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In memory of my dear Father,
"that God protects him in his paradise",
to my dearest Mother.

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Notations and acronyms

1. ***Acronyms :***
   - LTI : Linear Time Invariant .
   - LPV : Linear Parameter Varying .
   - LMI : Linear Matrix Inequality .
   - BMI : Bilinear matrix inequality .
   - SLF : Switching Lyapunov Function.
   - GAS : Global Asymptotically Stable.
   - CQLF : Commun Quadratic Lyapunov Function.

2. ***Notations :***
   - $\mathbb{R}$ is the set of real numbers ;
   - $\mathbb{R}^n$ is the vector space of dimension n built on the body of the real ;
   - $(\star)$ is used for blocks induced by symmetry in a matrix or in a matrix inequality ;
   - A^T represents the transposed matrix of A ;
   - A^{-T} represents the inverse of matrix A^T ;
   - $\mathbb{R}^{n\times m}$ is the set of all real matrices of n rows and m columns ;
   - I is an identity matrix of appropriate size and I_r represents a matrix of identity of dimension r
   - The value 0 represents a zero matrix of appropriate dimension, and $0_{(n,m)}$ designates null matrix with n rows and m columns ;
   - For a square matrix S, $S > 0$ $(S < 0)$ means that this matrix is positive definite (defined negative) ;
   - $A \leq B$ means that the matrix $B-A$ is semi-definite positive ;
   - $\mathrm{H}e(A)$ represents the sum $A + A^T$.
   - $\|.\|$ is the usual Euclidean norm ;
   - l_2^s is the space of the summable square sequences, i.e.,

$$l_2^s = \{x = (x(k))_k \subset \mathbb{R}^s, \sum_{k=0}^{\infty} \|x(k)\|^2 < +\infty\}$$

     with the norm $\|x\|_{l_2^s} = \left(\sum_{k=0}^{\infty} \|x(k)\|^2\right)^{\frac{1}{2}}$;
   - $\mathrm{diag}(A_1,\ldots,A_i)$ is the block diagonal matrix with $A_1,\ldots,A_i$ on its main diagonal ;
   - $e_s(i) = \underbrace{(0,\ldots,\overbrace{1}^{i\text{th}},\ldots,0)}_{s \text{ composantes}} \in \mathbb{R}^s$, $s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$.
   - $\mathscr{C}^1(\mathbb{R}^n)$) is the set of continuously differentiable functions,
   - ImB is the subspace spanned by columns of matrix B,
   - r_σ is the joint spectral radius.

General Introduction

Preamble

In this thesis, we are concerned with a linear matrix inequality (LMI)- based approach studying an observer-based control problem for two classes of systems :

1. nonlinear discrete-time interconnected systems with nonlinear interconnections.
2. discrete-time linear systems in presence of parameter uncertainties and ℓ_2-bounded disturbances.

Due to its importance, the observer-based stabilization problem for systems with parameter uncertainties has aroused interest in the automatic control community, both in continuous and discrete-time cases [48], [7]. Many efforts have been devoted to robust stability and stabilization of linear and nonlinear systems with uncertain parameters for the following reason : the presence of uncertainties can lead to instability and poor performance on a controlled system. For a recent literature, we refer readers to [9] and [91]. These uncertainties may be due to the following reasons, which constitute one of the main motivations of this work : (1) variations in parameters with varying environmental conditions, for example the change in the tire-floor friction coefficient due to ice or snow on a winter road ; (2) changes in parameters over time (for example, the slow and constant change in tire stiffness over the life of the tir ; (3) variations in parameters due to the nonlinear nature of the model (for instance, the model of a quadruple tank process provided in Chapter 3 (Example 3). A framework providing partial solutions to some classes of systems has been established by using LMI approaches. However, due to the complexity of the problem, until now, it is not yet completely solved.
For discrete-time systems, one can mention the work of Daniel and Guoping [10] dealing with the equality constraint based method. Likewise, Ibrir and his coauthors have considered the same problem and used different techniques to linearize the bilinear terms, such as the decoupling technique [28] and the slack variables based me-

thod [27, 26]. A recent development related to this topic has been given in [37, 35] and [17], where the key idea consists in using the Young's inequality in a convenient way. Contrarily to the equality constraint based approach, where the equality constraint turns out to be equivalent to a particular choice of the Lyapunov matrix [10], the Young's inequality-based approach avoids this conservative equality constraint [37, 35] and [9]. Although the Young's relation based technique provides less conservative LMI conditions, nevertheless it has a drawback due to the a priori choice of some scalar variables coming from the use of the Young's inequality before solving the LMI conditions. To cope with this issue, many researchers have addressed the robust stabilization problem by using constant gain Luenberger observers for linear systems with parametric uncertainties. For an overview of the literature, we refer the reader to [22], [30], [29], [69], [76], [16], [26], [81], [50] and [10], [55], [8], [56], [14], and the references therein. Unfortunately the results remain conservative and the problem still remains open until now. This motivates us to extend the established results from [91], for designing robust observer-based controllers for linear discrete-time systems with parameter uncertainties. Indeed, the proposed approach is inspired from the classical two-step method introduced in [70] and the new two-step method given in [91]. The new variant of the classical two-step LMI approach we propose works as follows. In the first step, we use a slack variable technique to solve the optimization problem resulting from stabilization by a static state feedback. In the second step, a part of the slack variable obtained is incorporated in the $\mathscr{H}_\infty$ observer-based stabilization problem, to calculate simultaneously the Lyapunov matrix and the observer-based controller gains. **For interconnected nonlinear systems**, many important critical infrastructures such as energy, transport, and water networks are inherently complex interconnected nonlinear systems. The operation of these systems requires the ability to stabilize in the face of nonlinearities and uncertainties. Over the last decades, decentralized control strategies for interconnected systems have gained significant attention and become the subject of many research activities (see, for example, [63], [41], [42], [49], [58], [66], [72], [77], [78], [85] and the references therein). An important motivation for the design of decentralized control schemes is that the information exchange between subsystems is not needed ; thus, the individual subsystem controllers are simple and use only locally available information. However, in most of the decentralized state feedback control schemes the state variables are not available or costly to measure. Therefore, it is necessary to design a state observer in order to reconstruct the individual states of each subsystem. In this regard, considerable attention has been focused on the design of state observers for interconnected systems [1], [15], [53], [67], [68]. Especially, for the case of linear sub-systems coupled with nonlinear interconnections [31], [32], [53], [68], [71], [74], [84], [87], [75]. On the other hand, most efforts have been focused on the decentralization by using the powerful optimization tool of linear matrix inequalities (LMIs) to get sufficient conditions guaranteeing the asymptotic stability of the closed-loop system. However, even for the linear sub-systems case, the stability analysis often leads to bilinear matrix inequalities (BMIs) which are not solvable numerically. Hence, numerous LMI methods have been established in the literature. Each method provides a new LMI technique. For instance, some techniques are based on the use of the classical two-steps method [68], [93] ; others use Riccati equations [53], and finally some are based on the use of strong equality constraint [32]. It turned out that all these techniques provide conservative LMI synthesis conditions. In this thesis, we consider the problem of designing decentralized observer-based controllers in the convex optimization context for nonlinear interconnected discrete-time systems. The proposed method uses the tools of previous work in [18], [89] and [90]. We have proved that for discrete-time systems, the introduction of a slack variable G, in a triangular form, avoids many bilinear terms and leads to simpler and less conservative LMI conditions. Although this idea seems simple and inspired by [3], [2], [4], but it leads to better LMIs from feasibility point of view. This is the first time this idea is used to solve the decentralized observer-based stabilization problem for interconnected systems. This novel idea may open some new directions in this area to solve more complicated control design problems. On the other hand, notice that this idea of using a matrix G of triangular form is one of the main the contributions of this thesis but it is not the only one. There is a chain of new mathematical developments based on well known tools. This thesis is organized in three chapters that are structured as follows : **Chapter 1** : The first chapter is a literature survey. The general stability problem is recalled in order to establish fundamental concepts that are necessary to the comprehension of manuscript. Then, we present the results of some recent studies tackling the same type of problem which are parts of the comparison in the following chapters. The rest of the chapters constitutes our contributions.

Chapter 2 : This chapter presents a new decentralized observer-based controller design method for nonlinear discrete-time interconnected systems with nonlinear interconnections. Thanks to some algebraic transformations and the use of a new variant of Young's inequality, an LMI-based approach is provided to compute the observer-based controller gain matrices. Furthermore, the congruence principle is used under a judicious and new manner leading to include additional slack variables and to cancel some bilinear matrix coupling. The resulted final LMI conditions are less conservative compared to the techniques existing in the literature. The effectiveness of proposed methodology is shown through two illustrative examples. **Chapter 3 :** This chapter addresses the problem of observer-based stabilization of discrete-time linear systems in presence of parameter uncertainties and ℓ_2-bounded disturbances. We propose a new variant of the classical two-step LMI approach. In the first step, we use a slack variable technique to solve the optimization problem resulting from stabilization by a static state feedback. In the second step, a part of the slack variable obtained is incorporated in the $\mathscr{H}_\infty$ observer-based stabilization problem, to calculate simultaneously the Lyapunov matrix and the observer-based controller gains. The effectiveness of the proposed design methodology is demonstrated using the dynamic model of a quadruple-tank flow process system. Additional numerical illustrations are presented to show the superiority of the proposed Modified Two-Step Method (MTSM) from a LMI feasibility point of view.

Personal publications

Below is a list of publications related to the work of this thesis :

- **Journals**

  1. **C. Bennani**, F. Bedouhene, A. Zemouche, H.R. Karimi & H. Kheloufi, "Robust Observer-based Stabilization of Linear Discrete-time Systems with Parameter Uncertainties " International Journal of Control, Automation and Systems – **17 (9)** : 2261-2273 (2019).

     https ://link.springer.com/article/10.1007/s12555-018-0754-x

- **Elsevier Book chapter**

  1. **Bennani, C.**, Zemouche, A., Bedouhene, F., Kheloufi, H., Trinh, H., & Karimi, H. R. (2019). A robust decentralized observer-based stabilization method for interconnected nonlinear systems : Improved LMI conditions. In New Trends in Observer-Based Control (pp. 267-291). Academic Press (Elsevier). https ://doi.org/10.1016/B97 0-12-817038-0.00008-1

- **"IEEE Proceedings" CDC, ACC, ICSC**

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  2. **Bennani, C.**, Bedouhene, F., Zemouche, A., Bibi, H., & Aitouche, A. (2017, May). A modified two-step LMI method to design observer-based controller for linear discrete-time systems with parameter uncertainties. In 2017 6th International Conference on Systems and Control (ICSC) (pp. 279-284). IEEE. DOI : 10.1109/ICoSC.2017.7958
  3. **Bennani, C.** ; F. Bedouhene ; A. Zemouche ; H. Bibi ; K. Chaib-Draa ; A. Aitouche ; R. Rajamani, "Robust $\mathscr{H}_\infty$ Observer-Based Stabilization of Linear Discrete-Time Systems with Parameter Uncertaintes," 2018 Annual American Control Conference (ACC), Milwaukee, WI, 2018, pp. 4398-4402.

     DOI : 10.23919/ACC.2018.8431745

CHAPITRE 1

Generalities and brief state of the art on observer-based control of uncertain nonlinear systems

1.1 Introduction

This first chapter is devoted to the presentation of both the necessary and crucial reminders for the comprehension of this script, and previous literature of the different LMIs synthesis methods by observer-based controller for some classes of linear and interconnected nonlinear discrete-time systems with parameters uncertainties. First, a reminder of the stability (Lyapunov's direction) is given. We also state briefly the criteria of controllability, observability and stabilizability of systems to explore. We move to present the LMI approach, depicting the rationale of using it. Finally, we close this chapter exposing the results of some recent studies tackling the same class of systems which are parts of the comparison in the following chapters. Given the richness of the literature, we have contented ourselves with presenting the definitions that will help the understanding of this work.

1.2 Mathematical Model For Nonlinear Systems

In this section, the stability analysis of a system, the dynamics of which are represented in time domain by nonlinear time-invariant ordinary differential equations, is considered. A nonlinear system may mathematically be represented in the following form :

$$\begin{cases} \dot{x}_1 &= f_1(x_1,x_2,\cdots,x_n,u_1,u_2,\cdots,u_m,t), \\ \dot{x}_2 &= f_2(x_1,x_2,\cdots,x_n,u_1,u_2,\cdots,u_m,t), \\ \vdots &= \vdots \\ \dot{x}_n &= f_n(x_1,x_2,\cdots,x_n,u_1,u_2,\cdots,u_m,t), \end{cases} \tag{1.1}$$

where $\dot{x}_i, i = 1, 2, \cdots, n$ denotes the derivative of x_i (the ith state variable) with respect to the time variable t and $u_j,\ j = 1, 2, \cdots, m$ denote the input variables. Equation (1.1) could be written in the following state-space form :

$$\dot{x} = f(x, u, t), \tag{1.2}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \text{ and } f(x,u,t) = \begin{pmatrix} f_1(x,u,t) \\ f_2(x,u,t) \\ \vdots \\ f_n(x,u,t) \end{pmatrix}.$$

The measurable outputs (a p-dimensional vector) are functions of the states, the inputs, and the time such that :

$$\begin{array}{rcl} y_1 & = & h_1(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m, t), \\ y_2 & = & h_2(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m, t), \\ \vdots & = & \vdots \\ y_p & = & h_n(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m, t), \end{array} \tag{1.3}$$

or, in the following general form :

$$y = h(x, u, t). \tag{1.4}$$

Equations (1.2) and (1.4) together are called the mathematical dynamic equations, or :

$$\begin{array}{l} \dot{x} = f(x,u,t). \\ y = h(x,u,t). \end{array} \tag{1.5}$$

These equations could be simulated using for example operational amplifiers (integrators) as shown in Figures 1.1 and 1.2 :
It seems the dynamic systems could be simulated to obtain their responses, having signal generators f_i and h_i. However, there is a drawback with this approach, since for each initial condition the simulation must be repeated. To have the actual response of dynamic systems, (1.2) and (1.4), the system must be at least locally Lipschitz in $x, \forall x \in D \subset R^n$ and continuous in t, for every t. Throughout this thesis, wherever this type of dynamic equation occurs, the satisfaction of these conditions is assumed. Although in theory, the simulation could be proposed as a solution for the stability analysis, it is impractical or impossible, since in nonlinear system studies, every initial condition should be used. For more details, we refer the readers to [52].
Special Cases : If a system is a feedback system, then the system's inputs would be functions of the states, thus :

$$u \triangleq g(x,t) \tag{1.6}$$

Substituting (1.6) into (1.5) yields the following unforced dynamic equations :

$$\dot{x} = f(x,u,t) = F(x,t) \triangleq f(x,t),\ y = h(x,u,t) = H(x,t) \triangleq h(x,t), \tag{1.7a}$$

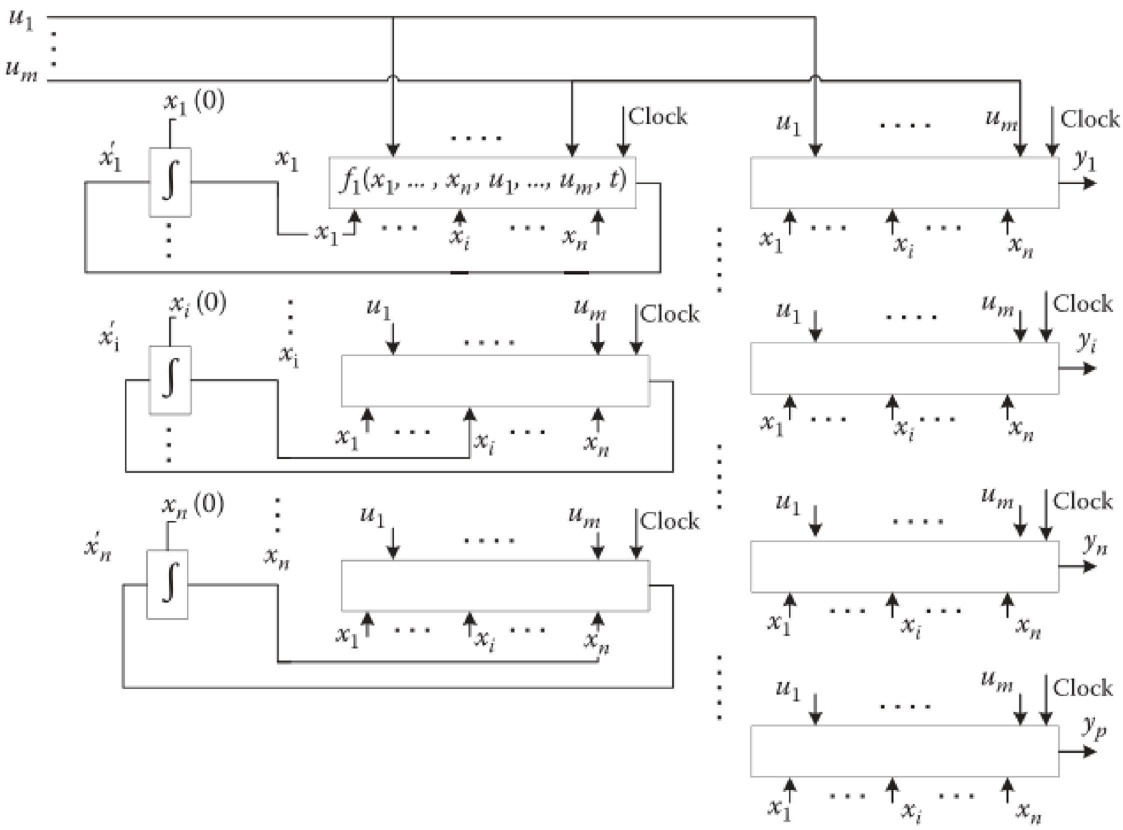


FIG. 1.1: System dynamic simulation (a) [52]

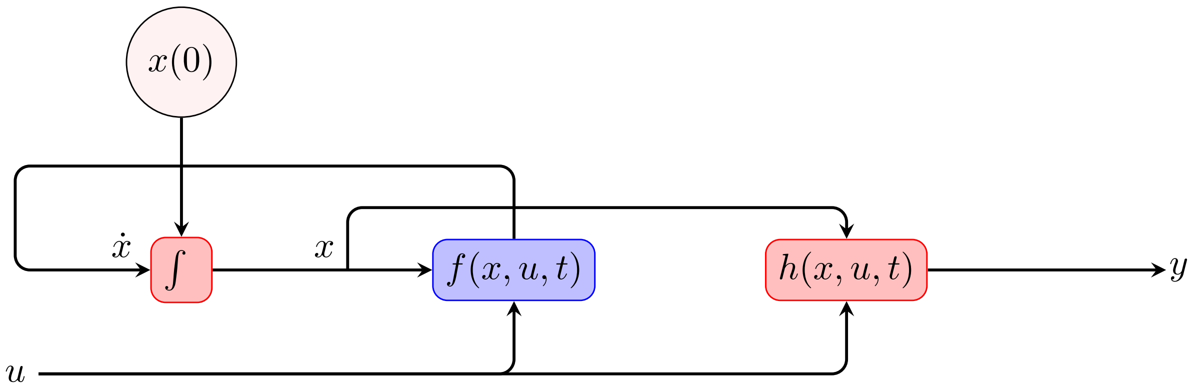


FIG. 1.2: System dynamic simulation (b)

If the dynamic system (1.7a) is time invariant, then the system is called an autonomous (either forced or unforced) system.

$$
\begin{aligned}
\dot{x} &= f(x, u), \quad \text{or} \quad f(x) \\
y &= h(x, u), \quad \text{or} \quad h(x).
\end{aligned}
\tag{1.7b}
$$

1.2.1 Uncertainties, Robusteness

The primary concern in command theory has always been the stability that was introduced by Routh and Hurwitz in the 19th century, then explored in detail by Lyapunov whose work is still used today. In addition to stability, a controlled system must meet various performance objectives such as robustness with respect to uncertainties that can affect the system. Indeed, depending on the type of information available a priori, we can distinguish two types of uncertainties :

– *Parametric uncertainties* : coming from modeling or measurement errors, and dependent on a parameter, and for which the mathematical model is well defined. In addition, depending on the mode of variation of the parameter, we can distinguish

  1. Polytopic uncertainties where the variant parameters are assumed to belong to intervals or convex sets, so that the uncertainty matrices are written as convex combinations of several matrices.
  2. Uncertainties structured in bounded norms where we have no information on the domain of the variant parameters, but, the matrices containing the uncertain parameter are subjected to boundedness constraints in norm.

– *Unstructured uncertainties* : these are the measurable Lebesgue functions, where only the possible amplitudes of the uncertainties are assumed to be known. Note that the target objective in the event of the presence of this kind of ℓ_2-bounded perturbations consists in optimizing the response of the system to the effect of a destabilizing input signal. Therefore, we get the robustness criteria $\mathscr{H}_2$ and $\mathscr{H}_\infty$. Indeed, the norm $\mathscr{H}_\infty$ or the gain $\mathscr{L}_2$ is the maximum amplification that the system can exert on the energy of the input signal. In other words, this standard favors the pulse for which the gain is maximum. If we seek to minimize such a standard, it means that we are interested in the most unfavorable frequency. The minimization effort therefore relates to this "worst" frequency and the resulting level of performance is then a fortiori guaranteed for other frequencies. It is quite different for the norm $\mathscr{H}_2$. It does not favor any frequency but rather reflects an energy of the outputs distributed over all the frequencies. To minimize it means to carry the effort of minimization on all the frequencies.
  Indeed, the norm $\mathscr{H}_\infty$ is none other than the norm $\|.\|_\infty$ of the transfer matrix G. If we know the expression of the maximum singular value of the transfer matrix, we therefore have

$$\|G\|_\infty = \sup_\omega \|G(i\omega)\|_2 = \sup_\omega \overline{\sigma}(G(i\omega)). \tag{1.8}$$

  For the practical cases encountered in automatic, if we assume that the signals are at finite energies (i.e. are therefore in L^2). So, if we suppose that Z is the image of w by an operator $\mathscr{R}$, then we can define the gain L^2 of $\mathscr{R}$ by

$$\mathscr{G}_{L^2}(\mathscr{R}) = \sup_{w\in L^2} \frac{\|Z\|_2}{\|w\|_2} \tag{1.9}$$

  or

$$\|X\|_2 \quad = \quad \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathrm{i}\omega)^* X(\mathrm{i}\omega)d\omega\right)^{1/2} \tag{1.10}$$

$$= \left(\int_0^\infty x(t)^* x(t) dt\right)^{1/2} \tag{1.11}$$

$$= \left(\int_0^\infty \|x(t)\|_2^2 dt\right)^{1/2} \tag{1.12}$$

$$\tag{1.13}$$

The gain $\mathscr{G}_{\mathscr{L}^2}(\mathscr{R})$ is in fact the greatest gain in energy associated with the operator $\mathscr{R}$. The norm $\mathscr{H}_\infty$ of a transfer matrix therefore corresponds to the gain $\mathscr{L}_2$ from the system. The general problem of the command $\mathscr{H}_\infty$, is then stated as follows :
At a performance level γ guaranteed for the standard $\mathscr{H}_\infty$, the command problem $\mathscr{H}_\infty$ corresponds to determining a stabilizing command under the optimality constraint $\mathscr{H}_\infty$ of level γ : $\|G\|_\infty < \gamma$.

1.2.2 An example of uncertain systems

Consider the mass-spring-damper system shown in Figure 1.3. One end of the spring

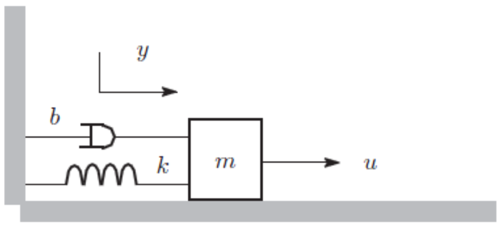


FIG. 1.3: Mass-Spring-Damper system [40]

and damper is fixed on the wall. The parameter m is the mass, b the viscous friction coefficient of the damper, and k the spring constant. $y(t)$ is the displacement of massm and $u(t)$ the external force. The state equation of this system is shown in Eq. (1.14) where the performance output is the displacement y :

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u, \ \ x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}. \tag{1.14}$$

Example 1.2.1 (Norm-Bounded Parametric System, [40]). *Recall the Mass-Spring-Damper system (1.14). Note that a feature of the coefficient matrices is that all uncertain parameters are in the second row. This makes it possible for us to put the uncertain parameters into a matrix and treat them as a matrix uncertainty. Set the nominal parameters as (m_0, k_0, b_0); the parameters can be written as*

$$\frac{k}{m} = \frac{k_0}{m_0}(1+\omega_1\delta_1), \quad \frac{b}{m} = \frac{b_0}{m_0}(1+\omega_2\delta_2), \quad \frac{1}{m} = \frac{1}{m_0}(1+\omega_3\delta_3), \quad |\delta_i| \le 1.$$

Due to

$$\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m_0} & -\frac{b_0}{m_0} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{k_0}{m_0}\omega_1\delta_1 & -\frac{b_0}{m_0}\omega_2\delta_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m_0} & -\frac{b_0}{m_0} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} [\delta_1 \quad \delta_2 \quad \delta_3] \begin{bmatrix} \frac{k_0}{m_0}\omega_1 & 0 \\ 0 & \frac{b_0}{m_0}\omega_2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m_0} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\delta_1 \quad \delta_2 \quad \delta_3] \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_0}\omega_3 \end{bmatrix},$$

the state equation can be rewritten as

$$\dot{x} = (A + B_1 \Delta C_1)x + (B_2 + B_1 \Delta D_{12})u$$

where $\Delta = [\delta_1 \quad \delta_2 \quad \delta_3]$ *and*

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m_0} & -\frac{b_0}{m_0} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{1}{m_0} \end{bmatrix}, \quad C_1 = -\begin{bmatrix} \frac{k_0}{m_0}\omega_1 & 0 \\ 0 & \frac{b_0}{m_0}\omega_2 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_0}\omega_3 \end{bmatrix}.$$

Naturally, the size of uncertainty vector δ *should be measured by vector norm. So, this system is called a* norm-bounded parametric system.

1.3 Lyapunov Theory for Discrete Time Autonomous Systems

There are different stability analysis methods for both linear and nonlinear systems that can be found in classical literature of dynamical systems. This section contains a collection of Lyapunov related theorems for discrete time systems. For the demonstration of the results presented in this section, we refer to the technical report [5], largely inspired by [19].

Définition 1.1. A function $f(t,x)$ is said to be **Lipschitz** in $(\bar{t},\bar{x})$ if

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|, \tag{1.15}$$

$\forall (t,x),(t,y)$ in a neighbourhood of $(\bar{t};\bar{x})$. The constant L is called **Lipschitz constant**. Consider $f(t,x) = f(x)$ independent of t.

– f is **locally Lipschitz** on a domain $D \subset \mathbb{R}^n$ open and connected if each point in D has a neighbourhood D_0 such that (1.15) is satisfied with Lipschitz constant L_0.
– f is **Lipschitz** on a set W if (1.15) is satisfied for all points in W with the same constant L. A function f locally Lipschitz on D is Lipschitz on every compact subset of D.
– f is **globally Lipschitz** if it is Lipschitz on $\mathbb{R}^n$.

If $f(t,x)$ depends on t, the same definitions hold, provided that (1.15) holds uniformly in t for all t in an interval of time (that is the Lipschitz constant do not vary due to the time).

A continuously differentiable function on a domain D is also Lipschitz in the same domain. Let us Consider a system of difference equations

$$\begin{cases} x(k+1) & = & f(x(k)),\ x \in U \\ f(0) & = & 0 \end{cases} \tag{1.16}$$

where U is a neighbourhood of the origin $\in \mathbb{R}^n$ and $f : R^n \to R^n$ is a continuous function. For each $p \in U$, let us denote by $f^k(p)$ the value at time k of the solution of (1.16) starting at p. We recall that

$$f^k(p) = f(f^{k-1}(p)), f^0(p) = p.$$

We recall that Lypschitz condition on f is sufficient to guarantee existence and uniqueness of solution for each initial state $p \in U$.
In general we are interested in point $x_0 \in U$ such that $f(x_0) = x_0$. They are usually called **equilibrium point** of the system. Corresponding to each equilibrium point x_0, we have a constant solution $f^k(x_0) = x_0$. of (1.16).
In what follows, suppose $f(0) = 0$, that is $x = 0$ is an equilibrium point for system (**??**) (all this can be extended for an equilibrium point different from 0).

Définition 1.2. The equilibrium point $x = 0$ of (1.16) is
- **stable** if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq 0$.
- **unstable** if it is not stable.
- **asymptotically stable** if it is stable and δ can be chosen such that $\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0$.
- **exponentially stable** if there exist $k > 0$, $\lambda > 0$ and $r_0 > 0$ such that the solutions of the system satisfy

$$\|x(t)\| \leq k\|x(0)\| \exp^{-\lambda t}, \ \forall x(0) \in D_0, \ \forall t \geq 0. \tag{1.17}$$

where $D_0 = \{x \in \mathbb{R}^n \ \|x\| < r_0\}$.

Theorem 1.3.1 (Existence of a Lyapunov function implies stability, [19]). *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t+1) = f(x(t))$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset R^n$ and $0 \in D$. Suppose there exists a function $V : D \to \mathbb{R}$ which is continuous and such that

$$V(0) = 0 \textit{ and } V(x) > 0, \forall x \in D - \{0\} \tag{1.18}$$

$$V(f(x)) - V(x) \leq 0, \forall x \in D. \tag{1.19}$$

Then $x = 0$ is stable. Moreover if

$$V(f(x)) - V(x) < 0, \forall x \in D - \{0\}. \tag{1.20}$$

then $x = 0$ is asymptotically stable.

Définition 1.3. A function $V : D \to \mathbb{R}$ satisfying (1.18) and (1.19) is called a **Lyapunov function**.

Theorem 1.3.2 (Global asymptotic stability from Lyapunov). *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t+1) = f(x(t))$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset R^n$ and $0 \in D$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$V(0) = 0 \text{ and } V(x) > 0,\ \forall x \in D - \{0\} \tag{1.21}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{1.22}$$

$$V(f(x)) - V(x) < 0, \forall x \in D. \tag{1.23}$$

then $x = 0$ is globally asymptotically stable.

If an equilibrium point is globally asymptotically stable, than it is the only possible equilibrium point of the system (1.16). The following gives a condition for instability

Theorem 1.3.3. ***(Instability condition from Lyapunov)*** *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t+1) = f(x(t))$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset R^n$ and $0 \in D$. Let $V : D \to \mathbb{R}$ be a continuous function such that $V(0) = 0$ and $V(x_0) > 0$, for some x_0 with arbitrary small $\|x_0\|$. Let $r > 0$ be such that $B_r \subset D$ and $U = \{x \in B_r | V(x) > 0\}$, and suppose that $V(f(x)) - V(x) > 0, \forall x \in U$. then $x = 0$ is unstable.

Theorem 1.3.4. ***(Exponential stability implies existence of a Lyapunov function)*** *Let $x = 0$ be an equilibrium point for the nonlinear autonomous system*

$$x(t+1) = f(x(t))$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable and $D \subset \{x \in \mathbb{R}^n\ \|x\| < r\}$. Assume that the equilibrium $x = 0$ is exponentially stable. Then there is a function $V : D_0 \to \mathbb{R}$ that satisfies

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$V(f(x)) - V(x) \le -c_3\|x\|^2$$

$$|V(x) - V(y)| \le c_4\|x - y\|(\|x\| + \|y\|)$$

for all $x, y \in D_0$ and for some positive constants c_1, c_2, c_3 and c_4.

1.4 Linear systems and Linearization

Consider the linear time-invariant system

$$x(t+1) = Ax(t), \ \ A \in \mathbb{R}^{n\times n}. \tag{1.24}$$

It has an equilibrium point in the origin $x = 0$. The solution of the linear system starting from $x_0 \in \mathbb{R}^n$ has the form

$$x(t) = A^t x(0)$$

We have the following result on the stability of linear systems

Theorem 1.4.1. *The equilibrium point $x = 0$ of the linear time-invariant system*

$$x(t+1) = Ax(t), \ \ A \in \mathbb{R}^{n\times n} \tag{1.25}$$

is stable if and only if all the eigenvalues of A satisfy $|\lambda_i| \leq 1$ and the algebraic and geometric multiplicity of the eigenvalues with absolute value 1 coincide. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all the eigenvalues of A are such that $\lambda_i < 1$.

A matrix A with all the eigenvalues in absolute value smaller than 1 is called a **Schur matrix**, and it holds that the origin is asymptotically stable if and only if matrix A is Schur.
To use Lyapunov theory for linear system we can introduce the following candidate $V(x) = x^T Px$ with P a symmetric positive definite matrix. It's total difference is

$$V(f(x)) - V(x) = x^T A^T PAx - x^T Px = x^T(A^T PA - P)x := -x^T Qx \tag{1.26}$$

Using Theorem 1.3.1 we have that if Q is positive-semidefinite the origin is stable, whether if Q is positive definite the origin is asymptotically stable. Fixing a positive definite matrix Q, if the solution of the Lyapunov equation

$$A^T PA - P = -Q \tag{1.27}$$

with respect to P is positive definite, then the trajectories converge to the origin.

Theorem 1.4.2 (Lyapunov for linear time invariant systems). *A matrix A is Schur if and only if, for any positive definite matrix Q there exists a positive definite symmetric matrix P that satisfies (1.27). Moreover if A is Schur, then P is the unique solution of (1.27).*

Let us consider again the nonlinear model

$$x(t+1) = f(x(t)) \tag{1.28}$$

with $f : D \to \mathbb{R}^n$ a continuously differentiable map from $D \subset R^n$, $0 \in D$ into R^n such that $f(0) = 0$. Using the mean value theorem, each component of f can be rewritten in the following form

$$f_i(x) = \frac{\delta f_i}{\delta x}(z_i)x$$

for some z_i on the segment from the origin to x. It is valid for any $x \in D$, where the line connecting x to the origin entirely belongs to D. We can also write

$$f_i(x) = \frac{\delta f_i}{\delta x}(0)x + \underbrace{[\frac{\delta f_i}{\delta x}(z_i) - \frac{\delta f_i}{\delta x}(0)]x}_{g_i(x)}$$

where each $g_i(x)$ satisfies

$$|g_i(x)| \leq \|\frac{\delta f_i}{\delta x}(z_i) - \frac{\delta f_i}{\delta x}(0)\|\|x\|.$$

Function f can be rewritten as

$$f(x) = Ax + g(x)$$

where $A = \frac{\delta f_i}{\delta x}(0)$. By continuity of $\frac{\delta f_i}{\delta x}$ we have that

$$\frac{\|g(x)\|}{\|x\|} \to 0 \text{ as } \|x\| \to 0.$$

Therefore, in a small neighbourhood of the origin the nonlinear system can be approximated by $x(t+1) = Ax(t)$. The following is known as Lyapunov's indirect method.

Theorem 1.4.3. ***(Linearised asympt. stable implies nonlinear asympt. stable)*** *Let $x = 0$ be an equilibrium point for the nonlinear autonomous system*

$$x(t+1) = f(x(t)) \tag{1.29}$$

where $f : D \to R^n$ is locally Lipschitz in $D \subset R^n$ and $0 \in D$. Let $A = \frac{\delta f_i}{\delta x}(x)|_{x=0}$. Then the origin is asymptotically stable if $|\lambda_i| < 0$ for all the eigenvalues of A. Instead, if there exists at least an eigenvalue such that $|\lambda_i| < 0$, then the origin is unstable.

