0, \qquad y^p_{ijk} = \begin{cases} 1 & \text{if } x^p_{ijk} > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall\, i, j, k, p.
\end{array}
\right.
$$

Crisp transformation of the objective function

Alefed [1] and Moore [74] introduced the concept of interval. The formulation of the original interval objective functions can be expressed as below.

We express the equivalent of the upper bounds Z^1_R, Z^2_R, Z^3_R and the lower bounds Z^1_L, Z^2_L, Z^3_L and the centers Z^1_C, Z^2_C, Z^3_C of the original problem (P_2) as follows:

$$Z^1_L = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left[(S^p_{L_j} - V^p_{R_i} - C^p_{R_{ijk}})x^p_{ijk} - F^p_{R_{ijk}}y^p_{ijk}\right]$$

$$Z^1_R = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left[(S^p_{R_j} - V^p_{L_i} - C^p_{L_{ijk}})x^p_{ijk} - F^p_{L_{ijk}}y^p_{ijk}\right]$$

$$Z^1_C = \frac{Z^1_L + Z^1_R}{2}$$

$$Z^2_L = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} T^p_{L_{ijk}}y^p_{ijk}$$

$$Z^2_R = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} T^p_{R_{ijk}}y^p_{ijk}$$

$$
\begin{aligned}
Z_C^2 &= \frac{Z_L^2 + Z_R^2}{2} \\
Z_L^3 &= \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} D_{L_{ijk}}^{p} y_{ijk}^{p} \\
Z_R^3 &= \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} D_{R_{ijk}}^{p} y_{ijk}^{p} \\
Z_C^3 &= \frac{Z_L^3 + Z_R^3}{2}
\end{aligned}
$$

Crisp transformation of the constraints

By using Hu and Wang's approach [42], we obtain the following crisp equivalent of the constraints:

$$
\begin{array}{lllll}
a_{L_i}^{p} + (1-\alpha) d_{a_{L_i}}^{p} & \leq & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^{p} & \leq & a_{R_i}^{p} + (1-\alpha) d_{a_{R_i}}^{p}, \quad \forall\, i, p, \\
b_{L_j}^{p} + (1-\alpha) d_{b_{L_j}}^{p} & \leq & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^{p} & \leq & b_{R_j}^{p} + (1-\alpha) d_{b_{R_j}}^{p}, \quad \forall\, j, p, \\
e_{L_k} + (1-\alpha) d_{e_{L_k}} & \leq & \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^{p} & \leq & e_{R_k} + (1-\alpha) d_{e_{R_k}}, \quad \forall\, k,
\end{array}
$$

$$
\begin{aligned}
&\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \Big(\frac{V_{L_i}^{p} + C_{L_{ijk}}^{p} + V_{R_i}^{p} + C_{R_{ijk}}^{p}}{2}\Big) x_{ijk}^{p} + \Big(\frac{F_{R_{ijk}}^{p} + F_{L_{ijk}}^{p}}{2}\Big) y_{ijk}^{p} \\
&\qquad\qquad \leq \Big(\tfrac{B_{R_j} + B_{L_j}}{2}\Big) + (1-\alpha) d_{\left(\frac{B_{R_j}+B_{L_j}}{2}\right)}, \quad \forall\, j,
\end{aligned}
$$

where $\alpha \in [0,1]$, and $d = \Big[d_{a_{L_i}}^{p}, d_{a_{R_i}}^{p}, d_{b_{L_j}}^{p}, d_{b_{R_j}}^{p}, d_{e_{L_k}}, d_{e_{R_k}}, d_{\left(\frac{B_{R_j}+B_{L_j}}{2}\right)}\Big] \geq 0.$

Methods to solve Crisp equivalent

In this section, we use three different techniques to solve a multi-objective linear program :

- Interval linear weighted method combined with flexible index;
- Interval fuzzy interactive satisfied method combined with flexible index;
- Interval global criteria method combined with flexible index.

Interval linear weighted method combined with flexible index

We can summarize this method in two steps:

Step 1: Transform an interval problem into a parametric multi-objective problem expressed with center bounds and lower bounds;

Step 2: Use the weighted sum of the objective functions to convert a parametric multi-objective problem to its equivalent parametric single objective optimization problem as described in the following model $(P_3)^{\alpha}$:

$$
(P_3)^\alpha \begin{cases}
\min Z^\alpha = \big\{ -\lambda_L^1 Z_L^1 - \lambda_C^1 Z_C^1 + \lambda_L^2 Z_L^2 + \lambda_C^2 Z_C^2 + \lambda_L^3 Z_L^3 + \lambda_C^3 Z_C^3 \big\} \\
a_{L_i}^p + (1-\alpha) d_{a_{L_i}}^p \le \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk}^p \le a_{R_i}^p + (1-\alpha) d_{a_{R_i}}^p, \quad \forall\, i, p, \\
b_{L_j}^p + (1-\alpha) d_{b_{L_j}}^p \le \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk}^p \le b_{R_j}^p + (1-\alpha) d_{b_{R_j}}^p, \quad \forall\, j, p, \\
e_{L_k} + (1-\alpha) d_{e_{L_k}} \le \sum_{p=1}^{P} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk}^p \le e_{R_k} + (1-\alpha) d_{e_{R_k}}, \quad \forall\, k, \\
\sum_{p=1}^{P} \sum_{i=1}^{m} \sum_{k=1}^{K} \Big(\frac{V_{L_i}^p + C_{L_{ijk}}^p + V_{R_i}^p + C_{R_{ijk}}^p}{2} \Big) x_{ijk}^p + \Big(\frac{F_{R_{ijk}}^p + F_{L_{ijk}}^p}{2} \Big) y_{ijk}^p \\
\qquad \le \Big(\frac{B_{R_j} + B_{L_j}}{2} \Big) + (1-\alpha) d_{\left(\frac{B_{R_j} + B_{L_j}}{2} \right)}, \forall\, j, \\
x_{ijk}^p \ge 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \\ 0 & \text{otherwise}, \end{cases} \quad \forall\, i, j, k, p,
\end{cases}
$$

where $\lambda_L^1, \lambda_C^1, \lambda_L^2, \lambda_C^2, \lambda_L^3, \lambda_C^3 \in [0,1]$, and $\lambda_L^1 + \lambda_C^1 + \lambda_L^2 + \lambda_C^2 + \lambda_L^3 + \lambda_C^3 = 1$.

Interval fuzzy interactive satisfied method combined with flexible index

Fuzzy Interactive Satisfied Method (FISM) was proposed by Sakawa [84] and used by Xu and Zhou [101]. We adapt this method to solve our model and we summarize it as follows:

Step 1: Use weighted the sum to aggregate the center bounds and the lower bounds into single objectives and obtain the expressions of Z^1, Z^2, Z^3 as described in the following (P_4^α) :

$$
(P_4^\alpha) \begin{cases}
\max Z^1 = \lambda_L^1 Z_L^1 + \lambda_C^1 Z_C^1 \\
\min Z^2 = \lambda_L^2 Z_L^2 + \lambda_C^2 Z_C^2 \\
\min Z^1 = \lambda_L^3 Z_L^3 + \lambda_C^3 Z_C^3 \\
a_{L_i}^p + (1-\alpha) d_{a_{L_i}}^p \le \sum_{j=1}^{n} \sum_{k=1}^{K} x_{ijk}^p \le a_{R_i}^p + (1-\alpha) d_{a_{R_i}}^p, \quad \forall\, i, p, \\
b_{L_j}^p + (1-\alpha) d_{b_{L_j}}^p \le \sum_{i=1}^{m} \sum_{k=1}^{K} x_{ijk}^p \le b_{R_j}^p + (1-\alpha) d_{b_{R_j}}^p, \quad \forall\, j, p, \\
e_{L_k} + (1-\alpha) d_{e_{L_k}} \le \sum_{p=1}^{P} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk}^p \le e_{R_k} + (1-\alpha) d_{e_{R_k}}, \quad \forall\, k, \\
\sum_{p=1}^{P} \sum_{i=1}^{m} \sum_{k=1}^{K} \Big(\frac{V_{L_i}^p + C_{L_{ijk}}^p + V_{R_i}^p + C_{R_{ijk}}^p}{2} \Big) x_{ijk}^p + \Big(\frac{F_{R_{ijk}}^p + F_{L_{ijk}}^p}{2} \Big) y_{ijk}^p \\
\qquad \le \Big(\frac{B_{R_j} + B_{L_j}}{2} \Big) + (1-\alpha) d_{\left(\frac{B_{R_j} + B_{L_j}}{2} \right)}, \quad \forall\, j, \\
x_{ijk}^p \ge 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \\ 0 & \text{otherwise}, \end{cases} \quad \forall\, i, j, k, p,
\end{cases}
$$

where $\lambda_L^1 + \lambda_C^1 = 1$, and $\lambda_L^2 + \lambda_C^2 = 1$, and $\lambda_L^3 + \lambda_C^3 = 1$.

Step 2: Formulate the Interval fuzzy interactive satisfied method as (P_5^α) which follows:

$$
(P_5^\alpha)\begin{cases}
\min \mu \\
\dfrac{Z_1 - \underline{Z_1}}{\overline{Z_1} - \underline{Z_1}} \geq (\mu - \epsilon) \\
\dfrac{\overline{Z_2} - Z_2}{\overline{Z_2} - \underline{Z_2}} \geq (\mu - \epsilon) \\
\dfrac{\overline{Z_3} - Z_3}{\overline{Z_3} - \underline{Z_3}} \geq (\mu - \epsilon) \\
a^p_{L_i} + (1-\alpha) d^p_{a_{L_i}} \leq \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} \leq a^p_{R_i} + (1-\alpha) d^p_{a_{R_i}}, & \forall\, i, p, \\
b^p_{L_j} + (1-\alpha) d^p_{b_{L_j}} \leq \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} \leq b^p_{R_j} + (1-\alpha) d^p_{b_{R_j}}, & \forall\, j, p, \\
e_{L_k} + (1-\alpha) d_{e_{L_k}} \leq \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} \leq e_{R_k} + (1-\alpha) d_{e_{R_k}}, & \forall\, k, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \Big(\frac{V^p_{L_i} + C^p_{L_{ijk}} + V^p_{R_i} + C^p_{R_{ijk}}}{2}\Big) x^p_{ijk} + \Big(\frac{F^p_{R_{ijk}} + F^p_{L_{ijk}}}{2}\Big) y^p_{ijk} \\
\qquad \leq \Big(\frac{B_{R_j}+B_{L_j}}{2}\Big) + (1-\alpha) d_{\left(\frac{B_{R_j}+B_{L_j}}{2}\right)}, & \forall\, j, \\
x^p_{ijk} \geq 0, \quad y^p_{ijk} = \begin{cases} 1 & \text{if } x^p_{ijk} > 0, \\ 0 & \text{otherwise}, \end{cases} & \forall\, i, j, k, p.
\end{cases}
$$

where $\underline{Z_1}, \underline{Z_2}, \underline{Z_3}$ are the lower bounds of Z_1, Z_2, Z_3 and $\overline{Z_1}, \overline{Z_2}, \overline{Z_3}$ are the upper bounds of Z_1, Z_2, Z_3 and ϵ is the reference value given by the decision maker.

