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Présentée par:

Adil Ania

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On Estimation and Observer Design in Nonlinear Systems: Theory and Application

Devant le jury d'examen composé de:

Mr. Djennoune Said	Professeur	UMMTO	Président
Mr. Hamaz Abdelghani	Maître de Conférences A	UMMTO	Rapporteur
Mr. Zemouche Ali	Maître de Conférences HDR	Université de Lorraine	Co-Rapporteur
Mr. Mansouri Rachid	Professeur	UMMTO	Examinateur
Mr. Ahmed Ali Sofiane	Maître de Conférences HDR	ESIGELEC Rouen	Examinateur
Mr. Zerrougui Mohamed	Maître de Conférences HDR	Université Aix Marseille	Examinateur
M^{me} Laleg-Kirati Taous-Meriem	Professeur	KAUST & INRIA	Invitée
Mr. N'Doye Ibrahima	Research Scientist	KAUST	Invité

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Abstract

On Estimation and Observer Design in Nonlinear Systems: Theory and Application

Ania Adil

The work conducted in this thesis contributes to the state of the art on observer design for nonlinear systems. The main objective is to design a high-gain like observer for a class of nonlinear systems with delayed output measurements. To overcome some limitations of the standard high-gain observer and enhance the time response, two high-gain like observer approaches have been proposed.

In the first part, we present a high-gain observer design for nonlinear systems with time-varying delayed output measurements. Based on a recent high-gain like observer design method, called HG/LMI observer, a larger bound of the time-delay is allowed compared to that obtained by using the standard high-gain methodology. Such a HG/LMI observer adopts a lower value of the tuning parameter, which results in the reduction of the observer's gain value and an increase in the maximum bound of the delay required to ensure exponential convergence. An application to nonlinear systems with sampled measurements is provided. Furthermore, the proposed methodology is extended to systems with nonlinear outputs.

The second part is dedicated to the design of a prescribed-time high-gain observer that achieves stability in a predefined time. The fixed time convergence of the observer within a prescribed time is shown through a Lyapunov differential inequality and a state transformation that involves a scaling function. The observer's gain depends on a time-varying function monotonically increasing to infinity as the time tends to the predefined time of convergence. Moreover, the proposed observer reduces the peaking phenomenon, which is one of the main limitations of the high-gain observer design.

In the last part, an application of the proposed high-gain approaches to estimate the water level in the Coupled Tanks system is presented. Extensive numerical simulations are provided to show the effectiveness of the HG/LMI observer and the prescribed-time high-gain observer. Furthermore, a comparison with the standard high-gain observer is established to demonstrate the superiority of the proposed observer design approaches in reducing the peaking phenomena and enhancing the estimation convergence error, respectively.

Keywords: High-gain observer, delayed output measurements, linear matrix inequalities (LMIs), Lyapunov–Krasovskii functionals, prescribed-time convergence, time-varying scaling function, fixed-time stability, nonlinear triangular systems, Coupled Tanks plant.

Resumé

Les travaux menés dans cette thèse est une contribution à la conception d'observateurs pour les systèmes non linéaires. L'objectif principal est de concevoir un observateur grand gain pour une classe de systèmes non linéaires avec des mesures de sortie retardées. Pour surmonter certaines limitations de l'observateur grand gain standard et améliorer sa réponse temporelle, deux approches de type observateur à grand gain ont été proposées.

Dans la première partie, nous proposons une conception d'observateur grand gain pour les systèmes non linéaires avec des mesures de sortie retardées par un retard variant dans le temps. En se basant sur une récente méthode de conception d'observateur grand gain, appelée observateur HG/LMI, une plus grande borne du retard est tolerée par rapport à celle obtenue en utilisant la méthodologie standard. Un tel observateur HG/LMI adopte une valeur inférieure du paramètre de réglage, ce qui entraîne la réduction de la valeur de gain de l'observateur et une augmentation de la borne maximale du retard nécessaire pour assurer la convergence exponentielle. Une application aux systèmes non linéaires avec des mesures échantillonnées est fournie. De plus, la méthodologie proposée est étendue aux systèmes avec des sorties non linéaires.

La deuxième partie est consacrée à la conception d'un observateur grand gain à temps prescrit qui atteint la stabilité dans un temps prédéfini. La convergence fixe de l'observateur en un temps prescrit est représentée par une inégalité différentielle de Lyapunov et une transformation d'état qui implique une fonction de transformation. Le gain de l'observateur dépend d'une fonction variable dans le temps augmentant de manière monotone jusqu'à l'infini lorsque le temps tend vers le temps de convergence prédéfini. De plus, l'observateur proposé réduit le phénomène de "peaking", qui est l'une des principales limitations de l'observateur grand gain.

Dans la dernière partie, les approches de l'observateur grand gain proposées sont appliquées pour estimer le niveau d'eau dans un procédé hydraulique avec réservoirs couplés. Des simulations numériques illustrent l'efficacité de l'observateur HG/LMI et de l'observateur grand gain à temps prescrit. En outre, une comparaison avec l'observateur grand gain standard est établie pour démontrer la supériorité des approches proposées à réduire le phénomène du "peaking" et améliorer la convergence de l'erreur d'estimation, respectivement.

Mots-Clés: Observateur grand gain, mesures de sortie retardées, inégalités matricielles linéaires, fonctionnelle de Lyapunov–Krasovskii, convergence en temps prescrit, fonction de transformation, stabilité en temps fixe, systèmes non linéaires triangulaires, procédé hydraulique avec réservoirs couplés.

"But it may be that you hate something while it is good for you,

and it may be that you love something while it is bad for you.

Allah knows, and you do not know."

– Ayah 216, *Surat Al-Baqara*

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CONTENTS

SYMBOLS AND ABBREVIATIONS

LIST OF SYMBOLS:

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+(\mathbb{R}_{\geq})$	The set of non-negative real numbers
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{R}^{n\times m}$	The set of $n\times m$-dimensional real matrices
$\mathbb{N}$	The set of natural numbers
$\mathcal{C}([a,b])$	The space of continuous functions over $[a,b]$
$\|.\|$	The Euclidean vector norm
$\mathrm{diag}(a,b,\dots)$	Diagonal matrix with diagonal elements $\{a,b,\dots\}$
$P>0$	Real symmetric positive-definite matrix
P^{-1}	The inverse of a matrix P
$A^T(x^T)$	The transpose of a matrix P (a vector x)
$\lambda_{\max}(P)$	The maximum eigenvalue of a symmetric matrix P
$\lambda_{\min}(P)$	The minimum eigenvalue of a symmetric matrix P
$I_{n\times n}$	The identity matrix of dimension $n\times n$
min	Minimum
sup	Supremum

LIST OF ABBREVIATIONS :

LMI	Linear matrix inequality
SHG	Standard high-gain observer
VVA	High-gain observer proposed in (Van Assche et al., 2011)
CT	Coupled Tanks plant

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  1. **Ania Adil**, Abdelghani Hamaz, Ibrahima N'Doye, Ali Zemouche, Taous-Meriem Laleg-Kirati, Fazia Bedouhene. *"On high-gain observer design for nonlinear systems with delayed output measurements"*. Automatica, Elsevier, 2022, 141, pp.110281.

- **International Conferences**

  1. **Ania Adil**, Ibrahima N'Doye, Abdelghani Hamaz, Ali Zemouche, Taous-Meriem Laleg-Kirati. *"Prescribed-time high-gain nonlinear observer design for triangular systems"*. 9th International Conference on Systems and Control, ICSC 2021, Nov 2021, Caen, France.
  2. **Ania Adil**, Ibrahima N'Doye, Ali Zemouche, Abdelghani Hamaz, Taous-Meriem Laleg-Kirati. *"Coupled Tanks state estimation using a high-gain like observer"*. 3rd IFAC Conference on Modelling, Identification and Control of Nonlinear Systems, MICNON 2021, Sep 2021, Tokyo, Japan.
  3. **Ania Adil**, Ania Adil, Ali Zemouche, Abdelghani Hamaz, Taous-Meriem Laleg-Kirati, Ibrahima N'Doye, Fazia Bedouhene, *"High-gain observer design for nonlinear systems with delayed measurements"*. 21st IFAC World Congress, IFAC-WC 2020, Jul 2020, Berlin.

CHAPTER 1

GENERAL INTRODUCTION

Time-delay is widely involved in various real-world engineering and physical systems such as industrial processes, chemical industry, mechanical and electrical engineering, communication networks, transportation, and biomedical engineering (MacDonald and Lags, 1978; Anthonis et al., 2007; Wu et al., 2010; Fridman, 2014) (see Figure 1.1). Time-delay is ubiquitous in most biological systems studying epidemics and diseases, for instance, implementing lockdowns due to the coronavirus disease 2019 pandemic is based on the latest data, but data includes delays in both diagnosis and reporting till the reduction of the transmission (Rong et al., 2020). Transmission delays in networked control systems are another family of time-delays. For example, a network delay may appear while transmitting the data from its origin to its destination, affecting the performance of the networked control systems (Tipsuwan and Chow, 2003). In everyday life, the time-delay may occur in several situations. For instance, while driving a car, the driver tries to avoid delay in his reaction if he needs to slam on the brakes upon receiving a stimulus (Abdallah et al., 2011). Another example is a delay between turning the faucet and what the person feels while trying to control the water temperature in the shower (Fridman, 2014).

In practice, dynamical systems' variables are not fully accessible for measurement and are not entirely known due to various physical and economic constraints on the dynamical model.

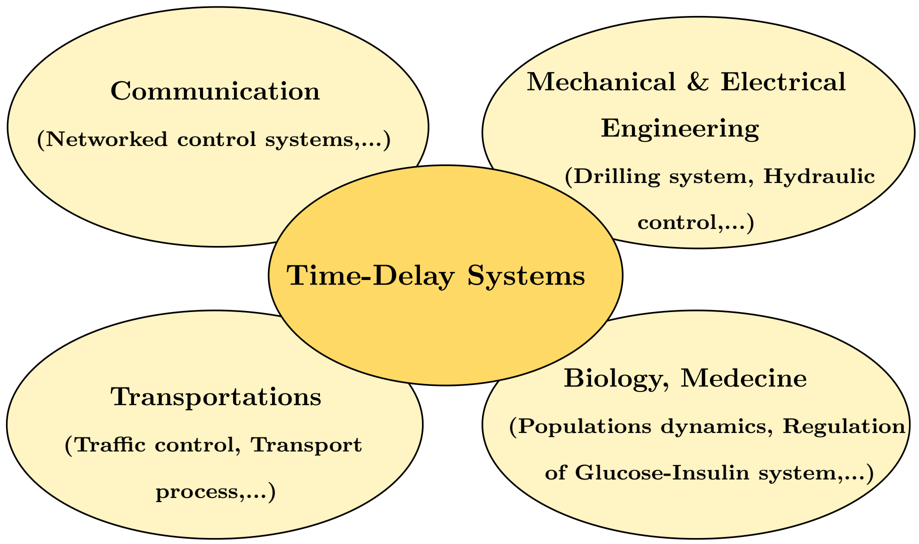

Figure 1.1: Examples of time-delay systems[1]

Moreover, dynamical systems may be corrupted by time-delays either in the state, the measurements, or the control input. It is crucial to consider the delay in performance evaluation and control system design. Indeed, the presence of time-delay leads to performance degradation or system instabilities. Therefore, it is important to develop efficient state estimation methods to extract the unavailable information from the accessible measurements and cope with the time-delay involved in the dynamical systems.

A common way of addressing the reconstruction of internal states problem is by using state observers. However, the presence of a time-delay renders the estimation process more challenging. Indeed, the delay can degrade the observer's performance and affect its stability and robustness if it is not appropriately compensated. Furthermore, time-delay systems belong to the class of functional differential equations which require a general type of functional functions for stability analysis. For these reasons, time-delay systems have increased the interest of the control community from different perspectives. Therefore, the stability analysis and control design of time-delay systems are of theoretical and practical importance.

There are different types of convergence for dynamical systems. Several applications behave diversely and require several time response constraints to complete a desired action or

[1] Modified from: $https://homepages.laas.fr/aseuret/Delsys/Links.html$

behavior in practice. In some applications, the model takes the time needed until the action is achieved. Such convergence is called asymptotic. In others, the model is often required to achieve action in a short and finite amount of time, called finite-time convergence. For example, a walking robot where the state must be provided before each impact with the ground, or the missile guidance that requires a short time to nullify the target (Holloway, 2018). Besides their practical importance, finite-time and asymptotic estimations also contribute different challenges to designing and analyzing time-delay systems in control theory. Hence, many control and estimation problems for time-delay systems remain a very active area of research.

1.1 Motivation and challenges

For many decades, the interest of automatic control community to nonlinear observers continues to grow because of their crucial role in the design of control schemes, namely trajectory tracking, fault diagnosis, and health monitoring (Parisini, 1997), (Alcorta-Garcia and Frank, 1997), (Gao and Ho, 2006). Due to the introduction of new technologies and the complexity of novel industrial infrastructures, the use of nonlinear observers has emerged in modern applications such as synchronization of multi-agent systems, cyber-attacks detection, and control of cyber-physical systems (Zhu and Basar, 2011), (Teixeira et al., 2010), (An and Liu, 2014). Despite the various methodologies of nonlinear observer in the literature, namely the extended Kalman observer (Kalman, 1960), Luenberger observer (Luenberger, 1971), (Huong et al., 2019), high-gain observer (Gauthier and Kupka, 1994), sliding mode observer (Alessandri, 2003), and LMI-based observers (Zemouche and Boutayeb, 2013), (Mazenc et al., 2017), these solutions are not general and can be applied only on a specific class of systems.

The high-gain observer is particularly interesting due to its easy implementation because it depends on only one single tuning parameter, which requires a specific condition, to ensure exponential convergence. Indeed, high-gain observer exhibits excellent global properties, which takes into consideration the nonlinear structure of the system. It ensures convergence and stability with adjustable convergence speed. Furthermore, it is usually applied to uniformly observable systems that have a complete or partial triangular structure, under the assumption that the nonlinearities satisfy Lipschitz condition. Despite this simplicity of implementation, the high-gain observer is far from being a perfect solution to nonlinear estimation, and it has three limitations that should be highlighted (Khalil, 2008; Khalil and Praly, 2014). First, it is computationally inefficient when dealing with large scale and high dimensional systems due to the high gain values. Second, it suffers from the peaking phenomenon. Last but not least, it is highly sensitive to output disturbances (measurement noise, delayed outputs, sampled data,. . .). Many research activities have been conducted in this research area aiming at proposing solutions overcoming such drawbacks of the high-gain observer (Zemouche et al., 2019; Alessandri and Rossi, 2015, 2013; Astolfi and Marconi, 2015; Khalil, 2017; Cacace et al., 2020; Zhang and Shen, 2017; Boizot et al., 2010; Nguyen and Trinh, 2016; Trinh et al., 2004).

In this thesis, we mainly focus on the use of high-gain methodologies for systems with delayed output measurements. Indeed, such a problem is complex when the goal is to provide an observer with a maximum allowable value of the time-delay. This latter can affect the performance of the observer and its stability and robustness if it is not appropriately compensated. More often, system's measurements are subject to delay due to the the system's complexity or to some communication delays. This measurement delay will certainly affect the estimation process. Various state estimation methods for nonlinear systems with time-delay have been developed in the literature. For instance, different types of observers have been extended from standard nonlinear systems to time-delay systems such as chain of observers (Cacace et al., 2014; Targui et al., 2018). These observers are based on a chain algorithm that allows estimating the system's states within the time-delay window. There are also observer-predictor based approaches to compensate the effects of the delay (Ahmed-Ali et al., 2013b; Khosravian et al., 2015). However, the determination of an implementable form for the observer-predictor feedback gains over the past time interval can be challenging. Therefore, these solutions are not general and can be applied only on a specific class of systems.

Besides, finite-time state estimation has recently attracted increasing attention from the automatic control community. Several approaches have been proposed in the literature (see (Basin, 2019) and the references therein). Examples of finite time observers include those based on sliding mode approaches which are widely used due to their robustness with respect to uncertainties and disturbances (Levant, 1998; Angulo et al., 2013; Cruz-Zavala et al., 2011; Hou et al., 2019). There are also homogeneity-based observers (Ménard et al., 2009; Lopez-Ramirez et al., 2018; Li et al., 2011; Ménard et al., 2017; Andrieu et al., 2009; Silm et al., 2018). Such observers exploit the homogeneity property of function in which a multiplicative scaling of the argument of the function results in a proportional scaling of the function (Lopez-Ramirez, 2019). Finite-time convergence estimation approaches also include modulating functions based methods (Jouffroy and Reger, 2015; Asiri et al., 2020; Ghaffour et al., 2020; Belkhatir et al., 2018; Djennoune et al., 2019).

The main challenge in finite time design is the strong effect of initial conditions on the convergence time function, which are often unknown. This may result in slow dynamic response when the initial error is large. In recent years, there has been significant interest into fixed time observer design where the observer converges within a prescribed time. The main feature of the prescribed-time convergence is that the convergence time is known in advance irrespectively to initial conditions and without the need to know the upper bound of uncertainties or the settling-time function as in the case of predefined-time convergence and finite/fixed-time convergence. Nevertheless, establishing finite-time and prescribed-time concepts remain challenging and few works are established in the literature (Espitia and Perruquetti, 2020; Espitia et al., 2022; Espitia and Perruquetti, 2021; Zhang and Efimov, 2021; Mazenc and Malisoff, 2021; Krishnamurthy and Khorrami, 2020).

1.2 Objectives and contributions

The work conducted in this thesis contributes to the state of the art on observer design for nonlinear systems in general and nonlinear systems with delayed output in particular. First, a high-gain like observer design for nonlinear triangular systems with time-varying delayed output measurements is designed, which covers the standard high-gain observer as a particular solution. Indeed, new and less conservative LMI conditions guaranteeing the stability of the proposed high-gain observer are developed. In addition to this, an extension to systems with nonlinear outputs and sampled measurements are established. Then, a new prescribed-time high-gain observer is designed allowing the convergence in prescribed time independently of initial conditions.

As mentioned previously, state estimation of nonlinear systems with delayed output measurements is challenging when the goal is to provide a high-gain observer with a lower tuning parameter and a higher maximum value of the delay. This latter is usually small due to the high value of the observer's tuning parameter, which means that the standard high-gain observer cannot guarantee the exponential convergence of the estimation error for larger value of the time-delay. To overcome this issue, in this thesis, we considered a new structure of high-gain observer, called HG/LMI observer, recently developed in (Zemouche et al., 2019). This is based on the utilization of the LMI-based approach combined with the standard high-gain technique which allows introducing a compromise index. This index offers the possibility to adjust the value of the tuning high-gain parameter. Such observer leads to significantly lower tuning parameter, which reduces the value of the observer gain and increases the maximum bound of the time-delay allowed to ensure exponential convergence. Furthermore, the peaking phenomenon is considerably reduced.

Moreover, as aforementioned, the initial conditions strongly affects on the convergence time in the finite-time design. To this end, we designed a new prescribed-time high-gain observer. Such observer achieves the prescribed-time stability of the estimation error within a predefined time, chosen a priori by the user. The advantage of the prescribed-time convergence is that the convergence time is known a priori independently of initial conditions and there is no need to upper-bound the settling-time function. The proposed observer has the structure of a high-gain observer where the gain depends on scaling function. The scaling function is monotonically increasing to infinity as the time tends to the predefined convergence time.

Through introducing a state transformation involving a time-scaling function, the estimation error is transformed into a fixed-time stable system. This feature allows to reduce the peaking phenomenon.

To show the relevance of the work conducted in this thesis, the main contributions are summarized in the following:

- A high-gain like observer design for nonlinear triangular systems with time-varying delayed output measurements is designed. The standard high-gain observer is covered as a particular solution.
- As compared to the standard high-gain observer, new and less conservative LMI conditions guaranteeing the stability of the proposed high-gain observer are developed.
- An extension to systems with nonlinear outputs is established by using the mean value theorem. Furthermore, an extension to nonlinear systems with sampled measurements is developed.
- A new prescribed-time high-gain observer for a class of nonlinear systems is proposed. Such observer achieves stability in a predefined time, freely set by the user independently of initial conditions.

1.3 Organization of the thesis

The structure of this thesis is briefly outlined in the following:

- Chapter 2 introduces some preliminary concepts on the Lyapunov stability of time-delay systems and a short reminder on observer design for nonlinear systems.
- Chapter 3 of this thesis is dedicated to the HG/LMI observer design for nonlinear system with delayed output measurements. We start by presenting the class of systems under consideration. The standard high gain observer is then recalled and its convergence is analyzed. We present and discuss the proposed observer design strategy using

the HG/LMI observer with a lower value of the tuning parameter highlighted with the observer design procedure. Extensions to systems with sampled measurements and systems with nonlinear outputs are also provided in the last section. Two numerical examples with comparisons to the standard high-gain observer and the observer presented in (Van Assche et al., 2011) are given.

- Chapter 4 is devoted to the design of the prescribed-time high-gain observer. In this chapter, we start by the problem formulation and presenting the class of systems under consideration. The prescribed-time observer is then designed and its convergence is presented. A numerical example is discussed to show the effectiveness of the proposed prescribed-time observer.

- Chapter 5 is concerned with the application of the established results in the previous chapters to estimate the water levels in the Coupled Tanks system. The numerical results are performed to illustrate the performance of the proposed HG/LMI observer and the prescribed-time high-gain observer in estimating the water levels in tanks.

- Chapter 6 summarizes the work of this thesis. Moreover, the future directions are outlined.