The proof of the instability part is based on following statement regarding the solvability of the discrete Lyapunov equation.

Proposition 1.4.4. *B The Lyapunov equation* (1.27) *admits a solution if and only if the eigenvalues λ_i of matrix A are such that*

$$\lambda_i \lambda_j \neq 1 \textit{ for all } i, j = 1, \cdots, n. \tag{1.30}$$

Moreover given a positive definite matrix Q, the corresponding solution P is positive definite if and only if $|\lambda_i| < 1$ for all $i = 1, \cdots, n$.

Theorem 1.4.3 allows to find a Lyapunov function for the nonlinear system in a neighbourhood of the origin, provided that the linearised system is asymptotically stable.

1.5 LMI vs BMI

Matrix inequality in the form of

$$F(x) = F_0 + \sum_{1}^{m} x_i F_i > 0 \tag{1.31}$$

is called a linear matrix inequality, or LMI for short. Here, $x \in \mathbb{R}^m$ is an unknown vector, and $Fi = F_i^T \in \mathbb{R}^{n\times n}$, $i = 1, \cdots, n$ is a constant matrix. $F(x)$ is an affine function of variable x. The inequality means that $F(x)$ is positive definite, that is, $u^T F(x) u > 0$ for all nonzero vector u. LMI can be solved numerically by methods such as the interior point method [40]. MATLAB has an LMI toolbox tailored for solving control problems.

1.5.1 Control Problem and LMI

In control problems, it is often the case that the variables are matrices. This is different from the LMI of (1.31) in form. However, it can always be converted to (1.31) equivalently via the introduction of matrix basis :

Exemple 1 ([40]). *According to the Lyapunov stability theory, the necessary and sufficient condition for the stability of a two-dimensional linear system*

$$\dot{x}(t) = Ax(t),\ x(0) \neq 0$$

is that there exists a positive definite matrix $P = P^T \in \mathbb{R}^{2\times 2}$ satisfying

$$AP + PA^T < 0.$$

This 2×2 symmetric matrix P has the following symmetric basis

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ P_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By using this symmetric basis, P can be expressed as

$$P = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} = x_1 P_1 + x_2 P_2 + x_3 P_3.$$

Substituting this equation into the inequality about P, we get an inequality about the vector $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ in the form of (1.31). Specifically, the result obtained is

$$x_1(AP_1 + P_1A^T) + x_2(AP_2 + P_2A^T) + x_3(AP_3 + P_3A^T) < 0.$$

In addition, when the dimension of symmetric matrix P is n, the number of its base matrices is $n(n+1)/2$. Therefore, linear inequality with matrix variables is also called LMI hereafter.

1.5.2 Typical LMI Problems

Now, we introduce several types of LMI problems (LMIP) which will be frequently encountered later.

1. **Feasibility Problem : LMIP** Given an LMIP, $F(x) < 0$. The so-called feasibility problem is to seek a vector x^* satisfying $F(x^*) < 0$, or make the judgment that the LMI has no solution.
2. **Eigenvalue Problem : EVP** The following optimization problem with constraints is called an eigenvalue problem (EVP) :

$$\text{minimize } \lambda$$
$$\text{subject to } \lambda I - A(x) > 0, B(x) > 0.$$

   Here, $A(x)$ and $B(x)$ are symmetric and affine matrix functions of vector x. Minimizing λ subject to $\lambda I - A(x) > 0$ can be interpreted as minimizing the largest eigenvalue of matrix $A(x)$. This is because the largest λ that does not meet $\lambda I - A(x) > 0$ is the maximal λ satisfying $\det(\lambda I - A(x)) = 0$. We may also regard λ as a variable and augment the vector x accordingly. Then, λ can be expressed as $c^T x$, and the two inequalities can be merged into one. Thus, we arrive at the following equivalent EVP ([40]) :

$$\text{minimize } \mathrm{c}^{\mathrm{T}}\mathrm{x}$$
$$\text{subject to } \mathrm{F}(\mathrm{x}) > 0.$$

3. **Generalized Eigenvalue Problem : GEVP** When λI in the EVP is generalized to $\lambda B(x)$, this problem becomes a generalized eigenvalue problem (GEVP). The detail is as follows :

$$\text{minimize } \lambda$$
$$\text{subject to } \lambda B(x) - A(x) > 0, B(x) > 0, C(x) > 0.$$

Définition 1.4 (Bilinear Matrix Inequality). A BMI constraint is a constraint on $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^r$ which is of the form

$$F(x,y) := F_{0,0} + \sum_{i=1}^{m}\sum_{j=1}^{r} x_i y_j F_{i,j} \geq 0, \tag{1.32}$$

with $F_{0,0} \in \mathbb{R}^{n\times n}$ and $F_{i,j} \in \mathbb{R}^{n\times n}$.

This is the generalization of the notion of LMI. A wide variety of automatic problems can be formulated as BMI. However, the BMI are not convex, which causes difficulties in their resolution.
The interest of LMIs is summarized in the four points described below.

1. **Convexity** : The LMI (1.31) defines a convex constraint in x. Indeed, $\forall x, y\, in \mathbb{E}$, $\mathbb{E} := \{x|F(x) > 0\}$, then

$$F(\alpha x + (1-\alpha y)) \leq \alpha F(x) + (1-\alpha F(y)) \ \forall \alpha \in]0,1[, \tag{1.33}$$

   where $F(x)$ is given by (1.31).

2. **Concatenation** : Multiple LMIs can be combined in one. Indeed, solve the two LMIs $F_1(x) > 0$ and $F_2(x) > 0$ is equivalent to solving $\bar{F}(x) > 0$ with $\bar{F} = \mathrm{diag}(F_1, F_2)$.
3. **Algorithms** : The algorithms used to solving LMI constraints are efficient : a good initialization guarantees the convergence of the algorithm, this convergence being at polynomial time. Currently, the main classes of algorithms are based on interior point methods [51].
4. **Applications** : Many conditions classics in automatic mode can be formulated under the form of LMI problem. In addition, it is possible to convert some nonlinear inequalities (in particular of Riccati) in LMI by the use of Schur's lemma.

1.5.3 Computer programming aspects

This subsection clarifies certain concepts that will appear in some chapters, especially when it comes to software aspects. We let's briefly state the various useful concepts. For more details, we can refer to [51].

Définition 1.5 (Polynomial time). A problem is said to be solvable in polynomial time when there is a resolution algorithm for which the required computation time is a polynomial function of the size [1] of the problem.

Définition 1.6 (NP-difficile). A problem is said to be NP-difficile when the computation time required is a function of the types a^n with $a > 1$ and n the size of the problem.

A BMI type problem is an example of an NP-hard problem while an LMI problem is not because it is solvable in polynomial time. However, a large number of Problems in automatic are by nature of the BMI type. To solve them, we must therefore reduce to a formulation of the LMI type, which introduces a *conservatism*.

Définition 1.7 (Conservatism). A condition (or a constraint) is said to be conservative when it is too restrictive in relation to the problem considered.

The dilemma between complexity and conservatism constantly arises and influences choices (both theoretical than practices) performed in solving a problem. Indeed, in order to obtain a corrector synthesis the least conservative possible, one increases the number of specifications of the problem. The problem can become NP-difficult. To solve it, that is to say to reduce the complexity of this problem, one is led either to approximate or remove some of the specifications. Consequently, we increase the conservatism of solutions obtained.

1.6 Controllability and Observability of an LTI System

Consider the LTI system :

$$x(k+1) = Ax(k) + Bu(k), \tag{1.34a}$$

[1] The size of a problem is in general defined as the number of variables involved in solving the problem.

$$y(k) = Cx(k) + Du(k), \tag{1.34b}$$

where $x(k)$, $u(k)$, and $y(k)$ are respectively the plant state vector, input vector, and output vector. Their dimensions are assumed to be respectively n, q, and p. Generally speaking, controllability is concerned with capabilities of a system in maneuvering its states through external inputs, whereas observability is concerned with capabilities of estimating its states using measurements of external outputs. Formally, they are defined as follows.

1.6.1 Controllability

Définition 1.8 ([92]). The matrix pair (A,B) or, equivalently, the discrete dynamical system described by Eqs. (1.34a) and (1.34b) is said to be controllable if for an arbitrary state vector pairs (x_0,x_1), there exist a positive integer k and an input vector sequence $u(0)$, $u(1),\cdots,u(k-1)$ such that under the drive of this sequence, the state vector of the system satisfies $x(k) = x_1$ when starting from $x(0) = x_0$. Otherwise, the system or the matrix pair is said to be uncontrollable.

In some literature, controllability of a discrete system is also called reachability. To be consistent with continuous systems and avoid possible confusions, in this thesis, we use the term controllability. Concerning a discrete LTI system, several criteria are available for the verification of its controllability using its system parameters. The following theorem summarizes some of the most widely used ones.

Theorem 1.6.1 ([92]). *The discrete dynamical system of equations* (1.34a) *and* (1.34b) *is controllable if and only if one of the following conditions is satisfied :*
– *The controllability matrix* $\mathscr{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$ *is of full row rank.*
– *The matrix* $\begin{bmatrix} \lambda I - A & B \end{bmatrix}$ *is of full row rank for every complex number* λ.

Note that except at an eigenvalue of the matrix A, the matrix $[\lambda I - AB]$ is always of full row rank and of full column rank. This implies that verification of the second criterion is only required at the eigenvalues of the matrix A.

Static output feedback control

The synthesis of static output feedback control is one of the central problems in control theory see e.g. [61]. The methods of reconstructing the observer-based state introduce an additional dynamic system. This is why, in the early 1970s, particular attention was paid to the problem of the design of output feedback regulators where a very limited number of state measurements are available [39]. The optimal solution to this control problem is obtained in terms of high order nonlinear matrix algebraic equations. The convergence complexities of the algorithms suggested for the solution of these equations have long prevented a wider application of this technique. In the case of an output return command, the input vector $u(t)$ is written in the following form :

$$u(t) = -K_y y(t) + \nu(t) \tag{1.35}$$

1.6.2 Feedback Control

Without control systems there can be no manufacturing, vehicles, computers, etc., in short, no technology. Control systems are those that make machines work. These systems are most often based on the feedback principle : the signal to be checked is compared with a desired reference signal. The difference between the signals is used to calculate the corrective action control. Three types of retroactive control exist :

1. State feedback control (see Figure 1.4),

2. Control by estimated state feedback (also called observer based control) (see Figure 1.6)

3. Output feedback control (see Figure 1.5).

State feedback control

The state feedback control is a closed loop control. It consists in implementing a control law $u(t)$ which takes into account the different values of the state at an instant t. Consider the system described by the set of equations (1.34), the control law applied to this system has the following form :

$$u(t) = -K_x x(t) + v(t) \tag{1.36}$$

where $K_x \in \mathbb{R}^{m\times n}$ and $v(t)$ represent respectively the control gain and the set point. On the other hand with this type of command, one needs to have the measurements of all the states to be able to apply the control, which is almost impossible from a practical point of view.

Control by estimated state feedback

The design of the optimal linear regulator requires knowledge of the measurements of all states of the system. In many practical applications, this is not possible due to the high cost of state measurements or the inaccessibility of measuring certain system states. The standard way to overcome these difficulties is to reconstruct the complete state vector from the measurements available by the design of a state observer (Luenberger for example, or the Kalman filter if the measurements are noisy).
The observer-based control law is written as follows :

$$u(t) = -K_{\hat{x}} \hat{x}(t) + v(t) \tag{1.37}$$

where $\hat{x}$ is the estimated state of x.

1.6.3 Synthesis the control gain by LMI method

We have seen that LMI method solves a system of matrix inequality. Considering, for example, the system described by the following equations : ?

$$x_{t+1} = Ax_t,\ x_0 = 0. \tag{1.38}$$

The system (1.38) is asymptotically stable if $\forall x \neq 0$, there exists a Lyapunov function $V(x_t) > 0$ such that $V(0) = 0$ and $V(x_{t+1}) - V(x_t) < 0$. Using the quadratic function $V(x) = x^T Px$, we get that P satisfies the two inequalities :

$$\begin{aligned} V(x_t) > 0 \quad &\Leftrightarrow \quad P = P^T > 0 \\ V(x_{t+1}) - V(x_t) < 0 \quad &\Leftrightarrow \quad A^T PA - P < 0. \end{aligned} \tag{1.39}$$

Before the appearance of LMI method, the resolution of this type of problem could be overcome by keeping equation (1.39) as an inequality. So, it turned out to be necessary to transform (1.39) into an equality using a matrix Q, assumed to be positive. Thus, we obtain asymptotic stability of (1.38) if and only if for any $Q = Q^T > 0$, there exists $P = P^T > 0$ solution of

$$A^T PA - P + Q = 0$$

see [7].

1.6.4 Observability

Another fundamental concept in system analysis and synthesis is observability of a system. While controllability and observability of a system are completely different from an engineering point of view, it is now well known that they are mathematically dual.

Définition 1.9 ([92]). The discrete dynamical system described by Eqs. (1.34a) and (1.34b) or, equivalently, the matrix pair (A,C), is said to be observable if there exits a positive integer k such that each initial system state vector $x(0)$ can be revealed from the input–output vector pairs $(u(s), y(s))_{s=0}^k$. Otherwise, this system or matrix pair is said to be unobservable.

Note that when the input sequence $(u(k))_{k\geq 0}$ is assumed to be known, it is clear from Eq. (1.34a) that all the uncertainties in the plant state vectors are caused by its initial values. This means that when the initial state vector of a plant is revealed from its output measurements, all of its state vectors can be revealed also from these measurements.
Similar to Theorem 1.6.1, we also have some algebraic criteria for the verification of observability of a system.

Theorem 1.6.2 ([92]). *The discrete dynamical system of equations* (1.34a) *and* (1.34b) *is observable if and only if one of the following conditions is satisfied :*
– *The observability matrix* $\mathcal{O} = \begin{bmatrix} C & A^T C^T & (A^2)^T C^T & \cdots & (A^{n-1})^T C^T \end{bmatrix}^T$ *is of full row rank.*
– *The matrix* $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ *is of full row rank for every complex number* λ.

1.6.5 Standard Types of Observers

The problem of estimating a state of a dynamical system driven by unknown inputs has been the subject of a large number of studies in the past three decades. A state observer is typically a computer-implemented mathematical model that provides the estimation of the internal states of the system. In most practical cases, the physical state of the system cannot be determined by direct observation. Instead, indirect effects of the internal state are observed by way of the system outputs.
The various models of state observers are summarized by :

- *Luenberger Observer (LO)*. This observer possesses a relatively simple design that makes it an attractive genaral design technique.

- *Kalman Observer (KO)*. In one view, KO can be viewed as an extension of LO. It could be argued that the KO (filter) is one of the good observers against a wide range of disturbances.

- *Unknown input observer (UIO)*. This is a good candidate for estimating the state of a linear system with unknown inputs, such as those treated in the problem of instrument fault detection, as well as fault detection and identification subject to plant parameter uncertainties.

- *Sliding Mode Observer (SMO)*. A key feature of SMO is the introduction of a switching function in the observer to achieve a sliding mode, and also stable error dynamics. This sliding mode characteristic, which is a consequence of the switching function, is claimed to result in system performance that includes insensitivity to parameter variations, and complete rejection of disturbances.

Consider the linear discrete-time system

$$x_{k+1} = Ax_k + Bu_k \tag{1.40}$$

$$y_k = Cx_k \tag{1.41}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the local control vector, and $y(k) \in \mathbb{R}^q$ is the output observation vector. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ are constants. The basic state estimation problem is defined by :

At each time k, construct an estimate $\hat{x}(k)$ of the state $x(k)$ by only measuring the output $y(k)$ and input $u(k)$.

A preliminary way to realize this construction is through an open-loop observer : Build an artificial copy of the system, fed in parallel with the same input signal $u(k)$. The result is depicted in Fig. 1.8. Notice that the "copy" is a numerical simulator

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) \tag{1.42}$$

reproducing the behavior of the real system. The dynamics of the real system and of the numerical copy given by (1.40),(1.42), respectively, yield the dynamics of the estimation error $\tilde{x}(k) = x(k) - \hat{x}(k)$ as

$$\tilde{x}(k+1) = A\tilde{x} \tag{1.43}$$

and then

$$\tilde{x} = A^k\left(x(0) - \hat{x}(0)\right).$$

This is not ideal, because

- The dynamics of the estimation error are fixed by the eigenvalues of A and cannot be modified.
- The estimation error vanishes asymptotically if, and only if, A is asymptotically stable.
- We are not exploiting $y(k)$ to compute the state estimate $\hat{x}(k)$! See Fig. 1.9.

1.6.6 Luenberger Observer

The LO possesses a relatively simple design that makes it an attractive general design technique. By correcting the estimation model (1.42) with feedback from the estimation error $y_k - \hat{y}_k$, we obtain

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L[y_k - C\hat{x}_k] \tag{1.44}$$

where the term $L[y(k) - \hat{y}(k)]$ is often called the residual, and the term $L[y(k) - \hat{y}(k)]$ designates a feedback on the estimation error with $L \in \mathbb{R}^{n\times q}$ is the observer gain. Introducing the state estimation error $\tilde{x}(k) = x(k) - \hat{x}(k)$, then we derive its dynamics as :

$$\tilde{x}(k+1) = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) - L[y(k) - C\hat{x}(k)] = [A - LC]\tilde{x}(k) \tag{1.45}$$

which leads to

$$\tilde{x}(k) = [A - LC]^k(x(0) - \hat{x}(k)).$$

It is interesting to record that the same idea applies to the continuous time system, depicted in Fig. 1.10 :

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t)$$

leading to

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + L[y(t) - C\hat{x}(t)] \tag{1.46}$$

and the associated state estimation error dynamics as

$$\frac{d}{dt}\hat{x}(t) = [A - LC]\tilde{x}(t). \tag{1.47}$$

A fundamental result is gained by the eigenvalue assignment of the state observer, which is summarized by Theorem 1.6.3.

Theorem 1.6.3. *[43] If the pair (A,C) is observable, then the eigenvalues of $[A - LC]$ can be placed arbitrarily.*

Proof : If the pair (A, C) is completely observable, then the dual system (A^T, C^T, B^T, D^T) is completely reachable. Thus we can design a compensator K for the dual system and place the eigenvalues of $[A^T + C^T K]$ arbitrarily. From basic matrix algebra, the eigenvalues of matrix $[A^T + C^T K]$ are the eigenvalues of its transpose $[A + K^T C]$. Finally, define $L = -K^T$ concludes the proof of the theorem. □

Remark 1.6.4. *It follows from Theorem 1.6.3 that the estimation error $\tilde{x}(t) \to 0$ as $t \to \infty$. This asymptotic behavior represents a basic property of linear Luenberger Observer.*

1.6.7 Building Observer-Based Controllers

The previous section showed how to obtain an observer that can make the estimation error approach zero asymptotically. However, our ultimate goal is to control the system. A natural ad hoc approach is to apply state feedback control, but use the estimate of the state rather than the actual state. That is, instead of using the feedback control $u(t) = Kx(t)$, we use $u(t) = K\hat{x}(t)$, where $\hat{x}$ is the estimated state.
To achieve our goal, we recall that the complete system under consideration composed of the plant, the observer, and the state estimate feedback controller as displayed in Fig. 1.11.
The combined dynamics can be cast into the form :

$$\begin{bmatrix} \frac{d}{dt}x(t) \\ \frac{d}{dt}\hat{x}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{1.48}$$

Making use of the fact

$$\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t) - \hat{x}(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{1.49}$$

Than algebraic manipulations of (1.48) using (1.49), lead to

$$\begin{bmatrix} \frac{d}{dt}x(t) \\ \frac{d}{dt}\hat{x}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{1.50}$$

from which we notice that the combined closed-loop eigenvalues are the union of the eigenvalues of $A - BK$ and $A - LC$. In return, these have negative real parts if the controller gain K and the observer gain L are both chosen to make $A - BK$ and $A - LC$ asymptotically stable.
We conclude that the observer-based controller (OBC) is described by

$$\begin{aligned} \frac{d}{dt}x(t) &= [A - BK - LC]\hat{x}(t) + Ly(t) \\ u(t) &= -K\hat{x}(t) \end{aligned} \tag{1.51}$$

The design task of the Observer-Based Controller amounts to determining the gain matrices K and L subject to some prescribed criteria using alternative techniques, such that (1.51) is asymptotically stable. For more details, see [46].

1.7 Observers of nonlinear systems

In view of the great variety of nonlinear systems, there are several types of observers.

1.7.1 Thau Observer (Lipschitz Observer)

This type of observer was proposed by Thau in 1973 [73]. This observer is suitable for nonlinear systems using Lyapunov methods. The work of [59] offers an application to Thau Observer. Thau approach is proposed for a system composed of two parts : a linear part which is assumed to be observable and a nonlinear part which often satisfies the Lipschitz condition. Let be the nonlinear system described by the following equations :

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & Ax(t)+Bu(t)+f(x(t)) \\ y(t) & = & Cx(t) \end{array}\right. \tag{1.52}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, and $y(t) \in \mathbb{R}^q$ is the output observation vector. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ are constants. $f : \mathbb{R}^n \to \mathbb{R}^n$ is a known γ_f-Lipschitizian function. Thau's observer has the following form :

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + f(x(t)) + L(y(t) - C\hat{x}(t)) \tag{1.53}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain to be determined and $\hat{x}_t \in \mathbb{R}^n$ is the estimate of x_t at time t. Let $e(t) := x(t) - \hat{x}(t)$ be the estimation error. Then the dynamics of the estimation error is described by the following equation :

$$\dot{e}(t) \quad = \quad (A-LC)e(t) + f(\hat{x}(t)) - f(x(t)).$$

If $(A - LC)$ is stable, therefore for any positive definite matrix Q, there exists a unique matrix $P > 0$ such that the following Lyapunov equality holds :

$$(A-LC)^T P + P(A-LC) = -2Q. \tag{1.54}$$

Theorem 1.7.1. *[73] If we choose L such that $(A - LC)$ can give a solution of Lyapunov's equality (1.54) and satisfying :*

$$\frac{\lambda_{\min}(Q)}{\|P\|} > \gamma_f \tag{1.55}$$

where γ_f is the Lipschitz constant, then Thau observer (1.53) is asymptotically stable.

1.7.2 High gain observer

The use of high gain observers has evolved as one of the most important techniques for the design of feedback control of nonlinear systems. This approach, initially mentioned by [73], was the subject of a great deal of research work. The high gain name is because the observer gain is chosen large enough to compensate for the nonlinearity of the system. The high gain observer applies for the following nonlinear systems :

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & f(x(t)) + h(x(t))u(t) \\ y(t) & = & g(x(t)) \end{array}\right. \tag{1.56}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control vector, and $y(t) \in \mathbb{R}^q$ is the output observation vector. $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^p$ are nonlinear functions.
The previous system can always be reduced by means of a transformation (a $\mathscr{C}^1$-diffeomorphism) to the following form

$$\begin{cases} \dot{x}(t) & = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \\ \psi(x(t)) \end{bmatrix} + \begin{bmatrix} h_1(x_1(t)) \\ h_2(x_1(t), x_2(t)) \\ \vdots \\ h_{n-1}(x_1(t), x_2(t), \cdots, x_{n-1}(t)) \\ h_n(x_1(t), x_2(t), \cdots, x_{n-1}(t), x_n(t)) \end{bmatrix} \\ & = F(x(t)) + H(x(t))u(t) \\ y(t) & = Cx(t) = x_1(t) \end{cases} \tag{1.57}$$

Such transformation still exists and is based on the Lie derivative. Here, ψ is a $\mathscr{C}^1$ globally Lipschitz function and $x_i \mapsto h_i(\bar{x}_i)$ is globally Lipschitz for each i, with $\bar{x}_i = \begin{bmatrix} x_1 & \cdots & x_i \end{bmatrix}^T$, If the system (1.57) is uniformly observable, then the high gain observer is written as :

$$\dot{\hat{x}}(t) = f(\hat{x}(t)) + h(\hat{x}(t)) + h(\hat{x}(t))u(t) - S_\infty^{-1} C^T (C\hat{x}(t) - y(t)) \tag{1.58}$$

where S_∞ is the solution to the following equation

$$-\omega S_\infty - A^T S_\infty - S_\infty A + C^T C = 0 \tag{1.59}$$

with

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^p \tag{1.60}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{1.61}$$

The variable ω is chosen large enough to adjust the speed of convergence and it satisfies the following inequality :

$$\|\hat{x}(t) - x(t)\| \leq K(\omega) e^{-\frac{\omega}{3}t} \|\hat{x}(0) - x(0)\| \tag{1.62}$$

where $K(\omega) \geq 0$. By increasing ω, the high gain observer gives a faster exponential response. This type of observer is often used since it presents sufficient conditions for convergence of $\hat{x}(t)$ to $x(t)$.

Unlike the extended Kalman filter, these observers are global with exponential convergence. However, they only apply to a small class of nonlinear systems. Also, despite the popularity of the high gain observer, the choice of the gain remains a bit difficult. In fact, by increasing the gain to gain speed, the observer becomes more and more sensitive to measurement noise.

1.7.3 Two-stage Observer for Descriptor Nonlinear System [20]

The class of systems we consider in this section is the same as in [20]. It is described by the following set of equations :

$$\left\{\begin{array}{rcl} \dot{\overbrace{E_\xi \xi(t)}} & = & A_\xi \xi(t) + B_f f\big(\xi(t), u(t)\big) + D_\xi \omega(t) \\ y(t) & = & C_\xi \xi(t) + B_h h\big(\xi(t), u(t)\big) + D_y \omega(t) \end{array}\right. \tag{1.63}$$

where $\xi \in \mathbb{R}^{n_\xi}, u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$ are respectively the state vector of the system, the known control input vector, and the output measurements vector. The input $\omega \in \mathbb{R}^{n_\omega}$ is the vector of unknown disturbances affecting the system (1.63). The matrices $E_\xi \in \mathbb{R}^{n_d \times n_\xi}$, $A_\xi \in \mathbb{R}^{n_d \times n_\xi}$, $B_f \in \mathbb{R}^{n_d \times n_f}$, $D_\xi \in \mathbb{R}^{n_d \times n_\omega}$, $C_\xi \in \mathbb{R}^{n_y \times n_\xi}$, $B_h \in \mathbb{R}^{n_y \times n_h}$, and $D_y \in \mathbb{R}^{n_y \times n_\omega}$ are known and constant. The nonlinear functions

$$\begin{array}{l} f : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_f} \\ h : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_h} \end{array}$$

are globally Lipschitz [2] with respect to ξ, uniformly on u, with Lipschitz constants γ_f and γ_h, respectively. At this stage, we assume that the following rank condition is satisfied :

$$\operatorname{rank}\left(\begin{bmatrix} E_\xi \\ C_\xi \end{bmatrix}\right) = n_\xi. \tag{1.64}$$

The objective is to find an observer which estimates the vector ξ while respecting a given optimality criterion. In case of systems with E_ξ is left-invertible (i.e. $\operatorname{rank}(E_\xi) = n_\xi$), a standard Luenberger observer can be used to solve easily the estimation problem with any optimality criterion from LMI point of view. However, in the descriptor case, the problem turns out to be difficult in some situations.

Two-stage observer structure

Inspired from previous results in the literature [21] and [57], the two-stage observer consider in [20] has the following structure :

$$\left\{\begin{array}{rcl} \dot{z}(t) & = & A_z z(t) + A_y y_{\eta_1}(t) + B_z f\big(\hat{\xi}(t), u(t)\big) \\ \hat{\xi}(t) & = & z(t) + Q_z y_{\eta_2}(t) \\ y_{\eta_i}(t) & = & y(t) - \eta_i(t),\ i = 1, 2 \end{array}\right. \tag{1.65}$$

[2]If f, h and g are only locally Lipschitz, we may consider their saturated versions f^s, h^s, and g^s, respectively, on an invariant compact set $\mathscr{X}$ in which f^s and h^s satisfy the global Lipschitz property [60].

where $\hat{\xi}(t)$ is the estimate of $\xi(t)$. The vectors $\eta_1(t), \eta_2(t)$ and the matrices A_z, A_y, B_z, and Q_z are the observer parameters to be determined so that the estimation error $\tilde{\xi}(t) = \hat{\xi}(t) - \xi(t)$ converges towards zero in $\mathscr{H}_\infty$ sense. By using (1.63) and (1.65), $\tilde{\xi}(t)$ is expressed as follows :

$$\begin{aligned}\tilde{\xi}(t) = z(t) + \left(Q_z C_\xi - \mathbb{I}_{n_\xi}\right)\xi(t) \\ + Q_z\left[B_h h(\xi(t), u(t)) - \eta_2(t)\right] + Q_z D_y \omega(t).\end{aligned} \tag{1.66}$$

Remark 1.7.1. *In the case $y(t) = C_\xi \xi(t)$, the problem becomes easy and the results are well known in the literature, because with $\eta_2(.) \equiv 0$, equation (1.66) is reduced to*

$$\tilde{\xi}(t) = z(t) + \left(Q_z C_\xi - \mathbb{I}_{n_\xi}\right)\xi(t) \tag{1.67}$$

because $B_h = 0$ and $D_y = 0$ in (1.66). Indeed, in such a case, we have $y_{\eta_2} = y$ in (1.65). Hence for every $t \in \mathbb{R}$,

$$\hat{\xi}(t) = z(t) + Q_z C_\xi \xi(t)$$

from which we deduce that

$$\tilde{\xi}(t) = \hat{\xi}(t) - \xi(t) = z(t) + Q_z C_\xi \xi(t) - \xi(t) = z(t) + \left(Q_z C_\xi - \mathbb{I}_{n_\xi}\right)\xi(t).$$

This means also that the nonlinear term $Q_z B_h\left[h(\xi(t), u(t))\right]$ and the noisy term $Q_z D_y \omega(t)$ vanish from (1.66).

From (1.64), there exist two matrices P_z and Q_z so that

$$P_z E_\xi + Q_z C_\xi = \mathbb{I}_{n_\xi}. \tag{1.68}$$

Then, the dynamics of the estimation error can be reformulated as follows :

$$\dot{\tilde{\xi}}(t) = \dot{z}(t) - P_z \overbrace{E_\xi \xi(t)}^{\cdot}. \tag{1.69}$$

Hence by substituting (1.63) and (1.65) in (1.69) we get straightforwardly appropriate and standard estimation error dynamics after some matrix decompositions. However, having $h(\xi(t), u(t))$ and $\omega(t)$ in the system output complicates the task. Some additional assumptions on the matrices B_h and D_y, related to the disturbance $\omega(t)$, are needed to compute the parameters A_z, A_y, B_z, Q_z.

Assumption 1. *The following condition is fulfilled :*

$$rank \begin{pmatrix} E_\xi & 0 \\ C_\xi & B_h \end{pmatrix} = n_\xi + n_h. \tag{1.70}$$

Condition (1.70) implies

$$n_d + n_y \geq n_\xi + n_h.$$

This ensures that :

$$\begin{cases} P_z E_\xi + Q_z C_\xi = \mathbb{I}_{n_\xi} \\ R_z E_\xi + S_z C_\xi = 0_{n_h \times n_\xi} \\ Q_z B_h = 0_{n_\xi \times n_h} \\ S_z B_h = \mathbb{I}_{n_h}. \end{cases} \tag{1.71}$$

We consider the two-stage observer (1.65) by specifying the variables $\eta_i, i = 1,2$. That is, we consider the following filter :

$$\begin{cases} \dot{z}(t) &=& A_z z(t) + A_y y_{\eta_1}(t) + B_z f\big(\hat{\xi}(t), u(t)\big) \\ \hat{\xi}(t) &=& z(t) + Q_z y_{\eta_2}(t) \\ y_{\eta_1}(t) &=& y(t) - B_h h\big(\hat{\xi}(t), u(t)\big) \\ y_{\eta_2}(t) &=& y(t), \end{cases} \tag{1.72}$$

which is identical to (1.65) with the following choice of η_1 and η_2 :

$$\begin{cases} \eta_1(t) &=& B_h h\big(\hat{\xi}(t), u(t)\big) \\ \eta_2(t) &=& 0_{\mathbb{R}^{n_y}}. \end{cases} \tag{1.73}$$

By considering (1.71), the dynamics (1.66) can be written as :

$$\tilde{\xi}(t) = z(t) - P_z E_\xi \xi(t) + Q_z D_y \omega(t). \tag{1.74}$$

Hence, we can write

$$\dot{\tilde{\xi}}(t) = \dot{z}(t) - P_z \overbrace{E_\xi \xi(t)}^{\cdot} + Q_z D_y \dot{\omega}(t). \tag{1.75}$$

From (1.63) and (1.72), we get the following detailed form of (1.75) :

$$\begin{aligned} \dot{\tilde{\xi}}(t) = A_z \tilde{\xi}(t) &+ \Big(A_z - \mathbb{A} + \mathbb{L} C_\xi\Big)\xi \\ &+ \Big(B_z - P_z B_f\Big) f(\xi, u) \\ &+ \Big(\mathbb{E}_\omega - \mathbb{L}\mathbb{D}_\omega\Big)\omega + B_z \delta f_t - A_y B_h \delta h_t + Q_z D_y \dot{\omega}(t) \end{aligned} \tag{1.76a}$$

$$\delta f_t = f(\hat{\xi}, u) - f(\xi, u) = \mathscr{H}\Phi(t)\tilde{H}, \tag{1.76b}$$

$$\delta h_t = h(\hat{\xi}, u) - h(\xi, u) = \mathscr{F}\Psi(t)\widetilde{F}, \tag{1.76c}$$

$$\mathbb{A} = P_z A_\xi, \mathbb{L} = A_y - A_z Q_z \tag{1.76d}$$

$$\mathbb{E}_\omega = -P_z D_\xi, \mathbb{D}_\omega = -D_y. \tag{1.76e}$$

where

$$
\begin{aligned}
\mathscr{H} &= \left[\left[H_{11}\dots H_{1n_f^1}\right]\left[H_{21}\dots H_{2n_f^2}\right]\dots\left[H_{n_f1}\dots H_{n_f n_f^{n_f}}\right]\right];\\
\mathscr{F} &= \left[\left[F_{11}\dots F_{1n_h^1}\right]\left[F_{21}\dots F_{2n_h^2}\right]\dots\left[F_{n_h1}\dots F_{n_h n_h^{n_h}}\right]\right];\\
\Phi(t) &= \mathrm{diag}\left\{\left(\phi_{11}\mathbb{I}_{n_f^1},\dots,\phi_{1n_f^1}\mathbb{I}_{n_f^1}\right),\dots,\left(\phi_{n_f1}\mathbb{I}_{n_f^{n_f}},\dots,\phi_{n_f n_f^{n_f}}\mathbb{I}_{n_f^{n_f}}\right)\right\};\\
\Psi(t) &= \mathrm{diag}\left\{\left(\psi_{11}\mathbb{I}_{n_h^1},\dots,\psi_{1n_h^1}\mathbb{I}_{n_h^1}\right),\dots,\left(\psi_{n_h1}\mathbb{I}_{n_h^{n_h}},\dots,\psi_{n_h n_h^{n_h}}\mathbb{I}_{n_h^{n_h}}\right)\right\};\\
\tilde{H} &= \left[\underbrace{\left[H_1^T\dots H_1^T\right]}_{n_f^1\ \text{times}}\dots\underbrace{\left[H_i^T\dots H_i^T\right]}_{n_f^i\ \text{times}}\dots\dots\underbrace{\left[H_{n_f}^T\dots H_{n_f}^T\right]}_{n_f^{n_f}\ \text{times}}\right]^T;\\
\widetilde{F} &= \left[\underbrace{\left[F_1^T\dots F_1^T\right]}_{n_h^1\ \text{times}}\dots\underbrace{\left[F_i^T\dots F_i^T\right]}_{n_h^i\ \text{times}}\dots\dots\underbrace{\left[F_{n_h}^T\dots F_{n_h}^T\right]}_{n_h^{n_h}\ \text{times}}\right]^T.
\end{aligned}
$$

From (**??**) and the fact that $Q_zB_h = 0$, the error dynamics (1.76) is reduced to the following one :

$$
\begin{aligned}
\dot{\tilde{\xi}}(t) = &\left(\mathbb{A}-\mathbb{L}C_\xi\right)\tilde{\xi}(t) + B_z\delta f_t - \mathbb{L}B_h\delta h_t\\
&+\left(\mathbb{E}_\omega - \mathbb{L}\mathbb{D}_\omega\right)\omega + Q_zD_y\dot{\omega}.
\end{aligned}
\tag{1.77}
$$

Therefore, equation (1.77) becomes

$$
\dot{\tilde{\xi}}(t) = \left(\left(\mathbb{A}-\mathbb{L}C_\xi\right)+\left[B_z\mathscr{H}\Phi(t)\tilde{H}-\mathbb{L}B_h\mathscr{F}\Psi(t)\widetilde{F}\right]\right)\tilde{\xi}(t)+\left(\mathbb{E}_\omega-\mathbb{L}\mathbb{D}_\omega\right)\omega+Q_zD_y\dot{\omega}.
\tag{1.78}
$$

To study the $\mathscr{H}_\infty$–criterion

$$
\|\tilde{\xi}\|_{\mathscr{L}_2^{n_\xi}} \le \sqrt{\mu^\infty\|\omega\|^2_{\mathscr{L}_2^{n_\omega}} + \nu^\infty\|\tilde{\xi}_0\|^2},
\tag{1.79}
$$

while avoiding the derivative of the disturbance $\omega(t)$, we use the Lyapunov function candidate

$$
V(\xi,w) = (\xi-\Upsilon\omega)^T\,\mathbb{P}\,(\xi-\Upsilon\omega),
$$

where $\mathbb{P}=\mathbb{P}^T>0$.
After using the Young's relation with the same manner and by following by analogy the same procedure of [90], we obtain the following theorem.