Interval Global criteria method

We can summarize this method as follows:

Step 1: Use weighted sum to aggregate the center bounds and the lower bounds in single objectives and obtain the expressions of Z^1, Z^2, Z^3 as described in the following (P_6^α) :

$$
(P_6^\alpha)\begin{cases}
\max Z^1 = \lambda^1_L Z^1_L + \lambda^1_C Z^1_C \\
\min Z^2 = \lambda^2_L Z^2_L + \lambda^2_C Z^2_C \\
\min Z^1 = \lambda^3_L Z^3_L + \lambda^3_C Z^3_C \\
a^p_{L_i} + (1-\alpha) d^p_{a_{L_i}} \leq \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} \leq a^p_{R_i} + (1-\alpha) d^p_{a_{R_i}}, & \forall\, i, p, \\
b^p_{L_j} + (1-\alpha) d^p_{b_{L_j}} \leq \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} \leq b^p_{R_j} + (1-\alpha) d^p_{b_{R_j}}, & \forall\, j, p, \\
e_{L_k} + (1-\alpha) d_{e_{L_k}} \leq \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} \leq e_{R_k} + (1-\alpha) d_{e_{R_k}}, & \forall\, k, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \Big(\frac{V^p_{L_i} + C^p_{L_{ijk}} + V^p_{R_i} + C^p_{R_{ijk}}}{2}\Big) x^p_{ijk} + \Big(\frac{F^p_{R_{ijk}} + F^p_{L_{ijk}}}{2}\Big) y^p_{ijk} \\
\qquad \leq \Big(\frac{B_{R_j}+B_{L_j}}{2}\Big) + (1-\alpha) d_{\left(\frac{B_{R_j}+B_{L_j}}{2}\right)}, & \forall\, j, \\
x^p_{ijk} \geq 0, \qquad y^p_{ijk} = \begin{cases} 1 & \text{if } x^p_{ijk} > 0, \\ 0 & \text{otherwise}, \end{cases} & \forall\, i, j, k, p.
\end{cases}
$$

where $\lambda^1_L + \lambda^1_C = 1$, and $\lambda^2_L + \lambda^2_C = 1$, and $\lambda^3_L + \lambda^3_C = 1$.

Step 2: Formulate the Interval global criteria method as (P_7^α) that follows:

$$
(P_7^\alpha)\left\{\begin{array}{lllll}
\multicolumn{5}{c}{\min \Big(\big(\frac{Z_1-\overline{Z}_1}{\overline{Z}_1}\big)^q + \big(\frac{Z_2-\underline{Z}_2}{\underline{Z}_2}\big)^q + \big(\frac{Z_3-\underline{Z}_3}{\underline{Z}_3}\big)^q \Big)^{(\frac{1}{q})}} \\
a_{L_i}^p + (1-\alpha) d_{a_{L_i}}^p & \leq & \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p & \leq & a_{R_i}^p + (1-\alpha) d_{a_{R_i}}^p, \quad \forall\, i, p, \\
b_{L_j}^p + (1-\alpha) d_{b_{L_j}}^p & \leq & \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p & \leq & b_{R_j}^p + (1-\alpha) d_{b_{R_j}}^p, \quad \forall\, j, p, \\
e_{L_k} + (1-\alpha) d_{e_{L_k}} & \leq & \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p & \leq & e_{R_k} + (1-\alpha) d_{e_{R_k}}, \quad \forall\, k, \\
\multicolumn{5}{l}{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \big(\frac{V_{L_i}^p + C_{L_{ijk}}^p + V_{R_i}^p + C_{R_{ijk}}^p}{2}\big) x_{ijk}^p + \big(\frac{F_{R_{ijk}}^p + F_{L_{ijk}}^p}{2}\big) y_{ijk}^p} \\
& \leq & \multicolumn{3}{l}{\big(\frac{B_{R_j}+B_{L_j}}{2}\big) + (1-\alpha) d_{\big(\frac{B_{R_j}+B_{L_j}}{2}\big)}, \quad \forall\, j,} \\
\multicolumn{5}{l}{x_{ijk}^p \geq 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall\, i, j, k, p.}
\end{array}\right.
$$

where $\underline{Z}_2, \underline{Z}_3$ are the lower bounds of Z_2, Z_3 and $\overline{Z}_1$ is the upper bound of Z_1, and $1 \leq q < \infty$ (a usual value of q is 2).

4.3.2 Formulation of the crisp equivalent using Expected value

Using the expected value, the model (P_1) will be converted to the following one denoted by (P_8^α):

$$
(P_8^\alpha)\left\{\begin{array}{lll}
\multicolumn{3}{l}{\max E[Z^1] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \Big\{ \Big(\Big(\big(\frac{S_{j1}^p + S_{j2}^p + S_{j3}^p + S_{j4}^p}{4}\big) - \big(\frac{V_{i1}^p + V_{i2}^p + V_{i3}^p + V_{i4}^p}{4}\big)} \\
\multicolumn{3}{r}{- \big(\frac{C_{ijk1}^p + C_{ijk2}^p + C_{ijk3}^p + C_{ijk4}^p}{4}\big)\Big)\Big) x_{ijk}^p - \big(\frac{F_{ijk1}^p + F_{ijk2}^p + F_{ijk3}^p + F_{ijk4}^p}{4}\big) y_{ijk}^p \Big\}} \\
\multicolumn{3}{l}{\min E[Z^2] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \Big(\frac{T_{ijk1}^p + T_{ijk2}^p + T_{ijk3}^p + T_{ijk4}^p}{4}\Big) y_{ijk}^p} \\
\multicolumn{3}{l}{\min E[Z^3] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \Big(\frac{D_{ijk1}^p + D_{ijk2}^p + D_{ijk3}^p + D_{ijk4}^p}{4}\Big) y_{ijk}^p} \\
\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p & \tilde{\leq} & \big(\frac{a_{i1}^p + a_{i2}^p + a_{i3}^p + a_{i4}^p}{4}\big), \quad \forall\, i, p, \\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p & \tilde{\geq} & \big(\frac{b_{j1}^p + b_{j2}^p + b_{j3}^p + b_{j4}^p}{4}\big), \quad \forall\, j, p, \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p & \tilde{\leq} & \big(\frac{e_{k1} + e_{k2} + e_{k3} + e_{k4}}{4}\big), \quad \forall\, k, \\
\multicolumn{3}{l}{\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \Big\{ \Big(\Big(\big(\frac{V_{i1}^p + V_{i2}^p + V_{i3}^p + V_{i4}^p}{4}\big) + \big(\frac{C_{ijk1}^p + C_{ijk2}^p + C_{ijk3}^p + C_{ijk4}^p}{4}\big)\Big)\Big) x_{ijk}^p} \\
+ \big(\frac{F_{ijk1}^p + F_{ijk2}^p + F_{ijk3}^p + F_{ijk4}^p}{4}\big) y_{ijk}^p \Big\} & \tilde{\leq} & \big(\frac{B_{j1} + B_{j2} + B_{j3} + B_{j4}}{4}\big), \quad \forall\, j, \\
\multicolumn{3}{l}{x_{ijk}^p \geq 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall\, i, j, k, p.}
\end{array}\right.
$$

Methods to solve Crisp equivalent

In this section, we use three different techniques to solve a linear multi-objective program with fuzzy equality:

- Linear weighted method combined with flexible index;
- Fuzzy interactive satisfied method combined with flexible index;
- Global criteria combined with flexible index.

A. Linear weighted method with flexible index

The stepwise of this method can be presented as follows:

Step 1: Convert the multi-objective problem into its equivalent single objective optimization problem by a weighted sum which reflects the importance of each objective given by the decision maker and obtain the following model (P_9):

$$
(P_9)\left\{
\begin{array}{llll}
\multicolumn{4}{c}{\min -\lambda_1 E[Z^1] + \lambda_2 E[Z^2] + \lambda_3 E[Z^3]} \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^{p} & \tilde{\leq} & \left(\dfrac{a_{i1}^{p}+a_{i2}^{p}+a_{i3}^{p}+a_{i4}^{p}}{4}\right), & \forall\, i,p, \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^{p} & \tilde{\geq} & \left(\dfrac{b_{j1}^{p}+b_{j2}^{p}+b_{j3}^{p}+b_{j4}^{p}}{4}\right), & \forall\, j,p, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^{p} & \tilde{\leq} & \left(\dfrac{e_{k1}+e_{k2}+e_{k3}+e_{k4}}{4}\right) & \forall\, k, \\
\multicolumn{4}{l}{\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\left\{\left(\left(\dfrac{V_{i1}^{p}+V_{i2}^{p}+V_{i3}^{p}+V_{i4}^{p}}{4}\right)+\left(\dfrac{C_{ijk1}^{p}+C_{ijk2}^{p}+C_{ijk3}^{p}+C_{ijk4}^{p}}{4}\right)\right)x_{ijk}^{p}\right.} \\
\left. +\left(\dfrac{F_{ijk1}^{p}+F_{ijk2}^{p}+F_{ijk3}^{p}+F_{ijk4}^{p}}{4}\right)y_{ijk}^{p}\right\} & \tilde{\leq} & \left(\dfrac{B_{j1}+B_{j2}+B_{j3}+B_{j4}}{4}\right), & \forall\, j, \\
\multicolumn{3}{c}{x_{ijk}^{p}\geq 0, \quad y_{ijk}^{p}=\begin{cases}1 & \text{if } x_{ijk}^{p}>0,\\ 0 & \text{otherwise,}\end{cases}} & \forall\, i,j,k,p.
\end{array}
\right.
$$