CHAPTER 2
THEORETICAL BACKGROUND

Contents

2.1 Introduction

The purpose of this chapter is to recall the fundamental concepts and tools required in this thesis. Stability definitions of time-delay systems and different notions of finite-time stability are presented. Then, the concept of state observer and the standard high-gain observer methodology are recalled.

2.2 Lyapunov stability of time-delay systems

Stability is a crucial for time-delay systems. For stability analysis, the Lyapunov method for time-delay systems needs an adaptation of the second Lyapunov method for ordinary differential equations to the functional differential equations, since Lyapunov function for time-delay systems depends on x_t and thus is a functional. Hereafter, we will present stability notions of time-delay systems. Consider the following retarded differential equation:

$$\dot{x}(t) = f(t, x_t), \quad t \geq t_0, \tag{2.1}$$

where $f : \mathbb{R} \times \mathcal{C}([-h, 0]) \to \mathbb{R}^n$ is continuous and is locally Lipschitz continuous in x and $x_t = x(t + \nu), \nu \in [-h, 0]$.

2.2.1 Fundamental concepts of stability

In this part, we will present some notions of stability for time-delay systems.

Definition 1 *(Fridman, 2014; Hale and Lunel, 2013) The trivial solution of* (2.1) *is*

- **Stable***: if for any $t_0 \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\varrho = \varrho(t_0, \epsilon)$ such that*

$$\|x_{t_0}\| < \varrho \Rightarrow \|x(t)\| < \epsilon, \quad t \geq t_0,$$

- **Asymptotically stable***: if it is stable, and if, for any $t_0 \in \mathbb{R}$, there exists a $\varrho = \varrho(t_0) > 0$ such that*

$$\|x_{t_0}\| < \varrho \Rightarrow \lim_{t \to \infty} \|x(t)\| = 0,$$

- **Global exponential stable***: if there exists constants $a > 0$ and $b > 0$ such that*

$$\|x(t)\| \leq b \sup_{-h \leq \nu \leq 0} \|x(\nu)\| e^{-at}.$$

The main approaches to investigate the stability of time-delay systems are the Lyapunov-Razumikhin method and the Lyapunov-Krasovskii method. The first method has been developed by Razumikhin (Razumikhin, 1956). It uses the classical Lyapunov function with additional conditions. The latter approach has been introduced by Krasovskii (Krasovskii, 1963) who proposed to use a functional that depends on x_t instead of the classical Lyapunov function that depends on $x(t)$. Both approaches will be recalled here.

2.2.2 Lyapunov-Razumikhin method

The objective of Lyapunov-Razumikhin method is to consider a classical Lyapunov function $V(t, x(t)) = x(t)^T P x(t), P > 0$ where the derivative is not necessary negative along all the trajectories of the system.

Theorem 1 (Lyapunov-Razumikhin Theorem) *(Fridman, 2014)*

Suppose $f : \mathbb{R} \times \mathcal{C}([-h, 0]) \longrightarrow \mathbb{R}^n$ takes $\mathbb{R}\times$(bounded set in $\mathcal{C}([-h, 0])$) into bounded sets of $\mathbb{R}^n$ and that $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exists a continuous functional $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^+$, which is positive definite such that

$$1) \quad u(\|x\|) \le V(t, \phi) \le v(\|x\|), \tag{2.2}$$

$$2) \quad \dot{V}(t, x(t)) \le -w(\|x(t)\|) \quad \textit{if} \quad V(t+\nu, x(t+\nu)) \leqslant V(t, x(t)), \forall \nu \in [-h, 0]. \tag{2.3}$$

then the trivial solution of (2.1) is uniformly stable.
Additionally, if $w(s) > 0$ for $s > 0$, and there exists a continuous nondecreasing function $\rho(s) > 0$ for $s > 0$ such that

$$\dot{V}(t, x(t)) \le -w(\|x(t)\|) \quad \textit{if} \quad V(t+\nu, x(t+\nu)) \leqslant \rho V(t, x(t)), \forall \nu \in [-h, 0]. \tag{2.4}$$

then the trivial solution is uniformly asymptotically stable. Moreover, if $\lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

The Lyapunov-Razumikhin method provides conservative results in general. Nevertheless, it allows to consider time-varying delays without constraints on the derivative of the delay which yields to delay-independent stability conditions.

2.2.3 Lyapunov-Krasovskii method

The Lyapunov-Krasovskii method is a generalization of the Lyapunov method for ordinary differential equations to the stability analysis of time-delay systems. It consists of choosing a functional $V(t, x_t)$ that is positive definite and decreasing along (2.1).

Theorem 2 (Lyapunov-krasovskii Theorem) *(Fridman, 2014; Hale and Lunel, 2013)*

Suppose that the function $f : \mathbb{R} \times \mathcal{C}([-h, 0]) \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded set in $\mathcal{C}([-h, 0])$) into bounded sets of $\mathbb{R}^n$ and that $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exists a continuous functional $V : \mathbb{R} \times \mathcal{C}([-h, 0]) \to \mathbb{R}^+$, which is positive definite such that

$$u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|), \tag{2.5}$$

$$\dot{V}(t, \phi) \leq -w(\|\phi(0)\|). \tag{2.6}$$

Then the trivial solution of (2.1) is uniformly stable.
If $w(s) > 0$ for $s > 0$, then the trivial solution is uniformly asymptotically stable. If $\lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

The selection of Lyapunov-Krasovskii functional is crucial to acquire the stability criteria. A candidate functional may lead to delay-dependent and delay-independent conditions. Delay-independent criteria means that the system is stable for any delays. If the stability is guaranteed for some values of the delay and unstable for others, then stability criteria is delay-dependent.

2.2.4 Delay-independent criteria

A simple Lyapunov-Krasovskii functional candidate for delay-independent analysis has the form (Fridman, 2014; Seuret et al., 2016)

$$V(x_t) = x^\top(t)Px(t) + \int_{t-h}^{t} x^T(s)Rx(s)ds \tag{2.7}$$

where $P > 0$ and $R > 0$ are $n \times n$ matrices. The requirement of delay-independent criteria arises while establishing stability analysis. Indeed, the obtained condition is independent from the delay h, which means that the system is stable for any arbitrary delay. Such functionals yield to restrictive conditions as a wide class of time-delay systems remain stable only for some values of the delay. Several studies have considered other functional candidates which yields to delay-dependent conditions.

2.2.5 Delay-dependent criteria

A simple method to derive delay-dependent stability criteria is by using Newton-Leibnitz formula

$$x(t-d) - x(t) = -\int_{t-h}^{t} \dot{x}(s)ds \tag{2.8}$$

Usually, a double integral term is added to the classical Lyapunov function $x^{\top}(t)Px(t)$. Such structure is considered to compensate the extra terms $\int_{t-h}^{t} \dot{x}(s)ds$ coming from the derivative of the first function. Several functionals have been proposed in the literature to handle various terms (Fridman and Shaked, 2003; Fridman, 2014), for example:

$$V(x_t) = V_1(x_t) + V_2(x_t), \tag{2.9}$$

where

$$V_1(t) = x^{\top}(t)Px(t),$$

and

$$V_2(t) = \int_{-h}^{0}\int_{t+\nu}^{t} x^T(s)Rx(s)dsd\nu.$$

To enrich the Lyapunov-Krasovskii functionals and deriving less restrictive stability conditions, the researchers developed and introduced more terms to include in the Lyapunov-Krasovskii functional to obtain better stability conditions (Fridman and Shaked, 2002, 2003; Seuret et al., 2016; Wu et al., 2010; Fridman and Shaked, 2003), for instance:

$$V(t, x_t, \dot{x}_t) = x^{\top}(t)Px(t) + \int_{t-h}^{t} x^T(s)Rx(s)ds + h\int_{-h}^{0}\int_{t+\nu}^{t} \dot{x}^T QR\dot{x}(s)dsd\nu$$

2.2.6 Finite-time stability

In the following, we will recall definitions on finite-time stability. The difference between the stability notions will be explained followed by a representative scheme for illustration.

Definition 2 *(Efimov et al., 2021; Djennoune et al., 2022) The origin of the system* (2.1) *is*

- **Finite-time stable**: *if it is stable and if for any* $x_0 \in \mathbb{R}^n$ *there exists* $0 \le T < +\infty$ *such that* $x(t, x_0) = 0$ *for all* $t \ge T$.

  The function $T(x_0) = \inf\{T \ge 0 : x(t, x_0) = 0, \quad \forall t \ge T\}$ *determines the settling time of the system* (2.1).

- **Fixed-time stable**: *if it is finite-time stable and if the settling time function* $T(x_0)$ *is bounded, i.e., there exists* $T_{\max} > t_0$ *such that, for every initial condition* $x_0 \in \mathbb{R}^n$,

$$T(x_0) \le T_{\max}.$$

- **Predefined-time stable**: *denote by* ρ *the system parameters of* (2.1) *and let* $T_c = T_c(\rho)$ *a design parameter. The origin of* (2.1) *is globally weakly predefined-time stable if it is globally fixed-time stable and if the settling time function satisfies*

$$T(x_0) \le T_c, \forall x_0 \in \mathbb{R}^n \tag{2.10}$$

  In addition, the origin of (2.1) *is globally strongly predefined-time stable if it is globally fixed-time stable and if the settling-time function satisfies*

$$\sup_{x_0 \in \mathbb{R}^n} T(x_0) = T_c, \forall x_0 \in \mathbb{R}^n \tag{2.11}$$

- **Prescribed-time stable**: *if it is globally fixed-time stable and if every nonzero trajectory reaches the origin at exactly a desired finite time* T_p *after* t_0, *i.e.,* $T(x_0) = T_p, \forall x_0 \neq 0$.

Remark 1 *From the previous definitions, we distinguish different finite-time stability criterions:*

- *When the settling time function depends on the initial conditions of the system, the convergence is called finite-time.*

- *If the settling time function does not depends on the initial conditions of the system but bounded, such convergence refers to be fixed-time.*

- *When the settling time is bounded but may depend on system's parameters involved in the design, the convergence is called predefined.*

- *Prescribed-time convergence refers to the situation when the user can establish a priori a desired time of convergence. Such time of convergence is independent on the system's initial conditions and the parameters involved in the design.*

Figure (2.1) summaries the types of convergence exposed in this chapter.

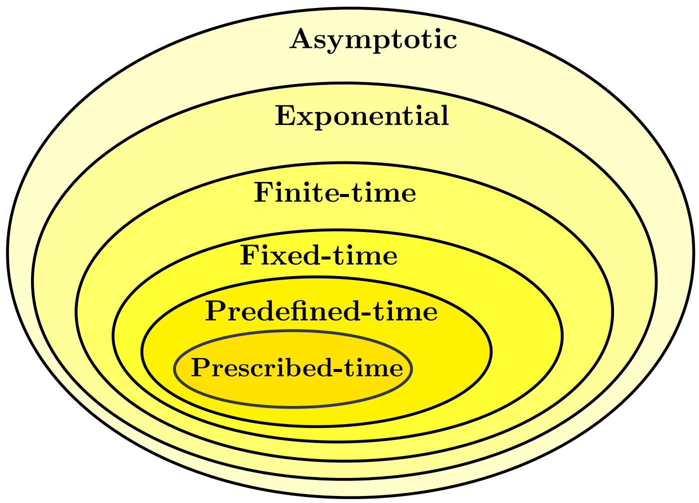

Figure 2.1: Types of convergence

2.3 Observer design for nonlinear systems

In several applications, estimation of the real state of a dynamical system is important for control and monitoring. A common way of addressing this problem is by using sensors in the physical system, called a state observer. After receiving some information from the sensors, the observer processes and reconstructs a reliable estimate of the system's states. This section presents the principle of a state observer and a short state of the art on the different techniques for designing observers for nonlinear systems is given.

2.3.1 State observer

Let us consider the following nonlinear system stated as follows

$$\begin{cases} \dot{x}(t) = f(x(t), u), \\ y(t) = g(x(t)), \end{cases} \tag{2.12}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^m$ represents the external inputs, and $y(t) \in \mathbb{R}^p$ is the vector of output measurements.

Definition 3 (Observer) *(Besançon, 2007) Let be a system (2.12). An observer is given by an auxiliary system:*

$$\begin{cases} \dot{z}(t) = F(z(t), u(t), y(t), t), \\ \hat{x}(t) = G(z(t), u(t), y(t), t), \end{cases} \tag{2.13}$$

such that

(1) $\hat{x}(0) = x(0) \implies \hat{x}(t) = x(t),\ \forall t \geqslant 0$*;*

(2) $\|\hat{x}(t) - x(t)\| \to 0$ *as* $t \to \infty$*;*

If (2) holds for any $x(0), \hat{x}(0)$*, the observer is global.*

If (2) holds with exponential convergence, the observer is exponential.

If (2) holds with a convergence rate which can be tuned, the observer is tunable.

An observer for system (2.12) can be illustrated by the block-diagram in Figure 2.2.

In control theory, the observer is known as a method to reconstruct the inaccessible states of a dynamical system using the inputs and outputs measured from this last. The classical state observer was proposed and developed by Luenberger for the first time in the early seventies of the last century (Luenberger, 1971). Since then, several approaches of nonlinear observers have been developed in the literature. Some approaches have been extended from linear systems to deal with nonlinear dynamical systems, for instance:

- Extended Kalman observer (Reif and Unbehauen, 1999),

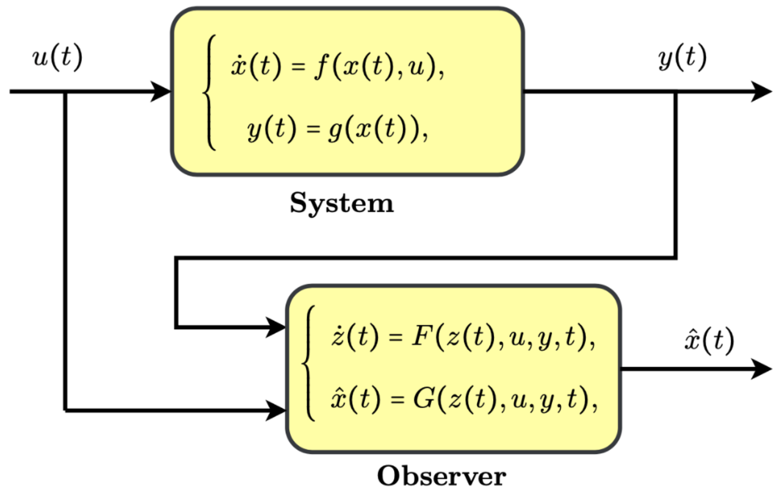

Figure 2.2: Block-diagram of a state observer

- Luenberger observer (Zeitz, 1987),
- High-gain observer (Khalil and Praly, 2014),
- Adaptive observer (Cho and Rajamani, 1997),
- Sliding mode observer (Fridman et al., 2008).

In the next section, we will only recall the standard high-gain observer methodology.

2.3.2 Standard high-gain observer methodology

We consider the class of nonlinear systems described by

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) \\ y(t) = Cx(t), \end{cases} \tag{2.14}$$

where the matrices A and C are defined by

$$C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \quad (A)_{i,j} = \begin{cases} 1 & \text{if} \quad j = i+1, \\ 0 & \text{if} \quad j \neq i+1, \end{cases} \tag{2.15}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system and $y(t) \in \mathbb{R}$ is the measured output. The function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the Lipschitz property. We consider the Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + L[y(t) - C\hat{x}(t)], \tag{2.16}$$

where $\hat{x}(t)$ represents the state estimation and L is the observer gain.

The dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$ is given as follows

$$\dot{e}(t) = (A - LC)e(t) + [f(x(t)) - f(\hat{x}(t))]. \tag{2.17}$$

In the standard high-gain methodology, as in (Gauthier and Kupka, 1994), the gain L of the observer is written as follows:

$$L \triangleq T(\theta)K; \quad \theta \geqslant 1, \tag{2.18}$$

where

$$T(\theta) := \mathrm{diag}(\theta, \dots, \theta^n) \quad \text{and} \quad K \in \mathbb{R}^n.$$

Moreover, the observer' estimation error is transformed into

$$\bar{e}(t) := T^{-1}(\theta)e(t),$$

where $T^{-1}(\theta)$ is the inverse of $T(\theta)$ given by

$$T^{-1}(\theta) := \mathrm{diag}\left(\frac{1}{\theta}, \dots, \frac{1}{\theta^n}\right).$$

The dynamics of the transformed error is given by

$$\dot{\bar{e}}(t) = \theta(A - KC)\bar{e}(t) + T^{-1}(\theta)\Delta f, \tag{2.19}$$

with

$$\Delta f := f(x) - f(x - T(\theta)\bar{e}).$$

From the fact that $\theta \geqslant 1$ and by using the Lipschitz condition, it was shown in (Alessandri and Rossi, 2013) that there exists a positive constant k_f, independent of θ, such that

$$\|T^{-1}(\theta)\Delta f)\| \leqslant k_f \|\bar{e}\|. \tag{2.20}$$

The exponential stability of the transformed error is guaranteed by the following conditions established in the theorem cited below:

Theorem 3 *(Zemouche et al., 2019) If there exist $P > 0$, $\lambda > 0$, Y, and $\theta \geq 1$ such that*

$$A^T P + PA - C^T Y - Y^T C + \lambda I < 0 \tag{2.21}$$

$$\theta > \theta_0 = \frac{2k_f \lambda_{\max}(P)}{\lambda} \tag{2.22}$$

then the estimation error e is exponentially stable with $K = P^{-1}Y^T$, where $\lambda_{\max}(P)$ is the maximum eigenvalue of the matrix P.

Proof *The proof of this theorem can be found in (Zemouche et al., 2019).*

2.4 Conclusion

The objective of this chapter was to give an overview related to the stability notions of time-delay systems and some existing methods about the stability were summarized (Lyapunov stability and finite-time stability). On the other hand, we presented a reminder of a state observer and the standard high-gain observer methodology.

CHAPTER 3

ON HIGH-GAIN OBSERVER DESIGN FOR NONLINEAR SYSTEMS WITH DELAYED OUTPUT MEASUREMENTS

Contents

3.1 Introduction

State estimation for nonlinear systems is a very important topic as the state is often unknown but needed for control design or fault detection. Many papers proposed state observer solutions for different classes of nonlinear systems. In particular, there has been a recent interest in the class of nonlinear systems with delayed output measurements. Indeed, system's measurements are often subject to delay due to the underlying complexity of the system itself or to some communication delays if the measurements need to be sent to another location. This measurement delay will obviously affect the estimation and the control performance.

In (Ahmed-Ali et al., 2009), the authors proposed a cascade high-gain observer based on the Lyapunov method in which explicit relations between the delay and the number of cascade observers have been proposed. Subsequently, an extension to high-gain observer design with time varying-delay and sampled data cases where the delay is sufficiently small has been presented in Van Assche et al. (2011). Although the maximum value of the allowable delay is improved, however, it still remains small due to the high value of the tuning parameter required by the standard high-gain observer. On the other hand, the maximum bound of the delay is obtained by solving an ordinary differential equation that depends on the design tuning parameter. The proposed work in this chapter has been motivated by this issue, namely establishing a high-gain like design method with a lower tuning parameter, which leads to a higher maximum value of the delay.

To this end, we proposed an observer design method, which covers the standard high-gain observer, used in the above papers, as a particular case. This is based on the exploitation of the LMI-based approach combined with the standard high-gain technique, called HG/LMI observer, which allows introducing a compromise index j_s. This index j_s offers the possibility to adjust the value of the tuning high-gain parameter ensuring exponential convergence for larger values of the delay. The convergence analysis is performed by using a Lyapunov-Krasovskii functional. To reach less conservative bounds, the Halanay inequality is applied on the integral term containing the error, instead of developing strong upper bounds to make it vanish from the Lyapunov analysis. The obtained results show explicitly the effectiveness of the proposed HG/LMI observer-based technique, due to the mathematical relation between the tuning parameter of the observer and the maximum bound of the delay. An extension to nonlinear systems with sampled measurements is given as an application of the result. Furthermore, the proposed methodology is extended to systems with nonlinear outputs.

3.2 System description and problem formulation

In this section, we introduce the class of systems under consideration and formulate the estimation problem.

We consider the class of nonlinear systems described by:

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 + f_1(x_1) \\ x_3 + f_2(x_1, x_2) \\ \vdots \\ x_{n-1} + f_{n-1}(x_1, x_2, \dots, x_{n-1}) \\ f_n(x_1, \dots, x_n) \end{bmatrix} \\ y = x_1(t - \tau(t)), \end{cases} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system and $y(t) \in \mathbb{R}$ is the measured output. We assume that $\tau(t)$ is a known time-varying delay satisfying

$$0 < \tau(t) \leqslant \tau_M.$$

The functions $f_i : \mathbb{R}^i \longrightarrow \mathbb{R}$, i=1,..., n, satisfy the following Lipschitz property:

$$|f_i(x_1 + \Delta_1, \dots, x_i + \Delta_i) - f_i(x_1, \dots, x_i)| \leqslant \sum_{j=1}^{i} k_j |\Delta_j| \tag{3.2}$$

where k_j is the Lipschitz constant and $\Delta_j \in \mathbb{R}, \forall j = 1, \dots, i$.

For simplicity of the presentation, system (3.1) can be rewritten under the following compact form:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)), \\ y(t) = Cx(t - \tau(t)), \end{cases} \tag{3.3}$$

where the matrices A and C are defined as in (2.15) and

$$f(x(t)) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_{n-1}(x_1, x_2, \ldots, x_{n-1}) \\ f_n(x_1, \ldots, x_n) \end{bmatrix} \tag{3.4}$$

Let us introduce the following Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + L[y(t) - C\hat{x}(t - \tau(t))], \tag{3.5}$$

where $\hat{x}(t)$ represents the state estimation and L is the observer gain.

The dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$ is given as follows

$$\dot{e}(t) = Ae(t) + [f(x(t)) - f(\hat{x}(t))] - LCe(t - \tau(t)). \tag{3.6}$$

The objective consists in designing a high-gain observer for system (3.1) that provides stability of the estimation error. We also provide an expression of the maximum bound of the allowable delay and the design parameter under which the proposed observer converges exponentially. To establish the exponential convergence, the following lemma is useful.

Lemma 1 (Halanay (1966)) *If there exist a positive Lyapunov-Krasovskii functional $V(e(t))$ such that*

$$\frac{d}{dt}V(e(t)) \leqslant -\alpha V(e(t)) + \beta \sup_{s\in[t-\tau_M,t]} V(e(s)),$$

where $\alpha > \beta > 0$, then there exist two scalars $\eta > 0$ and $\delta > 0$ such that

$$V(e(t)) \leqslant \eta e^{-\delta(t-t_0)} \quad \textit{for all} \quad t \geqslant t_0.$$

In order to show the improvement of the results by using the the HG/LMI observer based design, we first start by deriving new LMI conditions based on the use of the standard high-gain observer methodology.

3.3 Standard high-gain observer based design

This section is devoted to the preliminary results shared into several intermediate results. Indeed, after some preliminary and useful mathematical lemmas and propositions related to time-delay systems theory, we propose a new LMI-based observer design procedure. The latter provides conditions on the maximum upper bound of the allowed delay guaranteeing exponential convergence of the observer. The design technique is based on the use of the standard high-gain observer methodology.

By using the standard high-gain methodology recalled in Chapter 2, the dynamics of the transformed error is given by

$$\dot{\bar{e}}(t) = \theta(A - KC)\bar{e}(t) + T^{-1}(\theta)\Delta f - \theta KC(\bar{e}(t - \tau(t)) - \bar{e}(t)), \tag{3.7}$$

with

$$\Delta f := f(x(t)) - f(x(t) - T(\theta)\bar{e}(t)).$$

Before presenting the main theorem corresponding to the use of the standard high-gain observer, we will introduce first a series of intermediate results. Such intermediate results will improve the clarity and readability of the first main theorem.

Lemma 2 *Let f be a nonlinear function satisfying (2.20). Then for any symmetric and positive definite matrix P and a vector $\bar{e}$ of appropriate dimensions, we have*

$$\mathcal{H}e\Big\{\Big(T^{-1}(\theta)\Delta f\Big)^{\top}\bar{e}(t)\Big\} \leq 2k_f\lambda_{\max}\Big(P\Big)\bar{e}^{\top}(t)\bar{e}(t), \tag{3.8}$$

where $\mathcal{H}e\Big\{\mathcal{S}\Big\} := \mathcal{S} + \mathcal{S}^{\top}$.

Proof *The proof is omitted. It is straightforward by applying the well-known Cauchy-Schwarz inequality and the fact that $\theta \geq 1$.*

Proposition 1 *Let Y be an arbitrary matrix of appropriate dimension. Define Υ as*

$$\Upsilon \triangleq \bar{e}^{\top}(t)Y^{\top}C\Big(\bar{e}(t) - \bar{e}(t - \tau(t))\Big) + \Big(\bar{e}(t) - \bar{e}(t - \tau(t))\Big)^{\top}C^{\top}Y\bar{e}(t),$$

where $\bar{e}(t)$ is the transformed estimation error satisfying (3.7). Then there exist three positive scalars $\mu_i > 0, i = 1, \ldots, 3$ such that Υ satisfies the following inequality:

$$
\begin{aligned}
\Upsilon \leqslant\ & \frac{1}{\mu_1}\bar{e}^{\top}(t)Y^{\top}Y\bar{e}(t) \\
& + \mu_1\Big(1+\frac{1}{\mu_2}\Big)\tau_M\theta^2\int_{t-\tau_M}^{t}(\bar{e}_2(s))^2 ds \\
& + \mu_1\Big(1+\frac{1}{\mu_3}\Big)\Big(1+\mu_2\Big)k_1^2\tau_M\int_{t-\tau_M}^{t}(\bar{e}_1(s))^2 ds \\
& + \mu_1\Big(1+\mu_2\Big)\Big(1+\mu_3\Big)\tau_M K_1^2\theta^2\int_{t-\tau_M}^{t}\left(\bar{e}_1(s-\tau(s))\right)^2 ds.
\end{aligned} \tag{3.9}
$$

where $\bar{e}_1$ and $\bar{e}_2$ are the first and second components of the estimation error vector $\bar{e}(t)$, and K_1 is the first component of K.

Proof *By applying the Young inequality on Υ, we obtain*

$$\Upsilon \leqslant \mu_1\Big(\bar{e}(t)-\bar{e}(t-\tau(t))\Big)^{\top}C^{\top}C\Big(\bar{e}(t)-\bar{e}(t-\tau(t))\Big)+\frac{1}{\mu_1}\bar{e}^{\top}(t)Y^{\top}Y\bar{e}(t), \tag{3.10}$$

for any given scalar $\mu_1 > 0$. By using the Newton-Leibniz integration formula

$$\bar{e}(t)-\bar{e}(t-\tau(t)) = \int_{t-\tau(t)}^{t}\dot{\bar{e}}(s)ds,$$

inequality (3.10) can be rewritten as follows:

$$\Upsilon \leqslant \mu_1\mathbb{J}(t)+\frac{1}{\mu_1}\bar{e}^{\top}(t)\mathcal{Y}^{\top}\mathcal{Y}\bar{e}(t), \tag{3.11}$$

where

$$\mathbb{J}(t) \triangleq \left(\int_{t-\tau(t)}^{t}\dot{\bar{e}}(s)ds\right)^{\top}C^{\top}C\left(\int_{t-\tau(t)}^{t}\dot{\bar{e}}(s)ds\right).$$

From Jensen Inequality and the fact that $C\bar{e}(s) = \bar{e}_1(s)$, and

$$\dot{\bar{e}}_1(s) = \Big(\theta\bar{e}_2(s)+\frac{1}{\theta}\Delta f_1-\theta K_1\bar{e}_1(s-\tau(s))\Big), \tag{3.12}$$

we obtain an upper bound of $\mathbb{J}(t)$ *as follows:*

$$
\begin{aligned}
\mathbb{J}(t) &\leqslant \tau(t)\int_{t-\tau(t)}^{t} \dot{\bar{e}}(s)^{\top} C^{\top} C \dot{\bar{e}}(s) ds \\
&\leqslant \tau_M \int_{t-\tau_M}^{t} \left(\dot{\bar{e}}_1(s)\right)^2 ds \\
&= \tau_M \int_{t-\tau_M}^{t} \left(\theta \bar{e}_2(s) + \frac{1}{\theta}\Delta f_1 - \theta K_1 \bar{e}_1(s-\tau(s))\right)^2 ds \\
&\leqslant \left(1+\frac{1}{\mu_2}\right)\tau_M \theta^2 \int_{t-\tau_M}^{t} (\bar{e}_2(s))^2 ds \\
&\quad + \left(1+\mu_2\right)\tau_M \int_{t-\tau_M}^{t} \left(\frac{1}{\theta}\Delta f_1 - \theta K_1 \bar{e}_1(s-\tau(s))\right)^2 ds \\
&\leqslant \left(1+\frac{1}{\mu_2}\right)\tau_M \theta^2 \int_{t-\tau_M}^{t} (\bar{e}_2(s))^2 ds \\
&\quad + \left(1+\frac{1}{\mu_3}\right)\left(1+\mu_2\right) k_1^2 \tau_M \int_{t-\tau_M}^{t} (\bar{e}_1(s))^2 ds \\
&\quad + \left(1+\mu_2\right)\left(1+\mu_3\right)\tau_M K_1^2 \theta^2 \int_{t-\tau_M}^{t} \left(\bar{e}_1(s-\tau(s))\right)^2 ds
\end{aligned} \tag{3.13}
$$

where $\mu_2 > 0$ *and* $\mu_3 > 0$ *come from the application of Young inequality. By substituting* (3.13) *in* (3.11)*, inequality* (3.9) *is inferred. This ends the proof of Proposition 1.*

Lemma 3 *Let* $X(t)$ *be a positive and continuous function. Let* $\tau(.)$ *be a positive time-delay with* $\tau(t) \leq \tau_M$*. Then the following inequality holds:*

$$
\int_{t-\tau(t)}^{t} X\left(s-\tau(s)\right) ds \leq \tau_M \sup_{s\in[t-2\tau_M,\ t]} \left\{X\left(s\right)\right\}. \tag{3.14}
$$

Proof *The proof is straightforward. Indeed, if* $s \in [t-\tau_M,\ t]$ *and* $\tau(s) \geq 0$*, then we have*

$$
s - \tau(s) \in [t-2\tau_M,\ t].
$$

Proposition 2 *Consider the Lyapunov function* $V_1(\bar{e}(t)) \triangleq \bar{e}^{\top}(t) P \bar{e}(t)$*, where* $P = P^{\top} > 0$*. Then the derivative of* V_1 *along the trajectories of* (3.7) *satisfies the following inequality:*

$$
\begin{aligned}
\frac{d}{dt}V_1 &\leqslant \theta \bar{e}^{\top}(t)\left[A^{\top}P + PA - C^{\top}\mathcal{Y} - \mathcal{Y}^{\top}C + \frac{1}{\mu_1}\mathcal{Y}^{\top}\mathcal{Y}\right]\bar{e}(t) + 2k_f \lambda_{\max}\left(P\right)\bar{e}^{\top}(t)\bar{e}(t) \\
&\quad + \mu_1\left(1+\frac{1}{\mu_2}\right)\tau_M \theta^3 \int_{t-\tau_M}^{t} (\bar{e}_2(s))^2 ds \\
&\quad + \varpi\left(\mu_1, \mu_2, \mu_3\right)\tau_M^2 \sup_{s\in[t-2\tau_M,\ t]} V_1\left(s\right),
\end{aligned} \tag{3.15}
$$

where

$$\varpi(\mu_1, \mu_2, \mu_3) \triangleq \frac{\mu_1}{\lambda_{\min}(P)} \left[\left(1 + \frac{1}{\mu_3}\right)\left(1 + \mu_2\right) k_1^2 \theta + \left(1 + \mu_2\right)\left(1 + \mu_3\right) K_1^2 \theta^3 \right], \tag{3.16}$$

and

$$\mathcal{Y} = K^\top P.$$

Proof *Let us start by computing the derivative of* V_1 *along the trajectories of* (3.7). *We obtain*

$$\begin{aligned} \frac{d}{dt} V_1(t) &= \dot{\bar{e}}^T(t) P \bar{e}(t) + \bar{e}^T(t) P \dot{\bar{e}}(t) \\ &= \theta \bar{e}^T(t) \left[(A - KC)^T P + P(A - KC) \right] \bar{e}(t) \\ &+ \left(T^{-1}(\theta) \Delta f \right)^T P \bar{e}(t) + \bar{e}^T(t) P \left(T^{-1}(\theta) \Delta f \right) + \theta \Upsilon, \end{aligned}$$

where

$$\Upsilon \triangleq \bar{e}^\top(t) \mathcal{Y}^\top C \Big(\bar{e}(t) - \bar{e}(t - \tau(t)) \Big) + \Big(\bar{e}(t) - \bar{e}(t - \tau(t)) \Big)^\top C^\top \mathcal{Y} \bar{e}(t),$$

and

$$\mathcal{Y} = K^T P.$$

The rest of the proof can be obtained straightforwardly from Lemma 2, Proposition 1, and Lemma 3.

Lemma 4 *Let us define the function*

$$\vartheta(z) \triangleq \int_{t-\tau_M}^{t} \int_{\xi}^{t} z(s) ds d\xi,$$

where $z(s) \in \mathbb{R}^+$ *and* $\tau_M > 0$ *is a scalar constant. Then the two following equations hold:*

$$\vartheta(z) \leqslant \tau_M \int_{t-\tau_M}^{t} z(s) ds, \tag{3.17a}$$

$$\frac{d}{dt} \vartheta(z(t)) = \tau_M z(t) - \int_{t-\tau_M}^{t} z(s) ds. \tag{3.17b}$$

Proof *The first inequality* (3.17a) *is obvious and can be obtained by integrating with respect to* ξ *and the fact that* $t - s \leq \tau_M$ *for* $s \in [t - \tau_M\ ,\ t]$.

Now we are ready to state the preliminary result summarized in the following theorem providing new LMI conditions ensuring exponential convergence of the standard high-gain observer.

Theorem 4 *Assume there exist a positive definite matrix* P*, a matrix* $\mathcal{Y}$ *of appropriate dimension and real constants* $\mu_i > 0, i = 1, \ldots, 3,\ \lambda > 0,\ \tau_M > 0$ *such that the following conditions hold*

$$\begin{bmatrix} \mathcal{H}e\Big\{PA - \mathcal{Y}^\top C\Big\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \leqslant 0, \tag{3.18}$$

$$\theta > \max\left(1, \frac{2k_f \lambda_{\max}(P)}{\lambda}\right), \tag{3.19}$$

$$\tau_M \leq \min(\tau_1, \tau_2), \tag{3.20}$$

where

$$\tau_1 \triangleq \sqrt{\frac{\Big[\lambda\theta - 2k_f \lambda_{\max}(P)\Big]}{\lambda_{\max}(P)\varpi\big(\mu_1, \mu_2, \mu_3\big)}}, \tag{3.21}$$

$$\tau_2 \triangleq \frac{1}{\mu_1\Big(1 + \frac{1}{\mu_2}\Big)\theta^2 + \frac{[\lambda\theta - 2k_f \lambda_{\max}(P)]}{\lambda_{\max}(P)}}, \tag{3.22}$$

with

$$\varpi\big(\mu_1, \mu_2, \mu_3\big) \triangleq \frac{\mu_1}{\lambda_{\min}\big(P\big)}\left[\Big(1 + \frac{1}{\mu_3}\Big)\Big(1 + \mu_2\Big)k_1^2\theta + \Big(1 + \mu_2\Big)\Big(1 + \mu_3\Big)K_1^2\theta^3\right], \tag{3.23}$$

$$\mathcal{R} = [0 \quad 1 \quad 0_{1\times n-2}], \tag{3.24}$$

$$K = P^{-1}\mathcal{Y}^\top = [K_1 \ldots K_n]^\top. \tag{3.25}$$

Then the observer (3.5) *converges exponentially and the observer gain is given by* $L = T(\theta)K$.

Proof *Consider the following Lyapunov-Krasovskii functional*

$$V(t) = V(\bar{e}(t)) = V_1(\bar{e}(t)) + \theta V_2(\bar{e}(t)), \tag{3.26}$$

where

$$V_1(t) = \bar{e}^\top(t) P \bar{e}(t),$$

and

$$V_2(t) = \int_{t-\tau_M}^{t} \int_{\xi}^{t} (\bar{e}_2(s))^2 ds d\xi.$$

From Schur lemma, LMI (3.18) is equivalent to

$$A^T P + PA - C^T \mathcal{Y} - \mathcal{Y}^T C + \frac{1}{\mu_1} \mathcal{Y}^T \mathcal{Y} + \tau_M \mathcal{R}^T \mathcal{R} < -\lambda I.$$

Hence, by substituting this inequality in the derivative of $V(\bar{e}(t))$ and from Proposition 2 and Lemma 4, we get the following inequality:

$$\begin{aligned} \frac{d}{dt} V(t) \leqslant &-\Big(\theta\lambda - 2k_f \lambda_{\max}\big(P\big)\Big) \bar{e}^\top(t) \bar{e}(t) \\ &- \theta \tau_M \left(\frac{1}{\tau_M} - \mu_1 \Big(1 + \frac{1}{\mu_2}\Big) \theta^2 \right) \int_{t-\tau_M}^{t} (\bar{e}_2(s))^2 ds \\ &+ \varpi\big(\mu_1, \mu_2, \mu_3\big) \tau_M^2 \sup_{s \in [t-2\tau_M,\ t]} V_1\big(s\big), \end{aligned}$$

for any positive scalars $\mu_i, i = 1, \ldots, 3$, where $\varpi\big(\mu_1, \mu_2, \mu_3\big)$ is defined in (3.16). From (3.22), we deduce that

$$\frac{1}{\tau_M} - \mu_1 \Big(1 + \frac{1}{\mu_2}\Big) \theta^2 > 0.$$

Then by applying Lemma 4, Lemma 3, and since $V_1(s) \leq V(s)$, we obtain

$$\begin{aligned} \frac{d}{dt} V(t) \leqslant &-\Big(\theta\lambda - 2k_f \lambda_{\max}\big(P\big)\Big) \bar{e}^\top(t) \bar{e}(t) - \theta \left(\frac{1}{\tau_M} - \mu_1 \Big(1 + \frac{1}{\mu_2}\Big) \theta^2 \right) V_2(t) \\ &+ \varpi\big(\mu_1, \mu_2, \mu_3\big) \tau_M^2 \sup_{s \in [t-2\tau_M,\ t]} V\big(s\big). \end{aligned}$$

Finally, from (3.19) and the inequality below

$$-\bar{e}^\top(t) \bar{e}(t) \leq -\frac{1}{\lambda_{\max}\big(P\big)} V_1(t),$$

we get

$$\frac{d}{dt}V(t) \leqslant -\alpha V(t) + \varpi\big(\mu_1,\mu_2,\mu_3\big)\tau_M^2 \sup_{[t-2\tau_M,t]} V(s),$$

where

$$\alpha \triangleq \min\left(\frac{\theta\lambda - 2k_f\lambda_{\max}\big(P\big)}{\lambda_{\max}\big(P\big)}, \frac{1}{\tau_M} - \mu_1\Big(1+\frac{1}{\mu_2}\Big)\theta^2\right),$$

$\varpi\big(\mu_1,\mu_2,\mu_3\big)$ *is defined in* (3.16). *From* (3.21) *and* (3.22) *we have* $\varpi\big(\mu_1,\mu_2,\mu_3\big) < \alpha$. *Consequently, we deduce from Lemma 1 that there exist two positive scalars* η *and* δ *such that*

$$V(\bar{e}(t)) \leqslant \eta e^{-\delta(t-t_0)}, \quad \forall\, t \geqslant t_0,$$

which means that the estimation error is exponentially stable to zero. This ends the proof.

Remark 2 *All the scalar positive quantities* $\mu_i, i = 1,3$ *come from the application of Young inequality appropriately. The variable* μ_1 *is considered as a decision variable in the LMI* (3.18), *while the* μ_2 *and* μ_3 *appear in* (3.21)-(3.22). *They should be fixed adequately to increase the value of the tolerated* τ_M. *However, since they are involved nonlinearly, it is difficult to optimize their computation. To simplify the computation of* τ_1 *and* τ_2, *we chose as values of* μ_2 *and* μ_3 *those often used in the Young inequality, namely* $\mu_2 = \mu_3 = 1$. *Therefore,* (3.21)-(3.22) *are simplified as follows:*

$$\tau_1 \triangleq \sqrt{\frac{\lambda_{\min}(P)\Big[\lambda\theta - 2k_f\lambda_{\max}(P)\Big]}{4\mu_1\theta\lambda_{\max}(P)\Big[k_1^2 + K_1^2\theta^2\Big]}}, \tag{3.27}$$

$$\tau_2 \triangleq \frac{1}{2\mu_1\theta^2 + \dfrac{[\lambda\theta - 2k_f\lambda_{\max}(P)]}{\lambda_{\max}(P)}}. \tag{3.28}$$

Remark 3 *In the particular case where the system contains only one nonlinearity in the last component* ($f_i \equiv 0, i = 1,\ldots,n-1$), *then Theorem 4 with* $k_1 = 0$ *can be applied to get larger values of the tolerated* τ_M. *Indeed, in such a situation, the dynamics of* $\bar{e}_1$ *in* (3.12) *does not contain the term* $\frac{1}{\theta}\Delta f_1$.

Notice that the upper bound of the delay can be very small for high values of θ. This means that for relatively important delays, the considered observer cannot guarantee the exponential convergence of the estimation error. Indeed, an exponential observer with a high value of θ tolerates small upper bound τ_M of the delay. In the next section, to overcome this problem, we propose to use the high-gain like-observer with lower tuning parameter introduced in (Zemouche et al., 2019).

3.4 HG/LMI observer based design

In this section, we extend the HG/LMI observer methodology developed in (Zemouche et al., 2019) for nonlinear systems to system (3.1) with the objective of improving the allowable value of τ_M while ensuring exponential convergence of the observer.