Theorem 1.7.2. *[20] Assume that there exist symmetric positive definite matrices $\mathbb{P}\in\mathbb{R}^{n_\xi\times n_\xi}$, $\mathbb{S}\in\mathbb{R}^{n_f\times n_f}$, $\mathbb{M}\in\mathbb{R}^{n_h\times n_h}$ and a matrix $\mathscr{R}\in\mathbb{R}^{n_y\times n_\xi}$ such that the following convex optimization problem is solvable :*

$$
\min \mu_\infty \quad \textit{subject to } (1.81)
\tag{1.80}
$$

$$\begin{bmatrix} \mathrm{He}\left[\mathbb{P}\mathbb{A}-\mathscr{R}C_\xi\right]+\mathbb{I} & \mathbb{P}\mathbb{E}_\omega-\mathscr{R}\mathbb{D}_\omega-\left(\mathbb{A}^T\mathbb{P}\Upsilon-{C_\xi}^T\mathscr{R}^T\Upsilon\right) & \mathbb{P}B_z\mathscr{H}+\tilde{H}^T\mathbb{S} & \mathscr{R}B_h\mathscr{F}-\widetilde{F}^T\mathbb{M} \\ (\star) & -\mathrm{He}\left[\Upsilon^T\left(\mathbb{P}\mathbb{E}_\omega-\mathscr{R}\mathbb{D}_\omega\right)\right]-\mu_\infty^2\mathbb{I} & \Upsilon^T\mathbb{P}B_z\mathscr{H} & \Upsilon^T\mathscr{R}B_h\mathscr{F} \\ (\star) & (\star) & -\Lambda_f\mathbb{S} & 0 \\ (\star) & (\star) & (\star) & -\Lambda_h\mathbb{M} \end{bmatrix} < 0 \tag{1.81}$$

where $\Upsilon = Q_z D_y$. Then, the $\mathscr{H}_\infty$ criterion (1.79) is satisfied with

$$\nu^\infty = \lambda_{\max}(\mathbb{P}) \;\; \textit{and} \;\; \mu_\infty = \frac{\mu_\infty^\star}{\min\left(1, \lambda_{\min}(\mathbb{P})\right)} \tag{1.82}$$

where $\mu_\infty^\star$ is the minimum value returned by the convex optimization problem (1.81).

1.7.4 Sliding mode observer

The concept of a sliding mode emerged from the Soviet Union in the late sixties where the effects of introducing discontinuous control action into dynamical systems were explored. By the use of a judicious switched control law it was found that the system states could be forced to reach and subsequently remain on a pre-defined surface in the state space. Whilst constrained to this surface, the resulting reduced-order motion – referred to as the sliding motion – was shown to be insensitive to any uncertainty or external disturbance signals which were implicit in the input channels. This inherent robustness property has resulted in world wide interest and research in the area of sliding mode control. These ideas have subsequently been employed in other situations including the problem of state estimation via an observer. Consider a system of the following form :

$$\begin{cases} x(t) &= f(x(t);u(t)) \\ y(t) &= g(x(t)) \end{cases} \tag{1.83}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ denote, respectively, the state, the input and the output. $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are two nonlinear functions. The sliding mode observer (see Figure 1.12) associated with the system (1.84) is described by :

$$\begin{cases} \dot{\hat{x}}(t) &= f(\hat{x}(t), u(t)) - \Lambda \mathrm{sign}(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= g(\hat{x}(t)) \end{cases} \tag{1.84}$$

where $\hat{x}$ denotes the estimated state. Λ is the correction term proportional to the sign function applied to the output error e_y. The sliding surface is given by $S = y(t) = y(t)? - \hat{y}(t)$. The sliding mode observer has several advantages :

1. convergence towards a zero surface S with an evolution according to a dynamic n ;

2. guaranteed robustness against external disturbances and modeling errors thanks to the sign function ;
3. the dimensions of the observation system can be reduced to $n-1$.

The estimation error is given by $\varepsilon_y(t) = x(t) - ?\hat{x}(t)$. This error converges to the equilibrium value, from the initial value 0, in two steps. The first is called the approach mode, during which the trajectory of $\varepsilon(t)$ evolves towards the sliding surface such that $y(t) = 0$. During the second stage, often referred to as the slip mode, the estimation error trajectory slides with imposed dynamics and cancels out all estimation errors. In the presence of a fault, this observer is best suited to solve this kind of problem [13].

1.8 Observer-based stabilization : state of art on LMI methods

Here, we recall some LMI methods concerned with Observer-based stabilization. First, we recall the so-called Young's inequality-based approach established in [35] for asymptotic stability of the system (3.7). After this, we summarize the LMI design established in [28]. The end of this section will be devoted to a useful linearization Lemma [28] and [27], which gives a sufficiency condition for linearizing some BMI. We announce it and give the necessary condition part.

1.8.1 Young's relation based-approach [35]

Consider the same class of systems investigated in [35] :

$$x_{t+1} = (A + \Delta A(t))x_t + Bu_t \tag{1.85a}$$
$$y_t = (C + \Delta C(t))x_t \tag{1.85b}$$

where $t = 0, 1, \ldots$, $x_t \in \mathbb{R}^n$ is the state vector, $y_t \in \mathbb{R}^p$ is the output measurement, and $u_t \in \mathbb{R}^m$ is the control input vector. The nominal matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are known. The pairs (A,B) and (A,C) are assumed to be stabilizable and detectable, respectively. The uncertain terms $\Delta A(t) \in \mathbb{R}^{n\times n}$ and $\Delta C(t) \in \mathbb{R}^{p\times n}$ are unknown matrices that account for time-varying parameter uncertainties and are assumed to be of the form

$$\Delta A(t) = M_A F_A(t) N_A, \quad \Delta C(t) = M_C F_C(t) N_C \tag{1.86}$$

where M_A, N_A, M_C, and N_C are known real constant matrices ; $F_A(t)$ and $F_C(t)$ are unknown real time-varying matrices satisfying the inequalities

$$F_A^T(t) F_A(t) \leq \alpha^2 I, \quad F_C^T(t) F_C(t) \leq \beta^2 I, \quad \forall t \in \mathbb{N}. \tag{1.87}$$

For the sake of simplicity, from now on we will denote $\Delta A(t)$ and $\Delta C(t)$ by ΔA and ΔC, respectively. Such a notation will be adopted also for all the matrices that depend on $\Delta A(t)$ and $\Delta C(t)$.

The asymptotic stabilization of the discrete-time linear system (3.1) is addressed using the Luenberger observer

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + L(y_t - C\hat{x}_t) \tag{1.88}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain to be determined and $\hat{x}_t \in \mathbb{R}^n$ is the estimate of x_t at time t. Let $e_t := x_t - \hat{x}_t$ be the estimation error. Then, under the feedback

$$u_t = -K\hat{x}_t \tag{1.89}$$

where $K \in \mathbb{R}^{m\times n}$ is the controller gain, the closed-loop dynamics is described by

$$z_{t+1} = \begin{bmatrix} A - BK + \Delta A & BK \\ \Delta A - L\Delta C & A - LC \end{bmatrix} z_t = \Pi z_t \tag{1.90}$$

where $z_t^T = (x_t^T, e_t^T) \in \mathbb{R}^{2n}$ is the augmented state vector.
The goal is to find suitable controller and observer gains K and L that guarantee the asymptotic stability of (3.7). Toward this end, let $V_t := V(z_t) = z_t^T P z_t$ with $P > 0$ be a Lyapunov function. The closed-loop system (3.7) is asymptotically stable if there exists a matrix $P > 0$ such that the difference between the Lyapunov functions at two consecutive time instants $\Delta V_t := V_{t+1} - V_t$ is negative definite, that is,

$$\Delta V_t = z_t^T(\Pi^T P \Pi - P)z_t < 0, \forall z_t \neq 0, \forall t \in \mathbb{N}. \tag{1.91}$$

Using Schur Lemma, (3.10) turns out to be equivalent to :

$$\begin{bmatrix} -P & \Pi \\ (\star) & -P^{-1} \end{bmatrix} < 0. \tag{1.92}$$

Theorem 1.8.1 ([35]). *If there exist $\varepsilon_2 > 0$, two symmetric positive definite matrices $\mathscr{P} = \begin{bmatrix} \mathscr{P}_{11} & \mathscr{P}_{12} \\ \mathscr{P}_{12}^T & \mathscr{P}_{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ and $\mathscr{G}_1 \in \mathbb{R}^{n\times n}$, $G_2 \in \mathbb{R}^{n\times n}$ invertible, $\hat{K} \in \mathbb{R}^{m\times n}$, and $\hat{L} \in \mathbb{R}^{n\times p}$ such that the LMI*

$$\begin{bmatrix} \begin{bmatrix} \mathscr{W}_{11} & \mathscr{W}_{12} \\ (\star) & -\mathscr{P} \end{bmatrix} & \begin{bmatrix} \mathscr{W}_2 & \mathscr{W}_3 & \mathscr{W}_4 & \mathscr{W}_5 \end{bmatrix} \\ (\star) & \operatorname{diag}\begin{bmatrix} \mathbb{D}_{11}, & -\varepsilon_2 I, & \mathbb{D}_{22}, & \mathbb{D}_{33} \end{bmatrix} \end{bmatrix} < 0 \tag{1.93}$$

with

$$\begin{aligned}
\mathscr{W}_{11} &= \begin{bmatrix} \mathscr{P}_{11} - 2\mathscr{G}_1 + \varepsilon_2 M_A M_A^T & \mathscr{P}_{12} \\ \mathscr{P}_{12}^T & \mathscr{P}_{22} - (G_2 + G_2^T) \end{bmatrix} \\
\mathscr{W}_{12} &= \operatorname{diag}\begin{bmatrix} A\mathscr{G}_1 - B\hat{K}, & G_2^T A - \hat{L}C \end{bmatrix} \\
\mathscr{W}_2 &= \begin{bmatrix} \hat{K}^T B^T & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}^T, \mathbb{D}_{11} = \operatorname{diag}\begin{bmatrix} -\varepsilon_1 & -\varepsilon_1^{-1} \end{bmatrix} \mathscr{G}_1 \\
\mathscr{W}_3 &= \begin{bmatrix} 0 & 0 & N_A \mathscr{G}_1 & 0 \end{bmatrix}^T, \mathbb{D}_{22} = \operatorname{diag}\begin{bmatrix} -\varepsilon_3 & -\varepsilon_3^{-1} \end{bmatrix} I \\
\mathscr{W}_4 &= \begin{bmatrix} 0 & M_A^T G_2 & 0 & 0 \\ 0 & 0 & N_A \mathscr{G}_1 & 0 \end{bmatrix}^T, \mathbb{D}_{33} = \operatorname{diag}\begin{bmatrix} -\varepsilon_4, \frac{-1}{\varepsilon_4} \end{bmatrix} I \\
\mathscr{W}_5 &= \begin{bmatrix} 0 & M_C^T \hat{L}^T & 0 & 0 \\ 0 & 0 & N_C \mathscr{G}_1 & 0 \end{bmatrix}^T
\end{aligned}$$

is feasible for some positive constants ε_1, ε_3, and ε_4, then (3.7) is asymptotically stable with $K = \hat{K}\mathscr{G}_1^{-1}$ and $L = (G_2^T)^{-1}\hat{L}$.

The proof of Theorem 1.8.1 is based on the slack variables technique combined with the judicious handling of the famous Young's inequality.

1.8.2 Ibrir's approach based on separation principle

The separation principle for the stabilization of a dynamical system at an equilibrium point is known in the context of linear systems where the closed-loop eigenvalues under an observer-based controller are the union of the eigenvalues under state feedback and the observer eigenvalues ; hence, stabilization under output feedback can be achieved by solving separate eigenvalue placement problems for the state feedback and the observer.

Theorem 1.8.2 ([28]). *Consider system (3.1) and observer (3.5). If there exist two symmetric and positive definite matrices $P_1 \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n}$, two real matrices $Y_1 \in \mathbb{R}^{m\times n}$, $Y_2 \in \mathbb{R}^{n\times p}$, and five positive constants α, β, ε_1, ε_2, ε_3 such that the following holds :*

$$\begin{bmatrix} P_1 & I \\ \star & (2\beta-\alpha)I \end{bmatrix} > 0, \tag{1.94}$$

$$\begin{bmatrix} \sigma_{11} & AP_1+BY_1 & BY_1 & 0 & 0 & 0 \\ \star & -P_1 & 0 & P_1N_A^T & P_1N_C^T & P_1N_A^T \\ \star & \star & -\alpha I & 0 & 0 & 0 \\ \star & \star & \star & -\varepsilon_1 I & 0 & 0 \\ \star & \star & \star & \star & -\varepsilon_2 I & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_3 I \end{bmatrix} < 0, \tag{1.95}$$

$$\begin{bmatrix} -P_2 & \Gamma_{12} & \beta I & 0 & 0 \\ \star & -P_2 & 0 & P_2M_A & Y_2M_C \\ \star & \star & -P_1 & 0 & 0 \\ \star & \star & \star & -(2-\varepsilon_2)I & 0 \\ \star & \star & \star & \star & -(2-\varepsilon_3)I \end{bmatrix} < 0, \tag{1.96}$$

where $\sigma_{11} = -P_1 + \varepsilon_3 M_A M_A^T$ and $\Gamma_{12} = A^TP_2 + C^TY_2^T$, then there exist two gains $L = P_2^{-1}Y_2$ et $K = Y_1P_1^{-1}$ such that systems (3.1) and (3.5) are globally asymptotically stable under the feedback (3.6).

1.8.3 Ibrir's Linearization Lemma [27]

Beside some congruence transformation, one of the main ideas adopted in the proof of the main result in [27] is the following so-called Ibrir's Linearization Lemma, which proposes an interesting solution to the problem of coexistence of dependent variables in LMI framework, such as a variable and its inverse, as in (3.15).

Lemma 1.8.3 ([27]). *Given the matrices X, Y , and Z of appropriate dimensions where $X = X^\top > 0$ and $Z = Z^\top > 0$. Then, the following linear matrix inequality holds :*

$$\begin{bmatrix} -X & Y^\top \\ Y & -Z^{-1} \end{bmatrix} < 0 \tag{1.97}$$

if there exists a positive constant $\alpha > 0$ *such that*

$$\begin{bmatrix} -X & \alpha Y^\top & 0 \\ \alpha Y & -2\alpha I & Z \\ 0 & Z & -Z \end{bmatrix} < 0. \tag{1.98}$$

This lemma proposed and proved by Ibrir [27] has also exploited in the nonlinear context. Let us mention that (3.35) and (3.36) are not equivalent, see chapter ? ? for the proof.

1.8.4 The Two-step approach proposed in [91]

The proposed approach in [91] deals with observer-based controller design method for Lipschitz nonlinear systems with uncertain parameters and $\mathscr{L}_2$ bounded disturbance inputs, in the continuous case. A new LMI based design technique is developed to solve the problem for both linear and Lipschitz nonlinear systems.
Hence, consider the system (1.99) under the effect of additive disturbances and Lipschitz nonlinearities described by the following equations :

$$\dot{x} = \Big(A + \Delta A(t)\Big)x + Bu + \phi(x,u) + D\omega \tag{1.99a}$$
$$y = \Big(C + \Delta C(t)\Big)x + \psi(x,u) + E\omega \tag{1.99b}$$

where $\omega \in \mathbb{R}^s$ is the vector of the noises, D, and E, are $n \times s$, and $p \times s$ real matrices, respectively. The functions ϕ and ψ in (1.99) are globally Lipschitz uniformly with respect to the second variable, that is there exist $\gamma_\phi > 0$ and $\gamma_\psi > 0$ such that

$$\begin{aligned} \|\phi(x,u) - \phi(z,u)\| &\le \gamma_\phi \|x - z\|, \forall x, z \in \mathbb{R}^n \\ \|\psi(x,u) - \psi(z,u)\| &\le \gamma_\psi \|x - z\|, \forall x, z \in \mathbb{R}^n. \end{aligned}$$

Moreover, without loss of generality let us assume $\phi(0,u) = 0$ for all $u \in \mathbb{R}^m$. As in the linear case, by applying the transformation T
Since B is full column rank, there always exists a non singular matrix T so that $TB = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$. Hence, using the similarity transformation T, system (3.1) can be transformed under the following equivalent form :

$$A := TAT^{-1},\ C := CT^{-1},\ B := TB = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \tag{1.100a}$$
$$\Delta A(t) := T\Delta A(t)T^{-1},\ \Delta C(t) := \Delta C(t)T^{-1}, \tag{1.100b}$$
$$M_1 := TM_1,\ N_1 := N_1T^{-1},\ M_2 := M_2,\ N_2 := N_2T^{-1}. \tag{1.100c}$$

on the system (1.99), the later takes the following equivalent form :

$$D := TD, E := E,\ \phi(x,u) := T\phi(T^{-1}x,u),\ \psi(x,u) := \psi(T^{-1}x,u). \tag{1.101}$$

The rest of the parameters are given by (1.100).

The observer-based controller we consider in this part is under the form :

$$\dot{\hat{x}} = A\hat{x} + Bu + \phi(\hat{x},u) + L\Big(y - C\hat{x} - \psi(\hat{x},u)\Big) \tag{1.102a}$$
$$u = -K\hat{x} \tag{1.102b}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of x, $K \in \mathbb{R}^{m\times n}$ is the control gain, $L \in \mathbb{R}^{n\times p}$ is the observer gain.
Our problem is then reduced to find simultaneously the observer gain L and the state feedback gain K so that the closed loop system is $\mathscr{H}_\infty$ asymptotically stable with attenuation level $\mu > 0$. Under the feedback $u = -K\hat{x}$, the closed-loop system has the form :

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \Big(A - BK + \Delta A(t)\Big) & BK \\ \Big(\Delta A(t) - L\Delta C(t)\Big) & \Big(A - LC\Big) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} \phi(x,u) \\ \Delta\phi - L\Delta\psi \end{bmatrix} + \begin{bmatrix} D \\ D - LE \end{bmatrix} \omega \tag{1.103}$$

where $e = x - \hat{x}$ represents the estimation error, and

$$\Delta\phi = \phi(x,u) - \phi(\hat{x},u) \tag{1.104a}$$
$$\Delta\psi = \psi(x,u) - \psi(\hat{x},u). \tag{1.104b}$$

The nonlinearities in (1.103) are handled as in [89]. Hence, according to [89, Lemma 6], the Lipschitz property on ϕ and ψ system (1.103) can be rewritten under the form

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \Big(\mathscr{A}(\Lambda) - BK + \Delta A(t)\Big) & BK \\ \Big(\Delta A - L\Delta C(t)\Big) & \Big(\mathscr{A}(\Upsilon) - L\mathscr{C}(\Xi)\Big) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} D \\ D - LE \end{bmatrix} \omega \tag{1.105}$$

which takes the compact form

$$\dot{z} = \Pi_1(\Lambda,\Upsilon,\Xi)z + \Pi_2\omega \tag{1.106}$$

where $z^T = [x^T \; e^T]$ and the detailed expression of the matrices in $\Pi_1(\Lambda,\Upsilon,\Xi)$ derived from the nonlinerities are respectively

$$\mathscr{A}(\Lambda) := A + \Lambda = A + \sum_{i=1}^{n}\sum_{j=1}^{n} \phi_{ij}(x,0)H_{ij}^{n}$$

$$\mathscr{A}(\Upsilon) := A + \Upsilon = A + \sum_{i=1}^{n}\sum_{j=1}^{n} \phi_{ij}(x,\hat{x})H_{ij}^{n}$$

$$\mathscr{C}(\Xi) := C + \Xi = C + \sum_{i=1}^{p}\sum_{j=1}^{n} \psi_{ij}(x,\hat{x})H_{ij}^{p,n}.$$

and the parameters Λ and Υ (resp. Ξ) belong to bounded convex set $\mathscr{H}_n$ (resp. $\mathscr{H}_p$) for which the set of vertices is defined by :

$$\mathscr{V}_{\mathscr{H}n} = \Big\{\Pi \in \mathbb{R}^{n\times n}, \Pi_{ij} \in \{\underline{\gamma}_{\phi_{ij}}, \bar{\gamma}_{\phi_{ij}}\}\Big\}, \Big(\text{resp.}\, \mathscr{V}_{\mathscr{H}p} = \Big\{\Gamma \in \mathbb{R}^{p\times n}, \Gamma_{ij} \in \{\underline{\gamma}_{\psi_{ij}}, \bar{\gamma}_{\psi_{ij}}\}\Big\}\Big).$$

The objective consists in finding matrices K and L so that the augmented closed-loop system (1.105) with $\omega(t) = 0$ is asymptotically stable; and the effect of $\omega(t)$ on the tracking error $Z(t) = Hx(t)$, where H is a known matrix of appropriate dimension, is attenuated in the $\mathscr{H}_\infty$ sense. More precisely, it is required that

$$\|Z\|_2 < \mu\|\omega\|_2,\ \forall\omega \in \mathscr{L}_2[0,\infty) \tag{1.107}$$

where $\mu > 0$ is the disturbance attenuation level to be determined. To this end, let's consider the following index $\mathscr{J}$

$$\mathscr{J} = \int_0^\infty [Z(t)^T Z(t) - \mu^2\omega(t)^T\omega(t)]dt. \tag{1.108}$$

Under zeros-initial conditions, the Lyapunov function satisfies $V(0) = 0$ and $V(\infty) \geq 0$, which leads to

$$\begin{aligned}\mathscr{J} &= \int_0^\infty [Z^T(t)Z(t) - \mu^2\omega(t)^T\omega(t) + \dot{V}(t)]dt - V(\infty) \\ &\leq \int_0^\infty [Z^T(t)Z(t) - \mu^2\omega(t)^T\omega(t) + \dot{V}(t)]dt.\end{aligned} \tag{1.109}$$

To satisfy the attenuation level in (1.107), it suffices that inequality (1.110) holds :

$$Z^T(t)Z(t) - \mu^2\omega(t)^T\omega(t) + \dot{V}(t) < 0,\ \forall t \in [0,\infty). \tag{1.110}$$

Using Lyapunov function candidate

$$V(z) = z^T \mathrm{diag}\{P,R\}z, \tag{1.111}$$

inequality (1.110) is then satisfied if the following quadratic form

$$\begin{bmatrix} x \\ e \\ \omega \end{bmatrix}^T \begin{bmatrix} \Omega_{11} + H^T H & PBK & PD \\ (\star) & \Omega_{22} & R(D-LE) \\ (\star) & (\star) & -\mu^2 I \end{bmatrix} \begin{bmatrix} x \\ e \\ \omega \end{bmatrix} < 0 \tag{1.112}$$

holds for all $(\Lambda,\Upsilon,\Xi) \in \mathscr{H}_n \times \mathscr{H}_n \times \mathscr{H}_p$, or equivalently

$$\Omega(\Lambda,\Upsilon,\Xi) < 0,\ \forall(\Lambda,\Upsilon,\Xi) \in \mathscr{H}_n \times \mathscr{H}_n \times \mathscr{H}_p \tag{1.113}$$

with

$$\Omega(\Lambda,\Upsilon,\Xi) = \begin{bmatrix} \tilde{\Omega}_{11} & PBK & PM_1 & 0 & 0 & PD \\ (\star) & \tilde{\Omega}_{22} & 0 & RM_1 & \hat{L}M_2 & RD-\hat{L}E \\ (\star) & (\star) & -\varepsilon_1 I & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & -\varepsilon_2 I & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -\varepsilon_3 I & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\delta I \end{bmatrix}, \tag{1.114}$$

and

$$\tilde{\Omega}_{11} = \mathrm{He}\Big\{P\mathscr{A}(\Lambda) - PBK\Big\} + H^T H + (\varepsilon_1+\varepsilon_2)N_1^T N_1 + \varepsilon_3 N_2^T N_2 \tag{1.115}$$

$$\tilde{\Omega}_{22} \quad = \quad \mathrm{He}\Big\{R\mathscr{A}(\Upsilon) - \hat{L}\mathscr{C}(\Xi)\Big\}, \hat{L} = RL, \, \delta = \mu^2. \tag{1.116}$$

The general two-step algorithm proposed in [91] to linearize the bilinear term PBK in (1.114) combines the Young's inequality based approach when it works and the two-step approach. We begin by recalling the Young's inequality based approach applied to (1.114).

Theorem 1.8.1 ([91]). *System (1.99) is asymptotically stabilizable by (1.102) if for fixed scalars $\varepsilon_2 > 0$, $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$, there exist two positive definite matrices $\mathscr{X} \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, two matrices $\hat{K} \in \mathbb{R}^{m\times n}$, $\hat{L} \in \mathbb{R}^{n\times p}$, and a scalar $\varepsilon_1 > 0$ so that the following LMI optimization problem has a solution*

minimize(δ)
subject to the following set of LMIs

$$\begin{bmatrix} \mathscr{K}_{11}(\Lambda,\Upsilon,\Xi) & \mathscr{K}_{12} \\ (\star) & -\mathrm{diag}\left\{\frac{1}{\varepsilon_4}\mathscr{X},\, \varepsilon_4\mathscr{X},\, \varepsilon_1 I,\, \varepsilon_2 I,\, \frac{1}{\varepsilon_2}, \varepsilon_3 I,\, \frac{1}{\varepsilon_3}I,\, \delta I\right\} \end{bmatrix} < 0 \tag{1.117}$$

for all $(\Lambda,\Upsilon,\Xi) \in \mathscr{H}_n \times \mathscr{H}_n \times \mathscr{H}_p$

with

$$\mathscr{K}_{12} = \begin{bmatrix} B\hat{K} & 0 & \mathscr{X}N_1^T & 0 & \mathscr{X}N_1^T & 0 & \mathscr{X}N_2^T & D \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & RM_1 & 0 & \hat{L}M_2 & 0 & RD-\hat{L}E \end{bmatrix}$$

$$\mathscr{K}_{11}(\Lambda,\Upsilon,\Xi) = \begin{bmatrix} \mathscr{K}_{11}^{11}(\Lambda) & \mathscr{X}H^T & 0 \\ (\star) & -I & 0 \\ (\star) & (\star) & \mathscr{K}_{11}^{33}(\Upsilon,\Xi) \end{bmatrix}$$

$$\mathscr{K}_{11}^{11}(\Lambda) = \mathrm{He}\left(\mathscr{A}(\Lambda)\mathscr{X} - B\hat{K}\right) + \varepsilon_1 M_1 M_1^T$$

$$\mathscr{K}_{11}^{33}(\Upsilon,\Xi) = \mathrm{He}\left(R\mathscr{A}(\Upsilon) - \hat{L}\mathscr{C}(\Xi)\right).$$

Hence, the $\mathscr{H}_\infty$ stabilizing observer-based control gains are given by $K = \hat{K}\mathscr{X}^{-1}$ and $L = R^{-1}\hat{L}$, and the optimal disturbance attenuation level is given by $\mu_{\min} = \sqrt{\delta}$.

Remark 1.8.4. *Inequality (1.117) is a LMI only when a priori fixes the parameter ε_4. For this approach to work, it is therefore necessary to find such ε_4 making (1.117) feasible.*

When the Young's inequality based approach fails, we appeal to the new two-step method proposed in [91], which consists in solving in a first step the problem of stabilization by static state feedback, which gives as a solution part of the Lyapunov matrix, which we which we incorporate into the bilinear term PBK in (1.114).
Since the matrix P_{11} is symmetric positive definite, then invertible, there always exists a matrix $S \in \mathbb{R}^{m\times(n-m)}$ so that

$$P_{12} = P_{11}S. \tag{1.118}$$

Starting from inequality (1.114). Taking into account the detailed structure of the term PBK, and the structure (3.22), inequalities (1.113) become after using the change of variables $\hat{K} = P_{11}K,\ \hat{L} = RL,\ \delta = \mu^2$:

$$\begin{bmatrix} \tilde{\Omega}_{11} & \begin{bmatrix} \hat{K} \\ S^T\hat{K} \end{bmatrix} & PM_1 & 0 & 0 & PD \\ (\star) & \tilde{\Omega}_{22} & 0 & RM_1 & \hat{L}M_2 & RD - \hat{L}E \\ (\star) & (\star) & -\varepsilon_1 I & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & -\varepsilon_2 I & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -\varepsilon_3 I & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\delta I \end{bmatrix} < 0 \tag{1.119}$$

for all $(\Lambda, \Upsilon, \Xi) \in \mathscr{V}_{\mathscr{H}n} \times \mathscr{V}_{\mathscr{H}n} \times \mathscr{V}_{\mathscr{H}p}$, and

$$\tilde{\Omega}_{11} = \mathrm{He}\left\{ P\mathscr{A}(\Lambda) - \begin{bmatrix} \hat{K} \\ S^T\hat{K} \end{bmatrix} \right\} + H^T H + (\varepsilon_1 + \varepsilon_2) N_1^T N_1 + \varepsilon_3 N_2^T N_2.$$

The rest of the approach proposed in [91] is reported in the following new two-step algorithm :

Algorithm NTSA : general algorithm for nonlinear systems [91]

step 1 : Solve the optimization problem : minimize δ subject to LMI (1.117) with the decision variables $\mathscr{X}, R, \hat{K}, \hat{L}$, and $\varepsilon_1 > 0$ and go to *step 2* ;

step 2 : If LMI (1.117) is found feasible, then compute the observer-based controller gains as $K = \hat{K}^T \mathscr{X}^{-1}$, $L = R^{-1}\hat{L}^T$ and $\mu_{\min} = \sqrt{\delta}$. Otherwise, go to *step 3* ;

step 3 : Solve the stabilization problem of (1.99) by a static state feedback $u = -\mathscr{K}x$:

$$\text{minimize } \hat{\delta} \text{ subject to :} \tag{1.120a}$$

$$\begin{bmatrix} \mathrm{He}\left(\mathscr{A}(\Lambda)\mathscr{X} - B\bar{\mathscr{K}}\right) + \varepsilon M_1 M_1^T & \mathscr{X}N_1^T & \mathscr{X}H^T & D \\ (\star) & -\varepsilon I & 0 & 0 \\ (\star) & (\star) & -I & 0 \\ (\star) & (\star) & (\star) & -\hat{\delta} I \end{bmatrix} < 0, \forall \Lambda \in \mathscr{V}_{\mathscr{H}n} \tag{1.120b}$$

where $\bar{\mathscr{K}} = \mathscr{K}\mathscr{X}$, $\hat{\delta} = \nu^2$ and $\mathscr{X}^{-1} = \mathscr{P} = \begin{bmatrix} \mathscr{P}_{11} & \mathscr{P}_{12} \\ \mathscr{P}_{12}^T & \mathscr{P}_{22} \end{bmatrix}$ and go to *step 4* ;

step 4 : Compute $\bar{S}$ by $\bar{S} = \mathscr{P}_{11}^{-1}\mathscr{P}_{12}$, put $S = \Theta\bar{S}$, with $\Theta = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and go to *step 5* ;

step 5 : Solve the optimization problem : minimize δ subject to (1.119) with $S = \Theta\bar{S}$ by using the gridding method on $\alpha_i, i = 1, \ldots, m$, with the decision variables P_{11}, P_{22}, R, $\hat{K}$, $\hat{L}$ and $\varepsilon_i, i = 1, 2, 3$, and go to *step 6* ;

step 6 Compute the observer-based controller gains as $K = P_{11}^{-1}\hat{K}$, $L = R^{-1}\hat{L}$ and the optimal disturbance attenuation level as $\mu_{\min} = \sqrt{\delta}$.

1.9 Interconnected systems

The term interconnected model is associated with any model showing a finite number of weakly coupled or strongly coupled subsystems by interconnections. This model generally depends on the nature of the physical system itself and the nature of the problem to be solved. In general, models that are not interconnected in nature may undergo basic changes in the state space in order to design an interconnected model.
Modeling In the case of an interconnected structure, the system is broken down into N sub-systems interacting with each other through internal signals. Each subsystem has its own inputs and outputs, and its dynamics are affected by both local inputs and interactions with other subsystems.