Step 2: Use the flexible index and obtain the model described in (P_{10}^{α}):

$$
(P_{10}^{\alpha})\begin{cases}
\min -\lambda_1 E[Z^1] + \lambda_2 E[Z^2] + \lambda_3 E[Z^3] \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p \leq \left(\frac{a_{i1}^p + a_{i2}^p + a_{i3}^p + a_{i4}^p}{4}\right) + (1-\alpha) d_{\left(\frac{a_{i1}^p + a_{i2}^p + a_{i3}^p + a_{i4}^p}{4}\right)}, & \forall\, i, p, \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p \geq \left(\frac{b_{j1}^p + b_{j2}^p + b_{j3}^p + b_{j4}^p}{4}\right) + (1-\alpha) d_{\left(\frac{b_{j1}^p + b_{j2}^p + b_{j3}^p + b_{j4}^p}{4}\right)} & \forall\, j, p, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p \leq \left(\frac{e_{k1} + e_{k2} + e_{k3} + e_{k4}}{4}\right) + (1-\alpha) d_{\left(\frac{e_{k1} + e_{k2} + e_{k3} + e_{k4}}{4}\right)} & \forall\, k, p, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \left\{ \left(\left(\frac{V_{i1}^p + V_{i2}^p + V_{i3}^p + V_{i4}^p}{4} \right) + \left(\frac{C_{ijk1}^p + C_{ijk2}^p + C_{ijk3}^p + C_{ijk4}^p}{4} \right) \right) x_{ijk}^p \right. \\
\displaystyle\left. + \left(\frac{F_{ijk1}^p + F_{ijk2}^p + F_{ijk3}^p + F_{ijk4}^p}{4} \right) y_{ijk}^p \right\} \leq \left(\frac{B_{j1} + B_{j2} + B_{j3} + B_{j4}}{4} \right) \\
\qquad + (1-\alpha) d_{\left(\frac{B_{j1} + B_{j2} + B_{j3} + B_{j4}}{4}\right)}, & \forall\, j \\
x_{ijk}^p \geq 0, \quad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \\ 0 & \text{otherwise}, \end{cases} & \forall\, i, j, k, p.
\end{cases}
$$

where $\alpha \in [0,1]$

and

$$
d = \left[d_{a_{L_i}}^p, d_{a_{R_i}}^p, d_{b_{L_j}}^p, d_{b_{R_j}}^p, d_{e_{L_k}}, d_{e_{R_k}}, d_{\left(\frac{B_{j1}+B_{j2}+B_{j3}+B_{j4}}{4}\right)} \right] \geq 0
$$

B. Fuzzy interactive satisfied method combined with flexible index

Fuzzy Interactive Satisfied Method (FISM) was proposed by Sakawa [84] and used by Xu and Zhou [101]. We adapt this method to solve our model and we summarize it as follows:

Step 1: Use weighted sum to aggregate the center bounds and the lower bounds in single objectives and obtain the expressions of Z^1, Z^2, Z^3 as described in the following model (P_{11}^{α}):

$$
(P_{11}^{\alpha})\begin{cases}
\max E[Z^1] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\Bigg\{\left(\dfrac{S_{j1}^p+S_{j2}^p+S_{j3}^p+S_{j4}^p}{4} - \dfrac{V_{i1}^p+V_{i2}^p+V_{i3}^p+V_{i4}^p}{4}\right. \\
\qquad \left. -\dfrac{C_{ijk1}^p+C_{ijk2}^p+C_{ijk3}^p+C_{ijk4}^p}{4}\right)x_{ijk}^p - \left(\frac{F_{ijk1}^p+F_{ijk2}^p+F_{ijk3}^p+F_{ijk4}^p}{4}\right)y_{ijk}^p\Bigg\} \\
\min E[Z^2] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left(\dfrac{T_{ijk1}^p+T_{ijk2}^p+T_{ijk3}^p+T_{ijk4}^p}{4}\right)y_{ijk}^p \\
\min E[Z^3] = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\left(\dfrac{D_{ijk1}^p+D_{ijk2}^p+D_{ijk3}^p+D_{ijk4}^p}{4}\right)y_{ijk}^p \\
\sum_{j=1}^{n}\sum_{k=1}^{K}x_{ijk}^p \le \left(\dfrac{a_{i1}^p+a_{i2}^p+a_{i3}^p+a_{i4}^p}{4}\right) + (1-\alpha)d_{\left(\frac{a_{i1}^p+a_{i2}^p+a_{i3}^p+a_{i4}^p}{4}\right)}, \qquad \forall\, i,p, \\
\sum_{i=1}^{m}\sum_{k=1}^{K}x_{ijk}^p \ge \left(\dfrac{b_{j1}^p+b_{j2}^p+b_{j3}^p+b_{j4}^p}{4}\right) + (1-\alpha)d_{\left(\frac{b_{j1}^p+b_{j2}^p+b_{j3}^p+b_{j4}^p}{4}\right)}, \qquad \forall\, j,p, \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk}^p \le \left(\dfrac{e_{k1}+e_{k2}+e_{k3}+e_{k4}}{4}\right) + (1-\alpha)d_{\left(\frac{e_{k1}+e_{k2}+e_{k3}+e_{k4}}{4}\right)}, \quad \forall\, k, \\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K}\Bigg\{\left(\dfrac{V_{i1}^p+V_{i2}^p+V_{i3}^p+V_{i4}^p}{4} + \dfrac{C_{ijk1}^p+C_{ijk2}^p+C_{ijk3}^p+C_{ijk4}^p}{4}\right)x_{ijk}^p \\
+\left(\dfrac{F_{ijk1}^p+F_{ijk2}^p+F_{ijk3}^p+F_{ijk4}^p}{4}\right)y_{ijk}^p\Bigg\} \le \left(\dfrac{B_{j1}+B_{j2}+B_{j3}+B_{j4}}{4}\right) \\
\qquad\qquad +(1-\alpha)d_{\left(\frac{B_{j1}+B_{j2}+B_{j3}+B_{j4}}{4}\right)}, \quad \forall\, j, \\
\qquad x_{ijk}^p \ge 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall\, i,j,k,p.
\end{cases}
$$

where $\lambda_L^1+\lambda_C^1=1$, and $\lambda_L^2+\lambda_C^2=1$, and $\lambda_L^3+\lambda_C^3=1$.

Step 2: Formulate the Fuzzy Program as the following model (P_{12}^{α}):

$$
(P_{12}^{\alpha})\left\{
\begin{array}{ll}
\min \mu & \\
\dfrac{Z_1 - \underline{Z_1}}{\overline{Z}_1 - \underline{Z}_1} \geq (\mu - \epsilon) & \\
\dfrac{\overline{Z}_2 - Z_2}{\overline{Z}_2 - \underline{Z}_2} \geq (\mu - \epsilon) & \\
\dfrac{\overline{Z}_3 - Z_3}{\overline{Z}_3 - \underline{Z}_3} \geq (\mu - \epsilon) & \\
\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^{p} \leq \left(\frac{a_{i1}^{p}+a_{i2}^{p}+a_{i3}^{p}+a_{i4}^{p}}{4}\right) + (1-\alpha) d_{\left(\frac{a_{i1}^{p}+a_{i2}^{p}+a_{i3}^{p}+a_{i4}^{p}}{4}\right)}, & \forall\, i, p, \\
\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^{p} \geq \left(\frac{b_{j1}^{p}+b_{j2}^{p}+b_{j3}^{p}+b_{j4}^{p}}{4}\right) + (1-\alpha) d_{\left(\frac{b_{j1}^{p}+b_{j2}^{p}+b_{j3}^{p}+b_{j4}^{p}}{4}\right)}, & \forall\, j, p, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^{p} \leq \left(\frac{e_{k1}+e_{k2}+e_{k3}+e_{k4}}{4}\right) + (1-\alpha) d_{\left(\frac{e_{k1}+e_{k2}+e_{k3}+e_{k4}}{4}\right)}, & \forall\, k, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \left\{ \left(\frac{V_{i1}^{p}+V_{i2}^{p}+V_{i3}^{p}+V_{i4}^{p}}{4} + \frac{C_{ijk1}^{p}+C_{ijk2}^{p}+C_{ijk3}^{p}+C_{ijk4}^{p}}{4} \right) x_{ijk}^{p} \right. & \\
\left. + \left(\dfrac{F_{ijk1}^{p}+F_{ijk2}^{p}+F_{ijk3}^{p}+F_{ijk4}^{p}}{4} \right) y_{ijk}^{p} \right\} & \\
\qquad \leq \left(\dfrac{B_{j1}+B_{j2}+B_{j3}+B_{j4}}{4}\right) + (1-\alpha) d_{\left(\frac{B_{j1}+B_{j2}+B_{j3}+B_{j4}}{4}\right)}, & \forall\, j, \\
x_{ijk}^{p} \geq 0, \qquad y_{ijk}^{p} = \left\{ \begin{array}{ll} 1 & \text{if } x_{ijk}^{p} > 0, \\ 0 & \text{otherwise,} \end{array} \right. & \forall\, i, j, k, p.
\end{array}
\right.
$$

where $\underline{Z}_1, \underline{Z}_2, \underline{Z}_3$ respectively are the lower bounds of Z_1, Z_2, Z_3 respectively, and $\overline{Z}_1, \overline{Z}_2, \overline{Z}_3$ respectively are the upper bounds of Z_1, Z_2, Z_3 respectively, and ϵ is the reference value given by the decision maker.