3.4.1 HG/LMI-based transformation

From the LPV/LMI method in (Zemouche and Boutayeb, 2013), each nonlinear component Δf_i in (3.7) can be rewritten under the following form (Zemouche et al., 2019):

$$\Delta f_i = \sum_{j=1}^{i-j_i} \theta^j \psi_{ij} \bar{e}_j + \sum_{j=1}^{j_i} \theta^{k_i(j)} \psi_{ik_i(j)} \bar{e}_{k_i(j)},$$

where

$$k_i(j) = i - (j_i - j), \quad 0 \leqslant j_i \leqslant i.$$

It follows that Δf is written as

$$\Delta f = \underbrace{\overbrace{\sum_{i=1}^{n}\sum_{j=1}^{i-j_i} \theta^j \psi_{ij} v_n(i) \bar{e}_j}^{\text{for HG}}}_{\Delta f_1} + \overbrace{\sum_{i=1}^{n}\sum_{j=1}^{j_i} \theta^{k_i(j)} \psi_{ik_i(j)} v_n(i) \bar{e}_{k_i(j)}}^{\text{for LPV/LMI}},$$

Therefore, the error dynamics (3.7) is rewritten as follows:

$$\dot{\bar{e}}(t) = \theta(\mathcal{A}(\Psi^\theta) - KC)\bar{e}(t) + T^{-1}(\theta)\Delta f_1 - \theta KC(\bar{e}(t - \tau(t)) - \bar{e}(t)), \tag{3.29}$$

with

$$\mathcal{A}(\Psi^\theta) = A + \sum_{i=1}^{n}\sum_{j=1}^{j_i} \psi_{ij}^\theta v_n(i) \mathrm{e}_n k_i(j),$$

$$
\Psi^{\theta} = \begin{pmatrix} \psi^{\theta}_{11} \\ \vdots \\ \psi^{\theta}_{1j_1} \\ \psi^{\theta}_{21} \\ \vdots \\ \psi^{\theta}_{2j_2} \\ \vdots \\ \psi^{\theta}_{nj_n} \end{pmatrix} \in \mathbb{R}^{\sum_{i=1}^{n} j_i},
$$

and

$$
\psi^{\theta}_{j} = \frac{\psi_{ik_i(j)}}{\theta^{1+(j_i-j)}}.
$$

Define the convex bounded set

$$
\mathcal{H}^{\sigma}_{j_s} = \left\{ \Phi \in \mathbb{R}^{\sum_{i=1}^{n} j_i} : \frac{\underline{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \leq \Phi_{ij} \leqslant \frac{\bar{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \right\},
$$

for which the set of vertices is defined by

$$
\mathcal{V}_{\mathcal{H}^{\sigma}_{j_s}} = \left\{ \Phi \in \mathbb{R}^{\sum_{i=1}^{n} j_i} : \Phi_{ij} \in \left\{ \frac{\underline{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}}, \frac{\bar{\gamma}_{\gamma_{ik_i(j)}}}{\sigma^{1+(j_i-j)}} \right\} \right\}
$$

where $\bar{\gamma}_{\gamma_{ik_i(j)}} \geq 0$ and $\underline{\gamma}_{\gamma_{ik_i(j)}} \leq 0$ are, respectively, the lower and the upper bounds of the bounded parameter $\psi_{ik_i(j)}$. Then, for two positive scalars σ_1, σ_2, we have

$$
\sigma_1 \leq \sigma_2 \Longrightarrow \mathcal{H}^{\sigma_1}_{j_s} \supset \mathcal{H}^{\sigma_2}_{j_s}.
$$

It follows that

$$
\lim_{\sigma \to +\infty} \mathcal{H}^{\sigma}_{j_s} = \{0\}.
$$

On the other hand, we can show that there exists a constant k_{j_s} independent from θ such that the following inequality holds:

$$
\|T^{-1}(\theta)\Delta f_1\| \leqslant \frac{k_{j_s}}{\theta^{j_s}} \|\bar{e}\|. \tag{3.30}
$$

3.4.2 HG/LMI synthesis conditions

This section is devoted to the main theorem, which provides sufficient synthesis conditions guaranteeing exponential convergence of the estimation error. The design is based on the use of the HG/LMI technique for the class of nonlinear systems with delayed outputs given in (3.1).

Theorem 5 *Assume there exist a positive definite matrix P, a matrix $\mathcal{Y}$ of appropriate dimension and real constants $\mu_i > 0, i = 1, \dots, 3$, $\lambda > 0$, $\tau_M > 0$ such that the following conditions hold*

$$\begin{bmatrix} \mathcal{H}e\Big\{P\mathcal{A}(\Psi) - \mathcal{Y}^\top C\Big\} + \tau_M \mathcal{R}^\top \mathcal{R} + \lambda I & \mathcal{Y}^\top \\ \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \leqslant 0, \tag{3.31}$$

$$\theta > \max\left(\sigma, \sqrt[1+j_s]{\frac{2k_{j_s}\lambda_{\max}(P)}{\lambda}}\right), \tag{3.32}$$

$$\tau_M \le \min(\tau_1, \tau_2), \tag{3.33}$$

where

$$\tau_1 \triangleq \sqrt{\frac{\left[\theta\lambda - \dfrac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right]}{\lambda_{\max}(P)\varpi\big(\mu_1, \mu_2, \mu_3\big)}}, \tag{3.34}$$

$$\tau_2 \triangleq \frac{1}{\mu_1\Big(1 + \dfrac{1}{\mu_2}\Big)\theta^2 + \dfrac{\left[\theta\lambda - \dfrac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right]}{\lambda_{\max}(P)}}, \tag{3.35}$$

with

$$\varpi\big(\mu_1, \mu_2, \mu_3\big) \triangleq \frac{\mu_1}{\lambda_{\min}\big(P\big)}\left[\Big(1 + \frac{1}{\mu_3}\Big)\Big(1 + \mu_2\Big)k_1^2\theta + \Big(1 + \mu_2\Big)\Big(1 + \mu_3\Big)K_1^2\theta^3\right], \tag{3.36}$$

$$\mathcal{R} = [0 \quad 1 \quad 0_{1\times n-2}], \tag{3.37}$$

$$K = P^{-1}\mathcal{Y}^\top = [K_1 \dots K_n]^\top. \tag{3.38}$$

Then the observer (3.5) converges exponentially and the observer gain is given by $L = T(\theta)K$.

Proof *Consider the following Lyapunov-Krasovskii functional*

$$V(t) = V_1(t) + \theta V_2(t),$$

where

$$V_1(t) = \bar{e}^\top(t) P \bar{e}(t),$$

and

$$V_2(t) = \int_{t-\tau_M}^{t} \int_{\xi}^{t} (\bar{e}_2(s))^2 ds d\xi.$$

From Schur lemma, LMI (3.31) *is equivalent to*

$$\mathcal{A}(\Psi^\sigma)^\top P + P\mathcal{A}(\Psi^\sigma) - C^\top \mathcal{Y} - \mathcal{Y}^\top C + \frac{1}{\mu_1}\mathcal{Y}^\top \mathcal{Y} + \tau_M \mathcal{R}^\top \mathcal{R} \leqslant -\lambda I.$$

By analogy to the proof of Theorem 4, the derivative of V along the trajectories of (3.29) *satisfies*

$$\begin{aligned}\frac{d}{dt}V(t) \leqslant & -\Big(\theta\lambda - \frac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\Big)\bar{e}^\top(t)\bar{e}(t) - \theta\left(\frac{1}{\tau_M} - \mu_1\Big(1 + \frac{1}{\mu_2}\Big)\theta^2\right)V_2(t) \\ & + \varpi\big(\mu_1, \mu_2, \mu_3\big)\tau_M^2 \sup_{s\in[t-2\tau_M,\ t]} V\big(s\big)\end{aligned}$$

for any positive scalars $\mu_i, i = 1, \ldots, 3$, where $\varpi\big(\mu_1, \mu_2, \mu_3\big)$ is defined in (3.16).

Therefore

$$\frac{d}{dt}V(t) \leqslant -\alpha V(t) + \varpi\big(\mu_1, \mu_2, \mu_3\big)\tau_M^2 \sup_{[t-2\tau_M, t]} V(s),$$

with

$$\alpha \triangleq \min\left(\frac{\theta\lambda - \dfrac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}}{\lambda_{\max}(P)}, \frac{1}{\tau_M} - \mu_1\Big(1 + \frac{1}{\mu_2}\Big)\theta^2\right).$$

From (3.34) *and* (3.35) *we have $\varpi\big(\mu_1, \mu_2, \mu_3\big) < \alpha$. Hence, Lemma 1 allows concluding exponential convergence to zero of the estimation error. This ends the proof.*

Remark 4 *The previous proof shows the role of the "compromise index" j_s. It allows reducing the tuning parameter value of the observer. Since the expression of the delay depends on the tuning parameter, reducing this later increments the value of the maximum delay τ_M compared to the standard high-gain observer.*

Remark 5 *One important property of the standard high gain observers is the fast exponential convergence. However, such fast convergence has several drawbacks: high sensitivity to measurement noise; peaking phenomenon; and time-delay in the output. Indeed, high values of delay affect the convergence of the observer. It is important to keep in mind that the standard high-gain observer is a particular solution to the proposed methodology. It corresponds exactly to the case* $j_s = 0$. *This means that if for a given value of the delay* τ, *the standard high-gain observer converges, then also the proposed HG/LMI based observer converges with the same convergence rate. The proposed observer offers the possibility to adjust the values of* j_s *and* θ *to have a good tradeoff between ensuring analytically the exponential convergence, with adjustable convergence rate, and allowing large values of the delay, while standard high-gain observer does not have this possibility. A large value of the delay could make the standard high-gain observer diverge, while the proposed observer can converge by increasing the index* j_s *to adjust the value of* θ. *This analysis comes from the interpretation of the design conditions given in Theorem 4 and Theorem 5, and supported by the numerical results provided in Section 3.5. Furthermore, by adjusting* θ *and* j_s, *the proposed observer is able to avoid the peaking phenomenon, to reduce the sensitivity to high-frequency measurement noise, and to enhance the convergence rate if necessary.*

3.4.3 Application to sampled-data case

Theorem 5 can be applied straightforwardly to the case of systems with sampled output measurements. The output is sampled at instants t_k satisfying

$$0 \leqslant t_0 < \ldots < t_k < t_{k+1} < \ldots$$

with $\lim\limits_{t \to +\infty} t_k = +\infty$. In this case, the sampling period $\tau_k = t - t_k$ is positive with $\tau_k \leqslant \tau_M, \forall k \geqslant 0$. To apply the results of the previous sections, we write the sampled-output as a delayed-output, where the delay satisfies all the required conditions. Indeed, the output $y(t_k)$ can be written as

$$y(t_k) = y\big(t - \tau(t)\big),$$

with $\tau(t) = t - t_k$. For all $t \in [t_k \quad t_{k+1}]$, we have

$$0 < \tau(t) \leq \tau_M.$$

Hence, Theorem 5 can be applied to build an observer for system (3.3) based on the sampled-measurements $y(t_k)$. This application is summarized in the following corollary.

Corollary 1 *Let us consider the following observer (3.39) corresponding to system (3.3):*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + L(y(t_k) - C\hat{x}(t_k)),\ t \in [t_k, t_{k+1}[, \tag{3.39}$$

where L is given by (2.18). Assume there exist a positive definite matrix P and a matrix $\mathcal{Y}$ of appropriate dimensions and real constants $\mu_i > 0, i = 1, \ldots, 3$, $\lambda > 0$, and $\tau_M > 0$ such that the conditions (3.31)-(3.35) of Theorem 5 hold. Then the observer (3.39) converges exponentially.

3.4.4 Extension to systems with nonlinear output

This section provides an extension of the result to nonlinear output case. Hence we consider system (3.1) with output measurement

$$y(t) = h(x_1(t - \tau(t))), \tag{3.40}$$

where $h : \mathbb{R} \mapsto \mathbb{R}$ is a strictly monotonic nonlinear function. That is we assume that there exists $0 < \delta \leq 1$ such that

$$\frac{\partial h}{\partial z}(z) \geq \delta,\ \ \forall z \in \mathbb{R}. \tag{3.41}$$

Notice that without condition (3.41), we lose local observability of the system. Assume also that h is γ_h–Lipschitz. Without loss of generality, we assume that

$$\gamma_h \triangleq \max_{z \in \mathbb{R}} \left(\left| \frac{\partial h}{\partial z}(z) \right| \right) = 1. \tag{3.42}$$

Indeed, if (3.42) is not satisfied, then instead of $y(t)$, we can use as measurement the new output

$$y_h(t) \triangleq \frac{y(t)}{\max\limits_{z \in \mathbb{R}} \left(\left| \frac{\partial h}{\partial z}(z) \right| \right)} \triangleq \bar{h}(x_1(t - \tau(t))), \tag{3.43}$$

with

$$\bar{h}(x_1(t-\tau(t))) \triangleq \frac{h(x_1(t-\tau(t)))}{\max\limits_{z\in\mathbb{R}}\left(\left|\dfrac{\partial h}{\partial z}(z)\right|\right)},$$

satisfying

$$\max_{z\in\mathbb{R}}\left(\left|\frac{\partial \bar{h}}{\partial z}(z)\right|\right) = 1.$$

Now consider the observer corresponding to (3.1) and (3.40) as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + L\left[y(t) - h(\hat{x}_1(t-\tau(t)))\right], \tag{3.44}$$

where $\hat{x}$ represents the state estimation and L is the observer gain.

From the differential mean value theorem, there exists $z(t) \in Co\big(x_1(t-\tau(t)), \hat{x}_1(t-\tau(t))\big)$ such that

$$h(x_1(t-\tau(t))) - h(\hat{x}_1(t-\tau(t))) = \theta\frac{\partial h}{\partial z}(z)C\bar{e}(t-\tau(t)),$$

where $Co\big(x_1(t-\tau(t)), \hat{x}_1(t-\tau(t))\big)$ is the convex hull defined by $x_1(t-\tau(t))$ and $\hat{x}_1(t-\tau(t))$. Then, the dynamics of the transformed error, $\bar{e}(t)$, is given by

$$\dot{\bar{e}}(t) = \theta\Big(A - \frac{\partial h}{\partial z}(z)KC\Big)\bar{e}(t) + T^{-1}(\theta)\Delta f + \theta\frac{\partial h}{\partial z}(z)KC\Big(\bar{e}(t) - \bar{e}(t-\tau(t))\Big), \tag{3.45}$$

By using the HG/LMI technique, the error dynamics (3.45) can be written under the form:

$$\dot{\bar{e}}(t) = \theta(\mathcal{A}(\Psi^\theta) - \frac{\partial h}{\partial z}(z)KC)\bar{e}(t) + T^{-1}(\theta)\Delta f_1 + \theta\frac{\partial h}{\partial z}(z)KC\Big(\bar{e}(t) - \bar{e}(t-\tau(t))\Big). \tag{3.46}$$

Following the same steps than the previous section, will lead to the following theorem.

Theorem 6 *Assume there exist a positive definite matrix P, a matrix $\mathcal{Y}$ of appropriate dimension and real constants $\mu_i > 0, i = 1,\ldots,3$, $\lambda > 0$, $\tau_M > 0$ such that the following conditions hold*

a) *The following LMI conditions (3.47)-(3.48) are satisfied:*

$$\begin{bmatrix} \mathcal{H}e\Big\{P\mathcal{A}(\Psi) - \mathcal{Y}^\top C\Big\} + \tau_M\mathcal{R}^\top\mathcal{R} + \lambda I & \mathcal{Y}^\top \\ \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \leqslant 0, \tag{3.47}$$

$$\begin{bmatrix} \mathcal{H}e\Big\{P\mathcal{A}(\Psi) - \delta\mathcal{Y}^\top C\Big\} + \tau_M\mathcal{R}^\top\mathcal{R} + \lambda I & \mathcal{Y}^\top \\ \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \leqslant 0, \tag{3.48}$$

b) θ satisfies (3.32), subject to (3.47)-(3.48);

c) τ_M satisfies, (3.33)-(3.35), subject to (3.47)-(3.48);

d) The observer gain matrix K is given by $K = P^{-1}\mathcal{Y}^\top$.

Then the observer (3.44) converges exponentially.

Proof *The proof follows exactly the same steps of the previous section. The proof is based on the same Lyapunov–Krasovskii functional as defined in (3.26).*

From Schur lemma, the inequality

$$\mathcal{H}e\Big\{P\mathcal{A}(\Psi^\sigma) - \frac{\partial h}{\partial z}(z)\mathcal{Y}^\top C\Big\} + \frac{1}{\mu_1}\mathcal{Y}^\top\mathcal{Y} + \tau_M\mathcal{R}^\top\mathcal{R} \leqslant -\lambda I, \quad \forall \Psi^\sigma \in \mathcal{H}^\sigma_{j_s}.$$

is satisfied if the following one hold:

$$\begin{bmatrix} \mathcal{H}e\Big\{P\mathcal{A}(\Psi^\sigma) - \frac{\partial h}{\partial z}(z)\mathcal{Y}^\top C\Big\} + \tau_M\mathcal{R}^\top\mathcal{R} + \lambda I & \mathcal{Y}^\top \\ \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \leqslant 0. \tag{3.49}$$

On the other hand, assuming (3.42) is important and allows using the same developments established in Theorem 5 with slight modifications. Indeed, due to (3.42), the term $\frac{\partial h}{\partial z}(z)$ appears only in the matrix block $\mathcal{H}e\Big\{P\mathcal{A}(\Psi) - \frac{\partial h}{\partial z}(z)\mathcal{Y}^\top C\Big\}$. Since this matrix is convex in $\frac{\partial h}{\partial z}(z)$, then from the convexity principle, it is sufficient to solve the LMIs with $\max_{z\in\mathbb{R}}\left(\frac{\partial h}{\partial z}(z)\right) = 1$ and $\min_{z\in\mathbb{R}}\left(\frac{\partial h}{\partial z}(z)\right) = \delta$. Hence, using the fact that

$$\delta \leq \frac{\partial h}{\partial z}(z) \leq 1, \ \ \forall z \in \mathbb{R}, \tag{3.50}$$

and $\Psi^\sigma \in \mathcal{H}^\sigma_{j_s}$, we deduce that (3.49) is satisfied if the following two of LMIs hold:

$$\begin{bmatrix} \mathcal{H}e\Big\{P\mathcal{A}(\Psi^\sigma) - \mathcal{Y}^\top C\Big\} + \tau_M\mathcal{R}^\top\mathcal{R} + \lambda I & \mathcal{Y}^\top \\ \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \leqslant 0,$$

$$\begin{bmatrix} \mathcal{H}e\Big\{P\mathcal{A}(\Psi^{\sigma}) - \delta\mathcal{Y}^{\top}C\Big\} + \tau_M\mathcal{R}^{\top}\mathcal{R} + \lambda I & \mathcal{Y}^{\top} \\ \\ \mathcal{Y} & -\mu_1 \end{bmatrix} \leqslant 0.$$

By following the steps of the proof of Theorem (5), we obtain $\dfrac{\mathrm{d}}{\mathrm{d}t}V(t) < 0, \forall x(t) \neq 0$. *Then the conditions on* θ *and* τ_M *are derived similarly as in the previous theorems. This ends the proof.*

Remark 6 *The extension to systems with nonlinear outputs is particularly a new contribution and a non-straightforward extension. The generalization is based on the use of the differential mean value theorem and some judicious mathematical arrangements. The extension leads to more general design conditions, which can be reduced easily to those of the linear case. This extension is simple to implement, and therefore it is appropriate for applications to real-world models.*

Remark 7 *The linear case can be deduced straightforwardly from the nonlinear output case by taking* $\delta = 1$. *Indeed,* $h(.)$ *is linear if and only if* $\dfrac{\partial h}{\partial z}(z) \equiv \text{Constant}$. $\dfrac{\partial h}{\partial z}(z)$ *is identically constant if and only if*

$$\max_{z\in\mathbb{R}}\left(\frac{\partial h}{\partial z}(z)\right) = \min_{z\in\mathbb{R}}\left(\frac{\partial h}{\partial z}(z)\right).$$

From (3.42), $\dfrac{\partial h}{\partial z}(z)$ *is constant if* $\delta = 1$. *In this case, LMIs* (3.47) *and* (3.48) *are identical, and then reduced to* (3.47) *only, which corresponds to the linear case.*

3.4.5 Numerical design procedure

This section is devoted to a numerical observer design procedure. Due to the presence of several decision variables as observer parameters, in the previous theorems, a well-structured numerical design procedure will help the users to implement the proposed methodology. The proposed design procedure is based on the use of the gridding method. We introduce a bijective change of variables $\varrho = \dfrac{\sigma}{1-\sigma}$, $(\sigma = \dfrac{\varrho}{1-\varrho})$ where the new variable $\varrho \in \left]\dfrac{1}{2}\, 1\right]$. The proposed procedure allows obtaining a lower design parameter, θ, and a larger upper bound on the delay, τ_M, provided by Theorem 5 (which is applicable also on Corollary 1). To solve

LMI (3.31), we use Matlab LMI Toolbox and YALMIP. Furthermore, LMI (3.31) are always feasible (Zemouche et al., 2019), however they have an infinite number of solutions and depend on σ (or equivalently, on ϱ). Then the gridding method will return the solution giving a lower value of θ, and a larger bound on τ_M. On the other hand, it is worth noting that LMI (3.31) depends on τ_M, which is computed by (3.33)-(3.35) after solving the (3.31). Then, to solve the LMI (3.31) independently from τ_M, we need to introduce a new variable $\bar{\tau} > 0$, and solve (3.31) with $\bar{\tau} > 0$ instead of τ_M, *i.e.*:

$$\begin{bmatrix} \mathcal{H}e\Big\{P\mathcal{A}(\Psi) - \mathcal{Y}^\top C\Big\} + \bar{\tau}\mathcal{R}^\top\mathcal{R} + \lambda I & \mathcal{Y}^\top \\ \\ \mathcal{Y} & -\mu_1 \end{bmatrix} < 0, \tag{3.51}$$

Hence, by chosing

$$\tau_M = \min(\bar{\tau}, \tau_1, \tau_2). \tag{3.52}$$

we guarantee exponential convergence of the observer because LMI (3.31) still feasible for any $\tau_M \leq \bar{\tau}$. Furthermore, inequality (3.51) can return a $\bar{\tau}$ value very close to zero, which is not suitable because the objective is to get a value of $\bar{\tau}$ large enough to force τ_M to take either the value of τ_1 or that of τ_2. To this end, a solution consists in fixing $\bar{\tau}$ in (3.51). Indeed, we cannot maximise $\bar{\tau}$ because it is obvious to show that if (3.51) is feasible for $\bar{\tau} = 1$, then it still remains feasible for any $\bar{\tau} > 0$. A change of variable to eliminate $\bar{\tau}$ from (3.51) cannot be performed since μ_1 is used to calculate τ_1 and τ_2. The procedure is summarized in Algorithm 1, which will be implemented in Section 3.5 to show the performances of the proposed methodology, compared to those of the standard high-gain observer.