Structures of interconnected systems

In order to better present the studied model, this section looks at the state representation of an interconnected large-scale system. Three types of representation are possible. The first is a so-called compact unstructured representation whose decomposition gives rise to the other two models : the input / output oriented model and the interconnection oriented model. The structures of these three models are shown in Figure ? ? ? ? :

1. **Unstructured model :** Let be the model given by (1.123). This model is qualified as large dimension if the number of inputs, outputs and states is large, which makes its study impracticable. Despite the very frequent use of this structure in the theory of multivariate systems, it is of minor importance for large-dimensional systems because it does not add anything on subsystems.
2. **Input/output oriented structure :** In the case of multi-agent systems, the input/output vectors are decomposed down into sub-vectors $u^T = \begin{bmatrix} u_1^T & \cdots & u_N^T \end{bmatrix}$ and $y^T = \begin{bmatrix} y_1^T & \cdots & y_N^T \end{bmatrix}$ while the system is represented by a compact structure. In this case, the decentralized control center generates the command u_i and access the output measurements y_i. Unlike the model (1.123), the model described below makes the structural constraints of decentralized control accessible.
$$\begin{cases} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{N} B_i u_i(t) \\ y_i(t) &= C_i x(t) \end{cases} \tag{1.121}$$
3. **Interaction oriented model :** Many large-scale systems result from complex relationships between agents of the same environment. These relationships are based on a number of interactions between different agents and can have several different forms, such as an exchange of energy or information flow . This type of model is used mainly in the modeling of social phenomena and in software programming. The interactions are represented by signals through which the subsystems interact with each other. These additional signals are the internal signals h_i and z_i represented in the subsystem model as follows :
$$\begin{cases} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + E_i h_i(t) \\ y_i(t) &= C_i x_i(t) + F_i h_i(t) \\ z_i(t) &= C_{zi} x_i(t) + F_{zi} h_i(t) \end{cases} \tag{1.122}$$

Consider a large-dimensional linear system S whose global state representation is as follows :

$$\begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) \\ y(t) & = Cx(t) \end{cases} \tag{1.123}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the input, control and output vectors, respectively.

1.9.1 Control of large systems

One of the best ways to control a large, interconnected system is to use decentralized control since it is based only on local information from each subsystem. The command must be robust to compensate for the imperfections related to the modeling of the subsystems.

1.10 Some approach dealing with Stabilization problem for large-scale interconnected non linear systems

1.10.1 Siljak & Stipanovic approach for stabilisation of LSS

The following class of large-scale interconnected non linear systems is considered in [65, 54] :

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + h_i(t,x), \ \cdots x_i(t_0) = x_{i0} \tag{1.124a}$$

$$y_i(t) = C_i x_i(t) \tag{1.124b}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{m_i}$, $h_i \in \mathbb{R}^{n_i}$, t_0 and x_{i0} are the state, input, output, non-linear interconnection function, initial time and the initial state of the ith subsystem, System (1.124) consists of N subsystems, that is, $i = 1 : N$. The interconnections are assumed to be piecewise-continuous functions in both variables, and satisfy the quadratic constraints [62] :

$$h_i^T(t,x) h_i(t,x) \le \alpha_i^2 x^T H_i^T H_i x \le \alpha_i^2 \nu_i x^\top x \tag{1.125}$$

where $\alpha_i > 0$ are interconnection bounds, H_i are bounding matrices, $\nu_i = \lambda_{\max}(H_i^\top H_i)$ and $x^T = [x_1^T, x_2^T, \cdots, x_N^T]$ is the state of the overall system. It is assumed that A_i is Hurwitz. One specific practical application whose system model conforms to (1.124) with the quadratic interconnection bounds (1.125) is a multimachine power system consisting of N interconnected machines with steam valve control ; the dynamic model is discussed in [62].
The overall system (1.124) can be rewritten as

$$\begin{cases} \dot{x}(t) & = & A_D x(t) + B_D u(t) + h(t,x), \, x(t_0) = x_0 \\ y(t) & = & C_D x(t) \end{cases} \tag{1.126}$$

where $A_D = \text{diag}(A_1, \cdots, A_N)$, $B_D = \text{diag}(B_1, \cdots, B_N)$, $C_D = \text{diag}(C_1, \cdots, C_N)$, $u^T = [u_1^T, \cdots, u_N^T]$, $y^T = [y_1^T, \cdots, y_N^T]$, and $h^T = [h_1^T, \cdots, h_N^T]$. The nonlinear interconnections $h(t,x)$ are bounded as follows :

$$h^T(t,x)h(t,x) \leq x^T \Upsilon x \tag{1.127}$$

where $\Upsilon = \sum_{i=1}^N \alpha_i^2 H_i^T H_i$. The pair (A_D, B_D) is controllable and the pair (A_D, C_D) is observable.

The following linear decentralized controller and observer are considered :

$$u(t) = K_D \hat{x}(t) \tag{1.128a}$$

$$\dot{\hat{x}}(t) = A_D \hat{x}(t) + B_D u(t) + L_D(y(t) - C_D \hat{x}(t)) \tag{1.128b}$$

where $K_D = diag(K_1, \cdots K_N)$ and $L_D = diag(L_1, \cdots L_N)$ are the controller and observer gain matrices, respectively. Rewriting (1.126) and (1.128) in the coordinates $x(t)$ and $\tilde{x}(t)$, where $\tilde{x}(t) = x(t) - \hat{x}(t)$ is the estimation error, the closed-loop dynamics is

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & (A_D + B_D K_D)x(t) - B_D K_D \tilde{x}(t) + h(t,x) \\ \dot{\tilde{x}}(t) & = & (A_D - L_D C_D)\tilde{x}(t) + h(t,x) \end{array} \right. \tag{1.129}$$

Two broad methods are used to design observer-based decentralized output feedback controllers for large-scale systems :

1. Design local observer and controller for each subsystem independently, and check the stability of the overall closed-loop system. In this method, the interconnection in each subsystem is regarded as an unknown input [80], [1].
2. Design the observer and controller by posing the output feedback stabilization problem as an optimization problem. The optimization approach using LMIs can be found in [65].

We give a brief overview of this approach. It is assumed that H_i is known. The controller gain K_D and the observer gain L_D are obtained from the following minimization problem, provided it is feasible.

Theorem 1.10.1 ([65]). *For the large-scale system given by* (1.126)*, the decentralized controller and observer as given by* (1.128) *will result in exponential stabilization of the overall system, if the optimization problem,* (1.146)*, is feasible,*

$$\begin{aligned}
&\text{Minimise } \sum_{i=1}^{N} \gamma_i \text{ subject to} \\
&\tilde{P}_1 > 0, \quad \tilde{P}_2 > 0; \\
&\begin{bmatrix}
A_i^T \tilde{P}_1 + \tilde{P}_1 A_D + M_D^T + M_D & -M_D & \tilde{P}_1 & H_1^T & \cdots & H_N^T \\
-M_D^T & A_i^T \tilde{P}_2 + \tilde{P}_2 A_D - C_D^T N_D^T - N_D C_D & \tilde{P}_2 & 0 & \cdots & 0 \\
\tilde{P}_1 & \tilde{P}_2 & -I & 0 & \cdots & 0 \\
H_1 & 0 & 0 & -\gamma_1 I & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
H_N & 0 & 0 & 0 & \cdots & -\gamma_N I
\end{bmatrix} < 0
\end{aligned} \tag{1.130}$$

where $\gamma_i = \frac{1}{\alpha_i^2}$ *and* $\tilde{P}_1 B_D K_D = M_D$, $\tilde{P}_2 L_D = N_D$. *The gain matrices* K_D *and* L_D *were extracted as follows :* $K_D = B_D^{-1}\tilde{P}_1^{-1} M_D$ *and* $L_D = \tilde{P}_2^{-1} N_D$.

The LMI formulation given above requires the invertibility of the input matrix B_D ; that is, it requires as many independent control inputs as the number of state variables in each subsystem. Although the formulation as an optimization problem using the LMI framework is quite elegant from a numerical perspective, as it not only computes the gains but also maximizes the interconnection bounds, it can be applied only to a restrictive class of systems in which the input matrix is invertible. Further, obtaining block diagonal positive definite solutions, $\tilde{P}_1$ and $\tilde{P}_2$, from the optimization problem (1.146) in itself is a challenging problem ; the solutions need to be block diagonal for the computed K_D and L_D to be block diagonal ; otherwise, one does not get a decentralized solution.
Notice that, because of the nature of the interconnection, $h_i(t,x)$, in some cases, system (1.124) may not be stabilizable even with full-state feedback control. For example :

Example 1.10.2 ([54]).

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + \gamma_1 \begin{bmatrix} x_{11} - x_{12} \\ x_{21} \end{bmatrix}, y_1 = [1 \cdots 0] x_1 \tag{1.131}$$

If $\gamma_1 = 1$, *then the first state of* x_1, x_{11}, *has the dynamics* $\dot{x}_{11} = x_{11}$, *which is unstable and we lose controllability of the system. One cannot design a controller to stabilize the system* (1.131) *with the given interconnection, although* (A_1, B_1) *is controllable.*

From the example, it is clear that the structure and bounds of the interconnections will affect controllability of subsystems. The same holds true for observability of the system. Further, we develop sufficient conditions for the existence of symmetric positive definite solutions.

1.10.2 Pagilla and Y. Zhu approach for The decentralized output feedback controller design

For simplicity define the following : $A_{B_i} = B_i K_i$ and $A_{C_i} = L_i C_i$, $\gamma^2 = \sum_{i=1}^{N} 2\alpha_i^2 \nu_i$.

Definition 1.10.3 ([54]). *The real number*

$$\delta(M,N) \triangleq \min_{\omega \in \mathbb{R}} \sigma_{\min} \begin{bmatrix} i\omega I - M \\ N \end{bmatrix} \tag{1.132}$$

where $i = \sqrt{-1}$, $M \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{p \times n}$.

Theorem 1.10.4 ([54]). *For the large-scale system given by* (1.126), *the decentralized controller and observer as given by* (1.128) *will result in exponential stabilization of the overall system, if*

$$\delta(A_i^T, \sqrt{2(\gamma^2 + \eta_i)}(B_i^T B_i)^{-\frac{1}{2}} B_i^T) > \sqrt{2(\gamma^2 + \eta_i)} \tag{1.133}$$

and

$$\sqrt{\lambda_{max}(\tilde{Q}_{i1})+\tilde{\eta}_i} < \delta(A_i,C_i). \tag{1.134}$$

are satisfied for all $i=1:N$. The gains K_i and L_i are respectively given by

$$K_i = -(B_i^\top B_i)^{-1}(B_i^\top P_i),\ L_i = \frac{\varepsilon}{2}\tilde{P}_i^{-1}C_i^\top. \tag{1.135}$$

Il should be noticed that (1.133) and (1.134) guarantee, for some $\eta_i > 0$ and $\tilde{\eta}_i > 0$ the existence of symmetric positive definite solutions of two algebraic Ricatti equations (AREs) :

$$A_i^T P_i + P_i A_i + 2P_i(I - B_i(B_i^T B_i)^{-1}B_i^T)P_i + \gamma^2 I + \eta_i I = 0 \tag{1.136}$$

$$A_i^T \tilde{P}_i + \tilde{P}_i A_i + \tilde{P}_i\tilde{P}_i + \tilde{Q}_{i1} + \tilde{\eta}_i I - \varepsilon_i C_i^T C_i = 0 \tag{1.137}$$

guaranteeing the negativity of a certain Lyapunov functions, hence ensuring the exponential stabilization of the set closed loop system.

Remark 1.10.5. *The control gain matrix K_i given by* (1.135) *requires that $(B_i^\top B_i)$ is invertible ; $(B_i^\top B_i)$ is invertible if B_i has full column rank, which is always possible.*

Remark 1.10.6. *If A_i is not stable, then we can stabilize A_i by changing K_i and L_i given by* (1.135)*, to the following :*

$$K_i = -(B_i^\top B_i)^{-1}B_i^\top P_i - \bar{K}_i,\ L_i = \frac{\varepsilon_i}{2}\tilde{P}_i^{-1}C_i^\top + \bar{L}_i$$

where $\bar{K}_i$ and $\bar{L}_i$ are pre-feedback gains such that $A_i^c \triangleq A_i - B_i\bar{K}_i$ and $A_i^o \triangleq A_i - \bar{L}_i C_i$ are Hurwitz. In such a case, A_i in (1.136) *and* (1.137) *must be replaced by A_i^c and A_i^o, respectively. Notice that we cannot design the controller and observer independently, that is, the separation principle does not hold ; the ARE* (1.137) *depends on the control gain matrix K_i. It should be noted that the above reduction procedure has yielded the following : One can design the controller gain independent of the observer and further, only the first ARE,* (1.136)*, explicitly depends on the interconnection bounds.*

1.10.3 Kalsi et al. approach for decentralized dynamic feedback control for nonlinear interconnected systems [33]

We consider the nonlinear interconnected system consist of N sub-systems modeled by

$$\dot{x}_i = A_i x_i + B_{i1}u_{i1} + B_{i2}u_{i2}(x) \tag{1.138a}$$

$$y_i = C_i x_i,\quad i = 1,\cdots,N \tag{1.138b}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_{i1} \in \mathbb{R}^{m}_{i1}$ and $y_i \in \mathbb{R}^{p_i}$ are the state input, output vectors, respectively, of the $i-th$ subsystem, $x = [x_1^T,\cdots,x_N^T]^T \in \mathbb{R}^n$, with $n = \sum_{i=1}^N n_i$, $u_{i2} \in \mathbb{R}^{m_{i2}}$ models the unknown nonlinear interconnection of the $i-th$ subsystem with others. The nonlinear interconnection of each subsystem is globally bounded, that is $\|u_{i2}\| \leq \rho_i$ for some $\rho_i > 0$, and satisfies the same quadratic constraint as, for example, [64] and [53] that is

$$(B_{i2}u_{i2}(x))^T(B_{i2}u_{i2}(x)) \leq v_i^2 x\Gamma_i^T\Gamma_i x \tag{1.139}$$

where v_i is a known positive constant and $v_i \in \mathbb{R}^{n_i \times n}$ is a known interconnection matrix. We assume for the $i-th$ subsystem that the input matrix $B_{i1} \in \mathbb{R}^{n_i \times m_{i1}}(m_{i1} \leq n_i)$ has full rank, the pair (A_i, B_{i1}) is controllable, and the pair A_i, C_i is observable. We also assume that

$$\operatorname{rank}(B_{i2}) = \operatorname{rank}(C_iB_{i2}) = r_i$$

where $r_i \leq m_{i2} \leq p_i$, and the system zeros of the system model given by the triple (A_i, B_{i2}, c_i) are in the open left-hand complex plane, that is

$$\operatorname{rank}\begin{bmatrix} sI_{n_i} - A_i & B_{i2} \\ C_i & 0 \end{bmatrix} = n_i + r_i \tag{1.140}$$

for all s such that $\mathscr{R}(s) \geq 0$. In [34], the design of the decentralized observer controller compensator was subject to the assumption that for the controllable pair (A_{ci}, B_{i1})

$$\delta(A_{ci}, \sqrt{2\beta}B_{i1}(B_{i1}^TB_{i1})\frac{1}{1}) > \sqrt{2\beta}$$

where

$$\beta = \sum_{i=1}^{N} \mu_i^2 \lambda_{max}(\Gamma_i^T\Gamma)$$

with μ_i and Γ_i defined in (1.139).
As spiculated in [33], the above condition is quite conservative as can be seen on the following example.

Remark 1.10.7 ([33]). *The dynamics of the first subsystem are given by*

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -125 & -22.5 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{11} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_{12}(x)$$

where

$$u_{12}(x) = 0.2\cos(x_4)\sum_{i=1}^{5} \frac{x_i}{\sqrt{10}}. \tag{1.141}$$

It satisfies (1.139) *with* $\mu_1 = 0.2$. *If we increase* μ_1 *from* 0.2 *to* 0.8 *then it is easy to verify that*

$$\delta(A_{c1}, \sqrt{2\beta}B_{11}(B_{11}^TB_{11})^{-1/2}) < \sqrt{2\beta}$$

which implies that the delta condition is not satisfied. The dynamics of the second subsystem are given by

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -37.5 & -50 & -13.5 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_{21} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u_{22}(x)$$

where

$$u_{22}(x) = 0.2\cos(x_1)\sum_{i=1}^{5}\frac{x_i}{\sqrt{15}}$$

It satisfies (1.139) *with* $\mu_2 = 0.2$. *If we increase* μ_2 *from* 0.2 *to* 0.8 *then it is easy to verify that*

$$\delta(A_{c2}, \sqrt{2\beta}B_{21}(B_{21}^T B_{21})^{-1/2}) < \sqrt{2\beta}$$

which implies that the delta condition is once again not satisfied.

Kalsi et al. [33] constructed a decentralized dynamic output feed-back based linear controller of the form

$$u_{i1} = K_i\hat{x}_i \tag{1.142}$$

where K_i is the feedback gain matrix for the $i-th$ subsystem and $\hat{x}_i$ is the estimate of the i-th subsystem's state vector, so that the closed-loop system is robustly stabilized with as large interconnection bounds as possible. They proposed to employ a Linear Matrix Inequality (LMI) approach to achieve the above objective using local observers for the system modeled by (1.138) and (1.138a). If the earlier stated assumptions are satisfied, then the local sliding mode observer for the $i-th$ subsystem has the form

$$\dot{\hat{x}}_i = (A_i - L_iC_i)\hat{x}_i + L_iy_i + B_{i1}u_{i1} - B_{i2}E_i(y_i, \hat{y}_i, \eta_i) \tag{1.143}$$

with $\hat{y}_i = C_i\hat{x}_i$ and

$$E_i(y, \hat{y}_i, \eta_i) = \begin{cases} \eta_i\dfrac{F_i(\hat{y}_i - y_i)}{\|F_i(\hat{y}_i - y_i)\|} & \text{if} \quad F_i(\hat{y}_i - y_i) \neq 0 \\ 0 & \text{if} \quad F_i(\hat{y}_i - y_i) = 0 \end{cases}$$

where η_i is a positive design parameter, and $L_i \in \mathbb{R}^{n_i \times p_i}$ and $F_i \in \mathbb{R}^{m_{i2} \times p_i}$ are matrices satisfying

$$(A_i - L_iC_i)^T P_i^\circ + P_i^\circ(A_i - L_iC_i) = -Q_i^\circ \tag{1.144}$$

and

$$F_iC_i = B_{i2}^T P_i^\circ \tag{1.145}$$

for some symmetric positive definite $P_i^\circ \in \mathbb{R}^{n_i \times n_i}$ and $Q_i^\circ \in \mathbb{R}^{n_i \times n_i}$. It follows from [25] that the $\hat{x}_i$ converges asymptotically to x_i for $\eta_i \geq \rho_i$. The detailed design procedures for the matrices L_i, F_i and P_i° that satisfy the conditions (1.144) and (1.145) are given in [34] :

1. Find the non-singular matrices T_i and S_i such that the triple A_i, B_{i2}, C_i can be transformed into the following form,

$$\hat{A} = T_iA_iT_i^{-1} = \begin{bmatrix} A_{i_{11}} & A_{i_{12}} \\ A_{i_{21}} & A_{i_{22}} \end{bmatrix},$$

$$\hat{B}_{i2} = T_iB_{i2} = \begin{bmatrix} B_{i_{21}} \\ 0 \end{bmatrix},$$

$$\hat{C}_i = S_iC_iT_i^{-1} = \begin{bmatrix} I_n & 0 \\ 0 & C_{i_{22}} \end{bmatrix}$$

where $A_{i_{11}} \in \mathbb{R}^{r_i \times r_i}$, $A_{i_{22}} \in \mathbb{R}^{(n_i - r_i)\times(n_i - r_i)}$, $B_{i_{21}} \in \mathbb{R}^{r_i \times m_{i2}}$ and $C_{i_{22}} \in \mathbb{R}^{(p_i - r_i)\times(p_i - r_i)}$.

2. Choose $L_{i_{22}}$ such that $A_{i_{22}} - L_{i_{22}} C_{i_{22}}$ is Hurwitz.
   Solve $P^{\circ}_{i_{22}}$ for

$$(A_{i_{22}} - L_{i_{22}} C_{i_{22}})^T P^{\circ}_{i_{22}} + P^{\circ}_{i_{22}}(A_{i_{22}} - L_{i_{22}} C_{i_{22}}) = -Q^{\circ}_{i_{22}},$$

   where $Q^{\circ}_{i_{22}} = \hat{Q}^{\circ}_{i_{22}} + 2\delta_i I_{n_i - r_i}$, $\hat{Q}^{\circ}_{i_{22}}$ is a symmetric positive definite matrix and $\delta_i = \sigma^2_{max}(B_{i1}K_i)$ with $K_i = K_{i1} + K_{i2}$.
3. Choose k_i such that

$$k_i > \frac{1}{2}\lambda_{max}\left(A^T_{i_{11}} + A_{i_{11}} + (A_{i_{12}} + A^T_{i_{21}} P^{\circ}_{i_{22}}) \times \hat{Q}^{\circ -1}_{i_{22}}(A^T_{i_{12}} + P^{\circ}_{i_{22}} A_{i_{21}})\right);$$

4. Construct $\hat{P}^{\circ}_i$, $\hat{F}_i$ and $\hat{L}_i$ as

$$\hat{P}^{\circ}_i = \begin{bmatrix} I_n & 0 \\ 0 & P^{\circ}_{i_{22}} \end{bmatrix},$$
$$\hat{F}_i = \begin{bmatrix} B^T_{i2_1} & 0 \end{bmatrix},$$
$$\hat{L}_i = \begin{bmatrix} (k_i + \delta_i) I_n & 0 \\ 0 & L_{i_{22}} \end{bmatrix}.$$

   and compute $P^{\circ}_i = T_i^T \hat{P}^{\circ}_i T_i, \quad F_i = \hat{F}_i S_i, \quad L_i = T_i^{-1} \hat{L}_i S_i.$

The controller gain that stabilize the nonlinear interconnected system (1.138) is given in the following theorem :

Theorem 1.10.8 ([33]). *If the optimization given by*

$$\text{minimize } \sum_{i=1}^{N} \gamma_i$$
$$\text{subject to } \tau > 0,\ Y_D > 0,\ M_D,\ \kappa_Y > 0, \text{ and } \kappa_M > 0$$
$$\begin{bmatrix} \tilde{W}_D & I & Y_D \Gamma_1^T & \cdots & Y_D \Gamma_N^T \\ I & -I & 0 & \cdots & 0 \\ \Gamma_1 Y_D & 0 & -\gamma_1 I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Gamma_N Y_D & 0 & 0 & \cdots & -\gamma_N I \end{bmatrix} < 0 \tag{1.146}$$
$$\begin{bmatrix} \kappa_M I & M_D^T \\ M_D & -I \end{bmatrix} < 0 \tag{1.147}$$
$$\begin{bmatrix} Y_D & I \\ I & \kappa_Y I \end{bmatrix} > 0, \tag{1.148}$$
$$\text{with } \tilde{W}_D = Y_D A_D^T + A_D Y_D + B_{1D} M_D + (B_{1D} M_D)^T + \tau I \tag{1.149}$$

is feasible, then the nonlinear interconnected system

$$\dot{x} = A_D x + B_{1D} u_1 + B_{2D} u_2(x)$$

and

$$y = C_D x$$

driven by the dynamic output feedback controller $u_1 = K_D \hat{x}$ *with* K_D *and* $\hat{x}$ *given as follows*

$$K_D = M_D y_D^{-1} \tag{1.150}$$

and

$$P_D^{\circ} = \frac{1}{\tau} y_D^{-1} \tag{1.151}$$

and (1.143)*, respectively, is robustly stable.*

1.10.4 Zemouche and Alessandri approach for Decentralized Observer-Based Control of Linear Systems with Nonlinear Interconnections [87]

Consider the following class of linear systems under nonlinear interconnections :

$$\dot{x}_i = A_i x_i + B_i u_i + h_i(t,x) \tag{1.152a}$$

$$y_i = C_i x_i \tag{1.152b}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state vector, $y_i \in \mathbb{R}^{p_i}$ is the output measurement and $u_i \in \mathbb{R}^{m_i}$ is the control input vector for $i = 1, 2, \cdots, N$. Moreover, let $x = [x_1^T \cdots x_N^T]^T \mathbb{R}^n$, with $n = \sum_{i=1}^N n_i$, which is the state of the overall system.
The nominal matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, and $C_i \in \mathbb{R}^{p_i \times n_i}$ are constant. The pairs (A_i, B_i) and (A_i, C_i) are assumed to be stabilizable and detectable, respectively.
Moreover, the uncertain nonlinear connections $h_i(x)$ satisfy the following condition :

$$h_i^T(x) h_i(x) \leq \alpha_i^2 x^T H_i^T H_i x, \forall x \in \mathbb{R}^n \tag{1.153}$$

where the coefficients $\alpha_i > 0$ are interconnection bounds and $H_i \in \mathbb{R}^{\nu_i \times n}$ are known constant matrices. In practice, we assume that the functions $h_i(x)$ are unknown but such that (1.153) is satisfied with α_i and H_i known in principle (a problem we will address is that of finding the maximum value of α_i for which closed-loop stability is ensured). The global system can be represented in the following compact form :

$$\dot{x} = Ax + Bu + h(t,x) \tag{1.154a}$$

$$y = Cx \tag{1.154b}$$

where $u = [u_1^T \cdots u_N^T]^T \in \mathbb{R}^m$ and $y = [y_1^T \cdots y_N^T]^T \in \mathbb{R}^p$ are the entire input and output vectors, respectively, with

$$m = \sum_{i=1}^N m_i, \;\; p = \sum_{i=1}^N p_i$$

The matrices A, B, C and the global nonlinear interconnection h are given by :
$A_D = \text{diag}(A_1,\cdots,A_N)$, $A = \text{diag}(A_1,\cdots,A_N)$, $B = \text{diag}(B_1,\cdots,B_N)$, $C = \text{diag}(C_1,\cdots,C_N)$, $h = (h_1^T,\cdots,h_N^T)^T$.
Let $H = [H_1^T,\cdots,H_N^T]$, $\Lambda = \text{diag}(\alpha_1^2 I_{v_1},\cdots,\alpha_N^2 I_{v_N})$, $\Gamma = \text{diag}(\tau_1 I_{v_1},\cdots,\tau_N I_{v_N}) > 0$, where $\tau_1,\cdots,\tau_N > 0$. The inequalities (1.153) can be expressed as follows :

$$h^T(t,x)\Gamma h(t,x) \leq x^T \mathscr{H}^T \Gamma\Lambda\mathscr{H} x,\ \forall x \in \mathbb{R}^n. \tag{1.155}$$

We aim to design a Luenberger observer-based controller described by the following structure :

$$\dot{x} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{1.156a}$$

$$u = -K\hat{x} \tag{1.156b}$$

where

$$L = \text{diag}(L_1,\cdots,L_N),\ K = \text{diag}(K_1,\cdots,K_N) \tag{1.157}$$

are the global observer and controller parameters and $L_i \in \mathbb{R}^{n_i\times p_i}$, $K_i \in \mathbb{R}^{m_i\times n_i}$ are the local observer and controller gain matrices. $\hat{x}$ is the estimate of the entire system state x. It is worth noting that (1.154), and (1.157) correspond to a complete decentralized setting, i.e., a set of local controllers fed by local observers.
The difficulty to have an LMI stems from the off-diagonal bilinear term $\mathscr{P}_1^{-1}BK$.
In line with [37], to avoid the bilinear term in

$$A^T P + PA = \begin{bmatrix} A_K^T P_1^{-1} + P_1^{-1} A_K & P_1^{-1}BK \\ (BK)^T P_1^{-1} & A_L^T P_2 + P_2 A_L \end{bmatrix}$$

where $A_K = A - BK$, $A_L = A - LC$, we use
– a simple congruence transformation
– Schur lemma
– the Young relation in a judicious manner
We obtain a convex optimization problem that consists in maximizing the bounds on the nonlinear interconnections. Specifically, the introduction of the diagonal matrix Γ plays a role on the maximization of the bounds α_i.

Theorem 1.10.9 ([87]). *The closed-loop system*

$$\dot{\zeta} = \mathscr{A}\zeta + \mathscr{B}h(t,x) \tag{1.158a}$$

$$x = \mathscr{C}\zeta \tag{1.158b}$$

where $\mathscr{A} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} I_n \\ I_n \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} I_n & 0 \end{bmatrix}$ *and* $\zeta = \begin{bmatrix} x^T & e^T \end{bmatrix}^T$ *with* $e = x - \hat{x}$ *is asymptotically stabilizable with the decentralized observer-based controller* (1.156), (1.157) *if there exist block-diagonal Lyapunov matrices* $\mathscr{P}_1 = \text{diag}(\mathscr{P}_{11},\cdots,\mathscr{P}_{1N})$, $\mathscr{P}_2 =$

$\mathrm{diag}(\mathscr{P}_{21},\cdots,\mathscr{P}_{2N})$, *matrices* $\mathscr{R}_1 = \mathrm{diag}(\mathscr{R}_{11},\cdots,\mathscr{R}_{1N})$, $\mathscr{R}_2 = \mathrm{diag}(\mathscr{R}_{21},\cdots,\mathscr{R}_{2N})$ *of appropriate dimensions, and positive scalars* γ_i, τ_i, $i = 1,\cdots,N$ *and* $\varepsilon > 0$ *that result from the solution of the problem :*

$$min \sum_{i=1}^{N} (\gamma_i + \tau_i) \tag{1.159}$$

subject to

$$\begin{bmatrix} L(\mathscr{P}_1,\mathscr{R}_1) & 0 & I_n & \mathscr{P}_1 H^T & B\mathscr{R}_1 & 0 \\ 0 & M(\mathscr{P}_2,\mathscr{R}_2) & \mathscr{P}_2 & 0 & 0 & I_n \\ I_n & \mathscr{P}_2 & -\Gamma & 0 & 0 & 0 \\ H\mathscr{P}_1 & 0 & 0 & -\theta & 0 & 0 \\ \mathscr{R}_1^T B^T & 0 & 0 & 0 & -\frac{1}{\varepsilon}\mathscr{P}_1 & 0 \\ 0 & I_n & 0 & 0 & 0 & -\varepsilon\mathscr{P}_1 \end{bmatrix} < 0 \tag{1.160}$$

where $L(\mathscr{P}_1,\mathscr{R}_1) = \mathscr{P}_1 A^T + A\mathscr{P}_1 - \mathscr{R}_1^T B^T - B\mathscr{R}_1$ *and* $M(\mathscr{P}_2,\mathscr{R}_2) = A^T\mathscr{P}_2 + \mathscr{P}_2 A - \mathscr{R}_2^T C - C^T\mathscr{R}_2$*. Therefore, the local decentralized observer-based controller parameters are given by*

$$L_i = \mathscr{P}_{2i}^{-1}\mathscr{R}_{2i}^T,\ K_i = \mathscr{R}_{1i}\mathscr{P}_{1i}^{-1}$$