C. Global criteria method

We can summarize this method as follows:

Step 1: Use weighted sum to aggregate the center bounds and the lower bounds in single objective and obtain the expressions of Z^1, Z^2, Z^3 as it is described in the following model (P_{13}^{α}):

$$
(P_{13}^{\alpha})\begin{cases}
\max Z^1 = \lambda_L^1 Z_L^1 + \lambda_C^1 Z_C^1\\
\min Z^2 = \lambda_L^2 Z_L^2 + \lambda_C^2 Z_C^2\\
\min Z^3 = \lambda_L^3 Z_L^3 + \lambda_C^3 Z_C^3\\
\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p \leq \left(\dfrac{a_{i1}^p + a_{i2}^p + a_{i3}^p + a_{i4}^p}{4}\right) + (1-\alpha) d_{\left(\frac{a_{i1}^p + a_{i2}^p + a_{i3}^p + a_{i4}^p}{4}\right)}, & \forall\, i, p,\\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p \geq \left(\dfrac{b_{j1}^p + b_{j2}^p + b_{j3}^p + b_{j4}^p}{4}\right) + (1-\alpha) d_{\left(\frac{b_{j1}^p + b_{j2}^p + b_{j3}^p + b_{j4}^p}{4}\right)}, & \forall\, j, p,\\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p \leq \left(\dfrac{e_{k1} + e_{k2} + e_{k3} + e_{k4}}{4}\right) + (1-\alpha) d_{\left(\frac{e_{k1} + e_{k2} + e_{k3} + e_{k4}}{4}\right)}, & \forall\, k,\\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \Bigg\{ \left(\dfrac{V_{i1}^p + V_{i2}^p + V_{i3}^p + V_{i4}^p}{4} + \dfrac{C_{ijk1}^p + C_{ijk2}^p + C_{ijk3}^p + C_{ijk4}^p}{4}\right) x_{ijk}^p\\
\qquad + \left(\frac{F_{ijk1}^p + F_{ijk2}^p + F_{ijk3}^p + F_{ijk4}^p}{4}\right) y_{ijk}^p \Bigg\}\\
\qquad \leq \left(\frac{B_{j1} + B_{j2} + B_{j3} + B_{j4}}{4}\right) + (1-\alpha) d_{\left(\frac{B_{j1} + B_{j2} + B_{j3} + B_{j4}}{4}\right)}, & \forall\, j,\\
x_{ijk}^p \geq 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0,\\ 0 & \text{otherwise,}\end{cases} & \forall\, i, j, k, p.
\end{cases}
$$

Step 2: Formulate global criteria method as (P_{14}^{α}) that follows:

$$
(P_{14}^{\alpha})\begin{cases}
\min \left(\left(\dfrac{Z_1 - \overline{Z_1}}{\overline{Z_1}}\right)^q + \left(\dfrac{Z_2 - \underline{Z_2}}{\underline{Z_2}}\right)^q + \left(\dfrac{Z_3 - \underline{Z_3}}{\underline{Z_3}}\right)^q \right)^{\left(\frac{1}{q}\right)}\\
\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p \leq \left(\dfrac{a_{i1}^p + a_{i2}^p + a_{i3}^p + a_{i4}^p}{4}\right) + (1-\alpha) d_{\left(\frac{a_{i1}^p + a_{i2}^p + a_{i3}^p + a_{i4}^p}{4}\right)}, & \forall\, i, p,\\
\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p \geq \left(\dfrac{b_{j1}^p + b_{j2}^p + b_{j3}^p + b_{j4}^p}{4}\right) + (1-\alpha) d_{\left(\frac{b_{j1}^p + b_{j2}^p + b_{j3}^p + b_{j4}^p}{4}\right)}, & \forall\, j, p,\\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p \leq \left(\dfrac{e_{k1} + e_{k2} + e_{k3} + e_{k4}}{4}\right) + (1-\alpha) d_{\left(\frac{e_{k1} + e_{k2} + e_{k3} + e_{k4}}{4}\right)}, & \forall\, k,\\
\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{k=1}^{K} \Bigg\{ \left(\dfrac{V_{i1}^p + V_{i2}^p + V_{i3}^p + V_{i4}^p}{4} + \dfrac{C_{ijk1}^p + C_{ijk2}^p + C_{ijk3}^p + C_{ijk4}^p}{4}\right) x_{ijk}^p\\
\qquad + \left(\frac{F_{ijk1}^p + F_{ijk2}^p + F_{ijk3}^p + F_{ijk4}^p}{4}\right) y_{ijk}^p \Bigg\}\\
\qquad \leq \left(\frac{B_{j1} + B_{j2} + B_{j3} + B_{j4}}{4}\right) + (1-\alpha) d_{\left(\frac{B_{j1} + B_{j2} + B_{j3} + B_{j4}}{4}\right)}, & \forall\, j,\\
x_{ijk}^p \geq 0, \qquad y_{ijk}^p = \begin{cases} 1 & \text{if } x_{ijk}^p > 0,\\ 0 & \text{otherwise,}\end{cases} & \forall\, i, j, k, p.
\end{cases}
$$

where $\underline{Z}_2, \underline{Z}_3$ respectively are the lower bounds of Z_2, Z_3 respectively, and $\overline{Z}_1$ are the upper bounds of Z_1, and $1 \leq q \leq \infty$ (a usual value of q is 2).

4.4 Numerical Examples

We use two different soft-computing tools, namely MATLAB and LINGO-17.0, to solve the models.

We give three different examples:

- The First example is given to illustrate the proposed models;
- The second and third examples are proposed to illustrate the efficiency of our proposed methods.

Because of the space limitation, the data in the second and third examples are omitted. For more details, see Chakraborty et al. [14].

To illustrate the proposed model, let us consider two items $p = 1, 2$, two supplies $i = 1, 2$, three demands $j = 1, 2, 3$ and two kinds of conveyance $k = 1, 2$. The total transportation profit is maximized and the total transportation time is minimized and the deterioration of items is minimized at the same time.

Moreover, we assume that all the parameters are expressed as trapezoidal fuzzy variables and the equality constraints are fuzzy.
$d = [(1, 2, 3, 4), (0, 0, 0, 0), (1, 2, 3, 4), (2, 4, 5, 8), (0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0), (2, 4, 5, 8), (0, 0, 0, 0), (2, 4, 5, 8)]$

Table 4.1: Unit purchase costs of items 1 and 2 at different sources.

	i	1	2
Fuzzy	v_i^1	(1,2,3,4)	(1,2,3,5)
value	v_i^2	(1,2,3,5)	(1,3,7,8)
Interval	v_i^1	[1.5,3.5]	[1.5,4]
value	v_i^2	[1.5,4]	[2,7.5]
Expected	v_i^1	2.5	2.75
Value	v_i^2	2.75	4.75

Table 4.2: Unit selling prices of the items at the destinations.

	j	1	2	3
Fuzzy	s_j^1	(1,2,10,11)	(1,2,6,8)	(4,5,6,7)
value	s_j^2	(2,3,6,7)	(4,5,6,8)	(4,5,6,8)
Interval	s_j^1	[1.5,10.5]	[1.5,2]	[4.5,6.5]
value	s_j^2	[2.5,6.5]	[4.5,7]	[4.5,7]
Expected	s_j^1	6	4.25	5.5
Value	s_j^2	4.5	5.75	5.75

Table 4.3: Budget availability at destinations.

	j	1	2	3
Fuzzy value	B_j	(400,420,825,830)	(390,410,810,815)	(400,490,894,900)
Interval value	B_j	[410,827.5]	[400,812.5]	[445,897]
Expected value	B_j	618.75	606.25	671

Table 4.4: Available amounts of the items.

	i	1	2
Fuzzy value	a_i^1	(19,21,26,28)	(28,32,35,37)
	a_i^2	(32,34,37,39)	(25,28,30,33)
Interval value	a_i^1	[20,27]	[30,36]
	a_i^2	[33,38]	[26.5,31.5]
Expected value	a_i^1	23.5	33
	a_i^2	35.5	29

Table 4.5: Demands for the items.

	j	1	2	3
Fuzzy value	b_j^1	(14,16,19,22)	(17,20,22,25)	(12,15,18,21)
	b_j^2	(20,23,25,28)	(16,18,19,22)	(15,17,19,21)
Interval value	b_j^1	[15,20.5]	[18.5,23.5]	[13.5,19.5]
	b_j^2	[21.5,26.5]	[17,20.5]	[16,20]
Expected value	b_j^1	17.75	21	16.5
	b_j^2	24	18.75	18

Table 4.6: Transportation capacities of the conveyances.

	k	1	2
Fuzzy value	e_k	(50,60,60,70)	(55,55,65,85)
Interval value	e_k	[55,65]	[55,75]
Expected value	e_k	60	65

4.5 Results and discussion

In this section, we present:

- a numerical example to illustrate the proposed model,
- a comparison of our proposed methods with similar methods (there are only Chakraborty et al. [14] who studied the fuzzy inequality constraints),

by using three different compromise methods for each model.

For nearest interval approximation of fuzzy numbers model the corresponding methods are:

- Interval linear weighted method combined with flexible index ;

Table 4.7: Transportation costs for the items.

$p=1$	i	j	j	j	k
		1	2	3	
	1	(5,8,9,11)	(4,6,9,11)	(10,12,14,16)	1
Fuzzy	1	(9,11,13,15)	(6,8,10,12)	(7,9,12,14)	2
value	2	(8,10,13,15)	(6,7,8,9)	(11,13,15,17)	1
	2	(10,11,13,15)	(6,8,10,12)	(14,16,18,20)	2
	1	[6.5,10]	[5,10]	[11,15]	1
Interval	1	[10,14]	[7,11]	[8,13]	2
value	2	[9,14]	[6.5,8.5]	[12,16]	1
	2	[10.5,14]	[7,11]	[15,19]	2
	1	8.25	13	12	1
Expected	1	9	10.5	9.5	2
value	2	11.5	7.5	14	1
	2	12.25	9	17	2
$p=2$	i	j	j	j	k
		1	2	3	
	1	(9,10,12,13)	(5,8,10,12)	(10,11,12,13)	1
Fuzzy	1	(11,13,14,15)	(6,7,9,11)	(8,10,11,13)	2
value	2	(11,13,14,16)	(7,9,12,14)	(12,14,16,18)	1
	2	(14,16,18,20)	(9,11,13,14)	(13,14,15,16)	2
	1	[9.5, 12.5]	[6.5, 11]	[10.5, 12.5]	1
Interval	1	[12,14.5]	[6.5,10]	[9,12]	2
value	2	[12,15]	[8,13]	[13,17]	1
	2	[15,19]	[10,13.5]	[13.5, 15.5]	2
	1	11	8.75	11.5	1
Expected	1	13.75	8.25	10.5	2
value	2	13.5	10.5	15	1
	2	17	11.75	14.5	2

- Interval fuzzy interactive satisfied method combined with flexible index;
- Interval Global criteria method;

and the results are reported in Table 12.