Remark 8 *There are several methods applicable for the same class of systems studied in this chapter that avoid bounds on the delay by either using chain of observers (Germani et al., 2002), (Cacace et al., 2014) or by using predictors (Ahmed-Ali et al., 2013b), (Khosravian et al., 2015). These papers proposed effective methods based on elegant mathematical arguments overcoming the problem of presence of arbitrarily long delay in the output. What we propose in this work can be viewed as an alternative method, which improves existing results in*

Algorithm 1: A numerical design procedure

Step 1. Choose a small $\epsilon > 0$ for the gridding, take $\varrho = \dfrac{1}{2}$, appropriate values for $\lambda > 0, \mu_1 > 0$, and a sufficiently high value $v_{gain} > 0$ and go to **Step 2**;

Step 2. While $\varrho + \epsilon < 1$, take $\varrho := \varrho + \epsilon$ and go to **Step 3**;

Step 3. Solve LMI (3.51) with respect to $\lambda, P > 0, \varrho > 0$, for a given $\bar{\tau} > 0$.

Step 4. Take $\sigma = \dfrac{\varrho}{(1-\varrho)}, K = P^{-1}\mathcal{Y}^\top$ and compute

- $\theta = \max\left(\sigma, \sqrt[1+j_s]{\dfrac{2k_{j_s}\lambda_{\max}(P)}{\lambda}}\right)$;
- $L = T(\theta)K$;

if $v_{gain} > \|L\|$ **then**
  put $v_{gain} := \|L\|$ and go to **Step 2**

else
  return

- $\tau_1 = \sqrt{\dfrac{\left[\theta\lambda - \dfrac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right]}{\lambda_{\max}(P)\varpi\big(\mu_1, \mu_2, \mu_3\big)}}$;
- $\tau_2 = \dfrac{1}{\mu_1\Big(1 + \dfrac{1}{\mu_2}\Big)\theta^2 + \dfrac{\left[\theta\lambda - \dfrac{2k_{j_s}\lambda_{\max}(P)}{\theta^{j_s}}\right]}{\lambda_{\max}(P)}}$;
- $\tau_M = \min(\bar{\tau}, \tau_1, \tau_2)$.

the literature. Anywhere the standard high-gain observer is used for systems with delayed outputs, the proposed methodology can be applied to improve the results while ensuring exponential convergence for large values of the delay. The choice of Lyapunov based-functional can lead to delay-independent stability conditions. For instance, the chain of observers (Germani et al., 2002), (Cacace et al., 2014) and observer-predictor in (Ahmed-Ali et al., 2013b), (Khosravian et al., 2015) can be effective for the compensation of the delay. However, the determination of an implementable form for the observer-predictor feedback gains over the past time interval can be challenging. Furthermore, the construction of Lyapunov-Krasovskii functional for ex-

ponential stability analysis of the observer error dynamics under the observer-predictor scheme is difficult to carry due to the output delayed measurement state. It is worth mentioning that the practical implementation of the high-gain predictor-observer requires the future values of the measurement state, which can significantly increase the computation. On the other hand, the proposed observer in this work is simple to implement on real-world models without any computational complexity. The observer design parameters are also easy to compute.

Remark 9 *The methodology established in this work may open the door to further contributions and new ideas to solve other control problems, namely output feedback stabilization; reference trajectory tracking; self-synchronization in networks of multi-agent systems. More importantly, the proposed methodology can be used as a design tool in several alternative approaches, like those using a chain of observers (Germani et al., 2002), (Ahmed-Ali et al., 2009), (Cacace et al., 2014), or those using prediction part (Ahmed-Ali et al., 2013b), (Khosravian et al., 2015).*

3.5 Numerical comparisons

To show the effectiveness of the proposed methodology, we present in this section two numerical examples. We will provide some comparisons between the standard high-gain observer, the high-gain observer method presented in (Van Assche et al., 2011) and the proposed HG/LMI based observer. The simulations will be carried out by using MATLAB.

3.5.1 Example 1

The aim of this example is to compare the proposed approach to the standard high-gain design and the high-gain observer proposed in (Van Assche et al., 2011). We consider the following fifth order nonlinear system with only a single nonlinearity in the last component including a delay in the output:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x(t)), \\ y(t) = Cx(t - \tau(t)), \end{cases} \tag{3.53}$$

where the matrices A and C are defined as in (2.15), and B is as follows

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

The nonlinearity is defined by

$$f(x) = \frac{k_f}{5}\sum_{i=1}^{5}\sin(x_i). \tag{3.54}$$

In the sequel, we will provide comparisons between the standard high-gain observer, the observer by (Van Assche et al., 2011) and the HG/LMI based design.

The comparison results are shown in Table 3.1, which illustrates how the values of the tuning parameter are decreased and the maximum bounds on the delay become larger. For $j_s = 2$, the value of the tuning parameter θ is significantly reduced to $\theta = 1.8571$, compared to $\theta = 3407$ obtained by (Van Assche et al., 2011), for $k_f = 0.1$. It is also reduced from $\theta = 170350$ with (Van Assche et al., 2011) to $\theta = 10.7647$ for $k_f = 5$. Meanwhile, in comparison to the standard high-gain observer, the design parameter θ is decreased from $\theta = 6.3638$ to $\theta = 1.8571$ for $k_f = 0.1$ and from $\theta = 186.6738$ to $\theta = 10.7647$ for $k_f = 5$. On the other hand, the maximum bound on the delay given by the observer in (Van Assche et al., 2011) and the standard high-gain observer considerably increased from $\tau_M = 3.3760\times10^{-11}s$ and $\tau_M = 2.3661\times10^{-6}s$, respectively, to $\tau_M = 1.6505\times10^{-4}s$ for $k_f = 0.1$ with HG/LMI approach. The previous values are decreased more by increasing the value of the compromise index j_s. We notice that the HG/LMI observer gain is considerably reduced compared to the standard high gain observer and the observer in (Van Assche et al., 2011). For instance, for $k_f = 0.1$, the norms of the gains obtained by the standard high-gain observer and the observer in (Van Assche et al., 2011) are significantly decreased from 15875 and 4.5906×10^{17}, respectively, to 89.7547 with the HG/LMI approach for $j_s = 2$.

Table 3.2 provides percentage of reduction/increment of the design parameter and the maximum bound of the delay, according to the following formulas:

$$\Delta_{\theta,i} = \frac{\theta_i - \theta_{HG/LMI}}{\theta_i}\%, \tag{3.55}$$

$$\Delta_{\tau,i} = \frac{\tau_{M,HG/LMI} - \tau_{M,i}}{\tau_{M,HG/LMI}}\%, \tag{3.56}$$

Table 3.1: Comparison between the proposed HG/LMI observer (HG/LMI), the standard high-gain observer (SHG), and the high-gain observer (VVA) proposed in (Van Assche et al., 2011) for different values of k_f

Methods	k_f	j_s	σ	θ	τ_M	$\|L\|$
VVA	0.1	0	/	3407	3.3760×10^{-11}	4.5906×10^{17}
SHG		0	/	6.3638	2.3661×10^{-06}	15875
HG/LMI		1	2.5088	2.5088	8.2499×10^{-05}	269.5916
		2	1.8571	1.8571	1.6505×10^{-04}	89.7547
VVA	1	0	/	34070	3.0255×10^{-12}	4.5906×10^{22}
SHG		0	/	42.3560	1.5305×10^{-07}	5.2321×10^{08}
HG/LMI		1	7	7	2.3810×10^{-05}	84446
		2	4.1282	4.1282	3.6851×10^{-05}	9330.5
VVA	5	0	/	170350	3.9542×10^{-13}	1.4346×10^{26}
SHG		0	/	186.6738	9.0240×10^{-09}	1.8313×10^{12}
HG/LMI		1	17.1818	17.1818	3.8231×10^{-06}	1.9828×10^{07}
		2	10.7647	10.7647	4.1180×10^{-06}	3.3268×10^{06}

where $\theta_{M,i}$, τ_i, $\theta_{HG/LMI}$, and $\tau_{M,HG/LMI}$ stand for the design parameter and the maximum bound of the delay of the observer in (Van Assche et al., 2011), the standard high-gain observer, and the HG/LMI observer, respectively. The index i refers to VVA or SHG. We notice that the percentage of reduction of the design parameter is up to 90.2536% for $j_s = 2$ and $k_f = 1$ in comparison to the initial value given by the standard high-gain observer. It is also up to 99.9879% in comparison to the initial value provided by the observer in (Van Assche et al., 2011). Meanwhile, the improvement of the maximum bound of the delay is greater than 99.5846% and 99.999991% for $j_s = 2$ and $k_f = 1$, compared to both the standard high-gain observer and the observer by (Van Assche et al., 2011), respectively.

Simulations have been carried out with $k_f = 1$, and comparisons between the HG/LMI based

Table 3.2: Percentage of reduction/increment for different values of $j_s = 0, 1, 2$

Methods	k_f	j_s	θ	$\Delta_{\theta,SHG}$	$\Delta_{\theta,VVA}$	τ_M	$\Delta_{\tau,SHG}$	$\Delta_{\tau,VVA}$
VVA		0	3407	/	0	3.3760×10^{-11}	/	0
SHG		0	6.3638	0	/	2.3661×10^{-06}	0	/
HG/LMI	0.1	1	2.5088	60.5770	99.9264	8.2499×10^{-05}	97.1319	99.999959
		2	1.8571	70.8178	99.9455	1.6505×10^{-04}	98.5664	99.999979
VVA		0	34070	/	0	3.0255×10^{-12}	/	0
SHG		0	42.3560	0	/	1.5305×10^{-07}	0	/
HG/LMI	1	1	6	85.8344	99.9824	2.3810×10^{-05}	99.3572	99.999987
		2	4.1282	90.2536	99.9879	3.6851×10^{-05}	99.5846	99.999991

observer and the standard high-gain observer are provided. The simulation with the approach proposed in (Van Assche et al., 2011) cannot occur due the the high value of the gain and numerical instabilities.

By using MATLAB, the obtained value of the design parameter of the standard high-gain observer is $\theta = 42.3560$. By choosing the compromise index as $j_s = 2$, the obtained value of the tuning parameter by applying the HG/LMI approach is $\theta = 4.1282$, which is significantly reduced compared to that obtained by the standard high-gain based approach. In addition, the maximum bound on the delay obtained with the standard high-gain based approach is $\tau_M = 1.5310{\times}10^{-7}s$, while with the HG/LMI observer we got a larger value, $\tau_M = 3.6851{\times}10^{-5}s$. Denote by $\hat{x}_{LMI} = [\hat{x}_{1,LMI}, \hat{x}_{2,LMI}, \hat{x}_{3,LMI}, \hat{x}_{4,LMI}, \hat{x}_{5,LMI}]^\top$ and $\hat{x}_{HG} = [\hat{x}_{1,HG}, \hat{x}_{2,HG}, \hat{x}_{3,HG}, \hat{x}_{4,HG}, \hat{x}_{5,HG}]^\top$ the state estimates for the system (3.53) by using the observer design method proposed in the present work and the standard high-gain observer, respectively. Let $\hat{x}_{LMI}(0) = [-1, -1, -1, -1, -1]^\top$ and $\hat{x}_{HG}(0) = [-2, -2, -2, -2, -2]^\top$. The simulation results are depicted in Figures 3.1-3.5, which provide the behaviors of x_i and their estimates $\hat{x}_{i,LMI}$, $\hat{x}_{i,HG}$, $i = 1, \ldots, 5$, respectively. It is quite clear that both estimated states converge to the actual states, however, the proposed HG/LMI based observer considerably reduces the peaking phenomenon.

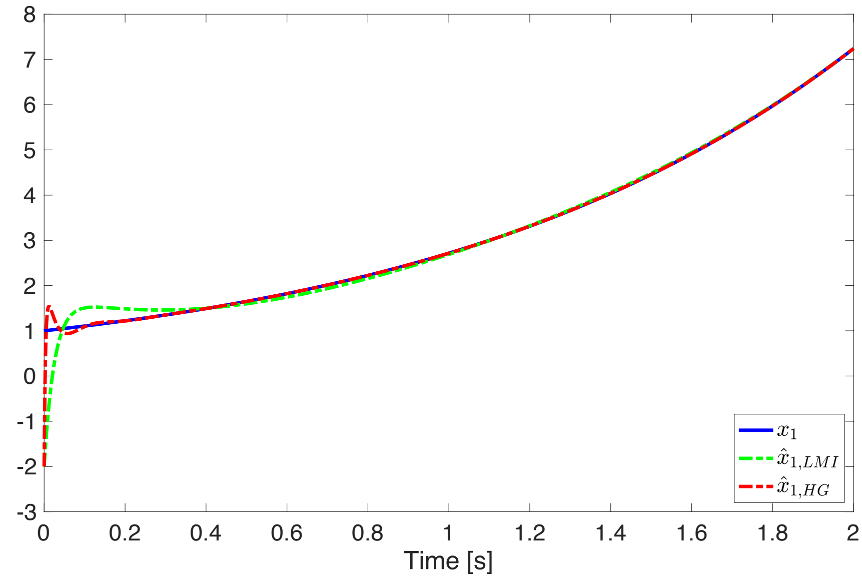

Figure 3.1: Behaviour of x_1 and its estimates for, $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$

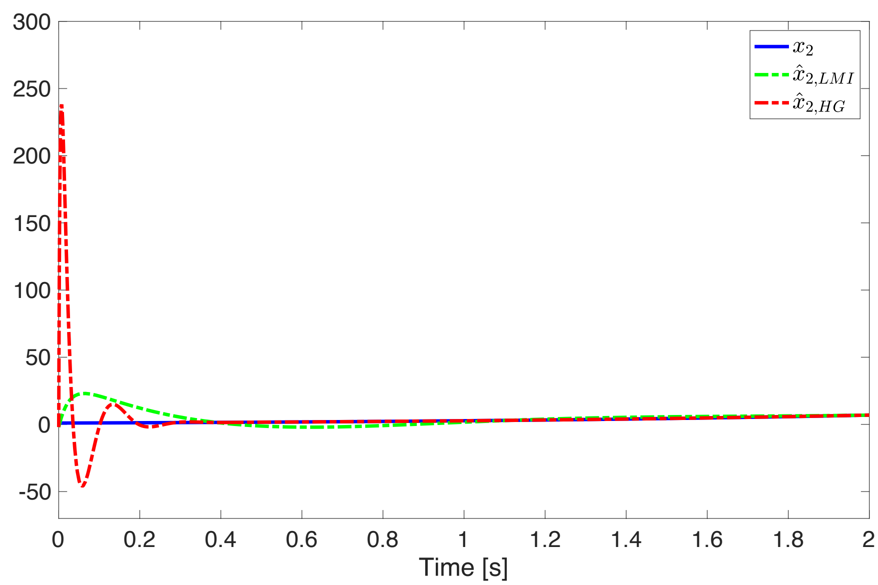

Figure 3.2: Behaviour of x_2 and its estimates for, $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$

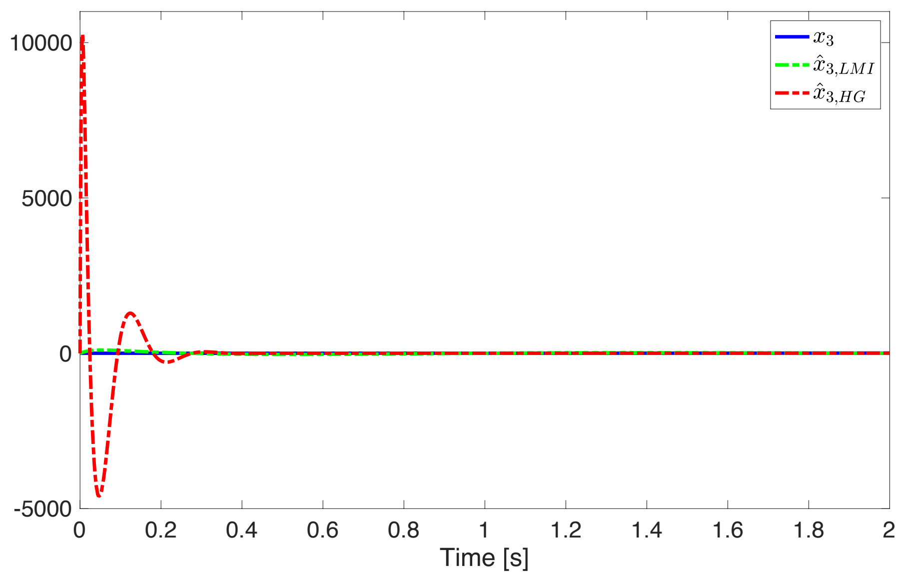

Figure 3.3: Behaviour of x_3 and its estimates for, $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$

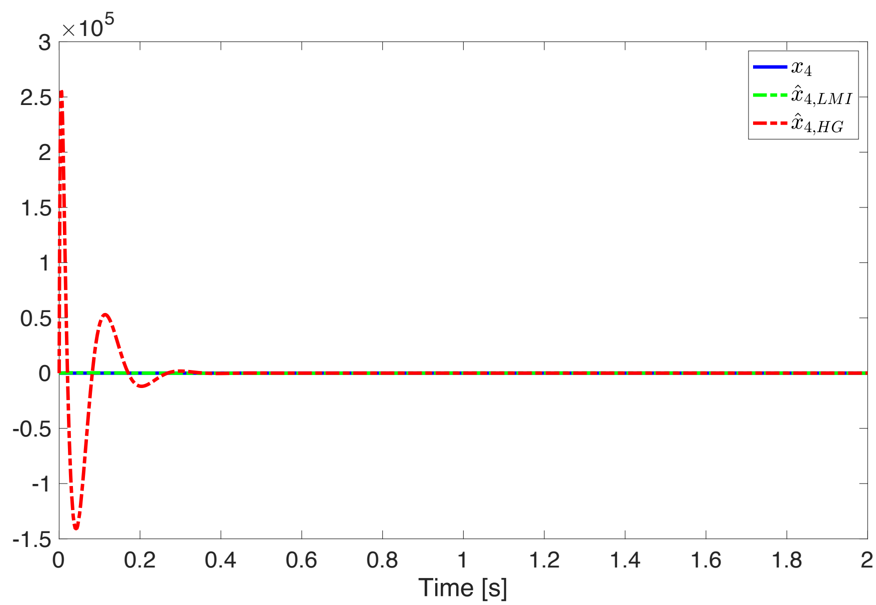

Figure 3.4: Behaviour of x_4 and its estimates for, $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$

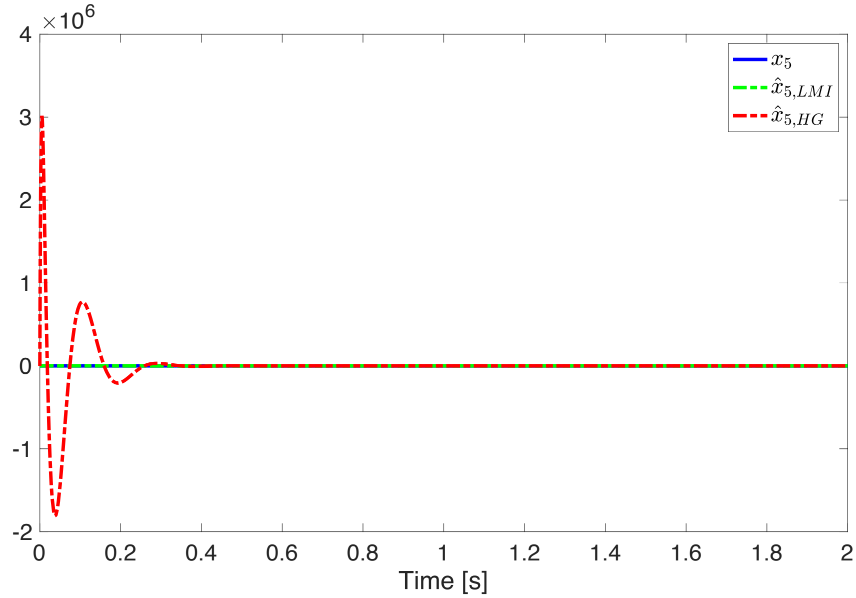

Figure 3.5: Behaviour of x_5 and its estimates for, $k_f = 1$, $\lambda = 1$ and $\mu_1 = 25$

3.5.2 Example 2

To evaluate the performance and superiority of the proposed HG/LMI observer based design method, compared to the observer developed in (Van Assche et al., 2011) for systems with multi-nonlinearities, we consider the example studied in (Van Assche et al., 2011) with a slight modification on the last component to cope with the same class of system investigated in this work. The system is described by the following equations:

$$\begin{cases} \dot{x}_1(t) = x_2(t) - l_1 x_1(t), \\ \dot{x}_2(t) = c_1 c_2 \sin(x_1(t)) + c_1 c_3 \cos(x_2(t)) - c_1 c_4 u(t) \\ y(t) = x_1(t - \tau(t)), \end{cases} \tag{3.57}$$

The values of the parameters are set to $c_1 = 1$, $c_2 = c_3 = 0.02$, $c_4 = 8$, $l_1 = 0.04$. The input function is $u(t) = \sin(0.35t)$. System (3.57) is in the canonical form

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u), \\ y(t) = x_1(t - \tau(t)), \end{cases} \tag{3.58}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f(x, u) = \begin{pmatrix} f_1(x_1(t), u) \\ f_2(x_1(t), x_2(t), u) \end{pmatrix}$$

where

$$\begin{cases} f_1(x_1(t), u) = -l_1 x_1(t), \\ f_2(x_1(t), x_2(t), u) = c_1 c_2 \sin(x_1(t)) + c_1 c_3 \cos(x_2(t)) - c_1 c_4 u(t). \end{cases}$$

The Lipschitz constants of the nonlinearities are $k_{f_1} = k_{f_2} = 0.04$. In (Van Assche et al., 2011), the following high gain observer design was proposed:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}, u) - \theta\Delta^{-1}S^{-1}C^T(C\hat{x}(t-\tau(t)) - y)), \\ y(t) = x_1(t-\tau(t)), \end{cases} \tag{3.59}$$

where

$$\Delta = \mathrm{diag}\Big(1, \frac{1}{\theta}\Big), \theta > 1,$$
$$S = S^T > 0,\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The matrix $S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ is determined by solving the following equation

$$SA + A^T S - C^T C = -S. \tag{3.60}$$

Table 3.3 illustrates the results obtained by the approach in (Van Assche et al., 2011) and the HG/LMI based observer design.