and the maximum bounds of the interconnections are

$$\alpha_i^2 = \frac{1}{\tau_i\gamma_i}.$$

0.7

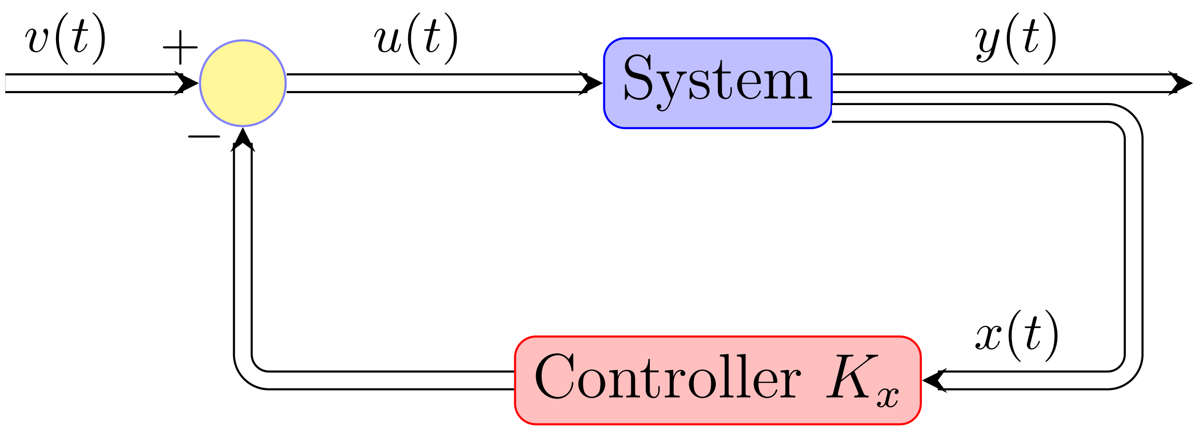


FIG. 1.4: State feedback control

0.7

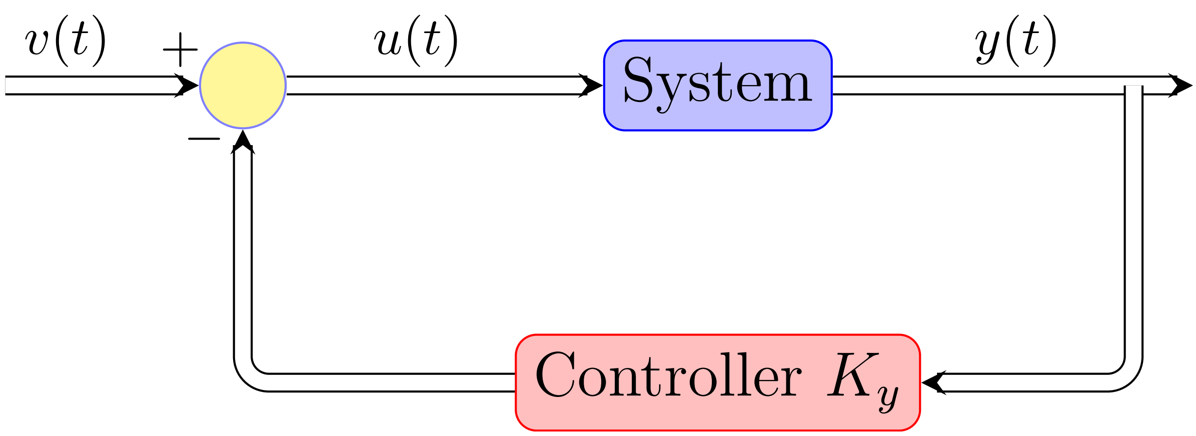


FIG. 1.5: Output feedback control

0.7

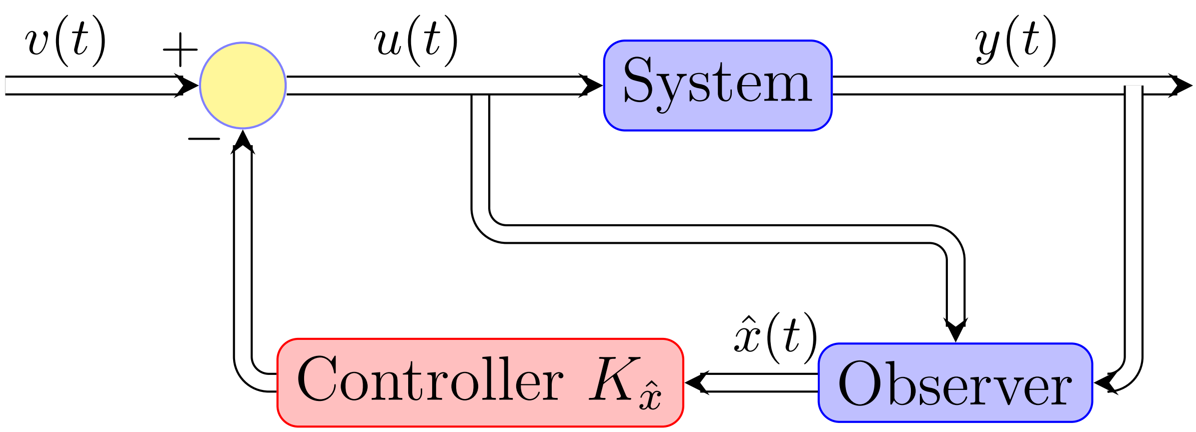


FIG. 1.6: Observer based control

FIG. 1.7: Types of retroactive control

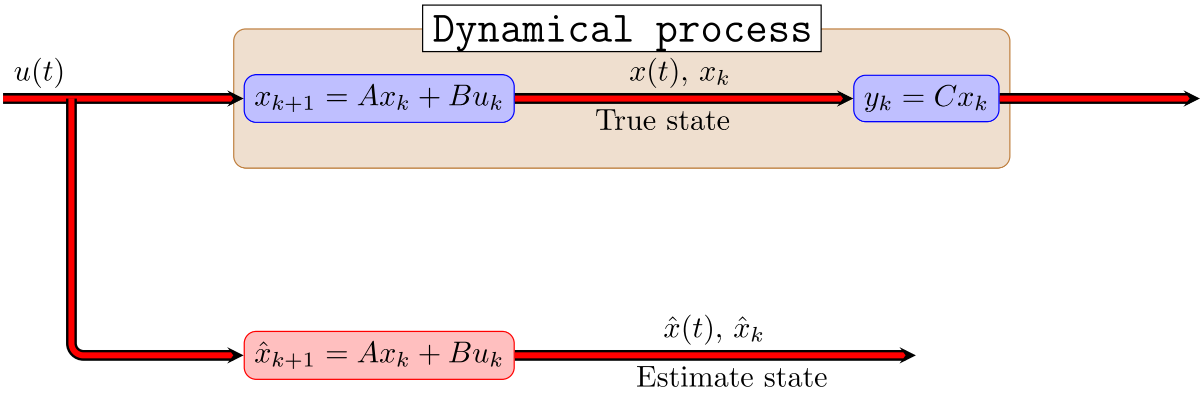


FIG. 1.8: Open-loop observer

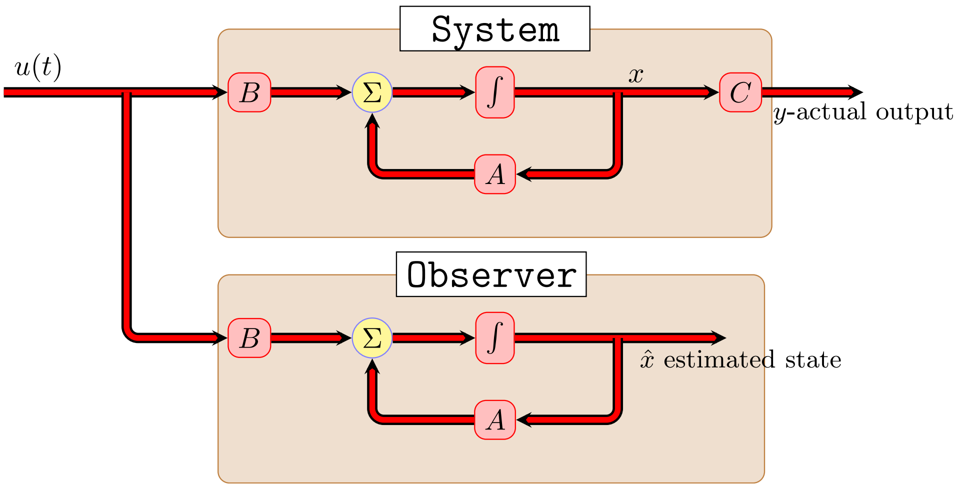


FIG. 1.9: Luenberger observer : continuous case

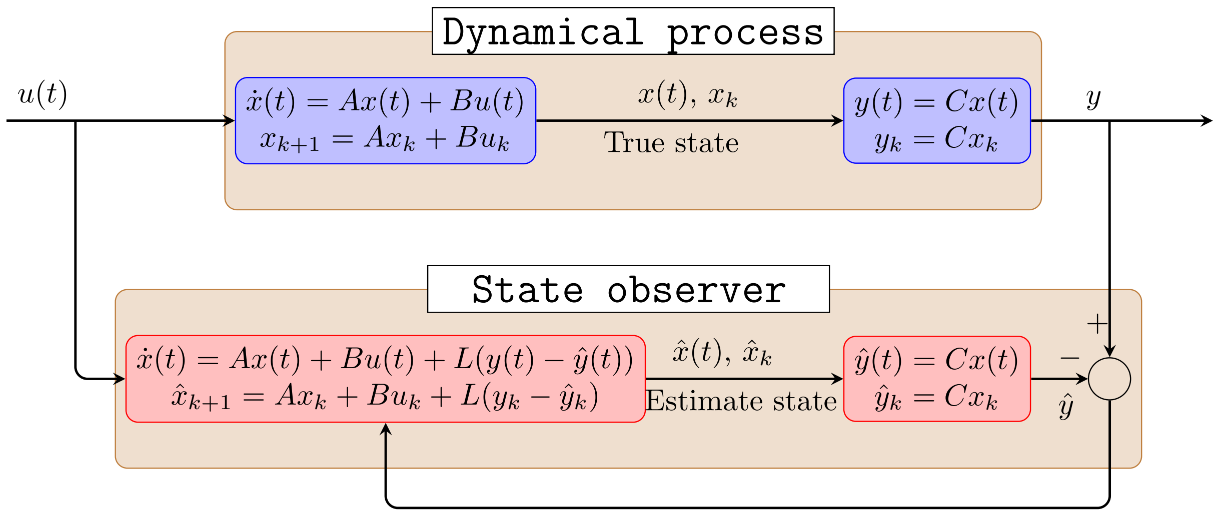


FIG. 1.10: Luenberger Observer

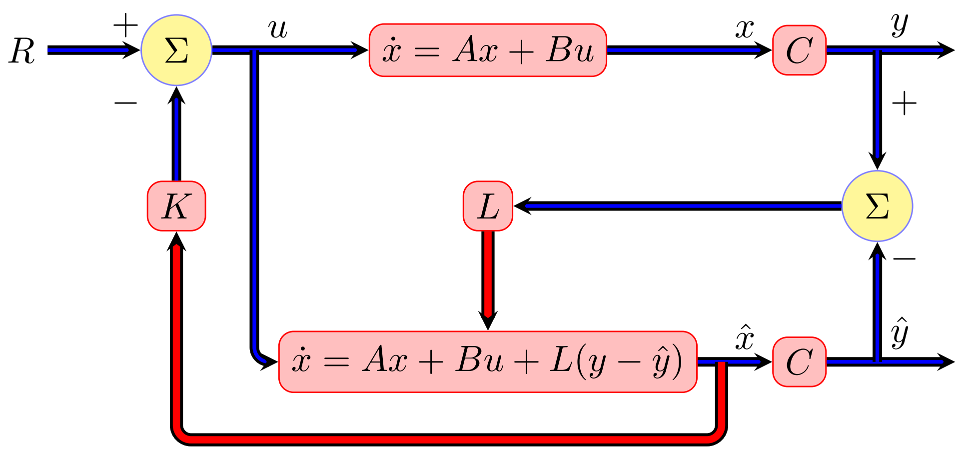


FIG. 1.11: Observer-based feedback : continuous case

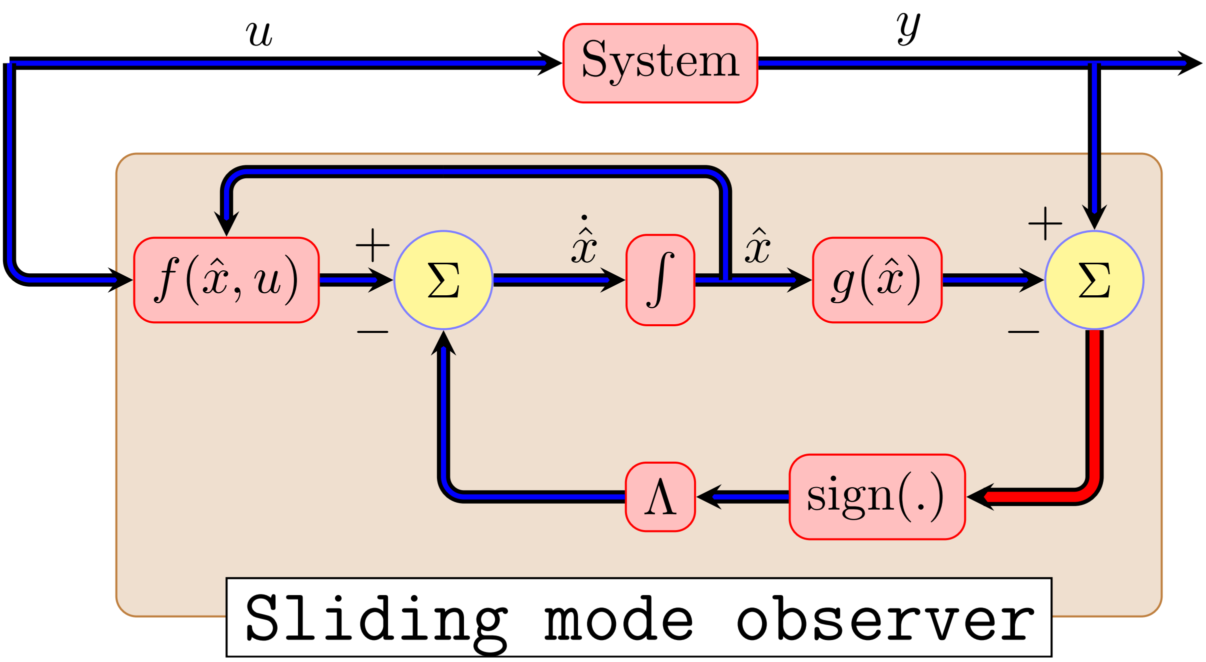


FIG. 1.12: Structure of a sliding mode observer

Chapitre 2

A Robust Decentralized Observer-Based Stabilization Method for Interconnected Nonlinear Systems. Improved LMI Conditions

2.1 Introduction

The stabilization problem of interconnected nonlinear systems using decentralized observer-based controllers in the convex optimization context for nonlinear interconnected discrete-time systems is considered in this chapter. The proposed method uses the tools of previous work in [18], [89] and [90]. We prove that for discrete-time systems, the introduction of a slack variable G, in a triangular form, avoids many bilinear terms and leads to simpler and less conservative LMI conditions. Although this idea seems simple but it leads to better LMIs from feasibility point of view. This is the first time this idea is used to solve the decentralized observer-based stabilization problem for interconnected systems.To be more clear, the contributions of this chapter may be summarized as follows :

- The introduction of the slack variable G in a triangular form to eliminate bilinear terms which are the source of conservatism of the resulted LMI conditions, and are generally unavoidable by standard LMI methods. This novel technique is specific for discrete-time systems for which the literature is limited.
- To get the final LMI condition, there is a chain of new mathematical developments based on well known tools, but exploited in new ways. Many linearization steps are needed to get an LMI condition.
- Contrarily to several methods in the literature where the considered models are linear with nonlinear interconnections, a known nonlinear function is included in the dynamics of the system in the goal to enlarge the applicability of the proposed method for a wide class of real-world applications. This known nonlinearity has been handled by

using as power tool the well-known differential mean value theorem, which leads to less conservative LMI conditions.
- The numerical comparisons presented in the chapter are significant and show the superiority of the proposed methodology and confirm the powerful role played by the novel structure of the slack variable G.

The rest of this chapter is organized as follows : Section 2.2 provides some notations and useful lemmas. The problem formulation is introduced in Section 3.2. The main contribution is presented and proved in Section 2.4. Two numerical examples are given in Section 2.5 to demonstrate the validity and effectiveness of the proposed methodology. Finally, we end the chapter by a conclusion.

2.2 Preliminary useful lemmas

Before formulation the problem we that we will investigate in this paper, we start by introducing some preliminaries, which will be used throughout this paper.

2.2.1 Preliminary useful lemmas

The following lemmas play an important role in establishing the main contribution of this paper.

Lemma 2.2.1 **(***Young's inequality***).** *Let X and Y be two given matrices of appropriate dimensions. Then, for any symmetric positive definite matrix S, the following inequality holds :*

$$X^TY+Y^TX \leq \underbrace{X^TS^{-1}X+Y^TSY}_{\mathrm{U}_1(X,Y)}. \tag{2.1}$$

Inequality (2.1) is the classical Young's inequality. It is often used in many control problems, especially to handle parametric uncertainties. Nevertheless, what we propose in this paper is not based only on inequality (2.1) like usually, but it is based on its reformulation.

Lemma 2.2.2 **([90]).** *Let X and Y be two given matrices of appropriate dimensions. Then, for any symmetric positive definite matrix S of appropriate dimension, the following inequality holds :*

$$X^TY+Y^TX \leq \underbrace{\frac{1}{2}\Big[X+SY\Big]^T S^{-1}\Big[X+SY\Big]}_{\mathrm{U}_2(X,Y)}. \tag{2.2}$$

Remark 2.2.3. *This new variant of Young's relation introduced firstly in [90] is better than the classical one (2.1). Inequality (2.2) provides smaller upper bound of $X^\top Y+Y^\top X$. Indeed, it is easy to show that*

$$\mathrm{U}_2(X,Y) \leq \mathrm{U}_1(X,Y), \forall X,Y, \text{ and } S=S^\top > 0. \tag{2.3}$$

From (2.3), it is obvious that the new variant of Young's inequality will lead to less conservative LMI conditions. In fact, for any matrix Λ of adequate dimensions, we have the following implication :

$$\Lambda + \mathrm{U}_1(X,Y) < 0 \implies \Lambda + \mathrm{U}_2(X,Y) < 0 \tag{2.4}$$

because from (2.3) we have

$$\Lambda + \mathrm{U}_2(X,Y) \leq \Lambda + \mathrm{U}_1(X,Y) < 0. \tag{2.5}$$

The following well-known result, introduced in [11], will be very useful.

Lemma 2.2.4 ([11]). *The following conditions are equivalent :*

1. *There exists a symmetric matrix $P > 0$ such that*

$$A^T PA - P < 0. \tag{2.6}$$

2. *There exist a symmetric matrix P and a matrix G such that*

$$\begin{bmatrix} P & A^T G^T \\ GA & G + G^T - P \end{bmatrix} > 0. \tag{2.7}$$

The next section is devoted to the formulation of the problem of decentralized observer-based stabilization for a class of nonlinear systems with unknown nonlinear interconnections. For more details about decentralized control design for interconnected nonlinear systems, we refer the reader to the pertinent references [82], [63], [41], [42], [83] [44], [12], [23], [47], [24]. What we study in this chapter consists in establishing a new LMI conditions ensuring the asymptotic stabilization of the closed-loop system. Thanks to the previous lemmas exploited in convenient ways and some new mathematical matrix decompositions and congruence principle, we will provide less conservative LMI conditions compared to the existing results in the literature.

2.3 Problem Formulation

Consider the following nonlinear interconnected system composed of N subsystems :

$$x_{t+1}^i = A_i x_t^i + B_i u_t^i + D_i f^i(x_t^i) + E_i \eta_i(x_t) \tag{2.8a}$$
$$y_t^i = C_i x_t^i, \tag{2.8b}$$

where, $x_t^i \in \mathbb{R}^{n_i}$, $u_t^i \in \mathbb{R}^{m_i}$, $y_t^i \in \mathbb{R}^{p_i}$ are the state, input, and the output vectors of the i^{th} subsystem, for $i = 1, 2, \ldots, N$. The nominal matrices $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m_i}, C_i \in \mathbb{R}^{p_i \times n_i}, D_i \in \mathbb{R}^{n_i \times q_i}, E_i \in \mathbb{R}^{n_i \times r_i}$ are constant known matrices, for each $i \in \{1, \ldots, N\}$.
Let $x_t = \begin{bmatrix} (x_t^1)^T & \ldots & (x_t^N)^T \end{bmatrix}^T \in \mathbb{R}^n$, with $n = \sum_{i=1}^N n_i$, be the state of the overall system. The function $f^i : \mathbb{R}^{s_i} \to \mathbb{R}^{q_i}$, are assumed to be continuously differentiable, and satisfy

$f(0)=0$. f^i can be rewritten under the detailed form :

$$f^i(x_t^i) := f^i(H_i x_t^i) = \begin{bmatrix} f_{i1}(H_{i1}x_t^i) \\ \vdots \\ f_{ij}(H_{ij}x_t^i) \\ \vdots \\ f_{iq_i}(H_{iq_i}x_t^i) \end{bmatrix} \in \mathbb{R}^{q_i}$$

with $H_{ij} \in \mathbb{R}^{s_{ij}\times n_i}$, and $H_i = \left[H_{i1}^T \dots H_{iq_i}^T\right]^T \in \mathbb{R}^{s_i \times n_i}$ for $1 \leq i \leq N$, $1 \leq j \leq q_i$, and $s_i := \sum_{j=1}^{q_i} s_{ij}$. Moreover, the uncertain nonlinear connections $\eta^i : \mathbb{R}^{n_i} \to \mathbb{R}^{r_i}$, depend on the entire sate vector x_t, and satisfy that for $i=1,\dots,N$, $\eta_i(.)$ is α_i-Lipschitz, otherwise, it is assumed that $\eta_i(.)$ satisfies the following condition

$$\eta_i^T(x_t)\eta_i(x_t) \leq \alpha_i^2 x_t^T M_i^T M_i x_t, \forall i = 1,\dots,N, \tag{2.9}$$

where $\alpha_i, i=1,\dots,N$, are the interconnection bounds and $M_i \in \mathbb{R}^{r_i \times n}$ are known constant matrices.

2.3.1 The global system

Let

$$m := \sum_{i=1}^N m_i,\ p := \sum_{i=1}^N p_i,\ q := \sum_{i=1}^N q_i,\ r := \sum_{i=1}^N r_i,\ s := \sum_{i=1}^N s_i = \sum_{i=1}^N \sum_{j=1}^{q_i} s_{ij}.$$

The global system is governed by the following equations

$$x_{t+1} = Ax_t + Bu_t + Df(Hx_t) + E\eta(x_t) \tag{2.10a}$$

$$y_t = Cx_t, \tag{2.10b}$$

where

$$u_t = \begin{bmatrix} (u_t^1)^T & \dots & (u_t^N)^T \end{bmatrix}^T \in \mathbb{R}^m,$$
$$y_t = \begin{bmatrix} (y_t^1)^T & \dots & (y_t^N)^T \end{bmatrix}^T \in \mathbb{R}^p,$$

$$f(Hx_t) := \begin{bmatrix} \begin{bmatrix} f_{11}(H_{11}x_t^1) \\ \vdots \\ f_{1q_1}(H_{1q_1}x_t^1) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} f_{N1}(H_{N1}x_t^N) \\ \vdots \\ f_{Nq_N}(H_{Nq_N}x_t^N) \end{bmatrix} \end{bmatrix} = \sum_{i=1}^N \sum_{j=1}^{q_i} e_q(j) f_{ij}(H_{ij}x_t^i) \in \mathbb{R}^q$$

are respectively the entire inputs, the outputs and known nonlinearities of the system with

$$H = \mathrm{diag}\left\{ \begin{bmatrix} H_{11} \\ \vdots \\ H_{1q_1} \end{bmatrix}, \dots, \begin{bmatrix} H_{i1} \\ \vdots \\ H_{iq_i} \end{bmatrix}, \dots, \begin{bmatrix} H_{N1} \\ \vdots \\ H_{Nq_N} \end{bmatrix} \right\} \in \mathbb{R}^{s \times n}. \tag{2.11}$$

The matrices A, B, C, D, E of the global system (2.10) are given by

$$\begin{aligned} A &= \mathrm{diag}\{A_1, \dots, A_N\} \in \mathbb{R}^{n \times n}, \\ B &= \mathrm{diag}\{B_1, \dots, B_N\} \in \mathbb{R}^{n \times m}, \\ C &= \mathrm{diag}\{C_1, \dots, C_N\} \in \mathbb{R}^{p \times n}, \\ D &= \mathrm{diag}\{D_1, \dots, D_N\} \in \mathbb{R}^{n \times q}, \\ E &= \mathrm{diag}\{E_1, \dots, E_N\} \in \mathbb{R}^{n \times r}. \end{aligned}$$

2.3.2 The structure of the observer

Among the different structures of the observers existing in the literature, we opted to the one presented in [18].

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + Df(v_t) + L(y_t - \hat{y}_t) \tag{2.13a}$$
$$v_t = H\hat{x}_t + K(y_t - \hat{y}_t) \in \mathbb{R}^s \tag{2.13b}$$
$$\hat{y}_t = C\hat{x}_t \tag{2.13c}$$
$$u_t = -F\hat{x}_t \tag{2.13d}$$

where $\hat{x}_t$ is the observer state. The ith subsystem of (2.13) is given by :

$$\hat{x}^i_{t+1} = A^i\hat{x}^i_t + B_iu_t + D_if^i(v^i_t) + L_i(y^i_t - \hat{y}^i_t), \tag{2.14a}$$
$$v^i_t = H_i\hat{x}^i_t + K_i(y^i_t - \hat{y}^i_t) \in \mathbb{R}^{s_i}. \tag{2.14b}$$
$$\hat{y}^i_t = C_i\hat{x}^i_t \tag{2.14c}$$
$$u^i_t = -F_i\hat{x}^i_t \tag{2.14d}$$

where $K_i \in \mathbb{R}^{s_i \times p_i}$, $F_i \in \mathbb{R}^{m_i \times n_i}$ and $L_i \in \mathbb{R}^{n_i \times p_i}$ for $i = 1, \dots, N$. Since $s_i = \sum_{j=1}^{q_i} s_{ij}$, we can express, for each $i = 1, \dots, N$, the vector v^i_t in its detailed form as follows :

$$v^{ij}_t = H_{ij}\hat{x}^i_t + K_{ij}(y^i_t - \hat{y}^i_t) \in \mathbb{R}^{s_{ij}}$$

where $K_{ij} \in \mathbb{R}^{s_{ij} \times p_i}$, for $i = 1, \dots, N,\ j = 1, \dots, q_i$. This allows us to put $K_i = \left[K^T_{i1}, \dots, K^T_{iq_i}\right]^T$. The choice of such observer is motivated by its efficiency proved in [18]. Indeed, as reported in [18], the introduction of the additional degree of freedom K plays an important role in the feasibility of the developed LMI condition. Now, since we deal with interconnected systems, we have to accommodate the parameters of (2.13) in such a way to decentralize the global system. Otherwise, it is assumed that each subsystem is controlled using its own local output. So, our problem is reduced to finding $F =$

$\operatorname{diag}\{F_1,\dots,F_N\} \in \mathbb{R}^{m\times n}, L = \operatorname{diag}\{L_1,\dots,L_N\} \in \mathbb{R}^{n\times p}$, and $K = \operatorname{diag}\{K_1,\dots,K_N\} \in \mathbb{R}^{s\times p}$ such that the closed loop system is stable. The closed loop system z_t composed by the state x_t and the estimation error $e_t = x_t - \hat{x}_t$ can be represented as

$$z_{t+1} = \begin{bmatrix} A-BF & BF \\ 0 & A-LC \end{bmatrix} z_t + \begin{bmatrix} Df(Hx_t) \\ D\big(f(Hx_t)-f(v_t)\big) \end{bmatrix} + \begin{bmatrix} E \\ E \end{bmatrix} \eta(x_t). \tag{2.15}$$

2.3.3 Handling the nonlinearity

Under the assumption of the differentiability of the nonlinearity f_{ij} and since $f_{ij}(0) = 0$, $i = 1,\dots,N$, $j = 1,\dots,q_i$, we obtain, thanks to the classical differential mean value theorem applied on each f_{ij}, $i = 1,\dots,N$, $j = 1,\dots,q_i$, the following reformulation of the ij^{th}-component of the nonlinearity :

$$f_{ij}(H_{ij}x_t^i) = \nabla^T f_{ij}(\xi_t^{ij}).(H_{ij}x_t^i - 0)$$

where for each $i = 1,\dots,N$ and $j = 1,\dots,q_i$,

$$\nabla^T = \left[\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_{s_{ij}}}\right]$$

is the transpose of the gradient operator, $\xi_t^{ij} \in \operatorname{co}(H_{ij}x_t^i, 0)$, and the partial derivatives, $\dfrac{\partial f_{ij}}{\partial z_k}$, satisfy for all $k = 1,\dots,s_{ij}$, the following boundedness condition :

$$-\infty < a_{ij}^k \le \frac{\partial f_{ij}}{\partial z_k}(\xi_t^{ij}) \le b_{ij}^k < +\infty, \tag{2.16}$$

where a_{ij}^k and b_{ij}^k are some known scalars. Without loss of generality, we can assume that $a_{ij}^k > 0$, for all the indices i, j and k, (see [88, Remark 1]).
Therefore, the expression of $f(Hx_t)$ becomes

$$\begin{aligned} f(Hx_t) &= \sum_{i=1}^{N}\sum_{j=1}^{q_i} e_q(j) f_{ij}(H_{ij}x_t^i) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{q_i} e_q(j)\nabla^T f_{ij}(\xi_t^{ij}) H_{ij}x_t^i \\ &= \sum_{i=1}^{N}\sum_{j=1}^{q_i}\sum_{k=1}^{s_{ij}} e_q(j) e_{s_{ij}}^T(k) \frac{\partial f_{ij}}{\partial z_k}(\xi_t^{ij}) H_{ij}x_t^i \end{aligned} \tag{2.17}$$

For the sake of brevity, $f(Hx_t)$ can be rewritten as the product

$$f(Hx_t) = \mathbb{H} F_t^{\nabla} H x_t$$

where

$$\mathbb{H} = \operatorname{diag}\left(\underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}}_{s_{ij}\text{times}}, i = 1,\dots,N,\ j = 1,\dots,q_i \right) \in \mathbb{R}^{q\times s}, \tag{2.18}$$

$$F_t^{\nabla} = \operatorname{diag}\left(\frac{\partial f_{ij}}{\partial z_k}(\xi_t^{ij}),\, i=1,\ldots,N,\, j=1,\ldots,q_i, k=1,\ldots,s_{ij}\right) \in \mathbb{R}^{s\times s}, \tag{2.19}$$

and $H \in \mathbb{R}^{s\times n}$ is defined in (2.11).
In the same way, we obtain the following reformulation of the difference $f(Hx_t) - f(v_t)$:

$$\begin{aligned} f(Hx_t) - f(v_t) &= \sum_{i=1}^{N}\sum_{j=1}^{q_i} e_q(j)\left(f_{ij}(H_{ij}x_t^i) - f_{ij}(v_t^{ij})\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{q_i} e_q(j)\nabla^T f_{ij}(\bar{\xi}_t^{ij})\left(H_{ij}x_t^i - v_t^{ij}\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{q_i}\sum_{k=1}^{s_{ij}} e_q(j)e_{s_{ij}}^T(k)\frac{\partial f_{ij}}{\partial z_k}(\bar{\xi}_t^{ij})\Big(H_{ij} - K_{ij}C_i\Big)(x_t^i - \hat{x}_t^i). \end{aligned} \tag{2.20}$$

where, for each $i = 1,\ldots,N$ and $j = 1,\ldots,q_i$, $\bar{\xi}_t^{ij} \in \operatorname{co}(H_{ij}x_t, v_t^{ij})$, and the partial derivatives, $\dfrac{\partial f_{ij}}{\partial z_k}(\bar{\xi}_t^{ij})$, $k = 1,\ldots,s_{ij}$, satisfy

$$-\infty < \bar{a}_{ij} \le \frac{\partial f_{ij}}{\partial z_k}(\bar{\xi}_t^{ij}) \le \bar{b}_{ij} < +\infty, \tag{2.21}$$

where $\bar{a}_{ij}$ and $\bar{b}_{ij}$ are some known constant scalars. Therefore,

$$f(Hx_t) - f(v_t) = \mathbb{H}\bar{F}_t^{\nabla}(H - KC)e_t$$

where $\bar{F}_t^{\nabla}$ is the time-depending matrix containing the partial derivative evaluating at the uncertain parameters $\bar{\xi}_t^{ij}$, that is

$$\bar{F}_t^{\nabla} := \operatorname{diag}\left(\frac{\partial f_{ij}}{\partial z_k}(\bar{\xi}_t^{ij}),\, i=1,\ldots,N,\, j=1,\ldots,q_i, k=1,\ldots,s_{ij}\right) \in \mathbb{R}^{s\times s}. \tag{2.22}$$

It should be noticed that the differentiability of f is assumed without loss of generality. Indeed, if f is not differentiable and satisfying the Lipschitz property, we can use the reformulation technique introduced in [89, Lemma 7] to get the same result.

2.3.4 Stability analysis of the closed-loop system

By substituting the terms $f(Hx_t)$, and $f(Hx_t) - f(v_t)$ by there equivalent matrices form in (2.15), we get the following equivalent structure of the closed loop system

$$z_{t+1} = \Pi_1 z_t + \Pi_2 \eta(x_t) \tag{2.23}$$

where

$$\Pi_1 = \begin{bmatrix} A - BF + D\mathbb{H}F_t^{\nabla}H & BF \\ 0 & A - LC + D\mathbb{H}\bar{F}_t^{\nabla}(H - KC) \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} E^T & E^T \end{bmatrix}^T.$$

Now to deal with the stability analysis, we choose the following Lyapunov function

$$V(z_t) = z_t^T P z_t,$$

with

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0.$$

Thus, the decreasing of the Lyapunov function between two successive instances $\Delta V(z_t)$ is expressed as follows

$$\Delta V(z_t) = V(z_{t+1}) - V(z_t) \tag{2.24}$$

$$= (\Pi_1 z_t + \Pi_2 \eta(x_t))^T P(\Pi z_t + \Pi_2 \eta(x_t)) - z_t^T P z_t$$

$$= \begin{bmatrix} z_t^T & \eta(x_t)^T \end{bmatrix} \begin{bmatrix} \Pi_1^T P \Pi_1 - P & \Pi_1^T P \Pi_2 \\ (\star) & \Pi_2^T P \Pi_2 \end{bmatrix} \begin{bmatrix} z_t \\ \eta(x_t) \end{bmatrix}. \tag{2.25}$$

2.3.5 Handing the interconnections

Now, let us introduce a new positive parameters $\{\gamma_i\}_{i=1}^N$. Inequality (2.9) is equivalent to the following one :

$$\eta_i^T(.)(\gamma_i I_{r_i})\eta_i(.) \le \alpha_i^2 x_t M_i^T(\gamma_i I_{r_i}) M_i x_t \tag{2.26}$$

Hence

$$\sum_{i=1}^N \eta_i^T(.)(\gamma_i I_{r_i})\eta_i(.) \le \sum_{i=1}^N \alpha_i^2 x_t M_i^T(\gamma_i I_{r_i}) M_i x_t \tag{2.27}$$

Put $\theta_i = \alpha_i^{-2}$. Let

$$M = \begin{bmatrix} M_1^T & \dots & M_N^T \end{bmatrix}^T \in \mathbb{R}^{r \times n},$$
$$\Gamma = \operatorname{diag}(\gamma_1 I_{r_1}, \dots, \gamma_i I_{r_i}, \dots, \gamma_N I_{r_N}) \in \mathbb{R}^{r \times r},$$
$$\Theta = \operatorname{diag}(\theta_1 I_{r_1}, \dots, \theta_i I_{r_i}, \dots, \theta_N I_{r_N}) \in \mathbb{R}^{r \times r}.$$

With these notations, inequality (2.27) becomes

$$\eta^T(x_t)\Gamma\eta(x_t) \le x^T M^T \Gamma \Theta^{-1} M x_t = z_t^T \hat{\Theta} z_t \tag{2.28}$$

where

$$\hat{\Theta} = \operatorname{diag}\left(M^T \Gamma \Theta^{-1} M, 0\right).$$

Equivalently to the more suitable form

$$\begin{bmatrix} z_t \\ \eta(x_t) \end{bmatrix}^T \begin{bmatrix} -\hat{\Theta} & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} z_t \\ \eta(x_t) \end{bmatrix} \le 0. \tag{2.29}$$

2.3.6 The BMI problem : Motivation and formulation

By taking into account constraint (2.29), the condition $\Delta V(z_t) < 0$ is satisfied if the following inequality holds :

$$\begin{bmatrix} \Pi_1^T P \Pi_1 - P & \Pi_1^T P \Pi_2 \\ \star & \Pi_2^T P \Pi_2 \end{bmatrix} < \begin{bmatrix} -\hat{\Theta} & 0 \\ 0 & \Gamma \end{bmatrix}. \tag{2.30}$$

This latter is identical to the following one

$$\begin{bmatrix} \Pi_1^T P \Pi_1 - P + \hat{\Theta} & \Pi_1^T P \Pi_2 \\ \star & \Pi_2^T P \Pi_2 - \Gamma \end{bmatrix} < 0. \tag{2.31}$$

Clearly, the BMI in its present form is not ready to be exploited, due to bilinear terms $\Pi_1^T P \Pi_1$, $\Pi_2^T P \Pi_2$, and $\Pi_1^T P \Pi_2$. In what follows, we proposed some rearrangements and transformations that lead to a more simplified BMI. Inequality (2.31) is equivalent by Schur lemma to :

$$\begin{cases} -P + \hat{\Theta} + \Pi_1^T P \Pi_1 - \Pi_1^T P \Pi_2 (\Pi_2^T P \Pi_2 - \Gamma)^{-1} \Pi_2^T P \Pi_1 < 0 \\ \Pi_2^T P \Pi_2 - \Gamma < 0 \end{cases} \tag{2.32}$$

that may be rewritten as below

$$\begin{cases} -P + \hat{\Theta} - \Pi_1^T [-P + P \Pi_2 (\Pi_2^T P \Pi_2 - \Gamma)^{-1} \Pi_2^T P] \Pi_1 < 0 \\ \Pi_2^T P \Pi_2 - \Gamma < 0 \end{cases} \tag{2.33}$$

and that comes back to the following one

$$\begin{bmatrix} [-P + P \Pi_2 (\Pi_2^T P \Pi_2 - \Gamma)^{-1} \Pi_2^T P]^{-1} & \Pi_1 \\ \star & -P + \hat{\Theta} \end{bmatrix} < 0. \tag{2.34}$$

From the inversion lemma [6], we can write for any invertible matrix $\mathscr{A}$, and matrices $\mathscr{B}, \mathscr{C}, \mathscr{D}$

$$(\mathscr{A} + \mathscr{B}\mathscr{C}\mathscr{D})^{-1} = \mathscr{A}^{-1} - \mathscr{A}^{-1}\mathscr{B}(I + \mathscr{C}\mathscr{D}\mathscr{A}^{-1}\mathscr{B})^{-1}\mathscr{C}\mathscr{D}\mathscr{A}^{-1}.$$

In particular, for $\mathscr{A} = -P, \mathscr{B} = P\Pi_2, \mathscr{C} = (\Pi_2^T P \Pi_2 - \Gamma)^{-1}, \mathscr{D} = B^T$, one obtains

$$[-P + P \Pi_2 (\Pi_2^T P \Pi_2 - \Gamma)^{-1} \Pi_2^T P]^{-1} = -P^{-1} + \Pi_2 \Gamma^{-1} \Pi_2^T. \tag{2.35}$$

Consequently, inequality (2.34) is equivalent to the following one

$$\begin{bmatrix} -P^{-1} + \Pi_2 \Gamma^{-1} \Pi_2^T & \Pi_1 \\ \Pi_1^T & -P + \hat{\Theta} \end{bmatrix} < 0. \tag{2.36}$$

Thanks again to Schur lemma, one obtains the equivalent formulation of (2.36) :

$$\begin{bmatrix} -P^{-1} & \Pi_2 & \Pi_1 & 0 \\ (\star) & -\Gamma & 0 & 0 \\ (\star) & (\star) & -P & \begin{bmatrix} M^T \\ 0 \end{bmatrix} \\ (\star) & (\star) & (\star) & -\Gamma^{-1}\Theta \end{bmatrix} < 0. \tag{2.37}$$

Even if inequality (2.37) is simpler than (2.36), since it is linear with respect to the variables K, L, and F, the synthesis of the gain matrices can not be done directly from inequality (2.37). This is due to the coexistence of the variable P and its inverse in the same constraint, as to the variable Γ and Γ^{-1} in the case of known interconnections, Θ. Adding to all that, the presence of time-varying uncertain matrices F_t^{∇}, and $\bar{F}_t^{\nabla}$ renders it more complicated. Indeed, even these latter can be treated using the linear parameter-varying (LPV) approach as in [38], i.e. by solving the obtained LMI for all the vertices of the convex sets including the partial derivatives $\frac{\partial f_{ij}}{\partial z_k}(\xi_t^{ij})$, and $\frac{\partial f_{ij}}{\partial z_k}(\bar{\xi}_t^{ij})$ respectively, for $i=1,\ldots,N$, $j=1,\ldots,q_i$ and $k=1,\ldots,s_{ij}$, this will lead to solving 4^s LMIs that may be solved in a non-polynomial time. Thus, linearizing such a BMI in such a way to derive an LMI condition numerically tractable in a polynomial time and ensuring existence of solutions of a wide class of systems with high values of interconnection bounds is a challenging task. In the next section, we present a simple and efficient design method for the decentralization of the system and the maximization of the bounds of the nonlinear interconnections. Our methodology is inspired from [?] and [3].