For the expected value model, the corresponding methods are:

- Linear weighted method combined with flexible index;
- Fuzzy interactive satisfied method combined with flexible index;

Table 4.8: Transportation time for the items.

$p = 1$	i	j	j	j	k
		1	2	3	
	1	(4,5,7,8)	(3,5,6,8)	(7,9,10,12)	1
Fuzzy	1	(6,7,8,9)	(4,6,7,9)	(5,7,9,11)	2
value	2	(6,8,9,11)	(5,6,7,8,)	(6,7,9,10)	1
	2	(4,6,8,10)	(7,9,11,13)	(9,10,11,12)	2
	1	[9.5,7.5]	[4,7]	[8,11]	1
Interval	1	[6.5,8.5]	[5,8]	[6,10]	2
value	2	[7,10]	[5.5,7.5]	[6.5,9.5]	1
	2	[5,9]	[8,12]	[9.5,11.5]	2
	1	6	5.5	9.5	1
Expected	1	7.5	6.5	8	2
value	2	8.5	6.5	8	1
	2	7	10	10.5	2
$p = 2$	i	j	j	j	k
		1	2	3	
	1	(5,7,9,10)	(4,6,7,9)	(9,11,12,13)	1
Fuzzy	1	(7,8,9,10)	(4,5,7,8)	(8,10,11,12)	2
value	2	(10,11,13,14)	(6,7,8,9)	(7,9,11,12)	1
	2	(6,8,10,12)	(5,7,9,11)	(9,10,12,14)	2
	1	[6, 9.5]	[5, 8]	[10, 12.5]	1
Interval	1	[7.5,9.5]	[4.5,7.5]	[9,11.5]	2
value	2	[10.5,13.5]	[6.5,8.5]	[8,11.5]	2
	2	[7,11]	[6,10]	[9.5, 13]	2
	1	7.75	6.5	11.25	1
Expected	1	8.5	6	10.25	2
value	2	12	7.5	9.75	1
	2	9	8	11.25	2

- Global criteria method;

and the results are reported in Table 13.

We consider the same weight values for all the objectives.

For the solutions generated by the nearest interval approximation of fuzzy numbers, we obtain the same results by interval linear weighted method combined with flexible index and Interval fuzzy interactive satisfied method combined with flexible index and Interval Global criteria method are the same.

Table 4.9: Fixed charge costs for the items.

$p = 1$	i	j	j	j	k
		1	2	3	
	1	(10,12,25,28)	(3,10,14,16)	(10,12,26,31)	1
Fuzzy	1	(5,10,17,18)	(8,10,21,23)	(1,10,13,14)	2
value	2	(8,10,22,24)	(5,10,18,19)	(9,11,22,25)	1
	2	(4,6,11,14)	(4,10,15,20)	(4,6,11,14)	2
	1	[11,26.5]	[6.5,15]	[11,28.5]	1
Interval	1	[7.5,17.5]	[9,22]	[5.5,13.5]	2
value	2	[9,23]	[7.5,18.5]	[10,23.5]	1
	2	[5,12.5]	[7,17.5]	[5,12.5]	2
	1	18.75	10.75	19.75	1
Expected	1	12.5	15.5	9.5	2
value	2	16	13	16.75	1
	2	8.75	12.25	8.75	2
$p = 2$	i	j	j	j	k
		1	2	3	
	1	(9,11,23,24)	(4,10,15,17)	(4,20,26,29)	1
Fuzzy	1	(4,10,17,19)	(9,10,20,24)	(2,10,14,17)	2
value	2	(9,10,22,25)	(1,20,24,26)	(8,10,20,23)	1
	2	(1,10,12,14)	(2,10,13,16)	(5,10,18,24) I	2
	1	[10, 23.5]	[7, 16]	[12, 27.5]	1
Interval	1	[7,126]	[9.5,22]	[6,15.5]	2
value	2	[9.5,23.5]	[10.5,25]	[9,21.5]	2
	2	[5.5,13]	[6,14.5]	[7.5, 21]	2
	1	16.75	11.5	19.75	1
Expected	1	12.5	15.75	10.75	2
value	2	16.5	17.75	15.25	1
	2	9.25	10.25	14.25	2

For the solutions generated by expected value model, the results obtained by linear weighted method combined with flexible index and Fuzzy interactive satisfied method combined with flexible index are non dominated to each other but are the same with Global criteria method.

Ranking fuzzy numbers is a key tool in decision making and other fuzzy analysis. Several strategies have been proposed for this task.

Recently, expected value and nearest interval approximation of fuzzy numbers are the most widely used and each has its advantages and disadvantages. For the expected value model, the fuzzy number is replaced by a single number but important information is lost. For the interval

Table 4.10: Deterioration of the items.

$p = 1$	i	j	j	j	k
		1	2	3	
	1	(3,10,11,20)	(2,5,10,11)	(4,6,10,15)	1
Fuzzy	1	(3,6,10,11)	(5,6,10,18)	(2,7,10,18)	2
value	2	(3,6,10,12)	(3,10,12,20)	(2,10,13,20)	1
	2	(2,5,9,10)	(2,10,14,15)	(2,9,14,15)	2
	1	[6.5,15.5]	[3.5,11.5]	[5,12.5]	1
Interval	1	[4.5,10.5]	[5.5,14]	[4.5,14]	2
value	2	[9,23]	[7.5,18.5]	[10,23.5]	1
	2	[5,12.5]	[7,17.5]	[5,12.5]	2
	1	11	7	8.75	1
Expected	1	7.5	9.75	9.25	2
value	2	7.75	11.25	11.25	1
	6.5	10.25	10	2	
$p = 2$	i	j	j	j	k
		1	2	3	
	1	(3,10,15,20)	(4,5,15,15)	(5,6,10,16)	1
Fuzzy	1	(3,10,15,20)	(4,5,15,15)	(5,6,10,16)	2
value	2	(9,10,22,25)	(1,20,24,26)	(8,10,20,23)	1
	2	(1,10,12,14)	(2,10,13,16)	(5,10,18,24)	2
	1	[6.5,17.5]	[4.5,15]	[5.5,13]	1
Interval	1	[5.5,16.5]	[4,13.5]	[5.5,14.5]	2
value	2	[6,16]	[5.5,14]	[5.5,15]	2
	2	[5,13]	[4.5,12]	[7,16]	2
	1	12	9.75	9.25	1
Expected	1	11	8.75	10	2
value	2	16.5	17.75	15.25	1
	2	11	9.75	10.25	2
	2	9	8.25	11.5	2

approximation of fuzzy numbers, we will have one step more than the expected value model, but it can represent the fuzziness. To prove the performance of the method we propose, we compare it with existing methods but the only existing method that yields with fuzzy inequality constraint with a flexible index is the one proposed by Chakraborty et al. [14] with deterministic parameters. Thus, we take all the parameters as deterministic and we omit the fixed charges, transportation times, total deterioration of goods, selling prices, purchase costs, budget at each destination.

Only Chakraborty et al. [14] have studied the multi-objective multi-item solid transportation

Table 4.11: Results obtained for the Nearest interval approximation by different methods combined with flexible index.

Transportation plan by Interval linear weighted method	Z^1, Z^2, Z^3
$x^1_{111} = 9, x^1_{212} = 6, x^2_{211} = 21.5$	[423.75,1638.3]
$x^2_{122} = 20.5, x^1_{221} = 22, x^1_{132} = 11$	[61,89]
$x^2_{132} = 12.5, x^1_{231} = 2.5$	
$x^2_{232} = 5$	[49.5,131.5]
Transportation plan by fuzzy interactive satisfied programming	Z^1, Z^2, Z^3
$x^1_{111} = 9, x^1_{212} = 6, x^2_{211} = 21.5$	[423.75,1638.3]
$x^2_{122} = 20.5, x^1_{221} = 22, x^1_{132} = 11$	[61,89]
$x^2_{132} = 12.5, x^1_{231} = 2.5$	
$x^2_{232} = 5$	[49.5,131.5]
Transportation plan by Interval global criteria	Z^1, Z^2, Z^3
$x^1_{111} = 9, x^1_{212} = 6, x^2_{211} = 21.5$	[423.75,1638.3]
$x^2_{122} = 20.5, x^1_{221} = 22, x^1_{132} = 11$	[61,89]
$x^2_{132} = 12.5, x^1_{231} = 2.5$	
$x^2_{232} = 5$	[49.5,131.5]

Table 4.12: Results obtained for expected value using different methods combined with flexible index.

Transportation plan by Linear weighted method	Z^1, Z^2, Z^3
$x^1_{111} = 12.63, x^2_{111} = 1.11, x^1_{212} = 5.11$	1009.1
$x^2_{122} = 16.38, x^1_{221} = 21, x^2_{211} = 22.88$	81.5
$x^2_{221} = 2.36, x^1_{132} = 10.86$	
$x^2_{132} = 18, x^1_{232} = 5.63$	99.5
Transportation plan by fuzzy interactive satisfied	Z^1, Z^2, Z^3
$x^1_{111} = 7, x^1_{131} = 16.5, x^1_{211} = 10.75$	1255.8
$x^2_{111} = 20.31, x^2_{131} = 5.43, x^1_{222} = 21$	91.5
$x^2_{132} = 9.75, x^2_{222} = 18.75$	
$x^2_{232} = 2.81, x^2_{212} = 3.68$	97.75
Transportation plan by using global criteria method	Z^1, Z^2, Z^3
$x^1_{111} = 7, x^1_{131} = 16.5, x^1_{211} = 10.75$	1255.8
$x^2_{111} = 20.31, x^2_{131} = 5.43, x^1_{222} = 21$	91.5
$x^2_{132} = 9.75, x^2_{222} = 18.75$	
$x^2_{232} = 2.81, x^2_{212} = 3.68$	97.75

Table 4.13: Results obtained by different methods combined with flexible index.