Table 3.3: Percentage of reduction/increment of θ and τ_M, respectively

Methods	θ	τ_M
VVA observer	1.55	0.01
HG/LMI observer	1.0202	0.0202
Relative error	34.18%	50.49%

We can see that the HG/LMI-based observer decreases the value of the tuning parameter θ by more than 34.18% compared to the one obtained by the observer proposed in (Van Assche et al., 2011). In addition, the maximum bound on the delay is increased by 50.49%. It should be mentioned that the reduction/increment of the design parameter θ and the maximum bound on delay τ_M using the HG/LMI observer is obtained with $j_s = 1$.

3.6 Conclusion

In this chapter, we considered the problem of observer design for a class of nonlinear systems with time-varying delayed output measurements. The delay is assumed to be time-varying. The objective was to develop a state observer with a small tuning parameter allowing a maximum bound of the delay as high as possible while ensuring exponential convergence. To this end, we extended the HG/LMI observer design introduced in (Zemouche et al., 2019) for delayed output measurements, which led to a considerably higher allowable maximum bound on the delay compared to the standard high-gain methodology. The convergence analysis is established by using a Lyapunov-Krasovskii functional, depending on the tuning parameter of the observer, jointly with the Halanay inequality. On the other hand, the explicit relation between the tuning parameter of the observer and the maximum bound of the delay shows analytically the superiority of the proposed method with respect to the standard high-gain observer design. Two examples are provided to illustrate the performance of the proposed observer design procedure in comparison to the standard high-gain and to the high-gain observer proposed by (Van Assche et al., 2011). Moreover, extensions to systems with nonlinear outputs and to systems with sampled measurements are established.

In the next chapter, we will introduce a new prescribed-time high-gain observer for a class of nonlinear systems. Such observer guarantees the convergence to the real state in a desired time chosen a priori by the user independently of the initial conditions.

CHAPTER 4

PRESCRIBED-TIME HIGH-GAIN NONLINEAR OBSERVER DESIGN FOR TRIANGULAR SYSTEMS

Contents

4.1 Introduction

In recent years, there has been significant interest into fixed time observer design where the observer converges within a prescribed time. This type of convergence is highly desirable in several applications, such as missile guidance (Holloway, 2018). Indeed, it is characterized by the property to enable achieving convergence within a short amount of time that can be arbitrarily prescribed irrespectively to initial conditions or any other design parameter of the system. Therefore, prescribed-time estimation has received increasing attention from control community during the recent years. For instance, (Holloway and Krstic, 2019) introduces the prescribed observer for class of linear system in canonical form. Motivated by (Song et al., 2017) and after using a change of coordinates of the observer error state and selecting the time-varying observer gains, the proposed observer achieves fixed-time estimation where the convergence time is defined a priori. However, the practical implementation of this observer through finding the high-gain parameter is challenging, and the algorithm is restricted to linear system. A state transformation is introduced in (Chen et al., 2020a) and (Chen et al., 2021) to address the prescribed-time feedback problem for a family of uncertain nonlinear multi-agent systems and lower triangular nonlinear systems, respectively. Using a state transformation through a time-varying scaling function, the prescribed-time feedback stabilization problem is transformed into designing appropriate gain parameters with global asymptotic stability guarantees. Similar ideas were used in (Chitour et al., 2020) to exploit the time-varying homogeneity.

Motivated by the aforementioned methods, in this chapter, we propose a new prescribed high-gain observer for a class of nonlinear systems, that achieves stability in a predefined fixed time. This effort is the first attempt on deriving prescribed-time observer for this class of systems. This is a step beyond the result in Holloway and Krstic (2019) that is limited to linear systems. The observer's gains depend on a time-varying function monotonically increasing to infinity as the time tends to the predefined convergence time. By introducing a state transformation involving a time-scaling function, the estimation error is transformed into a fixed-time stable system. Moreover, the proposed observer reduces the peaking phenomenon, which is one of the main limitations of the high-gain observer design.

4.2 Problem formulation

In this section, we formulate the estimation problem. We consider the class of nonlinear systems described by

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)), \\ y(t) = Cx(t), \end{cases} \tag{4.1}$$

where the matrices A and C are defined as in (2.15), $x(t) \in \mathbb{R}^n$ is the state vector of the system and $y(t) \in \mathbb{R}$ is the measured output. The nonlinearity f is defined as in (3.4).
The functions $f_i : \mathbb{R}^i \longrightarrow \mathbb{R}$, i=1,...,n, satisfy the Lipschitz property formulated under the following form

$$|f_i(x_1, \dots, x_i) - f_i(\bar{x}_1, \dots, \bar{x}_i)| \leqslant \gamma_{f_i} \sum_{j=1}^{i} |x_j - \bar{x}_j|, \tag{4.2}$$

where γ_{f_i} is the Lipschitz constant.
The objective is to estimate the state within a finite time $0 < T \leqslant t_f$ where T is prescribed independently of initial conditions and t_f is the terminal time. Hereafter, we propose a high-gain observer with an appropriate structure to estimate the state of system (4.1) while ensuring the prescribed-time asymptotic stability of the estimation error at predefined time T.

4.3 Prescribed-time high-gain observer design

This section is devoted to the design approach of the prescribed-time observer for nonlinear systems and to the proof of the stability of the estimation error.

4.3.1 Design approach

Motivated by the time-varying scaling function and the standard Lyapunov differential inequality (Holloway and Krstic, 2019), (Chen et al., 2020a) involving the observer design to achieve the prescribed-time stability of the observer error dynamics within a time T, we design the following observer for system (4.1) as follows

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) + \Gamma^{-1}K(y(t) - \hat{x}_1(t)), \tag{4.3}$$

where

$$\Gamma = \text{diag}\{1/\mu^{1+m}, \dots, 1/\mu^{n(1+m)}\},$$

is a scaling matrix, K is the observer gain, and m is a design parameter such that $m \geqslant 1$. The following function is introduced for our estimation design $\mu(t - t_0, T) : [t_0, t_0 + T) \to \mathbb{R}^+$ as (Holloway and Krstic, 2019):

$$\mu(t - t_0, T) := \frac{T}{T + t_0 - t}. \tag{4.4}$$

This function is monotonically increasing having the property that $\mu(t - t_0, T)$ tends to infinity as $t \to t_0 + T$ where T is predefined time.

In this approach, the dynamics of the observer error $e(t) = x(t) - \hat{x}(t)$ is given as follows

$$\dot{e}(t) = Ae(t) + \Delta f - \Gamma^{-1}Ke_1(t), \tag{4.5}$$

where

$$\Delta f := f(x(t)) - f(\hat{x}(t)). \tag{4.6}$$

In order to prove the prescribed-time convergence for system (4.1), the following state transformation is introduced

$$\bar{e}(t) = \Gamma(t)e(t). \tag{4.7}$$

The dynamics of the transformed error is given as follows

$$\dot{\bar{e}}(t) = \mu^{1+m}(t)(A - KC)\bar{e}(t) - (1 + m)\frac{\dot{\mu}(t)}{\mu(t)}D\bar{e}(t) + \Gamma\Delta f, \tag{4.8}$$

where D is a diagonal matrix defined as $D = \text{diag}\{1, 2, \dots, n\}$.

4.3.2 Stability analysis

This part is devoted to the main theorem, which provides synthesis conditions guaranteeing prescribed-time stability of the observer error.

Theorem 7 *Assume that there exist a symmetric positive definite matrix P, a matrix $\mathcal{Y}$ with appropriate dimension and real positive constants m, λ_1 and λ_2 such that the following conditions hold*

$$A^T P + PA - C^T \mathcal{Y} - \mathcal{Y}^T C + \lambda_1 I_n \leqslant 0, \tag{4.9}$$

$$D^T P + PD - \lambda_2 I_n \geqslant 0, \tag{4.10}$$

with

$$K = P^{-1}\mathcal{Y}^\top = [K_1 \dots K_n]^\top, \tag{4.11}$$

$$D = diag\{1, 2, \dots, n\}. \tag{4.12}$$

Then the observer error is asymptotically stable at defined time $T > 0$.

Proof *Consider the following Lyapunov function candidate*

$$V(\bar{e}(t)) = \bar{e}(t)^T P \bar{e}(t), \quad P > 0. \tag{4.13}$$

The derivative of V along the trajectories (4.8) can be calculated as

$$\begin{aligned}
\frac{\mathrm{dV(t)}}{\mathrm{dt}} &= \dot{\bar{e}}^T(t) P \bar{e}(t)(t) + \bar{e}^T(t) P \dot{\bar{e}}(t) \\
&= \mu^{1+m}(t) \bar{e}^T(t) \left[(A - KC)^T P + P(A - KC) \right] \bar{e}(t) \\
&- (1+m) \frac{\dot{\mu}(t)}{\mu(t)} \bar{e}^T(t) \left[D^T P + PD \right] \bar{e}(t) + 2\bar{e}^T(t) P(\Gamma \Delta f).
\end{aligned}$$

Let

$$\begin{aligned}
&(A - KC)^T P + P(A - KC) \leqslant -\lambda_1 I_n, \\
&D^T P + PD \geqslant \lambda_2 I_n.
\end{aligned}$$

Hence the derivative of $V(t)$ becomes

$$\frac{\mathrm{dV(t)}}{\mathrm{dt}} \leqslant -\lambda_1 \mu^{1+m} \bar{e}^T(t)\bar{e}(t) - \lambda_2(1+m)\frac{\dot{\mu}(t)}{\mu(t)}\bar{e}^T(t)\bar{e}(t) + 2\bar{e}^T(t)P(\Gamma\Delta f).$$

Using the fact that f is Lipshitz, we have

$$\begin{aligned} 2\bar{e}^T(t)P(\Gamma\Delta f) &\leqslant 2\bar{e}^T(t)P[\max_i \gamma_{f_i} n\Gamma(x(t) - \hat{x}(t))] \\ &\leqslant 2\gamma_f \bar{e}^T(t)P\Gamma \leqslant 2\gamma_f \lambda_{\max}(P)\bar{e}^T(t)\bar{e}(t), \end{aligned}$$

where $\gamma_f = \max_i \gamma_{f_i} n$, and knowing that

$$\frac{\dot{\mu}(t)}{\mu(t)} = \frac{1}{T}\mu(t). \tag{4.14}$$

We obtain the following inequality

$$\frac{\mathrm{dV(t)}}{\mathrm{dt}} \leqslant -\left[\lambda_1 \mu^{1+m}(t) + \lambda_2(1+m)\frac{\mu(t)}{T} - 2\gamma_f \lambda_{\max}(P)\right]\|\bar{e}(t)\|^2.$$

Similarly to the works (Chen et al., 2021), (Chen et al., 2020b), one can choose a time $t_e > t_0$ such that for all $t > t_e$

$$\lambda_2(1+m)\frac{\mu(t)}{T} \geqslant 2\gamma_f \lambda_{\max}(P), \tag{4.15}$$

which implies

$$\frac{\mathrm{dV(t)}}{\mathrm{dt}} \leqslant \begin{cases} 2\gamma_f \lambda_{\max}(P)\|\bar{e}(t)\|^2, & t \in [t_0, t_e], \\ -\lambda_1 \mu^{1+m}(t)\|\bar{e}(t)\|^2, & t \in (t_e, t_0 + T). \end{cases} \tag{4.16}$$

For $t \in [t_0, t_e]$, one can directly have

$$V(t) \leqslant V(0)\exp\left(\frac{2\gamma_f \lambda_{\max}(P)t}{\lambda_{\min}(P)}\right),$$

which means that the finite time escape phenomenon will not occur for the transformed observer error dynamic (4.8). For $t \in (t_e, T)$, one can get

$$\frac{\mathrm{dV(t)}}{\mathrm{dt}} \leqslant -\lambda_1 \mu^{1+m}(t)\|\bar{e}(t)\|^2, \tag{4.17}$$

and using the fact that

$$\lambda_{\min}(P)\|\bar{e}(t)\|^2 \leqslant V(t) \leqslant \lambda_{\max}(P)\|\bar{e}(t)\|^2, \tag{4.18}$$

we obtain

$$\frac{\mathrm{dV(t)}}{\mathrm{dt}} \leqslant -\frac{\lambda_1 \mu^{1+m}(t)}{\lambda_{\max}(P)} V(t). \tag{4.19}$$

By integrating over the time interval $[t_0, t)$, *we get (Chen et al., 2021)*

$$\int_{t_0}^{t} \frac{\dot{V}(s)}{V(s)} ds \leqslant -\int_{t_0}^{t} \frac{\lambda_1 \mu^{1+m}(t)}{\lambda_{\max}(P)} ds = -\frac{\lambda_1 T}{m\lambda_{\max}(P)} \mu^m(s)\Big|_{t_0}^{t},$$

which implies that

$$V(t) \leqslant V(0) \exp\left(-\frac{\lambda_1 T}{m\lambda_{\max}(P)} (\mu^m(t) - 1)\right), \tag{4.20}$$

and by using (4.18), *we obtain*

$$\|\bar{e}(t)\| \leqslant \sqrt{\frac{V(0)}{\lambda_{\min}(P)}} \exp\left(-\frac{\lambda_1 T}{m\lambda_{\max}(P)} (\mu^m(t) - 1)\right). \tag{4.21}$$

Since $\mu(t)$ *is monotonically increasing to infinity as* $t \to t_0 + T$, *we have that* $\lim_{t \to t_0 + T} \|\bar{e}(t)\| = 0$. *Hence, according to definitions* (A.2.3) *and* (A.2.4), *we get that the dynamics of the transformed error is globally asymptotically stable at defined time* T. *Moreover, through the state transformation* (4.7) *and inequality* (4.21), *we have*

$$\|e(t)\| \leqslant n\nu_1 g(t) \exp\left(-\frac{\lambda_1 T}{2m\lambda_{\max}(P)} (\mu^m(t) - 1)\right), \tag{4.22}$$

where $\nu_1 = \sqrt{\frac{V(0)}{\lambda_{\min}(P)}}$ *and*

$$g(t) = \mu(t)^{n(1+m)} \exp\left(-\frac{\lambda_1 T}{2m\lambda_{\max}(P)} (\mu^m(t) - 1)\right).$$

From the derivative of $g(t)$, *we have (Chen et al., 2021)*

$$\dot{g}(t) = \left(\frac{n(1+m)}{T} - \frac{m\nu_2}{2} \mu^m(t)\right) \mu(t) g(t),$$

where $\nu_2 = \frac{\lambda_1}{m\lambda_{\max}(P)}$. *If* $\frac{2n(1+m)}{(m\nu_2 T)} \geqslant \mu^m(t)$, *let* $\dot{g}(t_1) = 0$, *one can have* $t_1 = T - \left(\frac{m\nu_2 T}{2n(1+m)}\right)^{\frac{1}{m}} T$. *Hence, the upper bound of* $g(t)$ *is* $g(t_1)$, *otherwise it should be* $g(t_0)$. *From* (4.22) *and the upper bound of* $g(t)$, *the observer error is bounded and satisfies* $\lim_{t \to t_0 + T} \|e(t)\| = 0$. *Then, the observer error* e *is also globally asymptotically stable at defined time* T *according to definitions* (A.2.3) *and* (A.2.4).

Remark 10 *Numerical instabilities may be faced during the implementation of the observer. Such problem comes from the gains that grows to infinity as the time t goes to the convergence time T. Hence, several solutions have been proposed in the literature to solve such issues (Holloway and Krstic, 2019; Espitia and Perruquetti, 2020; Song et al., 2017). One can suggest to saturate the observer's gains, setting the prescribed convergence time to a larger value to prevent the observer's gain from growing to infinity, or by stopping the convergence before the prescribed time to obtain the prescribed-time stability to a neighborhood of the origin. Similar thorough discussions can be also found in (Holloway and Krstic, 2019).*

4.4 Illustrative example

To show the performance of the proposed prescribed-time observer design, we present in this section a numerical example. The aim of this example is to show the performance of the prescribed-time observer for different values of the time T and how the design parameter m affects the convergence of the proposed observer. The simulations will be carried out by using MATLAB and YALMIP.

We consider the following second order nonlinear system

$$\begin{cases} \dot{x}_1(t) = x_2(t) - l_1 x_1(t), \\ \dot{x_2}(t) = c_1 c_2 \sin(x_1(t)) + c_1 c_3 \cos(x_2(t)) - c_1 c_4 u(t), \\ y(t) = x_1(t), \end{cases} \tag{4.23}$$

The values of the system parameters are set to $c_1 = 1$, $c_2 = c_3 = 0.02$, $c_4 = 8$, $l_1 = 0.04$. The input function is $u(t) = \sin(0.35t)$. System (4.23) is in the canonical form

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u), \\ y(t) = x_1(t), \end{cases} \tag{4.24}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f(x,u) = \begin{pmatrix} f_1(x_1(t), u) \\ f_2(x_1(t), x_2(t), u) \end{pmatrix},$$

$$\begin{cases} f_1(x_1(t), u = -l_1 x_1, \\ f_2(x_1(t), x_2(t), u) = c_1 c_2 \sin(x_1) + c_1 c_3 \cos(x_2) - c_1 c_4 u(t). \end{cases}$$

The proposed prescribed-time observer can be constructed as follows

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) + f_1(\hat{x}) + \mu^{1+m}(t)K_1(x_1(t) - \hat{x}_1(t)), \\ \dot{\hat{x}}_2(t) = f_2(\hat{x}) + \mu^{2(1+m)}K_2(x_1(t) - \hat{x}_1(t)). \end{cases}$$

where the scaling function μ is defined as in (4.4). The initial time t_0 is set to zero, i.e $t_0 = 0$. By using Matlab and YALMIP, the matrices P and K are found equal to

$$P = \begin{bmatrix} 116.2897 & -33.3778 \\ -33.3778 & 61.2821 \end{bmatrix}, \ K = [1.4401, 2.6820]. \tag{4.25}$$

We set the prescribed-time as $T = 1$ and $T = 2$, and the design parameter $m = 1$. Denote by $\hat{x}(t) = [\hat{x}_1, \hat{x}_2]^\top$ the state estimates for the system (4.23) by using the observer design method proposed in the present chapter. Let $x(0) = [1, 1]^\top$ and $\hat{x}(0) = [2, 2]^\top$. The behaviours of x_i and its estimates $\hat{x}_i$, $i = 1, 2$, are illustrated in Figures 4.1-4.4. We can see that the proposed observer converges to the actual states at the chosen time.