2.4 New LMI Synthesis Condition

The following statement provides a sufficient condition for asymptotic stability of system (2.10), by the decentralized observer-based controller (2.13).

Theorem 2.4.1. *If there exists a positive definite matrix $\hat{P}\in\mathbb{R}^{2n\times 2n}$, invertible matrices $G_1,\mathcal{G}_2\in\mathbb{R}^{n\times n}$, bloc-diagonal matrices $\hat{L}\in\mathbb{R}^{n\times p},\hat{F}\in\mathbb{R}^{m\times n}$, a matrix $\hat{K}\in\mathbb{R}^{s\times p}$, and five diagonal matrices $\tilde{\Theta},\hat{\Gamma},\Upsilon\in\mathbb{R}^{r\times r}$, Φ, $\bar{\Phi}\in\mathbb{R}^{s\times s}$, such that the following optimization problem holds,*

$$\min \mathrm{Trace}\left(\tilde{\Theta}-\hat{\Gamma}\right) \textit{ subject to LMI } (2.38)$$

then, system (2.10) is asymptotically stabilizable by the decentralized observer-based controller (2.13), with $F=G_1^{-1}\hat{F}$, $L=\mathcal{G}_2^{-1}\hat{L}$, $K=\bar{\phi}^{-1}\hat{K}$ and interconnection bounds $\alpha_i=\sqrt{\hat{\gamma}_i/\tilde{\theta}_i}$.

Proof : To deal with the bilinearity P and P^{-1} in (2.37), many researchers chose in their frameworks a particular structure of the variable P (in general a diagonal one) see [93], [68], [53], [31], [74] to simplify the task and relax the BMI. In line with [36], we add a slack variable G by applying the congruence technique with $\mathrm{diag}(I,I,G,I)$ to obtain the following equivalent BMI of (2.36) :

$$\Omega := \begin{bmatrix} -P^{-1} & \Pi_2 & \Pi_1 G & 0 \\ \star & -\Gamma & 0 & 0 \\ \star & \star & -G^T P G & G^T \begin{bmatrix} M^T \\ 0 \end{bmatrix} \\ \star & \star & \star & -\Gamma^{-1}\Theta \end{bmatrix} < 0. \tag{2.39}$$

$$\begin{bmatrix} -\hat{P} & 0_{(2n,r)} & \Psi & 0_{(2n,r)} & \tilde{G}_2\begin{bmatrix}E\\E\end{bmatrix} & 0_{(2n,r)} & \begin{bmatrix}D\mathbb{H}\Phi\\0_{(n,s)}\end{bmatrix} & \begin{bmatrix}0_{(n,s)}\\\mathcal{G}_2 D\mathbb{H}\end{bmatrix} \\ \star & -\hat{\Gamma} & 0_{(r,2n)} & 0_r & 0_r & \hat{\Gamma} & 0_{(r,s)} & 0_{(r,s)} \\ \star & \star & \hat{P}-\tilde{G}_3 & \begin{bmatrix}G_1 M^T\\0_{(n,r)}\end{bmatrix} & 0_{(2n,r)} & 0_{(2n,r)} & \begin{bmatrix}G_1 H^T\\H^T\end{bmatrix} & \begin{bmatrix}0_{(n,s)}\\(\Phi H-\hat{K}C)^T\end{bmatrix} \\ \star & \star & \star & -\tilde{\Theta} & 0_r & 0_r & 0_{(r,s)} & 0_{(r,s)} \\ \star & \star & \star & \star & -\Upsilon & 0_r & 0_{(r,s)} & 0_{(r,s)} \\ \star & \star & \star & \star & \star & -(2I_r-\Upsilon) & 0_{(r,s)} & 0_{(r,s)} \\ \star & \star & \star & \star & \star & \star & -\Lambda\Phi & 0_s \\ \star & \star & \star & \star & \star & \star & \star & -\bar{\Lambda}\bar{\Phi} \end{bmatrix} < 0 \qquad (2.38)$$

where

$$\Psi = \begin{bmatrix} AG_1 - B\hat{F} & A \\ 0 & \mathcal{G}_2 A - \hat{L}C \end{bmatrix}, \tilde{G}_1 = \begin{bmatrix} G_1 & 0 \\ I & I \end{bmatrix}, \tilde{G}_2 = \begin{bmatrix} I & 0 \\ 0 & \mathcal{G}_2 \end{bmatrix}, \tilde{G}_3 = \begin{bmatrix} 2G_1 & I \\ I & 2\mathcal{G}_2 \end{bmatrix}$$

A sufficient condition in order to get $\Omega < 0$ is that $\bar{\Omega} < 0$ (provided by bounding $-G^T PG$ by $P^{-1} - G - G^T$), that is

$$\bar{\Omega} := \begin{bmatrix} -P^{-1} & \Pi_2 & \Pi_1 G & 0 \\ \star & -\Gamma & 0 & 0 \\ \star & \star & P^{-1} - G - G^T & G^T \begin{bmatrix} M^T \\ 0 \end{bmatrix} \\ \star & \star & \star & -\Gamma^{-1}\Theta \end{bmatrix} < 0. \qquad (2.40)$$

Note that (2.36) $\Rightarrow$ (2.40) with $G = P^{-1}$. In addition, thanks to Lemma 2.2.4, (2.37) is equivalent to (2.40). At this stage, the non-convex problem is transferred from the co-existence of both P and its inverse in (2.37) to the coupling bilinear term $\Pi_1 G$. To enhance the clarity of the contributions, we shared the rest of the proof into three steps. We will present the linearization of the BMI (2.40) to get the LMI (2.38) step by step until a full linearization.

First step : Linearization with respect to F

Let $G = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}$ be the detailed form of G. With this general structure, the detailed version of $\Pi_1 G$ becomes

$$\Pi_1 G = \begin{bmatrix} (A-BF)G_1 + D\mathbb{H}F_t^{\nabla} HG_1 + BFG_3 & BFG_4 + (A-BF)G_2 + D\mathbb{H}F_t^{\nabla} HG_2 \\ (A-LC)G_3 + D\mathbb{H}\bar{F}_t^{\nabla}(H-KC)G_3 & (A-LC)G_4 + D\mathbb{H}\bar{F}_t^{\nabla}(H-KC)G_4 \end{bmatrix}.$$

Observe that the gain matrix F is attached to all components of G. Use of an eventual change of variables is impossible. Our strategy to overcome this difficulty is inspired from [3]. It consists in using the "*self-elimination of variables* ". That is, we impose that $G_2 = G_4$ in order to eliminate the two terms BFG_4 and $-BFG_2$. Obviously, we must not handle both the terms BFG_1 and $-BFG_3$ in the same way. Otherwise, the variable F will completely vanish from the (2.40). Therefore, it is sufficient to chose $G_3 = 0$. This leads to the following structure of G :

$$G = \begin{bmatrix} G_1 & G_2 \\ 0 & G_2 \end{bmatrix}. \tag{2.41}$$

Hence the term $\Pi_1 G$ becomes :

$$\Pi_1 G = \begin{bmatrix} (A-BF)G_1 + D\mathbb{H}F_t^{\nabla}HG_1 & AG_2 + D\mathbb{H}F_t^{\nabla}HG_2 \\ 0 & (A-LC)G_2 + D\mathbb{H}\bar{F}_t^{\nabla}(H-KC)G_2 \end{bmatrix}. \tag{2.42}$$

Second step : Linearization with respect to L

Observe that (2.40)-(2.42) remains a non convex constraint since the presence of coupling terms as LCG_2. To overcome this difficulty, we use the congruence technique with

$$\operatorname{diag}\left(\tilde{G}_2, I, \tilde{G}_2, I\right),$$

where $\tilde{G}_2 := \operatorname{diag}\left(I, G_2^{-1}\right)$. As a consequence, we obtain the equivalent BMI given in (2.43).

$$\bar{\Omega} := \begin{bmatrix} -\tilde{G}_2\mathscr{P}\tilde{G}_2 & \tilde{G}_2\Pi_2\Gamma^{-1} & \bar{\Omega}_{13} & 0 \\ \star & -\Gamma^{-1} & 0 & 0 \\ \star & \star & \tilde{G}_2\mathscr{P}\tilde{G}_2 - \begin{bmatrix} 2G_1 & I \\ I & 2G_2^{-1} \end{bmatrix} & \begin{bmatrix} G_1 M^T \\ 0 \end{bmatrix} \\ \star & \star & \star & -\Gamma^{-1}\Theta \end{bmatrix} < 0 \tag{2.43}$$

where

$$\bar{\Omega}_{13} = \begin{bmatrix} (A-BF)G_1 + D\mathbb{H}F_t^{\nabla}HG_1 & A + D\mathbb{H}F_t^{\nabla}H \\ 0 & G_2^{-1}(A-LC) + G_2^{-1}D\mathbb{H}\bar{F}_t^{\nabla}(H-KC) \end{bmatrix}.$$

To simplify the notations, we make the change of variables

$$\mathscr{G}_2 = G_2^{-1}, \hat{P} = \tilde{G}_2\mathscr{P}\tilde{G}_2, \hat{\Gamma} = \Gamma^{-1}, \hat{F} = FG_1, \hat{L} = \mathscr{G}_2 L, \tilde{\Theta} = \hat{\Gamma}\Theta.$$

Inequality (2.43), or equivalently the matrix $\bar{\Omega}$, is rewritten under the following form according to the new variables :

$$\bar{\Omega} = \begin{bmatrix} -\hat{P} & \underline{\tilde{G}_2\Pi_2\hat{\Gamma}} & \bar{\Omega}_{13} & 0 \\ \star & -\hat{\Gamma} & 0 & 0 \\ \star & \star & \hat{P} - \begin{bmatrix} 2G_1 & I \\ I & 2\mathscr{G}_2 \end{bmatrix} & \begin{bmatrix} G_1 M^T \\ 0 \end{bmatrix} \\ \star & \star & \star & -\hat{\Gamma}\Theta \end{bmatrix} < 0 \tag{2.44}$$

where

$$\bar{\Omega}_{13} = \begin{bmatrix} AG_1 - B\hat{F} + \underline{D\mathbb{H}F_t^{\nabla}HG_1} & A + \underline{D\mathbb{H}F_t^{\nabla}H} \\ 0 & \mathscr{G}_2 A - \hat{L}C + \underline{\mathscr{G}_2 D\mathbb{H}\bar{F}_t^{\nabla}(H - KC)} \end{bmatrix}. \tag{2.45}$$

Third step : Linearization with respect to F_t^{∇}, $\bar{F}_t^{\nabla}$ and $\hat{\Gamma}$

This step is dedicated to the linearization of the uncertain terms from the nonlinearity and the bilinear term $\tilde{G}_2\Pi_2\hat{\Gamma}$ coming from interconnections. To do this, we rewrite $\bar{\Omega}$ as $\bar{\Omega} \triangleq \bar{\Omega}_0 + \bar{\Omega}_\Delta$, with

$$\bar{\Omega}_0 = \begin{bmatrix} -\hat{P} & 0 & \begin{bmatrix} AG_1 - B\hat{F} & A \\ 0 & \mathscr{G}_2 A - \hat{L}C \end{bmatrix} & 0 \\ \star & -\hat{\Gamma} & 0 & 0 \\ \star & \star & \hat{P} - \begin{bmatrix} 2G_1 & I \\ I & 2\mathscr{G}_2 \end{bmatrix} & \begin{bmatrix} G_1 M^T \\ 0 \end{bmatrix} \\ \star & \star & \star & -\hat{\Gamma}\Theta \end{bmatrix}$$

and $\bar{\Omega}_\Delta$ is the matrix containing the uncertainties and the bilinear terms underlined in (2.44) and (2.45). The expression of $\bar{\Omega}_{13}$ can be rewritten under the convenient form :

$$\bar{\Omega}_\Delta = X^T Y + Y^T X + U^T V + V^T U + W^T Z + Z^T W \tag{2.46}$$

where

$$\begin{aligned}
X &= \begin{bmatrix} (D\mathbb{H}F_t^{\nabla})^T & 0 & 0 & 0 & 0 \end{bmatrix} \\
Y &= \begin{bmatrix} 0 & 0 & HG_1 & H & 0 \end{bmatrix} \\
U &= \begin{bmatrix} 0 & (\mathscr{G}_2 D\mathbb{H})^T & 0 & 0 & 0 \end{bmatrix} \\
V &= \begin{bmatrix} 0 & 0 & 0 & \bar{F}_t^{\nabla}(H - KC) & 0 \end{bmatrix} \\
W &= \begin{bmatrix} 0 & \hat{\Gamma} & 0 & 0 & 0 \end{bmatrix} \\
Z &= \begin{bmatrix} (\tilde{G}_2\Pi_2)^T & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Now, to get less conservative LMI conditions, we use both the reformulated Young's inequality and the classical one on (2.46). Indeed, it would be better to use the reformulated inequality to handle the $Z^T W + W^T Z$, but this will lead to a BMI. Hence, particularly for this term, we use the classical Young's inequality. This strategy allows us to get the following inequality :

$$\bar{\Omega}_\Delta \leq \frac{1}{2}(Y + \mathbb{S}X)^T \mathbb{S}^{-1}(Y + \mathbb{S}X) + \frac{1}{2}(U + \bar{\mathbb{S}}V)^T \bar{\mathbb{S}}^{-1}(U + \bar{\mathbb{S}}V) + Z^T \Upsilon Z + W^T \Upsilon^{-1} W, \tag{2.47}$$

where

$$\mathbb{S} = (F_t^{\nabla})^{-1}\Phi, \; \bar{\mathbb{S}} = (\bar{F}_t^{\nabla})^{-1}\bar{\Phi} \tag{2.48}$$

with Φ and $\bar{\Phi}$ are positive diagonal matrices defined by

$$\Phi = \operatorname{diag}\left(\phi_{ij}^k,\, i,\ldots,N,\, j = 1,\ldots,q_i,\, k = 1,\ldots,s_{ij}\right) \in \mathbb{R}^{s\times s}, \tag{2.49}$$

$$\bar{\Phi} = \operatorname{diag}\left(\bar{\phi}_{ij}^{k},\, i,\dots,N,\, j=1,\dots,q_i,\, k=1,\dots,s_{ij}\right) \in \mathbb{R}^{s\times s}, \tag{2.50}$$

and, Υ is the positive and diagonal matrix given by

$$\Upsilon = \operatorname{diag}\{\varepsilon_1,\dots,\varepsilon_r\},$$

ε_l, $l=1,\dots,r$, are scalar positive parameters to be determined. From the boundedness of $\bar{\Omega}_\Delta$, it follows that a sufficient condition to get $\bar{\Omega}<0$ is that the following be feasible

$$\begin{bmatrix} \bar{\Omega}_0 & Z^T & W^T & (Y+\mathbb{S}X)^T & (U+\bar{\mathbb{S}}V)^T \\ \star & -\Upsilon & 0 & 0 & 0 \\ \star & \star & -\Upsilon^{-1} & 0 & 0 \\ \star & \star & \star & -2(F_t^{\nabla})^{-1}\Phi & 0 \\ \star & \star & \star & \star & -2(\bar{F}_t^{\nabla})^{-1}\bar{\Phi} \end{bmatrix} < 0. \tag{2.51}$$

The last steps leading to LMI (2.38), consist in :
- Using inequalities (2.16) and (2.21) to bound $-2(F_t^{\nabla})^{-1}$ and $-2(\bar{F}_t^{\nabla})^{-1}$ by Λ and $\bar{\Lambda}$, respectively, where

$$\Lambda = \operatorname{diag}\left(-\frac{2}{b_{ij}^{k}},\, i,\dots,N,\, j=1,\dots,q_i,\, k=1,\dots,s_{ij}\right) \in \mathbb{R}^{s\times s}.$$

$$\bar{\Lambda} = \operatorname{diag}\left(-\frac{2}{\bar{b}_{ij}^{k}},\, i,\dots,N,\, j=1,\dots,q_i,\, k=1,\dots,s_{ij}\right) \in \mathbb{R}^{s\times s}.$$

- Using the inequality $-\Upsilon^{-1}$ by $-(2I-\Upsilon)$, coming from the fact that $-\varepsilon_l^{-1} \leq -(2-\varepsilon_l)$ for all $l=1,\dots,r$.
- Make the change of variable $\hat{K} = \bar{\phi}K$.

This ends the proof of Theorem 2.4.1.

Remark 2.4.2. *It is worth noting that the reformulations* (2.33)-(2.36) *obtained by Schur lemma and the inversion lemma play an important role for the case of known interconnections. Indeed, unlike some references [87], where the obtained LMI works only for the case of unknown interconnections (because of the existence of Γ and Γ^{-1} in the LMI in case of Θ known), this reformulation allows us to get a matrix inequality that remains LMI even for the case where Θ is considered as known.*

Remark 2.4.3. *As to the choice of the triangular structure of G, it should be noted that such a choice is crucial since it avoids the apparition of additional bilinearities as an isolated term BF that could be inferred from a diagonal structure choice [36], [38].*

Remark 2.4.4. *It is very important to mention that in LMI framework, a small but ingenious and novel idea may have a major consequence and lead to less conservative LMI conditions, which is very significant for robustness and performances with respect to the interconnection bounds and disturbances. For this reason, often in the LMI approach, the contributions seem to be very simple and not significant enough, although the conservatism of the concerned control problem is very significantly overcome from LMI feasibility* *point of view. All the LMI techniques established in the literature are based on a simple idea followed by a chain of mathematical developments based on matrix decompositions using congruence principle and the Schur Lemma in various versions. In this paper, the introduction of the triangular form of the slack variable G avoids many bilinear terms and leads to simpler and less conservative LMI conditions. Although this idea seems simple but it leads to better LMIs from feasibility point of view. This is the first time this idea is used to solve the decentralized observer-based stabilization problem for interconnected systems, and may help for facing more complicated control design problems.*

2.5 Illustrative Examples

In order to show the effectiveness of the proposed design methodology, we validate it on two real examples of interconnected pendulums. The first one is taken from [86]. It is dedicated to an application of our derived result for the case of known interconnection and in the presence of internal non-linear dynamics. Regarding the second example, we choose one of the most repeated used examples in the literature [31], [32], [53], [87] that deals with the case of unknown interconnections. The aim is to show the less conservativeness brought by the convex optimization problem (2.38).

2.5.1 Example 1

The dynamics of pendulums are as given in [86] and described by the following equations :

$$\dot{x}_{i1} = x_{i2} \tag{2.52a}$$

$$\dot{x}_{i2} = \underbrace{\left(\frac{m_i g \rho}{J_i} - \frac{\kappa \rho^2}{4J_i}\right)}_{\Delta_i} \sin(x_{i1}) + \frac{u_i + f_i^s}{J_i} + \sum_{j=1}^{3} \delta_{ij}(x_j) \tag{2.52b}$$

where $x_i = \begin{bmatrix} x_{i1} & x_{i2} \end{bmatrix}^T$, with x_{i1} and x_{i2} being angular displacement of the pendulums from vertical and angular velocity respectively, $m_1 = m_3 = 5kg, m_2 = 6.5kgs$ are the pendulum end masses, $J_1 = J_3 = 0.4kg, J_2 = 0.9kgs$ are the moments of inertia, $\kappa = 3N/m$ is the spring constant, $\rho = 0.5m$ is the pendulum height, $g = 9.81$ is the gravitational acceleration, δ_{ij} are given by

$$\begin{aligned}
\delta_{12}(x_2) &= \left(\frac{\kappa \rho^2}{4J_1}\right) \sin(x_{21}) \\
\delta_{21}(x_1) &= \left(\frac{\kappa \rho^2}{4J_2}\right) \sin(x_{11}) \\
\delta_{23}(x_3) &= \left(\frac{\kappa \rho^2}{4J_2}\right) \sin(x_{31}) \\
\delta_{32}(x_2) &= \left(\frac{\kappa \rho^2}{4J_3}\right) \sin(x_{21})
\end{aligned}$$

$$\delta_{13} \quad = \quad \delta_{31} = 0,$$

and $f_i^s = \varepsilon_i x_{i2}$ with $\varepsilon_i \in [0, 0.14]$ is a fault which is due to some unexpected friction in the actuator. Now, prior to formulating equation (2.52) in the form of (2.8), we distinguish two ways to address the failure of the actuator. The first one consists in adding the terms f_i^s to the interconnections δ_{ij}. The second one is to neglect f_i^s by considering that the objective is not the fault diagnosis problem. Note that in both cases our approach provide feasible LMI solutions. We focus on the first case where system (2.52) can be represented by equation (2.8), with the following parameters

– Subsystems 1 and 3 :

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ \frac{1}{J_i} \end{bmatrix}, C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \text{for } i = 1, 3,$$

$$D_i = \begin{bmatrix} 0 \\ \Delta_i \end{bmatrix}, E_i = \begin{bmatrix} 0 & 0 \\ \frac{1}{J_i} & \frac{\kappa\rho^2}{4J_i} \end{bmatrix}, \text{for } i = 1, 3,$$

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

– Subsystem 2 :

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ \frac{1}{J_2} \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 \\ \Delta_2 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{J_2} & \frac{\kappa\rho^2}{4J_2} & \frac{\kappa\rho^2}{4J_2} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The known nonlinearities are given by

$$f_1(x_{11}) = \sin \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H_1 = H_{11}} x_1,$$

$$f_2(x_{21}) = \sin \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H_2 = H_{21}} x_2,$$

$$f_3(x_{31}) = \sin \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H_3 = H_{31}} x_3,$$

from which we deduce that $H = \mathrm{diag}\left\{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\end{bmatrix}\right\}$.
The interconnections are represented by the following vectors :

$$\eta_1(x) = \begin{bmatrix}\varepsilon_1 x_{12} & \sin(x_{21})\end{bmatrix}^T,$$
$$\eta_2(x) = \begin{bmatrix}\varepsilon_2 x_{22} & \sin(x_{11}) & \sin(x_{31})\end{bmatrix}^T,$$
$$\eta_2(x) = \begin{bmatrix}\varepsilon_3 x_{32} & \sin(x_{21})\end{bmatrix}^T.$$

The integer numbers determining the dimensions of the parameters are $N = 3$, $n_i = 2$, for all $i = 1, \ldots, 3$, $n = 6$, $q_i = 1$, for all $i = 1, \ldots, 3$, $q = 3$, $s_{ij} = s_{i1} = s_i = 1$, for all $i = 1, \ldots, 3$, (in this example, the only value of the indices j and k is $j = k = 1$ since $q_i = 1$ and $s_{ij} = 1$), $s = 3$, $r_{1,3} = 2$, $r_2 = 3$, $r = 7$, $p_i = 1$, for all $i = 1, \ldots, 3$ and $p = 3$. The nonlinearities $f_i, i = \overline{1,3}$ satisfy inequality (2.16), with $a^k_{ij} = a = -1, b^k_{ij} = b = 1$, for all $i = 1, \ldots, 3$. Now, since $a^k_{ij} = -1 \neq 0$, we use Remark 2.1 in [18] to obtain (2.16) with

$$f_i(x_{i1}) := f_i(x_{i1}) - aH_i x_i$$

or equivalently

$$f(Hx_t) := f(Hx_t) - a\mathbb{H}Hx_t$$

to obtain (2.16). The state matrix A is then

$$A := A + aD\mathbb{H}H$$

where here $\mathbb{H} = I_3$. In addition, we suppose that system (2.52) is not entirely measurable contrarily to the measurability condition imposed in [86]. So, we add a linear output y_i ($y_i = C_i x_i$). Note that the objective here is to decentralize the Euler discretization of system (2.52) with known interconnection bounds, that are

$$\alpha_1 = \alpha_3 = \alpha_6 = 1, \alpha_2 = \alpha_4 = \alpha_5 = \alpha_7 = 0.14$$

provided by the Lipschitz property on each component of $\eta_i(x)$, $i = 1, \ldots, 3$. By solving LMI (2.38) (with sampling period $\delta = 0.01$), we get the following gain matrices

$$F_1 = \begin{bmatrix}411.6333 & 44.0926\end{bmatrix}, L_1 = \begin{bmatrix}4.0044\\ 200.3291\end{bmatrix}$$
$$F_2 = \begin{bmatrix}889.3523 & 98.6853\end{bmatrix}, L_2 = \begin{bmatrix}3.9577\\ 197.3441\end{bmatrix}$$
$$F_3 = \begin{bmatrix}411.6333 & 44.0926\end{bmatrix}, L_3 = \begin{bmatrix}4.0044\\ 200.3291\end{bmatrix}$$

and

$$K_1 = \begin{bmatrix}0.2218 & 0 & 0\end{bmatrix},$$
$$K_2 = \begin{bmatrix}0 & -0.1204 & 0\end{bmatrix},$$

and

$$K_3 = \begin{bmatrix}0 & 0 & 0.2218\end{bmatrix}.$$

The asymptotic stabilizability of the system is shown through Figures 3.3-3.5.

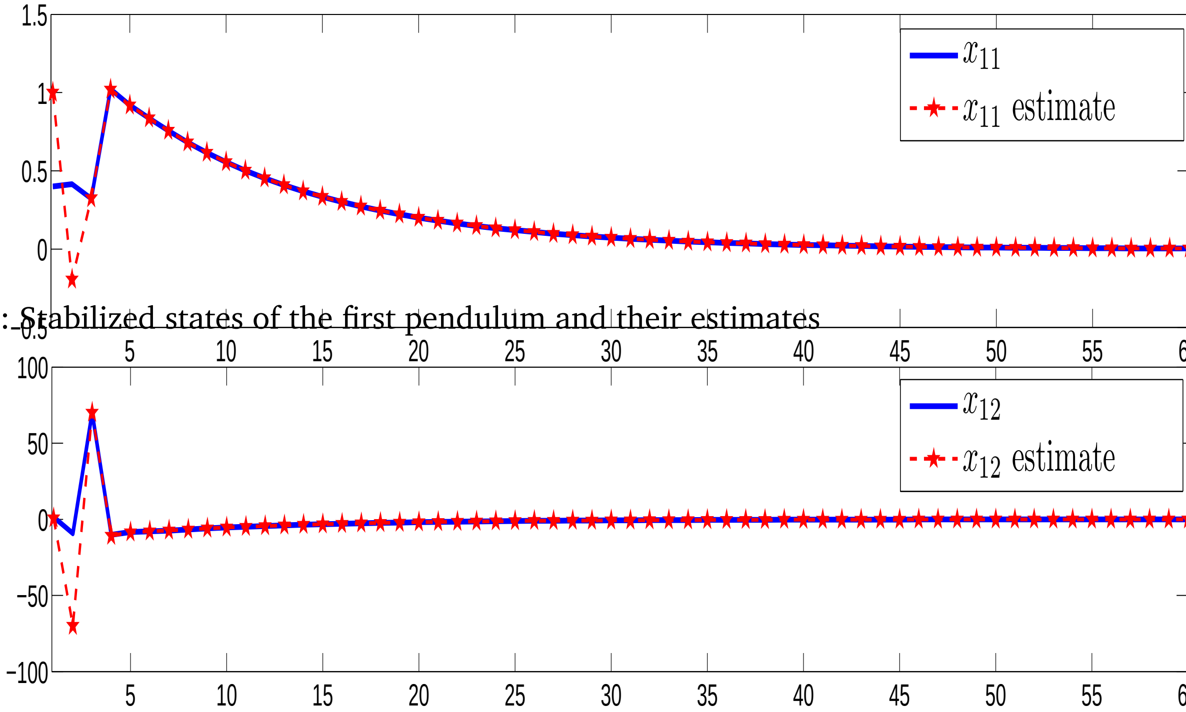


FIG. 2.1: Stabilized states of the first pendulum and their estimates

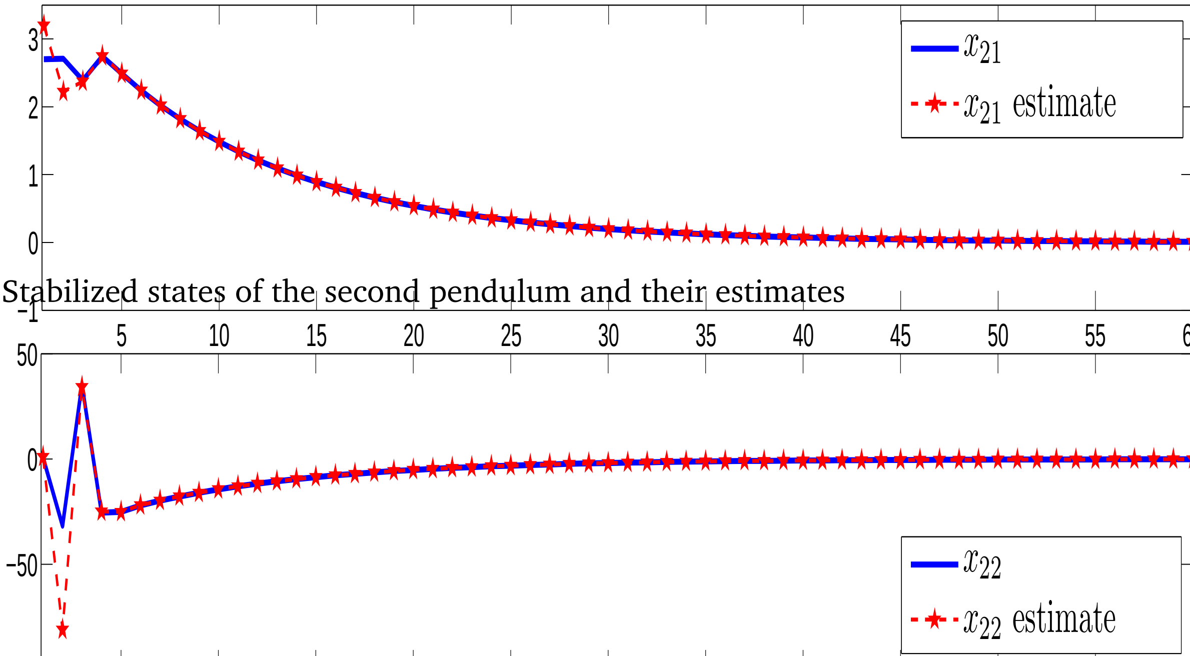


FIG. 2.2: Stabilized states of the second pendulum and their estimates

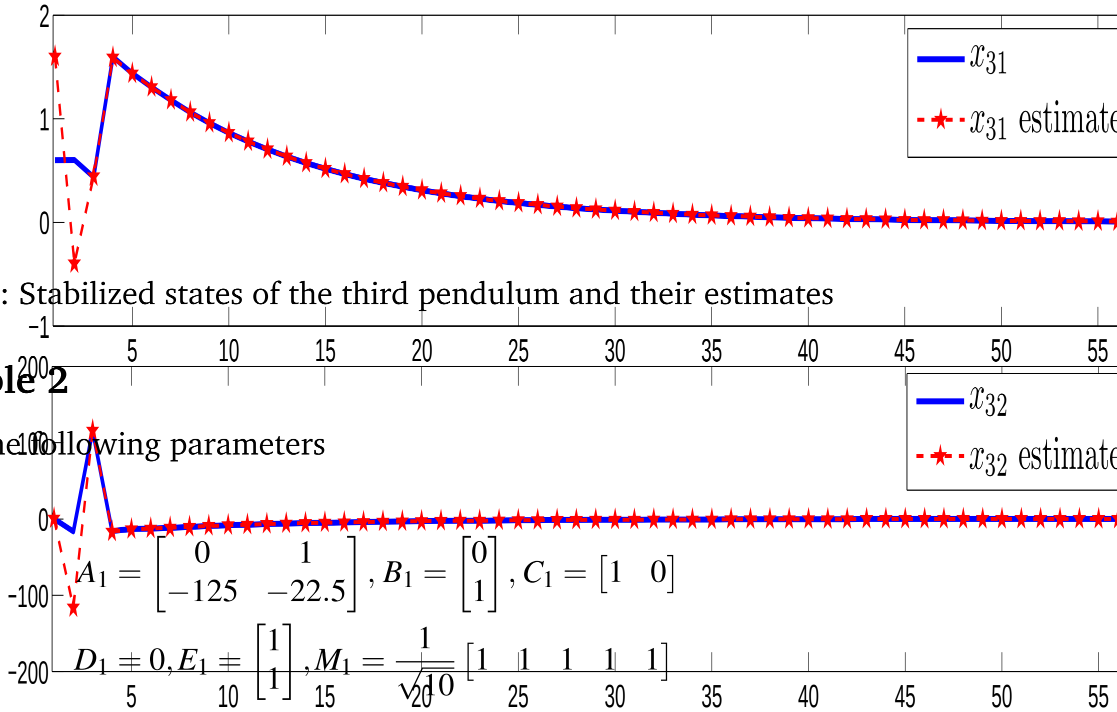


FIG. 2.3: Stabilized states of the third pendulum and their estimates

2.5.2 Example 2

The system has the following parameters

– Subsystem 1 :

$$A_1 = \begin{bmatrix} 0 & 1 \\ -125 & -22.5 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D_1 = 0, E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, M_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

– Subsystem 2 :

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -37.5 & -50 & -13.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D_2 = 0, E_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$M_2 = \frac{1}{\sqrt{15}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The interconnections $\eta_1(x)$ and $\eta_2(x)$ are given by

$$\eta_1(x) = \alpha_1 \cos(x_{22}) M_1 x,$$

and

$$\eta_2(x) = \alpha_2 \cos(x_{11}) M_2 x.$$

By solving LMI (2.38) for the discretized Euler system :

$$A := I + \delta A, B := \delta B,$$

$$E := \delta E, C := C$$

with sampling period $\delta = 0.01$, we obtained the following gain matrices

$$F_1 = \begin{bmatrix} -20.0904 \\ 83.4096 \end{bmatrix}^T, L_1 = \begin{bmatrix} 1.1221 \\ -0.3629 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 75.9918 \\ 64.4918 \\ 100.9918 \end{bmatrix}^T, L_2 = \begin{bmatrix} 1.2036 \\ 1.2036 \\ 0.1836 \end{bmatrix},$$

and the interconnection bounds

$$\alpha_1 = 9.2571, \alpha_2 = 10.2082.$$

The less-conservativeness is shown through Table I that includes the different maximum interconnection bounds provided by other techniques for the same example.