Transportation plan by linear weighted method	Z^1, Z^2
$x^1_{111} = 9, x^1_{131} = 8, x^1_{222} = 7$	550
$x^2_{212} = 7, x^2_{222} = 7, x^2_{231} = 6$	588
Transportation plan by fuzzy interactive satisfied method	**Z^1, Z^2**
$x^1_{111} = 9, x^1_{121} = 7, x^1_{131} = 8$	511
$x^2_{212} = 7, x^2_{222} = 7, x^2_{232} = 6$	745
Transportation plan by by global criteria method	**Z^1, Z^2**
$x^1_{111} = 9, x^1_{131} = 8, x^1_{222} = 7$	548
$x^2_{212} = 7, x^2_{221} = 2, x^2_{222} = 5, x^2_{231} = 6$	590

Table 4.14: Comparaison of results.

Methods	Chakraborty et al.[14]	Our proposed
FISM	(630.98,799.15)	(550,588)
Convex combination	(651,748.5)	(511,745)
Global criteria	(669,711.2)	(548,590)

Table 4.15: Results obtained by different methods combined with flexible index.

Transportation plan by linear weighted method	Z^1, Z^2
$x_{121} = 5, x_{131} = 2, x_{212} = 7$	150,193
Transportation plan by fuzzy interactive satisfied method	**Z^1, Z^2**
$x_{111} = 4.06, x_{121} = 5, x_{131} = 2, x_{211} = 2.93$	148.76,196.91
Transportation plan by global criteria method	**Z^1, Z^2**
$x_{111} = 7, x_{121} = 5, x_{131} = 2$	143,200

Table 4.16: Comparison of results.

Methods	Chakraborty et al.[14]	Our proposed
FISM	(273.4,330.07)	(148.76,196.91)
Convex combination	(252.1,344.2)	(150,193)
Global criteria	(259.06,337.24)	(143,200)

problem with fuzzy inequality constraints. In order to overcome the gaps in their method, we propose a model that meets market requirements in the first place, and we added constraints to improve the requirement market. Then, to prove that our method offers better results, we removed the constraints to make a comparison with the similar method of Chakraborty et al..

- The results obtained for Example 1 with different methods are shown in Table 14.
- The results obtained for Example 2 with different methods are shown in Table 16.

From the comparisons made in Table 15 and Table 17, it is easy to see that the methods we propose have the best solutions for both objectives and that we obtain the solution in fewer steps than Chakraborty et al.

4.6 Conclusion and future works

- For the first time in research, we have designed a multi-objective multi-item fixed charge solid transportation problem with budget constraint under fuzziness and we have introduced a deterioration of item in the objectives and the inequality constraints are considered fuzzy to tackle the real-life transporting system. In this investigation, we introduced the deterioration of item that plays an important role in transportation system, since during the transporting process, the goods/items have a high chance of breaking, being damaged, which has an effect on the profit resulting from solving the multi-objective problem.
- Next, we formulate two different models, namely nearest interval approximation of fuzzy numbers and expected value models, for fuzzy numbers. We also combined three different methods with a flexible index to solve the equivalent multi-objective problem for each model and proved that the nearest interval approximation method is a better tool for dealing with the fuzziness than the expected value model. Indeed, we loose important information whenever each time we replace a fuzzy set with a unique number.
- Linear weighted method combined with flexible index, Fuzzy interactive satisfied method combined with flexible index and Global criteria combined with flexible index methods can be considered as efficient methods and give better solutions than similar methods existing in literature.

The future extensions of our research work can be as follows:

- We have formulated the multi-objective multi-item fixed charge solid transportation problem with budget constraint under fuzziness but this model can be developed in a stochastic environment
- In our model, we took the maximum of central nodes as three, but it is a scope to formulate models with more central nodes.
- There are other parameters to consider in the model, and it is a scope to formulate and solve the model with discount cost and entropy function.

- We can use the soft computing tool to tackle the real-life situation with large scale models such as metaheuristics.
- We can integrate the proposed method with metaheuristic to solve large scale models.
- It is a scope to consider multi-stage multi-objective multi-item fixed charge solid transportation problem with budget constraints and deterioration of items under fuzziness.

Chapter 5

A Multi-Objective Multi-Item Solid Transportation Problem With Interval-Valued Trapezoidal Fuzzy Numbers

5.1 Introduction

This chapter propose a new model that consists of a Multi-Objective Multi-Item Solid Transportation Problem With Interval-Valued Trapezoidal Fuzzy Numbers.

This chapter is organized as follows. Section 2 presents a multi-objective multi-item solid transportation problem in terms of (λ, ρ) interval-valued fuzzy numbers. Section 3 presents the proposed method. Section 4 gives an illustration of the proposed method. In Section 5, we present our conclusion.

5.2 Multi-objective multi-item solid transportation problem in terms of (γ, δ) interval valued fuzzy numbers

In our study, we have formulated a new model which consists of a multi-objective multi-item solid transportation problem in terms of (γ, δ) interval valued fuzzy numbers which is formulated as follows:

$$(P_1)\begin{cases} \min Z^o = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} \tilde{\tilde{C}}_{ijk}^{po} x_{ijk}^{p} & & & \forall\, o \\ \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^{p} & \leq & \tilde{\tilde{A}}_i^{p}, & \forall\, i, p, \\ \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^{p} & \geq & \tilde{\tilde{B}}_j^{p}, & \forall\, j, p, \\ \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^{p} & \leq & \tilde{\tilde{E}}_k, & \forall\, k, \\ x_{ijk}^{p} & \geq & 0, & \forall\, i, j, k, p, \end{cases}$$

5.3 Crisp equivalent of the model

Shunmugapriya and Uthra [90] have recently proposed a new method for ranking the (γ, δ) interval valued fuzzy numbers based on the calculation of the balancing point of the membership function. The authors divide the membership function into three plane figures, namely a triangle, a quadrilateral(kit), and a triangle respectively, and then compute the centroids of these three plane figures for each right and lift limit of the Interval-Valued Trapezoidal. The centroid of these centroids is taken as the reference point that defines the ranking of interval valued fuzzy numbers. We develop here this theory to rank the Interval-Valued Trapezoidal Fuzzy Numbers Multi-Objective Multi-Item Solid Transportation Problem and present it in model (P_2).

$$(P_2)\begin{cases} \min Z^o = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} R(\tilde{\tilde{C}}_{ijk}^{po}) x_{ijk}^{p} & & & \forall\, o \\ \sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^{p} & \leq & R(\tilde{\tilde{A}}_i^{p}), & \forall\, i, p, \\ \sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^{p} & \geq & R(\tilde{\tilde{B}}_j^{p}), & \forall\, j, p, \\ \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^{p} & \leq & R(\tilde{\tilde{E}}_k), & \forall\, k, \\ x_{ijk}^{p} & \geq & 0,\ \forall\, i, j, k, p. & \end{cases}$$

Theorem 5.1. *Let $\tilde{\tilde{C}}_{ijk}^{po}, \forall\ o$ and $\tilde{\tilde{A}}_i^{p}, \tilde{\tilde{B}}_j^{p}, \tilde{\tilde{E}}_k$ be Interval-Valued Trapezoidal Fuzzy Numbers, associated with the ranking formulas: $R(\tilde{\tilde{C}}_{ijk}^{po}), \forall o$, $R(\tilde{\tilde{A}}_i^{p})$, $R(\tilde{\tilde{B}}_j^{p}), R(\tilde{\tilde{E}}_k)$ respectively. Then the crisp equivalent of (P_2) is presented in model (P_2).*

$$
(P_3)\begin{cases}
\min R(Z^o) = \\
\quad \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\gamma S\Big(\widetilde{\mu}_{C^{po}_{L_{ijk}}}\Big)+\delta S\Big(\widetilde{\mu}_{C^{po}_{R_{ijk}}}\Big)}{\gamma+\delta}x^p_{ijk}, \ \forall\, o \\
\quad \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K}x^p_{ijk} \ \le \ \frac{\gamma S\Big(\widetilde{\mu}_{A^p_{L_i}}\Big)+\delta S\big(\widetilde{A}^p_{R_i}\big)}{\gamma+\delta}, \quad \forall\, i,p, \\
\quad \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K}x^p_{ijk} \ \ge \ \frac{\gamma S\Big(\widetilde{\mu}_{B^p_{L_j}}\Big)+\delta S\big(\widetilde{B}^p_{R_j}\big)}{\gamma+\delta}, \quad \forall\, j,p, \\
\displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}x^p_{ijk} \ \le \ \frac{\gamma S\Big(\widetilde{\mu}_{E_{L_k}}\Big)+\delta S\big(\widetilde{E}_{R_k}\big)}{\gamma+\delta}, \quad \forall\, k, \\
\qquad x^p_{ijk} \ \ge \ 0, \qquad \forall\, i,j,k,p,
\end{cases}
$$

Proof. We can transform the objective functions:

$$Z^o = \sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K} R(\widetilde{\widetilde{C}}{}^{po}_{ijk})x^p_{ijk}, \ \forall\, o,$$

into the deterministic objectives:

$$\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{K}\frac{\gamma S(\widetilde{\mu}_{C^{po}_{L_{ijk}}})+\delta S(\widetilde{\mu}_{C^{po}_{R_{ijk}}})}{\gamma+\delta}x^p_{ijk}, \ \forall\, o.$$

Similarly, the constraint:

$$\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} \le R(\widetilde{\widetilde{A}}{}^p_i), \forall\, i,p,$$

can be transformed into the deterministic constraint:

$$\sum_{j=1}^{n}\sum_{k=1}^{K} x^p_{ijk} \le \frac{\gamma S(\mu_{\widetilde{A}^p_{L_i}})+\delta S(\widetilde{A}^p_{R_i})}{\gamma+\delta}, \ \forall\, i,p,$$

and the constraint:

$$\sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} \ge R(\widetilde{\widetilde{B}}{}^p_j), \forall\, j,p,$$

can be transformed into a deterministic constraint:

$$\sum_{i=1}^{m}\sum_{k=1}^{K} x^p_{ijk} \ge \frac{\gamma S(\mu_{\widetilde{B}^p_{L_j}})+\delta S(\widetilde{B}^p_{R_j})}{\gamma+\delta}, \ \forall\, j,p,$$

and the constraint:

$$\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} \le R(\widetilde{\widetilde{E}}_k), \forall\, k,$$

can be transformed into a deterministic constraint:

$$\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x^p_{ijk} \le \frac{\gamma S(\mu_{\widetilde{E}_{L_k}})+\delta S(\widetilde{E}_{R_k})}{\gamma+\delta}, \forall\, k.$$

□ □

5.4 Methodology for crisp equivalence

Here, we propose a new version of fuzzy goal programming for solving a multi-objective transportation problem. In the model formulation of the problem, the goals of each objective and membership function are defined by assigning the highest degree (unity) of a membership function as the aspiration level for the fuzzy goals. Then, introducing the deviational variables to each of them, then minimizing the total deviation. The stepwise of this method can be summarized as follows:

Step 1: Solve the multi-objective transportation problem as a single objective transportation problem, taking each time only one objective as an objective function and ignoring all others.