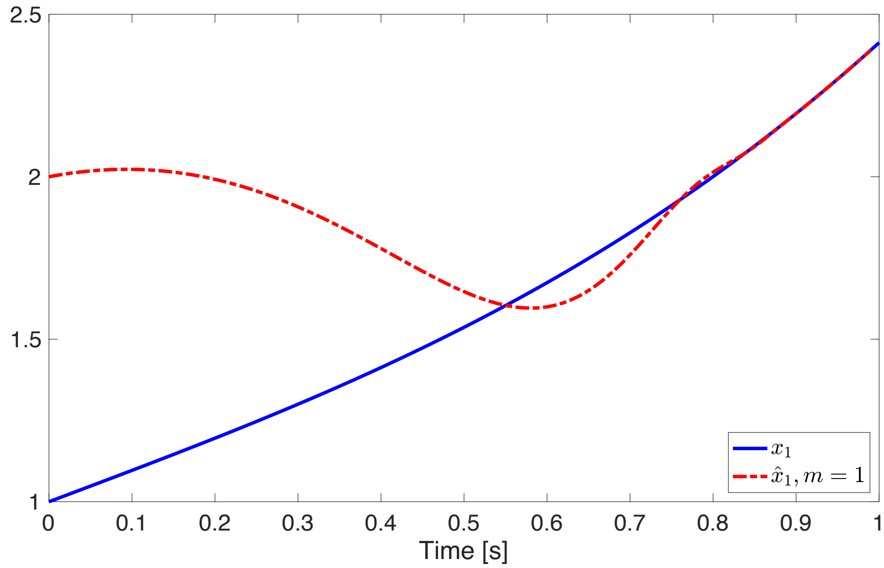

Figure 4.1: Behaviour of x_1 and its estimate for $T = 1s$ and $m = 1$

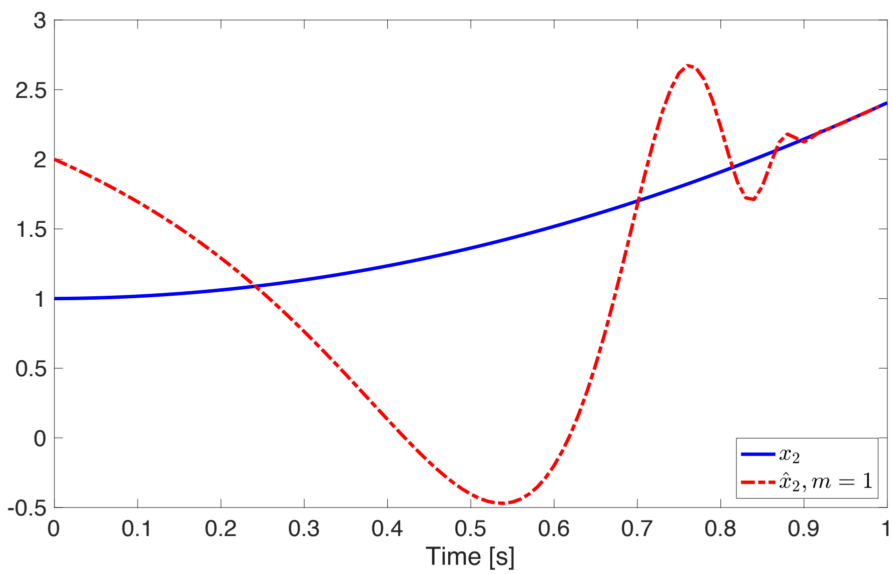

Figure 4.2: Behaviour of x_2 and its estimate for $T = 1s$ and $m = 1$

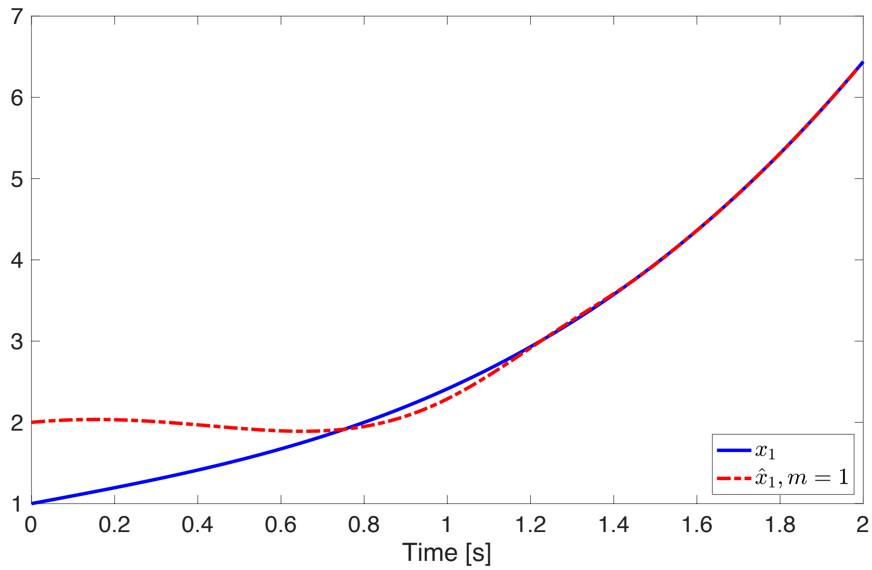

Figure 4.3: Behaviour of x_1 and its estimate for $T = 2s$ and $m = 1$

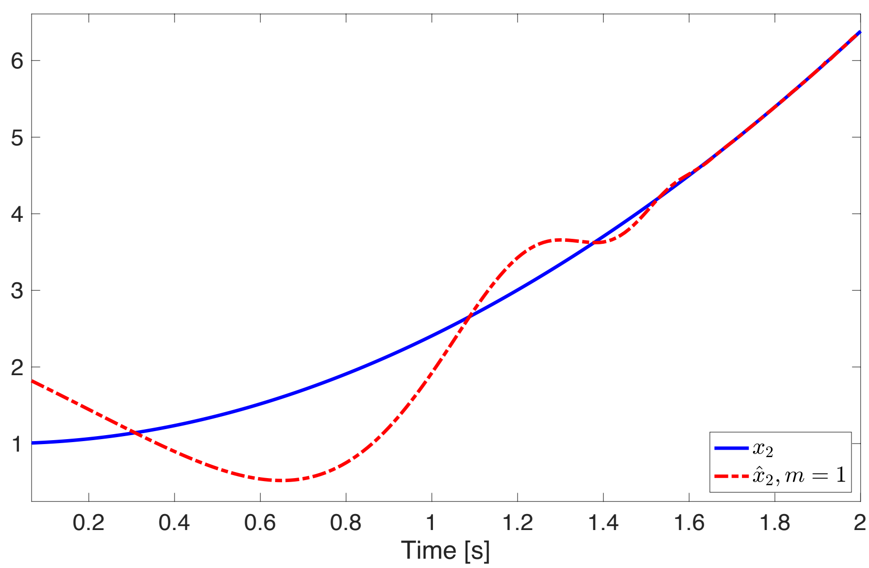

Figure 4.4: Behaviour of x_2 and its estimate for $T = 2s$ and $m = 1$

To analyze the influence of the design parameter m, three values are chosen $m = 1, m = 2$ and $m = 3$. The behaviours of x_i and its estimates $\hat{x}_i$, $i = 1, 2$ are illustrated in Figures 4.5-4.6. Indeed, the parameter m affects the convergence of the observer. By increasing m, the observer converges faster at the prescribed-time.

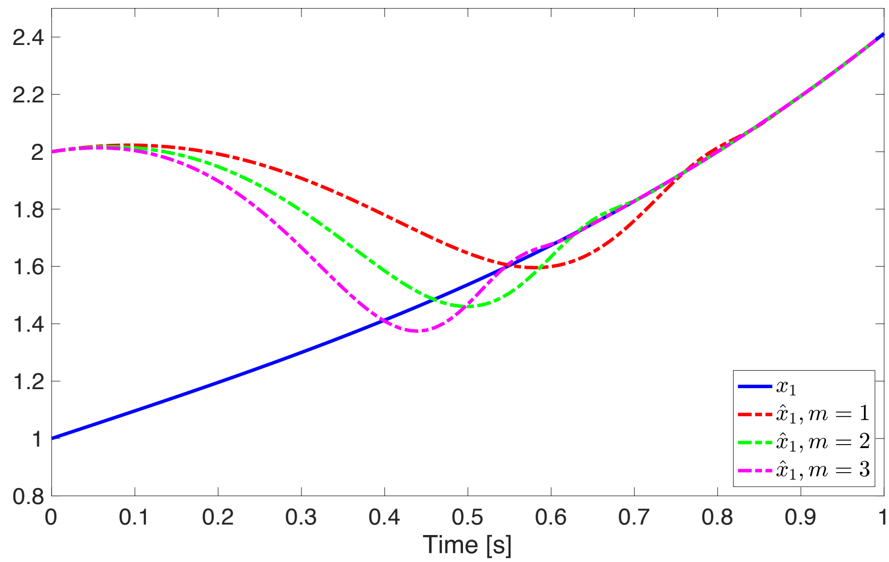

Figure 4.5: Behaviour of x_1 and its estimates for $T = 1s$ and $m = 1, 2, 3$

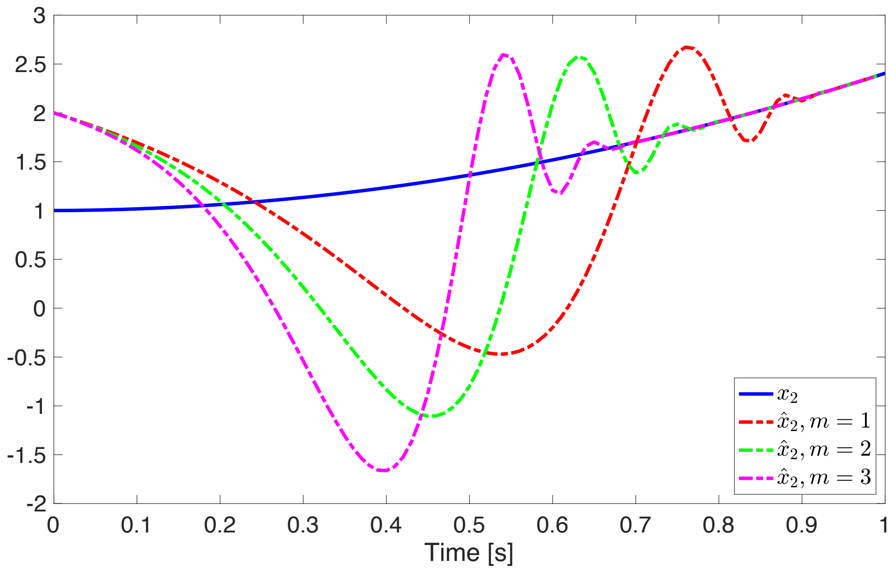

Figure 4.6: Behaviour of x_2 and its estimates for $T = 1s$ and $m = 1, 2, 3$

4.5 Conclusion

In this chapter, a new prescribed-time high-gain observer is designed for a class of nonlinear systems. The objective was to develop a state observer that achieves the prescribed-time stability of the estimation error within a predefined time T chosen a priory. The proposed observer has the structure of a high-gain observer where the gain depends on scaling function. A numerical example is provided to illustrate the performance of the proposed observer design procedure.

In the following chapter, we will apply the proposed observer approaches to estimate the water levels in the Coupled Tanks system. The effectiveness of the HG/LMI observer and the prescribed-time high-gain observer will be performed.

CHAPTER 5

COUPLED TANKS STATE ESTIMATION USING HIGH-GAIN LIKE OBSERVERS

Contents

5.1 Introduction

In the last decades, the Coupled Tanks plant has been widely used in industrial processes such as wastewater treatment, food industry, water desalination, chemical, and petrochemical plants. The control and regulation of the Coupled Tank plants are difficult due to their multi-variable/coupling structure, and nonlinear interactions between accessible and non-measured variables. In practice, the variables of most of these industrial processes are not fully accessible for measurements, and are not entirely known due to various physical and economic constraints on the dynamical model. Therefore, observer-based control algorithms are used in such applications to estimate and control the state, for example, the water level in the tanks (see (Guo et al., 2020), (Gouta et al., 2015a), (Turki et al., 2014), (Meng et al., 2020), (Gouta et al., 2015b), (Gouta et al., 2019) and references therein).

It should be noted that such application is usually selected for testing and validation of observer design approaches because it emerges in several industrial processes, which require several time response constraints to complete a desired action or behavior. This renders the prescribed-time convergence the main feature. Additionally, a large time delay can occur due to the interaction between the tanks, affecting the estimation and control design of the process. Moreover, a small change in the pump voltage might cause a delay during the measurement of the output (Li et al., 2020). Nevertheless, in order to apply advanced concepts of estimation and control to practical applications, the transformation of the dynamical system into an adequate canonical form is required. The presence of such constraints renders this application fits within the aim of the work proposed in this thesis.

In this chapter, we will apply the proposed HG/LMI observer and the prescribed-time high-gain observer for water level estimation in the Coupled Tanks system. We will start by describing the Coupled Tanks system and explain how the mathematical model is derived. Since the two proposed observers are under an observable canonical form, a transformation of the Coupled Tanks system into the desired form is provided using a suitable diffeomorphism. The effectiveness of the proposed observer design approaches in estimating the water levels in the Coupled Tanks system are given, providing a comparison to the standard high-gain observer.

5.2 Coupled Tanks system description

The Coupled-Tank (CT) plant consists of a pump with a water basin and two tanks. The two tanks are mounted on the front plate such that flow from the upper tank can flow, through an outlet orifice located at the bottom of the tank, into the lower tank (Apkarian et al., 2012). Flow from the second tank flows into the main water reservoir. The Coupled Tanks experiment and a schematic of the plant are shown in Figure 5.1.

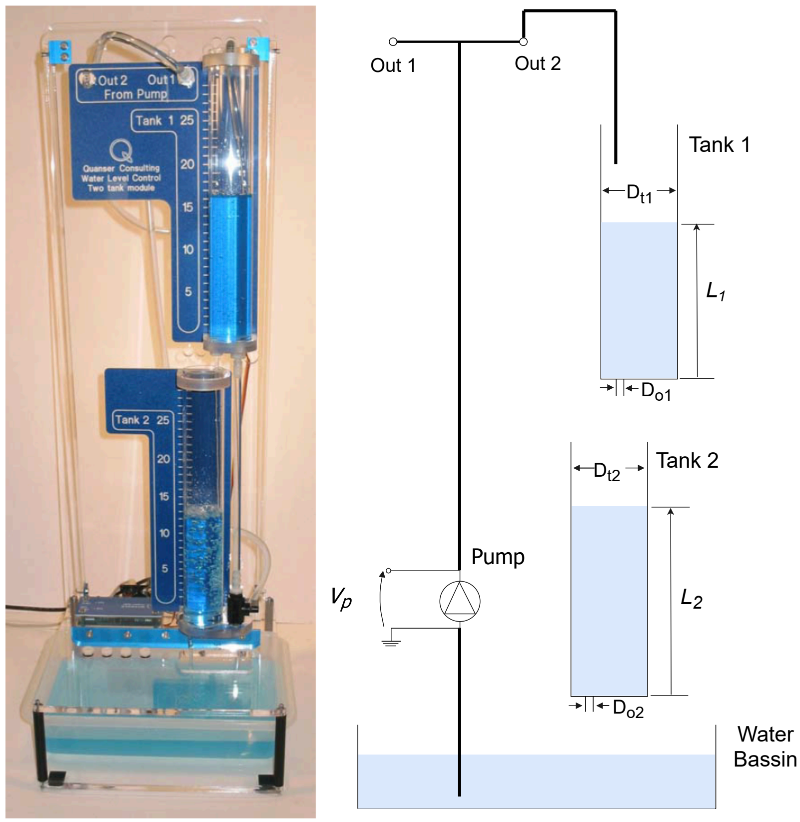

Figure 5.1: Coupled Tanks lab experiment with its schematic

5.3 Mathematical model of Coupled Tanks system

Let us start by deriving the Equation of Motion, EOM, characterizing the dynamics of tank 1. The input to the process is the voltage to the pump and its output is the water level in tank 1. The obtained Equation of Motion is a function of the system's input and output, expressed under the following format:

$$\frac{\mathrm{dL_1}}{\mathrm{dt}} = f(L_1, V_p). \tag{5.1}$$

In deriving the tank 1 EOM the mass balance principle can be applied to the water level in tank 1, i.e.,

$$A_{t1}\frac{\mathrm{dL_1}}{\mathrm{dt}} = F_{i1} - F_{o1}, \tag{5.2}$$

where A_{t1} is the area of tank 1. F_{i1} and F_{o1} are the inflow rate and outflow rate, respectively. The volumetric inflow rate to tank 1 is assumed to be directly proportional to the applied pump voltage, such that:

$$F_{i1} = K_p V_p, \tag{5.3}$$

where K_p is the pump volumetric flow constant and V_p is the pump voltage.

Applying Bernouilli's equation for small orifices, the outflow velocity from tank 1, v_{o1}, can be expressed by the following relationship:

$$v_{o1} = \sqrt{2gL_1}, \tag{5.4}$$

where g denotes the gravitational constant on Earth and L_1 is the height of the water level in the tank.

The outflow rate from tank 1, F_{o1}, can be expressed by:

$$F_{o1} = A_{o1}v_{o1}, \tag{5.5}$$

where A_{o1} is the cross-section area in tank 1 given by the following equation:

$$A_{o1} = \frac{1}{4}\pi D_{o1}^2, \tag{5.6}$$

Substituting in equation (5.2) F_{i1} and F_{o1} with their expressions, the equation of motion for the tank 1 is as follows

$$\frac{\mathrm{dL_1}}{\mathrm{dt}} = -\frac{A_{o1}}{A_{t1}}\sqrt{2gL_1} + \frac{K_p}{A_{t1}}V_p. \tag{5.7}$$

where D is the diameter of the tank.

As for tank 2, the input is the water level, L_1, in tank 1 (generating the outflow feeding tank

2) and its output is the water level, L_2, in tank 2. Hence, the inflow rate is equal to the outflow from tank 1:

$$F_{i2} = F_{i1}, \tag{5.8}$$

and the outflow rate from tank 2 is given by

$$F_{o2} = A_{o2}\sqrt{2gL_2}, \tag{5.9}$$

where A_{o2} is the cross-section area in tank 2 given by the following equation:

$$A_{o2} = \frac{1}{4}\pi D_{o2}^2, \tag{5.10}$$

So the equation of motion in tank 2 is as follows

$$\frac{\mathrm{dL_2}}{\mathrm{dt}} = \frac{A_{o1}}{A_{t2}}\sqrt{2gL_1} - \frac{A_{o2}}{A_{t2}}\sqrt{2gL_2}. \tag{5.11}$$

We obtain the following dynamic equations for Coupled Tanks system:

$$\begin{cases} \dfrac{\mathrm{dL_1}}{\mathrm{dt}} = -\dfrac{A_{o1}}{A_{t1}}\sqrt{2gL_1} + \dfrac{K_p}{A_{t1}}V_p, \\ \dfrac{\mathrm{dL_2}}{\mathrm{dt}} = \dfrac{A_{o1}}{A_{t2}}\sqrt{2gL_1} - \dfrac{A_{o2}}{A_{t2}}\sqrt{2gL_2}, \end{cases} \tag{5.12}$$

where L_1, L_2 are the water level in tank 1 and 2, respectively. V_p is the pump voltage.

For simpler reading of the model (5.13), we define the following notation $z_1 = L_2$ and $z_2 = L_1$. Hence the dynamical model of the Coupled Tanks plant can be rewritten as:

$$\begin{cases} \dfrac{\mathrm{dz_1}}{\mathrm{dt}} = \dfrac{A_{o1}}{A_{t2}}\sqrt{2gz_2} - \dfrac{A_{o2}}{A_{t2}}\sqrt{2gz_1}, \\ \dfrac{\mathrm{dz_2}}{\mathrm{dt}} = -\dfrac{A_{o1}}{A_{t1}}\sqrt{2gz_2} + \dfrac{K_p}{A_{t1}}V_p, \\ y = z_1, \end{cases} \tag{5.13}$$

where the measured output y of system (5.13) is the water level in tank 2. The values of the physical parameter of system (5.13) are given in Table (5.1).

In practice, the following specifications are required to monitoring the water levels in tanks 1 and 2:

Table 5.1: Coupled Tanks model's parameters (Apkarian et al., 2012)

Symbol	Description	Value	Unit
V_p	Pump voltage	12	V
K_p	Pump flow constant	3.3	cm^3/s/V
D_{o1}	Tank 1 outlet diameter	0.635	cm
D_{t1}	Tank 1 inside diameter	4.445	cm
A_{o1}	Tank 1 outlet section area	0.3167	cm^2
A_{t1}	Tank 1 inside cross-section area	15.1579	cm^2
D_{o2}	Tank 2 outlet diameter	0.45625	cm
D_{t2}	Tank 2 inside diameter	4.445	cm
A_{o2}	Tank 2 outlet section area	0.1781	cm^2
A_{t2}	Tank 2 inside cross-section area	15.1579	cm^2
$\bar{L}$	Maximum water levels in tanks $1, 2$	25	cm
g	Gravitational constant on earth	981	cm/s^2

Assumption 1

- *The minimum and maximum water levels in the two tanks should be always greater than* 1 *cm and less than* 25 *cm, respectively.*
- *Additionally, the pump voltage* V_p *is maintained between* 0 *V and* 12 *V.*

The Coupled Tanks system is a widely used nonlinear system for testing control and estimation methods. Among the challenging problems is the water level estimation which is needed for controlling the system. In this chapter, we will apply the HG/LMI observer and the prescribed-time observer to estimate the water level in tank 1. The design of such observers requires the transformation of system (5.13) into an observable canonical form using a suitable diffeomorphism. In the sequel, we will show that the Coupled Tanks system (5.13) is uniformly observable for any input and can be transformed into an observable canonical form.

5.4 Observability of Coupled Tanks system

A nonlinear single-input single-output system with linear output based on the mathematical model developed in the previous section can be written as follows:

$$\begin{cases} \dot{z}(t) = f(z(t)) + g(z(t))u, \\ y(t) = h(z(t)) = z_1, \end{cases} \tag{5.14}$$

where $z(t) \in \mathbb{R}^2$, $f(z)$ and $g(z)$ are 2-dimensional smooth vector fields; u is the control variable, y the output, $h(z)$ is a scalar function of z along with the followings:

$$z = [z_1, z_2]^T, \quad f(z) = \begin{bmatrix} \frac{A_{o1}}{A_{t2}}\sqrt{2gz_2} - \frac{A_{o2}}{A_{t2}}\sqrt{2gz_1} \\ -\frac{A_{o1}}{A_{t1}}\sqrt{2gz_2} \end{bmatrix}, \quad g(z) = \begin{bmatrix} 0 \\ \frac{K_p}{A_{t1}} \end{bmatrix} \tag{5.15}$$

In order to convert (5.14) into the observable canonical form, we choose the following nonlinear map $\Psi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$:

$$z \to x = \Psi(z) = \begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix} = \begin{bmatrix} z_1 \\ \frac{A_{o1}}{A_{t2}}\sqrt{2gz_2} - \frac{A_{o2}}{A_{t2}}\sqrt{2gz_1} \end{bmatrix} \tag{5.16}$$

The Jacobian matrix of $\Psi(z)$ is given as follows

$$J(\Psi(z)) = \begin{bmatrix} 1 & 0 \\ -\frac{A_{o2}g}{A_{t2}}\frac{1}{\sqrt{2gz_1}} & \frac{A_{o1}g}{A_{t2}}\frac{1}{\sqrt{2gz_2}} \end{bmatrix} \tag{5.17}$$

The determinant of the Jacobian matrix of $\Psi(z)$ is non zero, i.e, $\det(J(\Psi(z))) \neq 0$ for $z_1 \neq 0$ and $z_2 \neq 0$. By computing the rank of jacobian matrix $J(\Psi(z))$, we find that $\text{rank}(J(\Psi(z))) = 2$ and it is full row rank. It means that $\Psi(z)$ is invertible and $\Psi(z)$ is a diffeomorphism. The inverse of the change of coordinates $\Psi^{-1}(x)$ is given as follows:

$$z = \Psi^{-1}(x) = \begin{cases} z_1 = x_1, \\ z_2 = \frac{A_{t2}^2}{2gA_{o1}^2}\left[x_2 + \frac{A_{o2}}{A_{t2}}\sqrt{2gx_1}\right]^2. \end{cases} \tag{5.18}$$

Hence, system (5.13) is uniformly observable and can be transformed into an observable canonical form (Gauthier et al., 1992).