Method	[53]	[32]	[87]	LMI (2.38)
$\alpha_{1\,\max}$	0.2	5	8.6082	9.2571
$\alpha_{2\,\max}$	0.2	5	2.1669	10.2082

TAB. 2.1: Maximum allowable bounds of the interconnections

2.6 Conclusion

We have proposed in this paper a new decentralized observer-based stabilization design methodology for a class of nonlinear interconnected systems with nonlinear interconnections. Furthermore, a new and enhanced LMI design condition has been provided to guarantee asymptotic stability for systems with both known and unknown interconnection bounds. Indeed, in the previous results in the literature, the design conditions are often LMIs only if we consider the interconnection bounds as unknown. The mathematical developments used in this paper, such as the new variant of Young's inequality, the new way to use the congruence principle, particularly the new proposed structure of the slack matrix G, may be very useful for some control design problems for systems with uncertainties. Hence as for future work, we will aim to propose some enhanced techniques for other classes of systems, such as nonlinear switched systems. The validity and effectiveness of our design method have been shown through two numerical examples. The provided numerical comparisons demonstrate the superiority of the proposed methodology compared to some existing results in the literature.

CHAPITRE 3

Robust $\mathscr{H}_\infty$ Observer-Based Stabilization of Linear Discrete-Time Systems with Parameter Uncertainties

3.1 Introduction

This chapter addresses the problem of observer-based stabilization of discrete-time linear systems in presence of parameter uncertainties and ℓ_2-bounded disturbances. We propose a new variant of the classical two-step LMI approach. In the first step, we use a slack variable technique to solve the optimization problem resulting from stabilization by a static state feedback. In the second step, a part of the slack variable obtained is incorporated in the $\mathscr{H}_\infty$ observer-based stabilization problem, to calculate simultaneously the Lyapunov matrix and the observer-based controller gains. The effectiveness of the proposed design methodology is demonstrated using the dynamic model of a quadruple-tank flow process system. Additional numerical illustrations are presented to show the superiority of the proposed Modified Two-Step Method (MTSM) from a LMI feasibility point of view.

To clarify the contributions of the proposed design methodology, we summarize the advantages of the paper in the following items :

- We show analytically that Ibrir's approach [27, Lemma 1] based on linearization lemma is too restrictive for . Indeed, we propose a counter-example (Theorem 3.6.2) illustrating the inadequacies of the lemma.
- We reformulate Ibrir Lemma in terms of necessary and sufficient conditions, giving rise to Theorem 3.6.3, and which contains as a particular case the result by de Oliveira et al. result [11, Theorem 1].
- Based on Theorem 3.6.3, we propose several new variants of the two-step method, that can be considered as heuristic methods. The idea is to study the impact of the first stage decision variable on the feasibility of the proposed LMI conditions. Thus, instead of calculating the controller gain K as in [70], or part of the Lyapunov matrix

from the first step as in [91], we propose to introduce an additional slack variable (thanks to Theorem 3.6.3), that we calculate (a part) from the first step.
- Monte Carlo simulations show that calculating the main decision variables, namely, the controller and observer gains K and L, and the Lyapunov matrix P simultaneously remains the best strategy.
- The numerical examples presented reinforce this conclusion, and show the effectiveness and superiority of the new algorithm with respect to the Young inequality-based approach [35], and the existing two-step method [91] and [70].

The rest of this paper is organized as follows : Section 3.2 is devoted to the problem statement and a brief state of the art related to some LMI design methods available in the literature. The main contributions of this paper are introduced in Section 3.3. Section 3.4 is dedicated to numerical evaluation and comparisons to show the effectiveness and the superiority of the proposed methodology. Finally, we end the chapter by a conclusion and an appendix.

3.2 Problem statement

Let us consider the class of discrete-time systems described by the following equations :

$$x_{t+1} = (A + \Delta\mathbb{A}(t))x_t + (B + \Delta\mathbb{B}(t))u_t + E\omega_t \tag{3.1a}$$

$$y_t = (C + \Delta\mathbb{C}(t))x_t + D\omega_t \tag{3.1b}$$

$$z_t = Hx_t + \mathbb{S}\omega_t \tag{3.1c}$$

where $t \in \mathbb{N}$, $x_t \in \mathbb{R}^n$ is the state vector, $y_t \in \mathbb{R}^p$ is the output measurement, and $u_t \in \mathbb{R}^m$ is the control input vector, $\omega_t \in \mathbb{R}^v$ is an unknown exogenous disturbance, $z_t \in \mathbb{R}^q$ is the controlled output. The parameters $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $E \in \mathbb{R}^{n\times v}$, $D \in \mathbb{R}^{p\times v}$, $H \in \mathbb{R}^{q\times n}$ and $\mathbb{S} \in \mathbb{R}^{q\times v}$ are known matrices. We require the following assumptions :

Assumption 1. *1. The pairs (A,B) and (A,C) are stabilizable and detectable, respectively.*

2. The sequence $\omega := (\omega_t)_{t\in\mathbb{N}}$ is ℓ_2^v-bounded, namely

$$\omega \in \ell_2^v := \left\{ (w_t)_{t\in\mathbb{N}},\ \|w\|_{\ell_2^v} = \left(\sum_{s=0}^{\infty} \|\omega_s\|^2 \right)^{\frac{1}{2}} < +\infty \right\},$$

$\|.\|$ being the Euclidean norm.

3. The uncertain terms $\Delta\mathbb{A} := \Delta\mathbb{A}(t) \in \mathbb{R}^{n\times n}$, $\Delta\mathbb{C} := \Delta\mathbb{C}(t) \in \mathbb{R}^{p\times n}$ and $\Delta\mathbb{B}(t)$ are unknown matrices and are structured and norm-bounded in the following sense :

$$\Delta\mathbb{A}(t) = M_A \mathbb{F}_A(t) N_A, \quad \Delta\mathbb{C}(t) = M_C \mathbb{F}_C(t) N_C \tag{3.2}$$

where M_A, N_A, M_C, and N_C are known and constant matrices ; $\mathbb{F}_A(t)$ and $\mathbb{F}_C(t)$ are unknown time-varying matrices satisfying the inequalities :

$$\mathbb{F}_A^T(t)\mathbb{F}_A(t) \leq \alpha^2 I, \quad \mathbb{F}_C^T(t)\mathbb{F}_C(t) \leq \beta^2 I, \quad \forall t \in \mathbb{N}. \tag{3.3}$$

4. *The matrix B is full column rank and $\Delta\mathbb{B}(t) = 0$.*

It should be noticed that this last assumption is not restrictive since (3.1) can always be transformed into a system where $\Delta\mathbb{B}(t) = 0$, see for instance [28].

Remark 3.2.1. *Since B is full column rank, then there is a non singular matrix T such that $TB = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$. Hence, using the similarity transformation T, system (3.1) can be transformed and rewritten with the following matrices :*

$$A := TAT^{-1},\ C := CT^{-1},\ B := TB = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, E := TE,$$

$$H := HT^{-1}, \mathbb{S} := \mathbb{S}, D := D,\ \Delta\mathbb{A}(t) := T\Delta\mathbb{A}(t)T^{-1},$$
$$\Delta\mathbb{C}(t) := \Delta\mathbb{C}(t)T^{-1},\ M_A := TM_A,\ N_A := N_A T^{-1},$$
$$M_C := M_C,\ N_C := N_C T^{-1}.$$

It should be noted that such coordinate transformation is well known in the literature, see for instance [7] and [9]. Consequently, without loss of generality we assume that

$$B = \begin{bmatrix} I_m & 0 \end{bmatrix}^T. \tag{3.4}$$

The robust asymptotic stabilization of the discrete-time linear system (3.1) is addressed using the Luenberger observer

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + L(y_t - C\hat{x}_t), \tag{3.5}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain to be determined and $\hat{x}_t \in \mathbb{R}^n$ is the estimate of x_t at time t. Let $e_t := x_t - \hat{x}_t$ be the estimation error. Then, under the feedback

$$u_t = -K\hat{x}_t \tag{3.6}$$

where $K \in \mathbb{R}^{m\times n}$ is the controller gain, the dynamics of the augmented state $\xi_t^T = (x_t^T, e_t^T) \in \mathbb{R}^{2n}$ is written under the form :

$$\begin{aligned} \xi_{t+1} &= \begin{bmatrix} A - BK + \Delta\mathbb{A} & BK \\ \Delta\mathbb{A} - L\Delta\mathbb{C} & A - LC \end{bmatrix} \xi_t + \begin{bmatrix} E \\ E - LD \end{bmatrix} \omega_t \\ &= \Pi\xi_t + \Upsilon\omega_t. \end{aligned} \tag{3.7}$$

In this paper, we aim to design the observer-based controller gains K and L for the system (3.1), such that, in the presence of disturbance, the closed-loop system (3.7) is $\mathscr{H}_\infty$-asymptotically stable. More specifically :

Definition 3.2.2 ([17, 45]). *System (3.7) is said to be $\mathscr{H}_\infty$-asymptotically stable if there exist $\mu > 0$ and $\delta > 0$ such that*

$$\|\xi\|_{\ell_2^{2n}} \le \sqrt{\mu \|w\|^2_{\ell_2^{\nu}} + \delta \|\xi_0\|^2}. \tag{3.8}$$

The positive number $\sqrt{\mu}$ is then the disturbance attenuation level, to be minimized, representing the disturbance gain from w to the performance signal z, defined by (3.1c). We say that the $\mathscr{H}_\infty$ performance criterion is achieved if (3.8) holds.

Remark 3.2.3. *Notice that to solve the problem of $\mathscr{H}_\infty$-asymptotic stability it is sufficient to find a Lyapunov function $V_t := V(\xi_t) = \xi_t^T \mathbb{P} \xi_t$, with $\mathbb{P} = \mathbb{P}^T > 0$, so that*

$$W_t = V_{t+1} - V_t + z_t^T z_t - \mu \omega_t^T \omega_t < 0, \forall t \in \mathbb{N}. \tag{3.9}$$

Indeed, if (3.9) holds, by summation of (3.9) from 0 to t, we get for all $t \in \mathbb{N}$

$$V(\xi_t) - V(\xi_0) + \sum_{s=0}^{t} \|\xi_s\|^2 - \mu \sum_{s=0}^{t} \|\omega_s\|^2 \le 0.$$

From Rayleigh inequality $V(\xi_0) \le \lambda_{\max}(\mathbb{P})\|\xi_0\|^2$ and the fact that $V(\xi_t) \ge 0$ for all $t \ge 0$, then as $t \to +\infty$, we have

$$\sum_{s=0}^{\infty} \|\xi_s\|^2 \le \mu \sum_{s=0}^{\infty} \|\omega_s\|^2 + \lambda_{\max}(\mathbb{P})\|\xi_0\|^2.$$

Thus (3.8) is satisfied with $\delta = \lambda_{\max}(\mathbb{P})$.

Now, to investigate the $\mathscr{H}_\infty$-asymptotic stability of (3.7), in the LMI framework, let V_t be the candidate Lyapunov function defined in Remark 3.2.3. Then, for any $t \in \mathbb{N}$,

$$\begin{aligned} V_{t+1} - V_t = \xi_t^T (\Pi^T \mathbb{P} \Pi - \mathbb{P}) \xi_t + \xi_t^T \Pi^T \mathbb{P} \Upsilon \omega_t \\ + \omega_t^T \Upsilon \mathbb{P} \Pi \xi_t + \omega_t^T \Upsilon^T \mathbb{P} \Upsilon \omega_t. \end{aligned} \tag{3.10}$$

Furthermore,

$$z_t^T z_t = \begin{bmatrix} x_t^T & \omega_t^T \end{bmatrix} \begin{bmatrix} H^T H & H^T \mathbb{S} \\ (\star) & \mathbb{S}^T \mathbb{S} \end{bmatrix} \begin{bmatrix} x_t \\ \omega_t \end{bmatrix} \tag{3.11}$$

$$= \begin{bmatrix} \xi_t^T & \omega_t^T \end{bmatrix} \begin{bmatrix} \begin{bmatrix} H^T \\ 0 \end{bmatrix} \begin{bmatrix} H & 0 \end{bmatrix} & \begin{bmatrix} H^T \\ 0 \end{bmatrix} \mathbb{S} \\ (\star) & \mathbb{S}^T \mathbb{S} \end{bmatrix} \begin{bmatrix} \xi_t \\ \omega_t \end{bmatrix}. \tag{3.12}$$

Hence (3.9) can be rewritten under the quadratic form

$$\begin{bmatrix} \xi_t \\ \omega_t \end{bmatrix}^T \Sigma \begin{bmatrix} \xi_t \\ \omega_t \end{bmatrix} < 0 \tag{3.13}$$

where

$$\Sigma = \begin{bmatrix} \Pi^T \mathbb{P} \Pi - \mathbb{P} + \begin{bmatrix} H^T \\ 0 \end{bmatrix} \begin{bmatrix} H & 0 \end{bmatrix} & \Pi^T \mathbb{P} \Upsilon + \begin{bmatrix} H^T \\ 0 \end{bmatrix} \mathbb{S} \\ (\star) & \Upsilon^T \mathbb{P} \Upsilon - \mu I \end{bmatrix}.$$

Therefore, condition (3.9) holds if and only if $\Sigma < 0$. Now, using the following decomposition

$$\Sigma = \begin{bmatrix} -\mathbb{P} & 0 \\ (\star) & -\mu I \end{bmatrix} + \begin{bmatrix} \Pi^T & \begin{bmatrix} H^T \\ 0 \\ \mathbb{S}^T \end{bmatrix} \\ \Upsilon^T & \end{bmatrix} \begin{bmatrix} \mathbb{P} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Pi & \Upsilon \\ \begin{bmatrix} H & 0 \end{bmatrix} & \mathbb{S} \end{bmatrix} \tag{3.14}$$

of Σ and Schur Lemma, (3.9) turns out to be equivalent to the following BMI :

$$\begin{bmatrix} -\mathbb{P}^{-1} & \Pi & \Upsilon & 0 \\ (\star) & -\mathbb{P} & 0 & \begin{bmatrix} H^T \\ 0 \\ \mathbb{S}^T \end{bmatrix} \\ (\star) & (\star) & -\mu I & \\ (\star) & (\star) & (\star) & -I \end{bmatrix} < 0. \tag{3.15}$$

It should be noted that solving (3.15) is a very difficult task. Indeed, to our knowledge, there are no equivalent LMI conditions to (3.15), even in the uncertainty free case. The problem arises from the presence of the Lyapunov matrix P and its inverse, and the presence of unknown matrices in Π. Although many efforts have been dedicated to this issue [27], [35], [17], [30], the Young's inequality-based approach remains the more effective technique [35], however, it is expensive from a numerical point of view. In the following section, we will propose new and enhanced LMI conditions to handle the BMI (3.15). We will provide a new variant of the two-step algorithm introduced in [70], and improved recently in [91].

3.3 New LMI Design Algorithm

In this section, we present a numerically efficient technique to determine the observer-based controller gains ensuring robust asymptotic stability of (3.7). In order to get sufficient conditions that guarantee the $\mathscr{H}_\infty$-asymptotic stability of (3.7), we first linearize the BMI (3.15). Here, instead of using Lemma 3.6.1 [27], we use its relaxed version, namely Theorem 3.6.3, with :

$$Z = \mathbb{P}; Y = \begin{bmatrix} \Pi & \Upsilon & 0 \end{bmatrix}; X = \begin{bmatrix} -\mathbb{P} & 0 & \begin{bmatrix} H^T \\ 0 \\ \mathbb{S}^T \end{bmatrix} \\ (\star) & -\mu I & \\ (\star) & (\star) & -I \end{bmatrix}.$$

The BMI problem (3.15) is thus equivalent to find an invertible matrix G and a symmetric positive definite matrix $\mathbb{P}$ such that

$$\begin{bmatrix} \mathbb{P} - \mathrm{He}(G) & G\Pi & G\Upsilon & 0 \\ (\star) & -\mathbb{P} & 0 & \begin{bmatrix} H^T \\ 0 \\ \mathbb{S}^T \end{bmatrix} \\ (\star) & (\star) & -\mu I & \\ (\star) & (\star) & (\star) & -I \end{bmatrix} < 0, \tag{3.16}$$

where $\mathrm{He}(G) = G + G^T$. Hence, after developing $G\Pi$ in (3.16) and choosing $G = \mathrm{diag}\,(G_1, G_2)$, we get

$$\Omega := \begin{bmatrix} \mathbb{P} - \mathrm{He}(G) & \Phi & \begin{bmatrix} G_1 E \\ G_2(E - LD) \end{bmatrix} & 0 \\ (\star) & -\mathbb{P} & 0 & \begin{bmatrix} H^T \\ 0 \end{bmatrix} \\ (\star) & (\star) & -\mu I & \mathbb{S}^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} < 0, \tag{3.17}$$

where

$$\Phi := \begin{bmatrix} G_1(A - BK + \Delta\mathbb{A}) & G_1 BK \\ G_2(\Delta\mathbb{A} - L\Delta\mathbb{C}) & G_2(A - LC) \end{bmatrix}.$$

At first, we need to linearize the uncertainties in (3.17). This step is classical and uses the standard Young's relation. The details of this step are omitted since the objective is not the linearization of the uncertainties but rather how we handle the bilinear term $G_1 BK$ in Φ. Therefore, from the change of variables $\hat{L} = G_2 L$, $\hat{\alpha}^{-1} = \varepsilon_1 \alpha^2$, and $\hat{\beta}^{-1} = \varepsilon_2 \beta^2$, we deduce that inequality (3.15) holds if the following bilinear matrix inequality

$$\begin{bmatrix} \Omega_1 & \overbrace{\begin{bmatrix} G_1 M_A & 0 & 0 & \\ G_2 M_A & \hat{L} M_C & 0 & 0 \\ 0 & 0 & N_A^T & N_C^T \\ 0 & 0 & 0 & 0 \end{bmatrix}}^{\Omega_2} \\ \star & \mathrm{diag}\left[-\varepsilon_1 I, -\varepsilon_2 I, -\hat{\alpha} I, -\hat{\beta} I\right] \end{bmatrix} < 0, \tag{3.18}$$

where

$$\Omega_1 = \begin{bmatrix} \mathbb{P} - \mathrm{He(G)} & \Gamma & \begin{bmatrix} G_1 E \\ G_2 E - \hat{L} D \end{bmatrix} & 0 \\ (\star) & -\mathbb{P} & 0 & \begin{bmatrix} H^T \\ 0 \end{bmatrix} \\ (\star) & (\star) & -\mu I & \mathbb{S}^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix}, \tag{3.19}$$

$$\Gamma = \begin{bmatrix} G_1 A - \boxed{G_1 BK} & \boxed{G_1 BK} \\ 0 & G_2 A - \hat{L} C \end{bmatrix}. \tag{3.20}$$

is fulfilled. Now, it remains to linearize the bilinear term $G_1 BK$ involved in (3.18). For this purpose, taking into account the structure (3.4) of B, we have

$$G_1 B = \begin{bmatrix} G_{11}^1 & G_{12}^1 \\ G_{21}^1 & G_{22}^1 \end{bmatrix} \begin{bmatrix} I_m \\ 0 \end{bmatrix} = \begin{bmatrix} G_{11}^1 \\ G_{21}^1 \end{bmatrix} \tag{3.21}$$

and therefore

$$G_1 BK = \begin{bmatrix} G_{11}^1 K \\ G_{21}^1 K \end{bmatrix}.$$

$$\begin{bmatrix} \begin{bmatrix} \mathbb{P}-\mathrm{He(G)} & \begin{bmatrix} G_1A-\begin{bmatrix}\hat{K}\\ S\hat{K}\end{bmatrix} & \begin{bmatrix}\hat{K}\\ S\hat{K}\end{bmatrix} \\ 0 & G_2A-\hat{L}C \end{bmatrix} & \begin{bmatrix} G_1E \\ G_2E-\hat{L}D \end{bmatrix} & 0 \\ (\star) & -\mathbb{P} & 0 & \begin{bmatrix} H^T \\ 0 \end{bmatrix} \\ (\star) & (\star) & -\mu I & \mathbb{S}^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} & \begin{bmatrix} G_1M_A & 0 & 0 & \\ G_2M_A & \hat{L}M_C & 0 & 0 \\ 0 & 0 & N_A^T & N_C^T \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ (\star) & \mathrm{diag}\left[-\varepsilon_1 I, -\varepsilon_2 I, -\hat{\alpha} I, -\hat{\beta} I\right] \end{bmatrix} < 0 \tag{3.23}$$

By choosing the matrix G_{11}^1 invertible (and symmetric), there always exists a matrix $S \in \mathbb{R}^{(n-m)\times m}$ such that

$$G_{21}^1 = SG_{11}^1. \tag{3.22}$$

By the change of variable $\hat{K} = G_{11}^1 K$, the matrix Γ in (3.18) becomes :

$$\Gamma = \begin{bmatrix} G_1A-\begin{bmatrix}\hat{K}\\ S\hat{K}\end{bmatrix} & \begin{bmatrix}\hat{K}\\ S\hat{K}\end{bmatrix} \\ 0 & G_2A-\hat{L}C \end{bmatrix}.$$

By injecting this last expression of Γ into (3.18), we obtain (3.23), as the detailed form of the final BMI to be solved. Summing up, Inequality (3.23) remains still a BMI because of the coupling term $S\hat{K}$. It becomes LMI if we *a priori* fixe the structure of S. The next step is to propose an heuristic approach to calculate G_{11}^1 and G_{21}^1, and therefore

$$S = G_{21}^1 (G_{11}^1)^{-1}.$$

To this end, inspired by [91] and [70], we solve the optimization problem resulting from the stabilization of (3.1a) by a static state feedback $u = -\mathscr{K}x$. We consider the $\mathscr{H}_\infty$-stabilization problem of (3.1) by a static state feedback $u = -\mathscr{K}x$. System (3.1a) can be written in the closed-loop form :

$$x_{t+1} = \Big(A - B\mathscr{K} + M_A \mathbb{F}_A(t) N_A\Big)x_t + E\omega_t. \tag{3.24}$$

System (3.24) is $\mathscr{H}_\infty$ globally asymptotically stable by the Lyapunov function $\mathbb{V}(x) = x^T\mathscr{P}x$, with a minimum attenuation level η, if the following inequality holds :

$$\mathbb{V}(x_{t+1}) - \mathbb{V}(x_t) + z_t^T z_t - \eta\,\omega_t^T\omega_t < 0, \forall t \in \mathbb{N}. \tag{3.25}$$

Expoiting Eq. (3.11) and Schur Lemma, we get that (3.25) holds if and only if

$$\begin{bmatrix} -\mathscr{P}^{-1} & \Big(A - B\mathscr{K} + \Delta A\Big) & E & 0 \\ (\star) & -\mathscr{P} & 0 & H^T \\ (\star) & (\star) & -\eta I & \mathbb{S}^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} < 0. \tag{3.26}$$

Now, applying Theorem 3.6.3 with

$$\begin{cases} Z &= \mathscr{P}; \\ Y &= \left[\left(A - B\mathscr{K} + \Delta A\right) \quad E \quad 0\right] \end{cases}$$

and

$$X = \begin{bmatrix} -\mathscr{P} & 0 & H^T \\ (\star) & -\eta I & \mathbb{S}^T \\ (\star) & (\star) & -I \end{bmatrix}$$

there exists a matrix $\mathscr{G}$ so that (3.26) is equivalent to

$$\mathfrak{F} := \begin{bmatrix} \mathscr{P} - \mathscr{G} - \mathscr{G}^T & \mathscr{G}\left(A - B\mathscr{K} + \Delta A\right) & \mathscr{G}E & 0 \\ (\star) & -\mathscr{P} & 0 & H^T \\ (\star) & (\star) & -\eta I & \mathbb{S}^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} < 0. \tag{3.27}$$

By choosing $\mathscr{G} = \mathscr{G}^T$, and using the congruence principal, then (3.27) holds if, and only if, the following inequality is fulfilled : $\mathrm{diag}(\mathscr{G}^{-1}, \mathscr{G}^{-1}, I, I)\,\mathfrak{F}\,\mathrm{diag}(\mathscr{G}^{-1}, \mathscr{G}^{-1}, I, I) < 0$. Therefore, after developing the calculation and by making the change of variables $\mathscr{Z} = \mathscr{G}^{-1}\mathscr{P}\mathscr{G}^{-1}$, $\mathscr{G}^{-1} = \hat{G}$ and $\bar{\mathscr{K}} = \mathscr{K}\hat{G}$, we get

$$\begin{bmatrix} \mathscr{Z} - \hat{G} - \hat{G}^T & A\hat{G} - B\bar{\mathscr{K}} & E & 0 \\ (\star) & -\mathscr{Z} & 0 & \hat{G}H^T \\ (\star) & (\star) & -\eta I & \mathbb{S}^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} M_A \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbb{F}_A(t) \begin{bmatrix} 0 \\ (N_A\hat{G})^T \\ 0 \\ 0 \end{bmatrix}^T\right) < 0. \tag{3.28}$$

Inequalities (3.28) can be easily linearized by Young's inequality. Now, using the change of variable $\tilde{\alpha} = \alpha^{-2}\varepsilon^{-1}$, where ε is a positive scalar coming from the use of Young's inequality, we deduce that (3.24) is globally $\mathscr{H}_\infty$ asymptotically stable if the following LMI condition is fulfilled :

$$\begin{bmatrix} \mathscr{Z} - \hat{G} - \hat{G}^T & A\hat{G} - B\bar{\mathscr{K}} & E & 0 & M_A & 0 \\ (\star) & -\mathscr{Z} & 0 & \hat{G}H^T & 0 & \hat{G}N_A^T \\ (\star) & (\star) & -\eta I & \mathbb{S}^T & 0 & 0 \\ (\star) & (\star) & (\star) & -I & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -\varepsilon I & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\tilde{\alpha} I \end{bmatrix} < 0, \tag{3.29}$$

Since the objective consists in optimizing the bounds of the uncertainties, $\tilde{\alpha}$ is maximized if both ε and α are minimized, then we have to solve the following convex optimization problem :

$$\min\left(\eta+\tilde{\alpha}+\varepsilon\right) \text{ subject to } (3.29). \tag{3.30}$$

with respect to the decision variables $\mathscr{X}>0$, $\bar{\mathscr{K}}$, $\hat{G}>0$.

3.3.1 Computation of the matrix S from (3.30)

Once $\hat{G}$ and $\mathscr{X}$ are calculated from (3.30), we compute S using the Lyapunov matrix $\mathscr{P}=\hat{G}^{-1}\mathscr{X}\hat{G}^{-1}$, resulting from (3.30), by choosing

$$G_{11}^{1}:=\mathscr{P}_{11},\, G_{21}^{1}:=\mathscr{P}_{21},\, S=\mathscr{P}_{21}\mathscr{P}_{11}^{-1}.$$

To enhance the feasibility of (3.23), we introduce m additional degrees of freedom, by putting

$$S:=\operatorname{diag}(\theta_1,\ldots,\theta_m)G_{21}^{1}(G_{11}^{1})^{-1}.$$

The new LMI technique we proposed is summarized in the following algorithm.

3.3.2 New $\mathscr{H}_\infty$ Observer-Based Design Algorithm

Algorithm 1: Modified two-step method (MTSM)

if *Check if conditions in Assumption 1. hold. If yes,* **then**

a. Solve the optimization problem (3.30) :

$$\min_{\mathscr{Z}>0,\mathscr{K},\hat{G}>0,\mathscr{F}>0} \text{Trace}\left(\mathscr{F}\right) \text{ subject to (3.29)}$$

where $\mathscr{F} = \text{diag}\left(\eta, \tilde{\alpha}, \varepsilon\right)$, and go to **b.** ;

b. Compute $\bar{S}$ using the Lyapunov matrix $\mathscr{P} = \hat{G}^{-1}\mathscr{Z}\hat{G}^{-1}$, resulting from (3.30), by

$$G_{11}^1 := \mathscr{P}_{11},\ G_{21}^1 := \mathscr{P}_{21},\ \bar{S} = \mathscr{P}_{21}\mathscr{P}_{11}^{-1}$$

and go to **c.**

c. Put $S = \text{diag}(\theta_1, \ldots, \theta_m)\bar{S}$ and

$$G_1 = \begin{bmatrix} \mathscr{P}_{11} & \mathscr{P}_{21}^T \\ \mathscr{P}_{21} & G_{22}^1 \end{bmatrix}$$

in BMI (3.23),

if LMI (3.23) *is feasible for some fixed* $(\theta_i)_{i=1}^m$, **then**

d. Solve the optimization problem :

$$\min_{\mathbb{P}>0,G_{22}^1,G_2,\hat{K},\hat{L}} \text{Trace}\left(\Lambda\right) \text{ subject to (3.23)} \tag{3.31}$$

with $\Lambda = \text{diag}\left(\hat{\alpha}, \hat{\beta}, \varepsilon_1, \varepsilon_2, \mu\right) > 0$ and go to **e.** ;

e. Compute the observer-based controller gains as

$$K = (G_{11}^1)^{-1}\hat{K},\ L = G_2^{-1}\hat{L}$$

with optimal disturbance attenuation level $\mu_{\min}$.

else

Use the gridding method[a] with respect to the positive parameters $(\theta_i)_{i=1}^m$, and go to **d.**

else

The observer and controller gains can not be computed.

[a] Such a technique consists in scaling $\theta_i > 0$ by defining $\kappa_i \in]-1,1[$, via the change of variable $\theta_i = \kappa_i/1-\kappa_i^2$. Then, we assign a uniform subdivisions of the interval $]-1,1[$, and solve (3.31) for each subdivision.

Remark 3.3.1. *In item* **b.** *of Algorithm 1, we can compute the matrix $\bar{S}$ with three other possibilities. Each possibility allows linearizing the problem (3.31) :*

1. *Since $\mathscr{G}$ and $\hat{G}$ are congruent matrices, we can use*

$$\bar{S} := \hat{G}_{21}\hat{G}_{11}^{-1}$$

or

$$\bar{S} := \mathscr{G}_{21}\mathscr{G}_{11}^{-1}$$

2. *We can also compute $\bar{S}$ using $\mathscr{X}$:*

$$\bar{S} := \mathscr{X}_{21}\mathscr{X}_{11}^{-1}.$$

3. *In order to broaden the domain of search of the matrix S, we can be satisfied with its search in the open-loop system stability domain, namely, in the domain of stability of (3.24), without taking into account the $\mathscr{H}_\infty$ performance criterion. This amounts to directly solving the LMI constraint (3.29), with $E = 0$, $\mathbb{S} = 0$ and $H = 0$.*

Before proceeding with the next section, we briefly comment what has been presented so far to clarify the contribution given in Algorithm 1 as compared with the results available in the literature. In particular, comparison with the approaches in [91] and [70] are provided. It should be noticed that both approaches are introduced for continuous nonlinear systems. What we present here (Algorithm 2 and 3) can be seen as an adaptation to discrete-time systems case. The existing two-step method are different from the new design technique, as we benefit from the advantages of the introduced slack variable G and consequently Theorem 3.6.3. While in in [91] and [70], G is simply the Lyapunov matrix.
The classical two-step method (CTSM) proceeds as in the following algorithm :

Algorithm 2 : (classical two-step method (CTSM) [70])

(1) Solve the optimization problem (3.30) ;

(2) Compute $\mathscr{K}$ by $\mathscr{K} = \bar{\mathscr{K}}\mathscr{X}^{-1}$;

(3) Replace K by $\mathscr{K}$ in (3.18). Then, solve

$$\min_{\mathbb{P}>0,\hat{L},\text{LMI (3.18)}} \text{Trace}(\Lambda).$$

(4) Compute the observer gain as $L = \mathbb{P}_2^{-1}\hat{L}$.

As it can be seen from Algorithm 2, the gains K and L are computed separately. In addition, Algorithm 2 consists of fixing $n \times m$ variables, while with Algorithm 1, only $(n-m) \times m$ variables are fixed a priori.
The discrete version of the new two-step method (NTSM) introduced in [91] is summarized in the following Algorithm 3 :

Algorithm 3 (NTSM) [91]

1) Solve the optimization problem (3.30) ;

2) Compute $\bar{S}$ by $\bar{S} = (\mathscr{P}_{21})\mathscr{P}_{11}^{-1}$;

3) Inject $S = \bar{S}\Theta$, with $\Theta = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_m)$, and

$$\mathbb{P}_1 = \begin{bmatrix} \mathscr{P}_{11} & \mathscr{P}_{21}^T \\ \mathscr{P}_{21} & \mathbb{P}_{22}^1 \end{bmatrix}$$

in BMI (3.23). Then, solve

$$\min_{\mathbb{P}_2, \mathbb{P}_{22}^1 \hat{L}, \hat{K},\ \text{LMI (3.18)}} \quad \mathrm{Trace}(\Lambda)$$

using the gridding method on θ_i.

4) Compute the observer-based controller gains as $K = \mathscr{P}_{11}^{-1}\hat{K}$ and $L = \mathbb{P}_2^{-1}\hat{L}$.

As we have already pointed out, the introduction of the slack variable G (in both first step and second step) in Algorithm 1 renders it more general than NTSM algorithm. Indeed, if we take $\mathscr{G} = \mathscr{X}$ in item **a.** and $G = \mathbb{P}$ in item **d.** in Algorithm 1, we get exactly Algorithm 3 (NTSM). The advantage of the new method lies in the simultaneous search for the decision variable K, L, and $\mathbb{P}$ in the final LMI. Contrarily to Algorithm 3, where a block of the Lyapunov matrix $\mathbb{P}$ is fixed by the first step.

Remark 3.3.2. *Other variants of the two-step method can be obtained. Indeed,*

1. *In the classical two-step method, instead of solving (3.18) for $K = \mathscr{K}$, we can also consider $K = \bar{\mathscr{K}}$, or a more general structure by weighting $\bar{\mathscr{K}}$ by a slack variable $\Theta = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m)$, namely, $K = \Theta\bar{\mathscr{K}}$, and then solve (3.18) by the gridding method with respect to α_i, $i = 1, \ldots, m$.*

2. *In the stability analysis, instead of considering the closed loop system provided by $z_t = (x_t^T, e_t^T)^T$, we can also consider the augmented vector $z_t = (e_t^T, \hat{x}_t^T)^T$. In this case, the first step is devoted to the search of the gain matrice L, instead of K.*

3. *In Algorithm 3, we can propose two other possibilities. The first one requires no condition on the matrix B. It consists in evaluating (entirely) the Lyapunov matrix $\mathbb{P}$ from the first step, by replacing $\mathbb{P}$ by $\mathscr{X}$ (or by $\Theta\mathscr{X}$), and then compute simultaneously the gains K and L from the second step. The second one consists in replacing P by $\mathscr{P}$, or by $\Theta\mathscr{P}$.*

All these scenarios have a common point, namely the linearization of the coupling term $\mathbb{P}BK$ or G_1BK. The idea of calculating a part of P or G_1 is specific to particular structures of the matrix B, that must be full column rank. In this case, we need just to calculate a part of $\mathbb{P}$ or G_1 from the first step, instead of having them fully from the first step.