Step 2: Compute:

$$R(L^o) = \min R(Z^o), \qquad \forall\, o$$
$$R(U^o) = \max R(Z^o), \qquad \forall\, o$$

Step 3: Define the membership function:

$$\begin{cases} 1, & \text{if } R(Z^o) \leq R(L^o), \\ 1 - \dfrac{R(Z^o) - R(L^o)}{R(U^o) - R(L^o)}, & \text{if } R(L^o) \leq R(Z^o(x)) \leq R(U^o), \\ 0, & \text{if } R(Z^o(x)) \geq R(U^o). \end{cases}$$

Step 4 : Develop the proposed model as follows:

$$(P_4)\begin{cases} \min \displaystyle\sum_{o=1}^{O}(t_{o1}+t_{o2}) + \sum_{o=1}^{O} t_o \ , & & & \forall\, o \\ R(Z^o) - t_o & = & R(L^o), & \forall\, o \\ \dfrac{R(U^o)-R(Z^o)}{R(U^o)-R(L^o)} - t_{o1} + t_{o2} & = & 1, & \forall\, o \\ \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{K} x_{ijk}^p & \leq & R(\tilde{\tilde{A}}_i^p), & \forall\, i,p, \\ \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{K} x_{ijk}^p & \geq & R(\tilde{\tilde{B}}_j^p), & \forall\, j,p, \\ \displaystyle\sum_{p=1}^{P}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk}^p & \leq & R(\tilde{\tilde{E}}_k), & \forall\, k, \\ x_{ijk}^p & \geq & 0, & \forall\, i,j,k,p, \end{cases}$$

where

- t_{o1}, t_{o2}, t_o are deviational variables from the membership functions and the goal objectives $\forall\, o$.
- $\mu_o(Z^o(x)$ are the membership functions $\forall\, o$.

- L_o are the ideal objectives $\forall$ o.

Theorem 5.2. *A feasible solution of the crisp equivalent of Model* (P_3) *is:*

(i) an optimal solution of the compromise model (P_4) *if it is Pareto optimal to the multi-objective model* (P_3)*;*

(ii) a Pareto optimal solution of the multi-objective model (P_3) *if it is an optimal solution of the compromise model* (P_4)*.*

Proof.

(i) Let h^* be an optimal solution of the compromise model(P_4) , which is not Pareto optimal to the multi-objective model (P_3). Then, there exists a Pareto optimal solution h, which dominates h^* or $h \prec h^*$. This implies

$$\frac{R(U^o) - R(Z^o)^h}{R(U^o) - R(L^o)} - t_{o1} + t_{o2} - 1 \quad < \quad \frac{R(U^o) - R(Z^o)^{h^*}}{R(U^o) - R(L^o)} - t_{o1} + t_{o2} - 1$$

and

$$R(Z^o)^h - t_o - R(L^o) < R(Z^o)^{h^*} - t_o - R(L^o).$$

It implies that t^* is not the optimal solution of (P_4) which directly contradicts our previous assumption that t^* is the optimal solution of (P_4)

(ii) Let h^* be the Pareto optimal solution of the model (P_3), which is not an optimal solution of the model (P_4). Then, there exists an optimal solution $h^{'}$ of the model (P_4) such that;

$$\frac{R(U^o) - R(Z^o)^{h^{'}}}{R(U^o) - R(L^o)} - t_{o1} + t_{o2} - 1 \quad < \quad \frac{R(U^o) - R(Z^o)^{h^*}}{R(U^o) - R(L^o)} - t_{o1} + t_{o2} - 1$$

and

$$R(Z^o)^{h^{'}} - t_o - R(L^o) < R(Z^o)^{h^*} - t_o - R(L^o)$$

Hence, h^* is not a Pareto optimal solution of the model (P_3), which contradicts our initial hypothesis (that h^* is a Pareto optimal solution of (P_3)).

□ □

5.5 Application examples

There are several methods to find the optimal solution of fuzzy transportation problems. To overcome all the shortcomings mentioned by Ebrahimnejad [31], we present a new method based on a new ranking formula. Let us consider the following numerical examples presented by both Gupta and Kumar [38] and Ebrahimnjad [31].

5.5.1 Example 1

Let us consider the example given by Table 1 below:

Table 5.1: Data of Example 1

Origin	Destination 1	Destination 2	Destination 3	Supply
1	$\langle(10,20,30,40;\frac{2}{3}),$ $(5,15,35,45;1)\rangle$	$\langle(50,60,70,90;\frac{2}{3}),$ $(45,55,75,95;1)\rangle$	$\langle(80,90,110,120;\frac{2}{3}),$ $(75,85,115,125;1)\rangle$	$\langle(70,90,90,100;\frac{2}{3}),$ $(65,85,95,105;1)\rangle$
2	$\langle(60,70,80,90;\frac{2}{3}),$ $(55,65,85,95;1)\rangle$	$\langle(70,80,100,120;\frac{2}{3}),$ $(65,75,105,125;1)\rangle$	$\langle(20,30,50,60;\frac{2}{3}),$ $(15,25,55,65;1)\rangle$	$\langle(40,60,70,80;\frac{2}{3}),$ $(35,55,75,85;1)\rangle$
Dm	$\langle(30,40,50,70;\frac{2}{3}),$ $(25,35,55,75;1)\rangle$	$\langle(20,30,40,50;\frac{2}{3}),$ $(15,25,45,55;1)\rangle$	$\langle(40,50,50,80;\frac{2}{3}),$ $(35,45,55,85;1)\rangle$	

The following results were obtained by applying the Ebrahimnjad's method to the proposed example:

$$Z = \left\langle \begin{array}{c} (1700,3550,5850,8950;\frac{2}{3}), \\ (1325,3350,6400,9450;1) \end{array} \right\rangle$$

Applying our method, we get:

$$Z^* = \left\langle \begin{array}{c} (843.6,1189.1,1669.9,2106.1;\frac{2}{3}), \\ (670.85,1016.4,1842.7,2278.8;1) \end{array} \right\rangle$$

$$\begin{aligned} x_{11} &= 11.95, \\ x_{12} &= 9.07, \\ x_{23} &= 13.53. \end{aligned}$$

By applying the formula of ranking, our method gives a solution better than those found by Ebrahimnjad [31]:

$$R(Z) > R(Z^*), \qquad Z \succ Z^*.$$

5.5.2 Example 2

Let us consider the proposed example by Gupta and Kumar [38] as follows (Tables 2-3):

$$a_1 = \left\langle \begin{array}{c} (6,7,12;0.6), \\ (5,7,15;0.9) \end{array} \right\rangle, a_2 = \left\langle \begin{array}{c} (17,20,21;0.6), \\ (11,20,22;0.9) \end{array} \right\rangle,$$

$$a_3 = \left\langle \begin{array}{c} (15,16,21;0.6), \\ (14,16,24;0.9) \end{array} \right\rangle,$$

$$b_1 = \left\langle \begin{array}{c} (10,11,12;0.6), \\ (9,11,13;0.9) \end{array} \right\rangle, b_2 = \left\langle \begin{array}{c} (1.5,2,4.5;0.6), \\ (1,2,10;0.9) \end{array} \right\rangle,$$

Table 5.2: Data of Example 2: Objective 1.

Origin	Destination			
	1	2	3	4
1	$\langle(0.5,1,1.5;0.6),$ $(0.25,1,1.75;0.9)\rangle$	$\langle(0.5,1,5.5;0.6),$ $(0.25,1,7.75;0.9)\rangle$	$\langle(4,6,12;0.6),$ $(2,6,16;0.9)\rangle$	$\langle(6,7,8;0.6),$ $(5,7,9;0.9)\rangle$
2	$\langle(0.4,0.5,2.6;0.6),$ $(0.25,0.5,3.75;0.9)\rangle$	$\langle(6,8,12;0.6),$ $(5,8,18;0.9)\rangle$	$\langle(2,3,4;0.6),$ $(1,3,5;0.9)\rangle$	$\langle(2,3,4;0.6),$ $(1,3,13;0.9)\rangle$
3	$\langle(5,7,13;0.6),$ $(3,7,17;0.9)\rangle$	$\langle(8.5,9,9.5;0.6),$ $(7,9,11;0.9)\rangle$	$\langle(2,3,4;0.6),$ $(1,3,13;0.9)\rangle$	$\langle(5,6,7;0.6),$ $(3,6,9;0.9)\rangle$

Table 5.3: Data of Example 2: Objective 2.

Origin	Destination			
	1	2	3	4
1	$\langle(2,3,4;0.6)$ $(1,3,13;0.9)\rangle$	$\langle(3,4,5;0.6),$ $(2,4,6;0.9)\rangle$	$\langle(2.5,3,3.5;0.6),$ $(1,3,5;0.9)\rangle$	$\langle(1.5,2,4.5;0.6),$ $(1,2,10;0.9)\rangle$
2	$\langle(3,5,7;0.6),$ $(2,5,8;0.9)\rangle$	$\langle(6,7,8;0.6),$ $(5,7,17;0.9)\rangle$	$\langle(7,10,11;0.6),$ $(1,10,12;0.9)\rangle$	$\langle(9,10,11;0.6),$ $(8,10,12;0.9)\rangle$
3	$\langle(4,5,8;0.6),$ $(3,5,14;0.9)\rangle$	$\langle(0.5,1.5,5;0.6),$ $(0.25,1,7.75;0.9)\rangle$	$\langle(3,5,7;0.6),$ $(2,5,8;0.9)\rangle$	$\langle(0.5,1,1.5;0.6),$ $(0.25,1,1.75;0.9)\rangle$

$$b_3 = \left\langle \begin{array}{c} (13,14,15;0.6), \\ (11,14,17;0.9) \end{array} \right\rangle, b_4 = \left\langle \begin{array}{c} (14,15,20;0.6), \\ (13,15,23;0.9) \end{array} \right\rangle.$$

The following results were obtained by applying the Gupta and Kumar method [38] on this example:

$$Z^1 = \left\langle \begin{array}{c} (116.699, 152.0250, 218.611; 0.6), \\ (66.71, 152.025, 292.23; 0.9) \end{array} \right\rangle,$$

$$Z^2 = \left\langle \begin{array}{c} (134.765, 201.95, 245.555; 0.6), \\ (43.21, 201.95, 308.48; 0.9) \end{array} \right\rangle.$$

Applying our method, we get

$$Z^{1^*} = \left\langle \begin{array}{c} (81.33, 105.46, 152.742; 0.6), \\ (46.81, 105.46, 203.96; 0.9) \end{array} \right\rangle,$$

$$Z^{2^*} = \left\langle \begin{array}{c} (94.09, 141.190, 172.5; 0.6), \\ (31.74, 141.19, 221.16; 0.9) \end{array} \right\rangle.$$

$$\begin{array}{ll} Z^{1^*} \prec Z^1 & R(Z^{1^*}) < R(Z^1), \\ Z^{2^*} \prec Z^2 & R(Z^{2^*}) < R(Z^2). \end{array}$$

By applying the formula of ranking, our method gives results that dominate the results found by Gupta and Kumar [38].