3.4 Numerical Evaluation

In this section, numerical examples are presented to illustrate the effectiveness and the superiority of the proposed LMI technique compared to some existing results in the literature. The aim of the first example is to show that the proposed design methodology tolerates larger uncertainty level. We will also provide a Monte Carlo simulation to evaluate the superiority of the proposed Algorithm 1 and its different versions (see Remark 3.3.1), compared to Algo. 2 and Algo. 3. Finally, we end this section by an application to a quadruple tank process model.

3.4.1 Example 1

We consider the example of [28] and [35], described by the following system matrices :

$$A = \begin{bmatrix} 1 & 0.1 & 0.4 \\ 1 & 1 & 0.5 \\ -0.3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0.3 \\ -0.4 & 0.5 \\ 0.6 & 0.4 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, M_A = \gamma_1 \begin{bmatrix} 0 & 0 & 0 \\ 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0 \end{bmatrix},$$
$$N_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0 \end{bmatrix}, M_C = \gamma_2 \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0 & 0.8 \end{bmatrix},$$
$$N_C = \operatorname{diag}\begin{bmatrix} 0 & 0 & 0.2 \end{bmatrix}.$$

In order to transform this system into an equivalent system for which $B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, we use the transformation

$$T = \begin{bmatrix} 2.9412 & -1.7647 & 0 \\ 2.3529 & 0.5882 & 0 \\ 1.9327 & -0.5882 & -0.7143 \end{bmatrix},$$

as in Remark 1. By choosing $\gamma_1 = \gamma_2 = 1$, Algorithm 1 (MTSM) is found feasible ; we get the observer-based controller gains

$$K^T = \begin{bmatrix} 1.0311 & 0.7324 \\ 0.0544 & 1.5474 \\ -0.5672 & -1.8033 \end{bmatrix}, L = \begin{bmatrix} -1.7623 & 0.6609 \\ 1.5539 & -0.5827 \\ -0.9256 & 0.3471 \end{bmatrix}$$

Also Algorithms 2 and 3 perform successfully. Now, we searched for the maximum values of γ_1 and γ_2 that satisfy the above described LMIs. After solving the optimization problem (3.18)-(3.19) with $\theta_i = 1.5543$, we get $\gamma_{1\max} = 5.78$, $\gamma_{2\max} = 10^{15}$. The superiority of the Algorithm 1 (MTSM) is quite clear from Table 3.4.1. The obtained

Method	[35]	**Algo. 1**	**Algo. 2** [70]	**Algo. 3** [91]
				$\alpha_i = 1.42$
$\gamma_{1\,\max}$	4.64	$\boxed{5.78}$	5.08	5.5
$\gamma_{2\,\max}$	10^{13}	10^{15}	10^{15}	10^{15}

TAB. 3.1: Comparison between different LMI design methods

observer-based controller gains in this case are given by :

$$K^T = \begin{bmatrix} 0.1491 & 0.8117 \\ -0.5840 & 1.6815 \\ -0.1615 & -2.3640 \end{bmatrix}, \ L = \begin{bmatrix} -1.7788 & 0.6671 \\ 1.9689 & -0.7383 \\ -0.9227 & 0.3460 \end{bmatrix}.$$

Despite the fact that the proposed technique requires the gridding method with respect to θ_i, (and hence it is expensive from numerical complexity point of view), it remains better than the other considered algorithms even with the particular choice $\theta_i = 1$. Indeed, in this case, we get $\gamma_{1\,\max} = 5.36$ with Algorithm 1 and $\gamma_{1\,\max} = 3.5$ with Algorithm 3, while $\gamma_{2\max} = 10^{15}$ for both Algorithms 1 and 3.

3.4.2 Numerical evaluation by Monte Carlo

Now, to deepen the conservatism evaluation of the proposed design methodology as compared with Algorithm 2 and Algorithm 3, in the case where $B = I_{n,m}$. We generate randomly 1000 stabilizable and detectable systems of dimensions $n = 3$, $1 \leq m \leq 3$, and $1 \leq p \leq 2$ via a Monte Carlo simulation, and compute the percentage of feasible LMIs for each method. The parameters are generated as follows : $A = \text{randn}(n)$, $C = \text{randn}(p,n)$, $M_C = \text{rand}(p,n)$, and $N_C = \text{rand}(n)$, $\Delta\mathbb{A} = 0$.
The results are summarized in Table 3.4.2. It is clear, from Table 3.4.2, that Algorithm 1 with the choice

$$\boxed{\bar{S} = \mathscr{P}_{21}\left(\mathscr{P}_{11}\right)^{-1}}$$

is better than the other design methods for all m and p. This is the reason why we opted for this choice of $\bar{S}$ in Algorithm 1, Section 3.3.2.

3.4.3 Application to a quadruple tank process

The system we consider herein models a quadruple tank process composed of four water tanks that are interconnected and the flow through the pipes is controlled by two pumps as depicted in Figure 3.1 (taken from [45] and [79]). The two upper tanks drain freely into the two bottom tanks, and the two bottom tanks drain freely into the reservoir basin. The liquid levels in the bottom two tanks are directly measured with pressure transducers and constitute the measured outputs of the system. The states of the system

		3	3	3	3	3
n		3	3	3	3	3
m		2	2	1	1	3
p		2	1	2	1	2
Algo. 1	$\bar{S}$					
	$\mathscr{G}_{21}(\mathscr{G}_{11})^{-1}$	35.1%	15.6%	26%	14.8%	29.7%
	$\hat{G}_{21}(\hat{G}_{11})^{-1}$	20.5%	10.6%	10%	8%	29.7%
	$\mathscr{P}_{21}(\mathscr{P}_{11})^{-1}$	**100%**	**100%**	**100%**	**100%**	**73.9%**
	$\mathscr{Z}_{21}(\mathscr{Z}_{11})^{-1}$	24.9%	12.7%	12%	10%	29.7%
Algo. 2		29.9%	13.8%	22%	13.3%	21.2%
Algo. 3		32.1%	14.5%	22.4%	13.7%	28.7%

TAB. 3.2: Superiority of the proposed LMI methodology via Monte Carlo evaluation

consist of the deviations of heights in the 4 tanks from desired values and the system matrices are given by

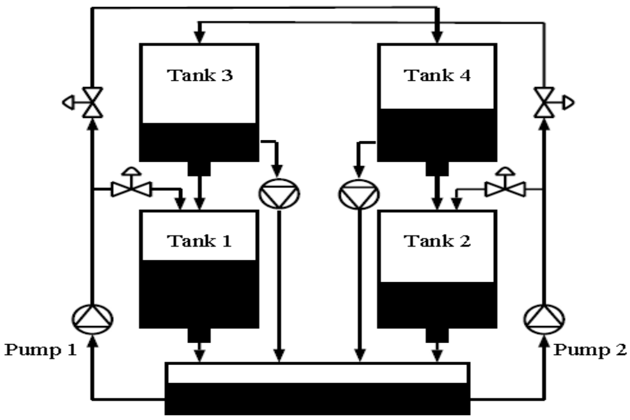


FIG. 3.1: Schematic of the four interconnected tank system [79] and [45].

$$A = \begin{bmatrix} -0.0278 & 0 & 0.0206 & 0 \\ 0 & -0.0233 & 0 & 0.0141 \\ 0 & 0 & -0.0206 & 0 \\ 0 & 0 & 0 & -0.0141 \end{bmatrix},$$

$$B = \begin{bmatrix} 5 & 0 \\ 0 & 6.667 \\ 0 & 10 \\ 11.667 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D = 0,$$

$$E = \begin{bmatrix} 0.0987 & 0.0863 & 0.1091 & 0.0999 \\ 0.1210 & 0.1000 & 0.1256 & 0.1100 \\ 0.0898 & 0.0918 & 0.1000 & 0.0888 \\ 0.1090 & 0.1220 & 0.1235 & 0.1156 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.0905 & 0.0703 & 0.1064 & 0.1064 \\ 0.1006 & 0.0108 & 0.1072 & 0.0539 \\ 0.0141 & 0.0309 & 0.0175 & 0.0889 \\ 0.1015 & 0.0608 & 0.1078 & 0.0158 \end{bmatrix},$$

$$\mathbb{S} = \begin{bmatrix} 0.0987 & 0.0863 & 0.1091 & 0.0999 \\ 0.1210 & 0.1000 & 0.1256 & 0.1100 \\ 0.0898 & 0.0918 & 0.1000 & 0.0888 \\ 0.1090 & 0.1220 & 0.1235 & 0.1156 \end{bmatrix}.$$

Due to the Lipschitz condition on the unknown nonlinearity considered in [45], this nonlinearity can be seen as uncertainty represented by the parameters

$$M_A = \begin{bmatrix} 2.9875 & 2.1215 & 4.7685 & 3.4080 \\ 1.6765 & 2.1470 & 2.2895 & 2.3165 \\ 1.4960 & 0.6245 & 1.2025 & 1.0610 \\ 2.2630 & 0.1220 & 3.8195 & 0.4925 \end{bmatrix}, N_A = 0.1I.$$

By using the similarity transformation

$$T = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \\ 0 & 1 & -0.6667 & 0 \\ 1 & 0 & 0 & -0.4285 \end{bmatrix},$$

Algo. 1 (MTSM) is found feasible. We obtain the optimal disturbance attenuation level $\sqrt{\mu}_{\min} = 1.4983$ and the following observer-based controller gains :

$$K^T = \begin{bmatrix} -0.0845 & -0.0200 \\ 0.0370 & -0.0490 \\ -0.0080 & 0.0019 \\ 0.0006 & -0.0035 \end{bmatrix}, L = \begin{bmatrix} -0.0346 & 0.0073 \\ -0.0097 & -0.0038 \\ -0.0416 & -0.0110 \\ -0.1304 & 0.0263 \end{bmatrix} \tag{3.33}$$

We also tested the feasibility of the LMI conditions in Algo. 2, Algo. 3 and [35]. All these methods perform successfully for $\alpha_{\max} = 1$. Table 3.4.3 summarizes the results obtained with different approaches. Table 3.4.3 shows that we obtain a better result with respect to the index of performance when using Algorithm 1, since $\sqrt{\mu}_{\min}$(Algo. 1) is the smallest one. To show simulations and improve comparisons between different LMI design methods, assume that the system is disturbed by the noise $\omega_t = \chi(t)\mathscr{N}_t$, where $\chi(.)$ is defined by

$$\chi(t) = \begin{cases} 5 & \text{if } t \in [30, 40[; \\ 3 & \text{if } t \in [60, 70[; \\ 0 & \text{elsewhere,} \end{cases}$$

Method	[35]	Algo. 1	Algo. 2 [70]	Algo. 3 [91]
	$\varepsilon_1 = 0.1$ $\varepsilon_3 = 0.81$	$\theta_i = 1.55$		$\theta_i = 0.93$
$\sqrt{\mu}_{\min}$	1.5624	$\boxed{1.4983}$	1.9159	1.5

TAB. 3.3: Comparison of $\sqrt{\mu}_{\min}$ between different LMI design methods.

and $\mathscr{N}_t$ is a normally distributed random, that is for each $t = 1, \ldots, 400$, $\mathscr{N}_t = randn(1)$. The simulation results corresponding to the observer-based controller gains (3.33), over an horizon $T = 400$, are given in Figure 3.2. The initial states used for the simulation are : $x = \begin{bmatrix} 1 & -1 & 2 & -2 \end{bmatrix}^T$, $\hat{x} = \begin{bmatrix} -1 & 1 & 0 & 2 \end{bmatrix}^T$ and the uncertain parameter is $\mathbb{F}_A(t) = \cos(t)$. Note that the condition (3.3) is satisfied, since $\cos^2(t) \leq \alpha_{\max} = 1$.

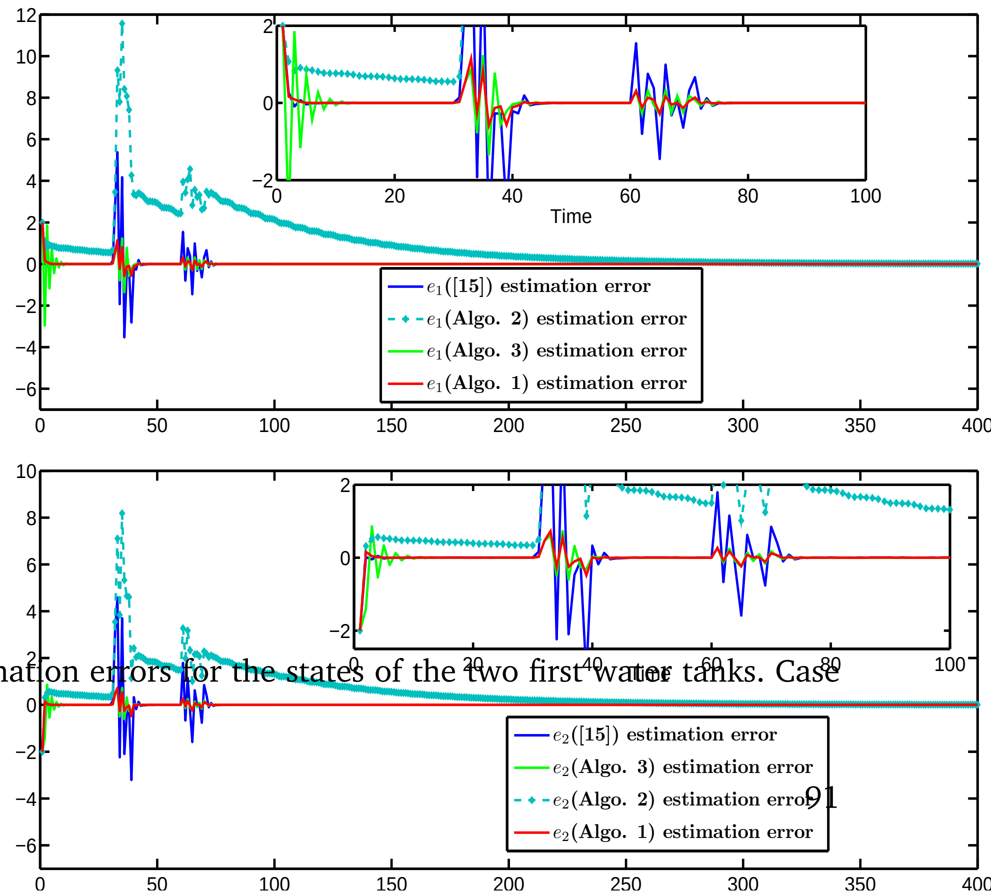
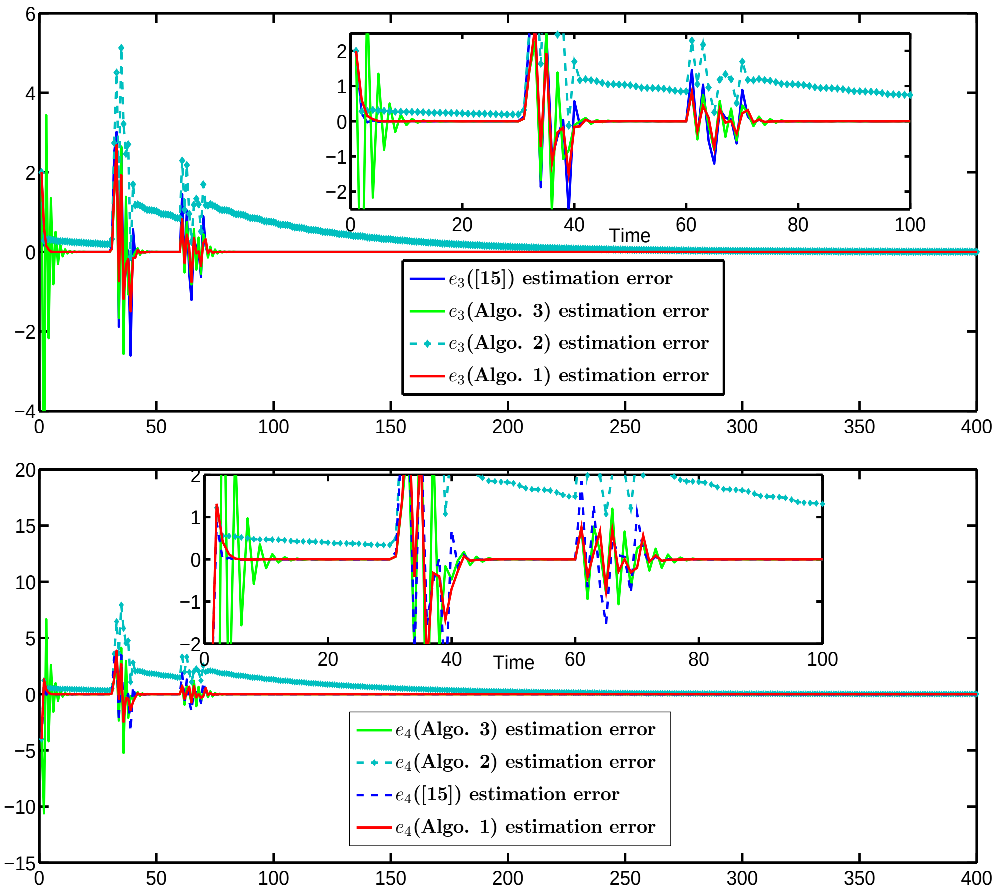


FIG. 3.2: Behavior of estimation errors for the states of the two first water tanks. Case when $\mathbb{F}_A(t) = \cos(t)$



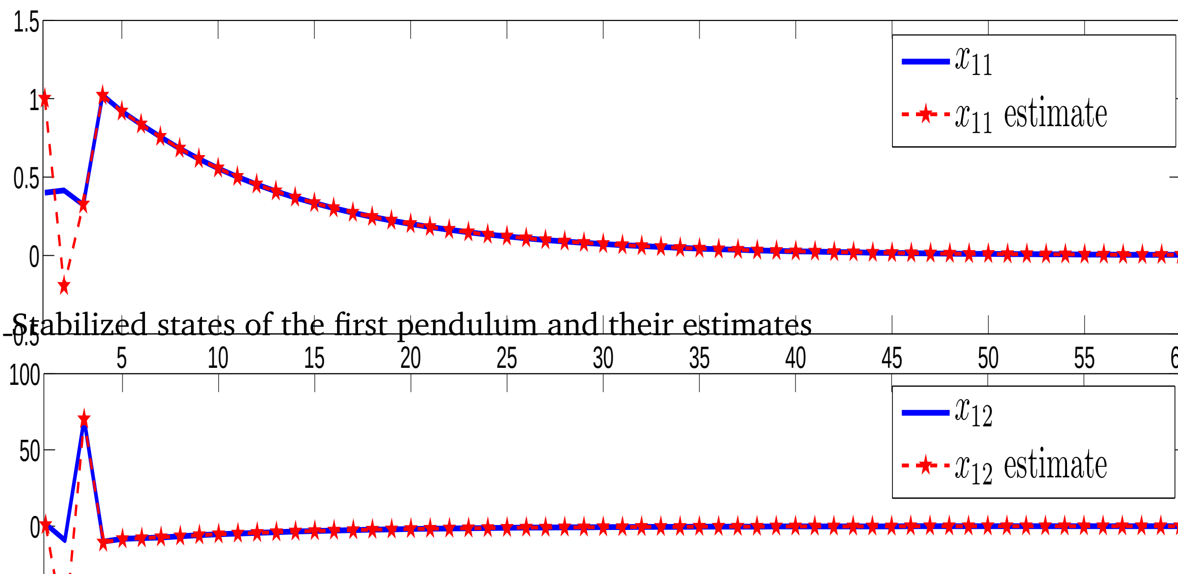


FIG. 3.3: Stabilized states of the first pendulum and their estimates

Figures 3.2-3.6 show clearly the $\mathscr{H}_\infty$–robustness of the observer-based controller for all LMI design methods in Table 3.4.3. Indeed, the estimation error, in the disturbance-free case, converges asymptotically towards zero. While, in the presence of disturbances, the magnitude of the estimation error is bounded for the four methods. Moreover, the magnitude of the estimation error, obtained via the proposed algorithm (the curves in red color in Fig. 3.2), is smaller compared to other methods. These figures perfectly reflect the order on the performance indices presented in Table 3.4.3. Indeed, the magnitude of the error of each of the methods is proportional to the corresponding performance index. Small is better. In addition, for Algo. 1, the estimation error reaches zero more quickly compared to other methods. This shows that Algo. 1 performs better than the other LMI design methods. To boost comparison with the above-mentioned methods, we consider the case where the uncertain parameter is

$$\mathbb{F}_A(t) = \alpha \arctan(t),$$

instead of $\mathbb{F}_A(t) = \cos(t)$. Condition (3.3) becomes

$$\mathbb{F}_A^2(t) \leq (\alpha)^2 . \frac{\pi^2}{4} = \alpha_{\max}^2 . \tag{3.34}$$

Although the results obtained by resolution of the LMI conditions of each method give $\alpha_{\max} = 1$, these conditions remain however sufficient. The simulation results show that the value of $\alpha_{\max}$ can be greater than 1. Indeed, we have manually increased the value of $\alpha_{\max}$ in (3.34) until robust stabilization is lost. The results obtained are illustrated in Table 3.4.3.

Method	[35]	**Algo. 1**	**Algo. 2** [70]	**Algo. 3** [91]
$\alpha_{\max}$	1.2967	$\boxed{1.3352}$	1.2608	1.2967

TAB. 3.4: Comparison of $\alpha_{\max}$ in (3.34) between different LMI design methods.

We can observe from Table 3.4.3 that Algo. 1 tolerates a larger value of $\alpha_{\max}$, which confirms the superiority of the new proposed Algorithm. The simulations results when

$$\mathbb{F}_A(t) = 0.84 \arctan(t).$$

are depicted in Figures 3.7 and 3.8.

Figure 3.7 illustrates the unstable behavior of the uncontrolled states, even if the matrix A is Schur stable. This is due to the presence of a nonlinearity that we handled as uncertain parameter. Hence the need to implement a robust observer-based controller to stabilize the system.

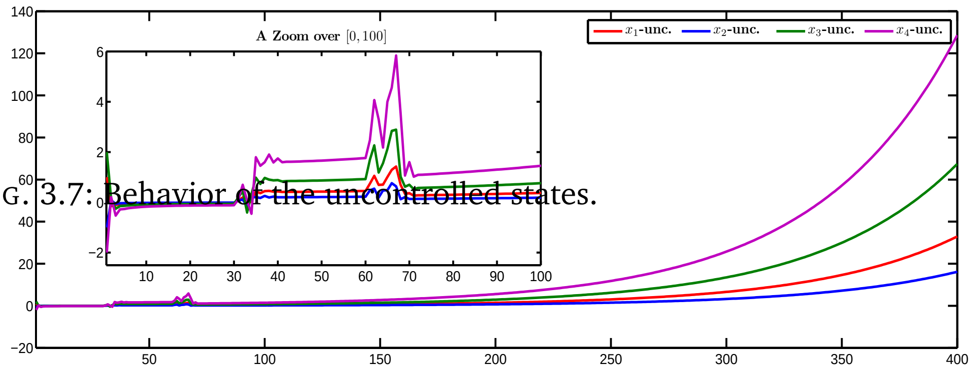


FIG. 3.7: Behavior of the uncontrolled states.

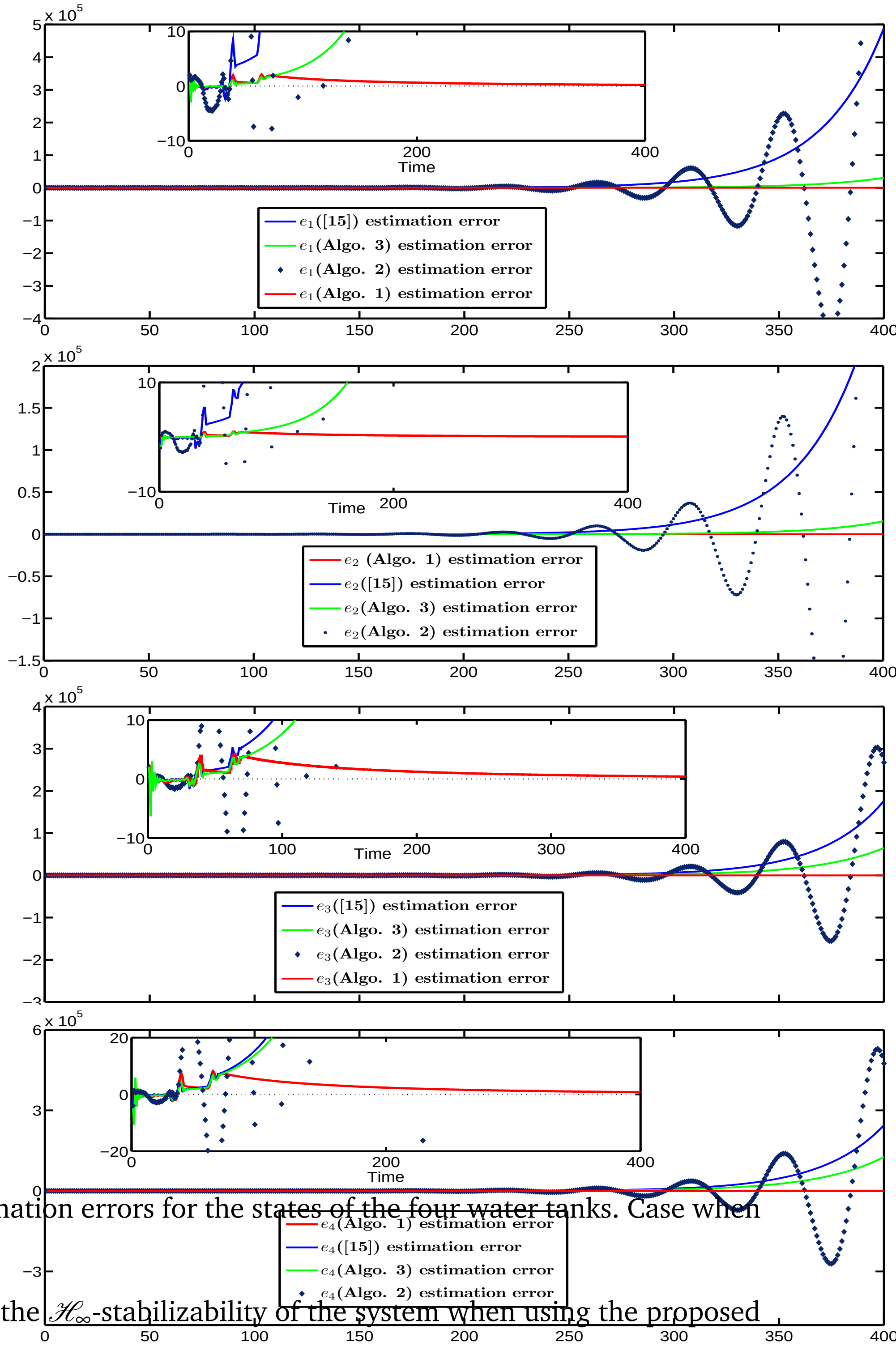


FIG. 3.8: Behavior of estimation errors for the states of the four water tanks. Case when $\mathbb{F}_A(t) = 0.85\arctan(t)$

Figure 3.8 reflects clearly the $\mathscr{H}_\infty$-stabilizability of the system when using the proposed algorithm, namely Algo. 1, while the other three methods failed to stabilize the system. The robustness of the proposed controller and observer follows then.

3.5 Conclusion

In this paper, a new variant of two-step algorithm to construct observer-based controllers for uncertain linear discrete-time systems is proposed. The connection with respect to the existing two-step algorithms in the literature are analyzed analytically and evaluated through numerical examples. The numerical results have shown a better feasibility as compared with the alternative design methods. Future work will concern possible extensions of the proposed approach to the design of observer-based controllers for nonlinear time-delay systems.

3.6 Appendix

3.6.1 Auxiliaries results

This appendix is devoted to some theoretical results. We will show first that [27, Lemma 1] is conservative and we will propose a new theorem, which can be viewed as a generalization of the famous linearization theorem in [11] for discrete-time systems.

Lemma 3.6.1 ([27]). *Given three matrices X, Y, Z of appropriate dimensions where $X = X^T > 0$ and $Z = Z^T > 0$. Then, the following inequality holds*

$$\begin{bmatrix} -X & Y^T \\ Y & -Z^{-1} \end{bmatrix} < 0 \tag{3.35}$$

if there exists a positive constant $\alpha > 0$ such that

$$\begin{bmatrix} -X & \alpha Y^T & 0 \\ \alpha Y & -2\alpha I & Z \\ 0 & Z & -Z \end{bmatrix} < 0. \tag{3.36}$$

This lemma proposed and proved by Ibrir [27] has also exploited in the nonlinear systems context.

Theorem 3.6.2. (3.36) *is conservative compared to* (3.35).

Proof : Indeed, we propose hereafter a simple example of matrices X, Y and Z satisfying (3.35), and for which there is no $\alpha > 0$ such that (3.36) holds. Let us choose

$$X = \begin{bmatrix} 4 & 1 \\ 1 & 6 \end{bmatrix}, Y^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \tag{3.37}$$

With these choices it is easy to show that inequality (3.35) is satisfied. Now, if we assume that (3.36) holds for some $\alpha > 0$, then using Sylvester's criterion, we will get that all principal minors of odd order of (3.36) are negative. In particular, it follows that

$$\Delta_5(\alpha) = 2\alpha(18\alpha^2 - 55\alpha + 46) < 0.$$

Clearly, $\Delta_5(\alpha)$ and $P(\alpha) := 18\alpha^2 - 55\alpha + 46$ have the same sign. However, we can check easily that $P(\alpha) > 0$ because its discriminant is negative. Hence $\Delta_5(\alpha) > 0$ for all $\alpha > 0$, which is a contradiction. □

In what follows, we introduce a more general result providing an equivalent condition to (3.35).

Theorem 3.6.3. *Given three matrices X, Y, Z of appropriate dimensions where $X = X^T > 0$ and $Z = Z^T > 0$. Then, the following inequality holds*

$$\begin{bmatrix} -X & Y^T \\ Y & -Z^{-1} \end{bmatrix} < 0 \tag{3.38}$$

if and only if there exists an invertible and symmetric matrix G such that

$$\begin{bmatrix} -X & Y^T G & 0 \\ GY & -2G & Z \\ 0 & Z & -Z \end{bmatrix} < 0. \tag{3.39}$$

Proof : The theorem is trivial. The sufficiency follows from well known results based on the Young's relation, namely $(G-Z)Z^{-1}(G-Z) \geq 0$. The necessity follows from the fact that, with the choice $G = Z$, (3.39) and (3.38) are equivalent from Schur lemma. □

Clearly, (3.39) is more general than (3.36).

Remark 3.6.4.
1. *Note that when $G = \alpha I$ in in Theorem 3.6.3, we retrieve Ibrir's lemma, namely Lemma 3.6.1.*
2. *Also note that when $Z = X$ in (3.38), Theorem 3.6.3 coincides with de Oliveira et al. result [11, Theorem 1].*

General conclusion and some research perspectives

This PHD thesis is dedicated to the robust stability analysis and control design for linear systems. The research is principally focused on discrete time linear systems with parameter uncertainties law. The stability problems are treated in the framework of the Lyapunov Theory and it is based on the resolution of linear matrix inequalities. In this context we showed how it is possible to take into account the uncertain parameters in order to design efficient control synthesis laws : the uses of Parameter Dependent Lyapunov Functions allowed us to derive less conservative stability and control design criteria. we revisited and corrected the approach proposed in .... that combines Finsler's lemma and the switched Lyapunov function approach. A general theoretical method was proposed, which leads to less conservative LMI conditions. This is due to the use of Finsler's inequality in a new and convenient way. new LMI conditions have been developed for the problem of the stabilization of a class of switching discrete-time linear systems with parameter uncertainties and l_2-bounded disturbances. We have shown that a judicious choice of slack variables coming from Finsler's lemma leads to less conservative LMIs. We have presented two new LMI synthesis methods to design observer-based controllers for a class of LPV systems with inexact but bounded parameters. The first approach is based on the use of Young's relation, while the second one uses a new congruence principle by pre- and post-multiplying the basic BMI by new and ingenious matrices.

In future work, we hope to extend our technique to more general classes of switching systems, namely nonlinear systems, systems with uncertainties in all the matrices of the model, linear parameter varying systems with inexact parameters.

Perspectives

Many questions remain in this area. As future research perspectives on the results presented in this manuscript, we intend to develop and improve the following points

- Extend the applicability of our approach to other types of systems, such as continuous time switching systems, Lipschitz nonlinear switching systems, switching systems without a priori knowledge of mode and switching law , and interconnected systems.

- Search for new Lyapunov functions resulting in less conservative synthesis conditions.

- Study of stochastic systems.

- Study of T-S fuzzy switched system.

Finally, we would like our contribution to be of interest and further study in this field, where, as we have said, many fundamental questions remain open.

Bibliographie

Bibliographie

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Résumé

Cette thèse aborde le problème de la stabilisation basée sur l'observateur de Luenberger pour une classe de systèmes linéaires à temps discret en présence d'incertitudes structurées et de perturbations bornées au sens de ℓ_2. Nous proposons une nouvelle variante de l ?approche LMI classique à deux étapes. Dans la première étape, nous utilisons une technique de variables artificielles pour résoudre le problème d'optimisation résultant de la stabilisation par un retour d'état statique. Dans la deuxième étape, une partie de la variable de artificielle obtenue est incorporée au problème de stabilisation basé sur l'observateur $\mathscr{H}_\infty$, afin de calculer simultanément la matrice de Lyapunov et les gains de contrôleur basés sur l'observateur. L'efficacité de la méthodologie de conception proposée est démontrée à l'aide du modèle dynamique d'un système de traitement de flux à quadruple cuve. Des illustrations numériques supplémentaires sont présentées pour montrer la supériorité de la méthode à deux étapes modifiée (MTSM) proposée du point de vue de la faisabilité des LMI.

Mots-clés: Systèmes à commutations à temps discret ; Fonction de Lyapunov commutée ; Lemme de Finsler ; Observateur de Luenberger ; Stabilisation ; Inégalités Matricielles Linéaires (LMI) ; Incertitudes

Abstract

This thesis addresses the problem of observer-based stabilization of discrete-time linear systems in presence of parameter uncertainties and ℓ_2-bounded disturbances. We propose a new variant of the classical two-step LMI approach. In the first step, we use a slack variable technique to solve the optimization problem resulting from stabilization by a static state feedback. In the second step, a part of the slack variable obtained is incorporated in the $\mathscr{H}_\infty$ observer-based stabilization problem, to calculate simultaneously the Lyapunov matrix and the observer-based controller gains. The effectiveness of the proposed design methodology is demonstrated using the dynamic model of a quadruple-tank flow process system. Additional numerical illustrations are presented to show the superiority of the proposed Modified Two-Step Method (MTSM) from a LMI feasibility point of view.