5.5.3 Example 3

A company has two different products to transport from two origins to two destinations using two different conveyances. By taking all parameters as interval-valued trapezoidal fuzzy numbers (Tables 4-7) and applying our method we get the following results:

$$Z^1 = \left\langle \begin{array}{c} (568.67, 1103.3, 1482.2, 2509; \frac{2}{3}), \\ (287.53, 761.34, 1673, 3184.7; 1) \end{array} \right\rangle,$$

$$Z^2 = \left\langle \begin{array}{c} (730.1, 1233.2, 1612.2, 2638.9; \frac{2}{3}), \\ (417.44, 749.41, 1802.9, 3341.7; 1) \end{array} \right\rangle,$$

$$\begin{array}{rcl} x^1_{111} &=& 11.95, \\ x^1_{122} &=& 9.07, \\ x^2_{111} &=& 13.53, \\ x^2_{222} &=& 15.75. \end{array}$$

There is not a method for solving Multi-Objective Multi-Item Solid Transportation Problem With Interval-Value.

Table 5.4: Data of Example 3: Costs C^{11}_{ijk}.

i	j		j	
	1	2	1	2
1	$\langle(10, 20, 30, 40; \frac{2}{3}),$ $(5, 15, 35, 45; 1)\rangle$	$\langle(50, 60, 70, 90; \frac{2}{3}),$ $(45, 55, 75, 95; 1)\rangle$	$\langle(25, 30, 40, 50; \frac{2}{3}),$ $(25, 35, 45, 75; 1)\rangle$	$\langle(15, 25, 30, 60; \frac{2}{3}),$ $(5, 15, 35, 80; 1)\rangle$
2	$\langle(60, 70, 80, 90; \frac{2}{3}),$ $(55, 65, 85, 95; 1)\rangle$	$\langle(60, 80, 90, 100; \frac{2}{3}),$ $(45, 65, 75, 105; 1)\rangle$	$\langle(25, 30, 40, 50; \frac{2}{3}),$ $(25, 35, 45, 75; 1)\rangle$	$\langle(15, 25, 30, 60; \frac{2}{3}),$ $(5, 15, 35, 80; 1)\rangle$
k	1		2	

Table 5.5: Data of Example 3: Costs C^{12}_{ijk}.

i	j		j	
	1	2	1	2
1 1	$\langle(8, 18, 28, 40; \frac{2}{3}),$ $(3, 12, 32, 43; 1)\rangle$	$\langle(45, 55, 65, 85; \frac{2}{3}),$ $(40, 55, 75, 95; 1)\rangle$	$\langle(25, 30, 40, 50; \frac{2}{3}),$ $(20, 35, 45, 75; 1)\rangle$	$\langle(15, 25, 30, 60; \frac{2}{3}),$ $(10, 25, 35, 85; 1)\rangle$
2	$\langle(60, 70, 80, 90; \frac{2}{3}),$ $(55, 65, 85, 95; 1)\rangle$	$\langle(55, 80, 90, 100; \frac{2}{3}),$ $(45, 65, 75, 105; 1)\rangle$	$\langle(25, 30, 40, 50; \frac{2}{3}),$ $(15, 35, 45, 75; 1)\rangle$	$\langle(13, 25, 30, 60; \frac{2}{3}),$ $9, 18, 32, 85; 1)\rangle$
k	1		2	

$$a^1_1 = \left\langle \begin{array}{c} (70, 90, 90, 100; \frac{2}{3}), \\ (65, 85, 95, 105; 1) \end{array} \right\rangle a^1_2 = \left\langle \begin{array}{c} (72, 92, 97, 100; \frac{2}{3}), \\ (64, 85, 92, 105; 1) \end{array} \right\rangle$$

$$a^2_1 = \left\langle \begin{array}{c} (40, 60, 70, 80; \frac{2}{3}), \\ (35, 55, 75, 85; 1), \end{array} \right\rangle a^2_2 = \left\langle \begin{array}{c} (42, 62, 72, 82; \frac{2}{3}), \\ (35, 57, 75, 88; 1) \end{array} \right\rangle,$$

Table 5.6: Data of Example 3: Costs C^{21}_{ijk}.

i	j		j	
	1	2	1	2
1	$\langle(12,22,32,42;\frac{2}{3}),$ $(7,17,37,47;1)\rangle$	$\langle(52,62,72,92;\frac{2}{3}),$ $(47,57,77,97;1)\rangle$	$\langle(27,32,42,52;\frac{2}{3}),$ $(27,37,47,77;1)\rangle$	$\langle(17,27,32,62;\frac{2}{3}),$ $(7,17,37,82;1)\rangle$
2 2	$\langle(62,72,82,92;\frac{2}{3}),$ $(57,67,87,97;1)\rangle$	$\langle(62,82,92,102;\frac{2}{3}),$ $(47,67,77,107;1)\rangle$	$\langle(27,32,42,52;\frac{2}{3}),$ $(27,37,47,77;1)\rangle$	$\langle(17,27,32,62;\frac{2}{3}),$ $(7,17,37,82;1)\rangle$
k	1		2	

Table 5.7: Data of Example 3: Costs C^{22}_{ijk}.

i	j		j	
	1	2	1	2
1	$\langle(11,21,31,43;\frac{2}{3}),$ $(6,15,35,48;1)\rangle$	$\langle(48,58,68,88;\frac{2}{3}),$ $(43,58,78,98;1)\rangle$	$\langle(28,33,43,53;\frac{2}{3}),$ $(23,38,48,78;1)\rangle$	$\langle(18,28,33,63;\frac{2}{3}),$ $(13,28,38,88;1)\rangle$
2 2	$\langle(63,73,83,93;\frac{2}{3}),$ $(58,68,88,98;1)\rangle$	$\langle(58,83,93,103;\frac{2}{3}),$ $(48,68,78,108;1)\rangle$	$\langle(28,33,43,53;\frac{2}{3}),$ $(18,38,48,78;1)\rangle$	$\langle(18,28,33,63;\frac{2}{3}),$ $(12,12,35,88;1)\rangle$
k	1		2	

$$b^1_1 = \left\langle \begin{array}{c} (30,40,50,70;\frac{2}{3}), \\ (25,35,55,75;1) \end{array} \right\rangle b^1_2 = \left\langle \begin{array}{c} (20,30,40,50;\frac{2}{3}), \\ (15,25,45,55;1) \end{array} \right\rangle$$

$$b^2_1 = \left\langle \begin{array}{c} (40,50,50,80;\frac{2}{3}), \\ (35,45,55,85;1) \end{array} \right\rangle b^2_2 = \left\langle \begin{array}{c} (45,52,52,82;\frac{2}{3}), \\ (35,57,75,88;1) \end{array} \right\rangle,$$

$$e_1 = \left\langle \begin{array}{c} (146,149,151,159;\frac{2}{3}), \\ (140,150,160,190;1) \end{array} \right\rangle$$

$$e_2 = \left\langle \begin{array}{c} (145,152,105,182;\frac{2}{3}), \\ (130,159,168,220;1) \end{array} \right\rangle.$$

5.5.4 Results and discussion

- It is obvious from both Example 1 and Example 2 that our method gives better results than the existing method.
- From all Example 1, Example 2 and Example 3, it is easy to see that our method can treat all kinds of transportation problems.
- In Example 3, we considered other not yet studied variants of transportation problems by considering different items and conveyances.

5.6 Conclusion

In our study, we investigated in multi-objective multi-item solid transportation problems in terms of (γ, δ) interval-valued fuzzy numbers in which we considered that the unit transportation costs, supplies at origins, demands at destinations and conveyances capacities are expressed as (γ, δ) interval-valued fuzzy numbers. We have proposed a new method based on a new ranking formula which gives better results than the existing ones.

The future extensions of our investigation can be summarized as follows:

- We will consider another parameters (that are not yet considered) in the model. It is a scope to formulate and solve the model with cost discount and entropy function.
- We can use the soft computing tool to tackle the real-life situation with large scale models such as metaheuristics.
- We can incorporate the proposed method with metaheuristic for solving large scale models.
- There is a scope when considering multi-stage multi-objective multi-item solid transportation problems in terms of (γ, δ) interval-valued fuzzy numbers.

Conclusion

The problem of transport (TP) is one of the most traditional problems of operational research. In its basic version, it refers to the transport of goods from certain sources to certain destinations at the lowest cost. In practice, other elements are involved, usually the type of product and/or the mode of transport. In the real world, transport problems involve not only minimizing costs, but also optimizing many other objectives, such as maximizing profits, minimizing delays, minimizing total environmental degradation. Moreover, the available data of a transport system, such as costs, sources, demands and transport capacities, are not always accurate but are given in an uncertain way.

In this thesis, we study different models of the multi-objective transport problem in an uncertain environment and introduce parameters that meet market requirements and propose new methods that allow to solve practical problems , in order to model uncertain data, especially when sample data are insufficient or non-existent, we use different theories to describe different situation which can arise.

Finally this work opens the way to several areas of research :

- Study metaheuristics and use them to solve this kind of problems,
- Application of the proposed models to real-world with big size.

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