We consider the hypotheses needed by the observer based on the map Ψ.

Assumption 2 *The system is Globally Uniformly Lipschitz Drift-Observable (GULDO), i.e. Ψ is a diffeomorphism in the domain of interest Ω and the maps Ψ, Ψ^{-1} are uniformly Lipschitz in Ω.*

In order to use the methods developed in this thesis, it will be necessary to determine a positively invariant compact set Ω within the nonlinearity is globally lipschitz. According to Assumption 1, since the input V_p is bounded, one can select the invariant set as following

$$\Omega = [\varsigma, \bar{L}] \times [\varsigma, \bar{L}]. \tag{5.19}$$

where ς is a positive constant satisfying $1 \leq \varsigma < \bar{L}$. Therefore, $x = \Psi(z)$ is a state transformation, and the observable canonical form of system in the x-coordinates is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \underbrace{L_f^2 h(\Psi^{-1}(x)) + L_g L_f h(\Psi^{-1}(x))u}_{\varphi(z,u)} \end{bmatrix}, \tag{5.20}$$

where

$$\varphi(z,u) = -\frac{A_{o1}^2 g}{A_{t1} A_{t2}} \left(1 + \frac{K_p V_p}{A_{t2} x_2 + A_{o2}\sqrt{2 g z_1}} \right) - \frac{A_{o2} g}{A_{t2}} \left(\frac{z_2}{\sqrt{2 g z_1}} \right).$$

Additionally, the following conditions hold:

- $L_f^2 h(\Psi^{-1}(x))$ is uniformly Lipschitz in $\Psi(\Omega)$, that is there exist a constant γ_1 such that for all $x_1, x_2 \in \Psi(\Omega)$

$$\|L_f^2 h(\Psi(x_1)^{-1}) - L_f^2 h(\Psi(x_2)^{-1})\| \leq \gamma \|x_1 - x_2\| \tag{5.21}$$

- $L_g L_f h(\Psi^{-1}(x))$ is uniformly Lipschitz in $\Psi(\Omega)$, that is there exist a constant γ_2 such that for all $x_1, x_2 \in \Psi(\Omega)$

$$\|L_g L_f h(\Psi(x_1)^{-1}) - L_g L_f h(\Psi(x_2)^{-1})\| \leq \gamma \|x_1 - x_2\| \tag{5.22}$$

System (5.20) can be rewritten under the following form

$$\dot{x}(t) = Ax(t) + \Phi(x(t)), \tag{5.23}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Phi(x(t)) = \begin{bmatrix} 0 \\ \varphi(x_1, x_2) \end{bmatrix},$$

where the state vector of the system $x(t) = [x_1(t) \quad x_2(t)]^T$ belongs to the subset $\Omega \subset \mathbb{R}^2_{>0}$.

The function $\varphi(.)$ is Lipschitz with respect to x in $\Psi(\Omega)$. The Lipschitz constant of $\varphi(.)$ is computed by calculating its partial derivatives in open-loop settings

$$\begin{aligned}
\frac{\partial \varphi(x_1, x_2)}{\partial x_1} &= -\frac{A_{o2} A_{o1}^2 g K_p V_p \sqrt{2gx_1}}{2A_{t1} A_{t2} (A_{t2} x_2 + A_{o2}\sqrt{2gx_1})^2} + \frac{A_{o2} g}{2A_{t2}} \frac{x_2}{\sqrt{2gx_1}}, \\
\frac{\partial \varphi(x_1, x_2)}{\partial x_2} &= -\frac{A_{o1}^2 g K_p V_p}{A_{t1} (A_{t2} x_2 + A_{o2}\sqrt{2gx_1})^2} - \frac{A_{o2}}{A_{t2}\sqrt{2gx_1}}.
\end{aligned}$$

and finding out the supremum of

$$\left\| \left[\frac{\partial \Phi(x_1, x_2)}{\partial x_1}, \frac{\partial \Phi(x_1, x_2)}{\partial x_2} \right] \right\| \tag{5.24}$$

over a sufficiently long time interval.

5.5 Coupled Tanks state estimation

The following sections are devoted to the application of the proposed approaches in Chapters 3 and 4. We will show the effectiveness of the proposed observers on estimating the water level in the tanks with a comparison to the standard high-gain observer.

5.5.1 HG/LMI observer

This section shows the effectiveness of the standard high gain observer and HG/LMI observer design on estimating the Coupled Tanks water levels under a larger value of the time delay. Resulting from the interaction between the tanks, the measured water level from tank 2 is assumed to be affected by a time delay. Note that, in practice, the time delay is widespread in most industrial chemical processes, including water level control. On the other hand, changing the pump voltage might lead to a delay while measuring the output (Li et al., 2020). Therefore, one can write

$$y(t) = x_1(t - \tau(t)). \tag{5.25}$$

Remark 11 *System (5.23) contains only one nonlinearity in the last component. Then, Theorems 4 and 5 with $k_1 = 0$ can be used to estimate the water level in the tanks and get larger values of the delay τ_M.*

The simulations are carried out by using MATLAB and YALMIP. Notice that $j_s = 0$ corresponds to standard high-gain while the HG/LMI observer is defined for $j_s = 1$. Table 5.2 illustrates the comparison results of both high gain observers' designs. Subsequently, the tuning parameter values are decreased, and the delay's maximum bounds become more significant.

Table 5.2: Comparison between the standard high-gain observer (SHG) and the proposed HG/LMI observer (HG/LMI)

Methods	k_f	j_s	σ	θ	τ_M	$\|L\|$
SHG		0	/	2984.9	1.2471×10^{-6}	74099
HG/LMI	0.4943	1	49	49	2.5×10^{-3}	24.6425

The Lipschitz constant $L = 0.4943$ is computed from (3.2). By taking $j_s = 1$ in the HG/LMI design, the value of the tuning parameter θ given by the standard high-gain observer is decreased from $\theta = 2984.9$ to $\theta = 49$. On the other hand, the maximum bound of the delay is considerably increased from $\tau_M = 1.2471{\times}10^{-6}s$ to $\tau_M = 2.5{\times}10^{-3}s$. We also notice that the HG/LMI observer gain is considerably reduced compared to the standard high gain. For instance, the norm of the standard high-gain observer's gains is reduced from 74099 to 24.6425. This is due to the compromise index j_s which reduces the values of the gain K and then the gain L becomes smaller.

The estimates of water levels in the tanks are plotted using the HG/LMI observer and the standard high-gain observer. Denote by $\hat{x}_{HGLMI} = [\hat{x}_{1,HGLMI}, \hat{x}_{2,HGLMI}]^T$ and $\hat{x}_{SHG} = [\hat{x}_{1,SHG}, \hat{x}_{2,SHG}]^T$ the state estimates for the system (5.23) by using the observer design method proposed in Chapter 3 and the standard high-gain observer, respectively. The initial water level in tanks 1 and 2 is $x(0) = [4,4]^T$. Let $\hat{x}_{HGLMI}(0) = [3,3]^T$ and $\hat{x}_{SHG}(0) = [3,3]^T$. The curves of the states x_i and their estimates $\hat{x}_i$ and $\hat{x}_{i,HG}$, $i = 1,2$ are illustrated in Figure 5.2.

From the plots of the estimated states, we observe both observer states converge to the actual states. Furthermore, the HG/LMI observer considerably reduces the peaking phenomenon. Nevertheless, the maximum value of the water level $x_{2,HG}$ in tank 1 given by the standard high gain observer exceeds the maximum level of water in tank 1, which is equal to $\bar{L} = 25$ cm and may damage the setup.

To analyze the effect of the maximum delay in both standard high gain observer and HG/LMI observer, we set $\tau_M = 2.5\times10^{-3}$ s, which corresponds to the maximum delay of the HG/LMI observer. This value is greater than the value found by the standard high gain observer. The curves of the states x_i and their estimates $\hat{x}_i$ and $\hat{x}_{i,HG}$, $i = 1,2$ are illustrated in Figure 5.3. The HG/LMI observer converges to the real states meanwhile the standard high-gain observer fails to reconstruct the water level in the tanks due to the large value of the time-delay. This shows the superiority of HG/LMI observer on estimating the water level in tank 1.

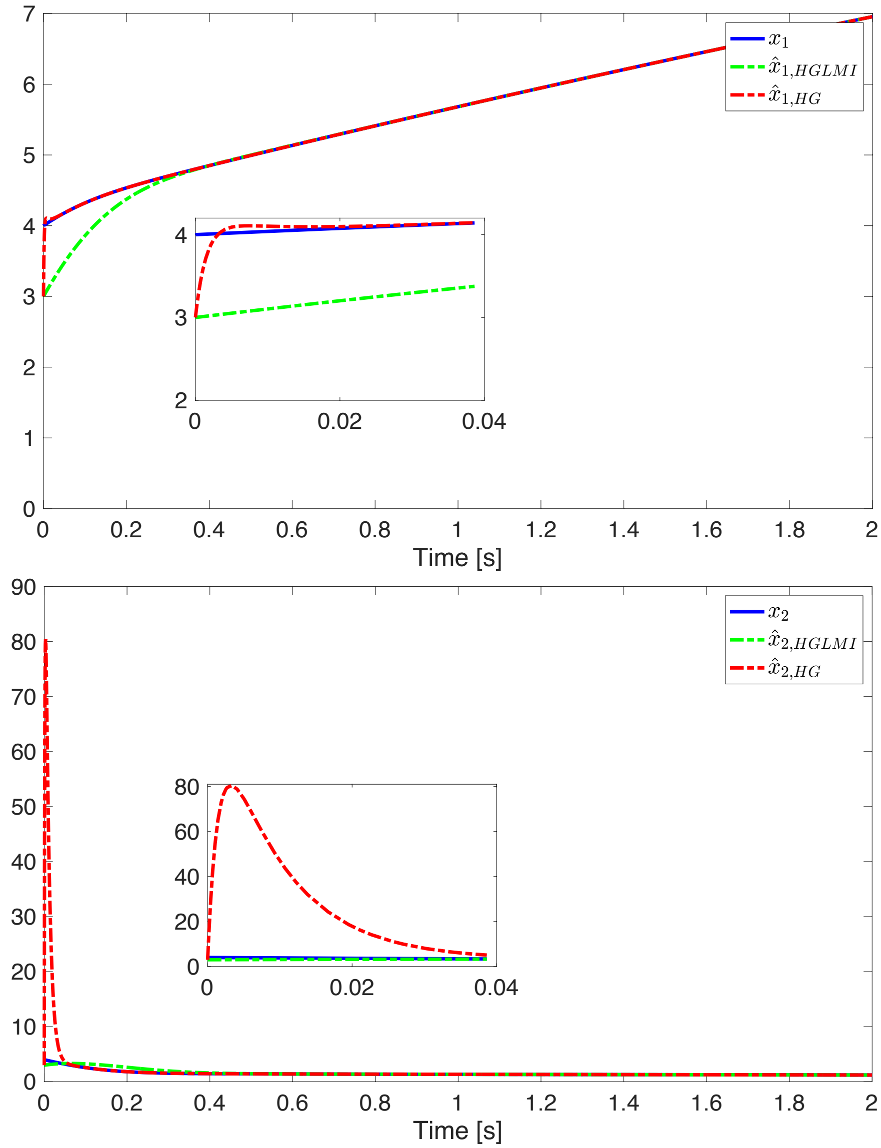

Figure 5.2: Behaviour of x_1, x_2 and their estimates by using the standard high-gain observer and the proposed HG/LMI observer

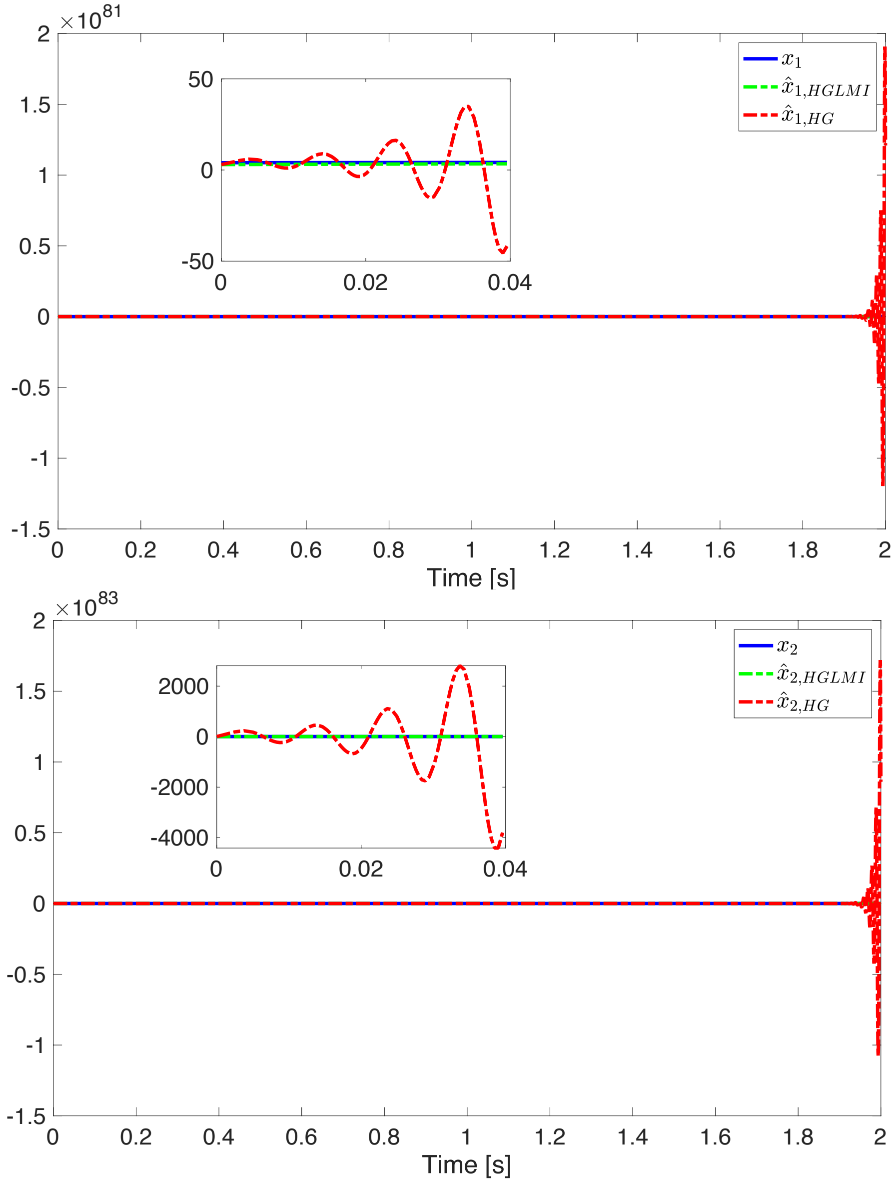

Figure 5.3: Behaviour of x_1, x_2 and their estimates by using the standard high-gain observer and the proposed HG/LMI observer for τ_M = 2.5×10^{-3}s

5.5.2 Prescribed-time high-gain observer

The aim of this section is to show the effectiveness of the proposed prescribed-time high-gain observer design on estimating the Coupled Tanks water levels. This example is motivated by the fact that some Coupled Tanks applications such as wastewater treatment, water desalination, pharmaceutical industries, and petrochemical plants may involve different time responses constraints, which makes prescribed-time convergence the key feature. The proposed prescribed-time observer will be used to estimate the water level in tank 1.

The proposed prescribed-time observer can be constructed as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) + \mu^{1+m} K_1(x_1(t) - \hat{x}_1(t)), \\ \dot{\hat{x}}_2(t) = \phi(\hat{x}_1, \hat{x}_2) + \mu^{2(1+m)} K_2(x_1(t) - \hat{x}_1(t)). \end{cases} \tag{5.26}$$

where $\hat{x}_1$ and $\hat{x}_2$ are the state estimates and the scaling function μ[1] is defined as follows:

$$\mu(t - t_0, T) = \frac{T}{T + t_0 - t}.$$

The initial time t_0 is set to zero, i.e $t_0 = 0$. By using Matlab and YALMIP, the matrices P and K are found equal to

$$P = \begin{bmatrix} 237.7077 & -68.4342 \\ -68.4342 & 04.4990 \end{bmatrix}, \quad K = [1.6616, 3.3629].$$

The estimates of water levels in the tanks are then plotted for $T = 0.5s$ and $T = 1s$. Denote by $\hat{x}(t) = [\hat{x}_1, \hat{x}_2]^T$ the state estimates for the system (5.13) by using the observer (5.26). The initial water level in tanks 1 and 2 is $x(0) = [4, 4]^T$. Let $\hat{x}(0) = [3, 3]^T$. The curves of the states x_i and their estimates $\hat{x}_i$ are illustrated in Figures 5.4-5.5-5.6 and 5.7. We can see that the proposed observer converges to the actual states at the prescribed-time.

[1]The notation is chapter specific. T here refers to the prescribed convergence time.

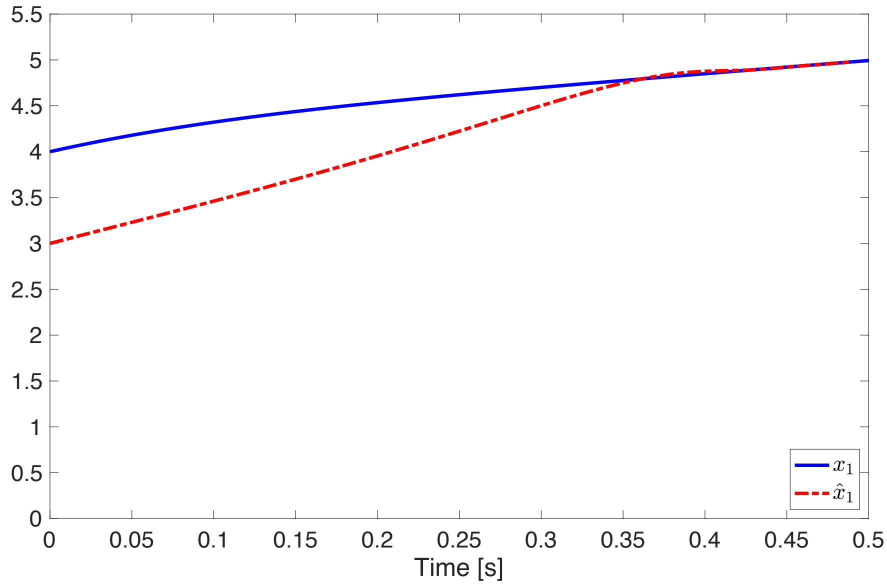

Figure 5.4: Behaviour of x_1 and its estimates for $T = 0.5s$ and $m = 1$

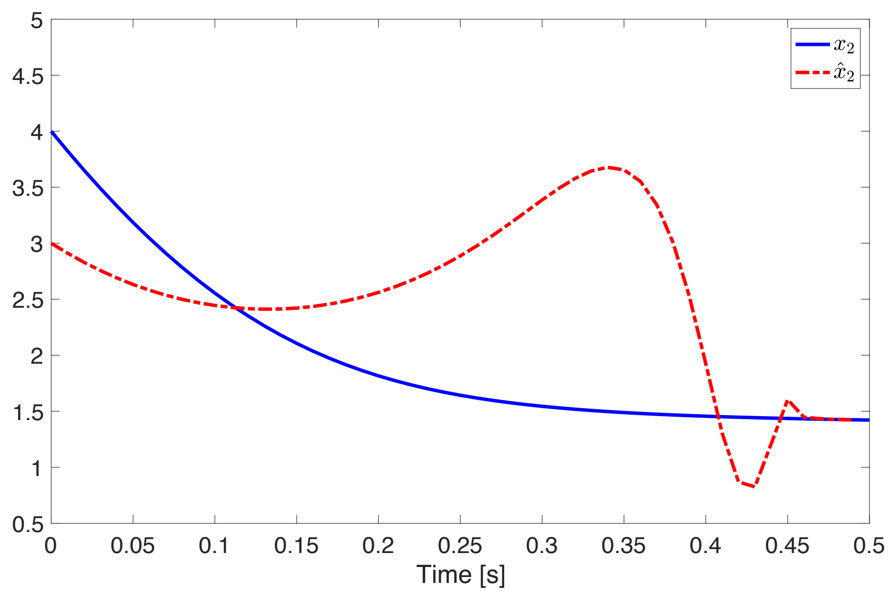

Figure 5.5: Behaviour of x_2 and its estimates for $T = 0.5s$ and $m = 1$

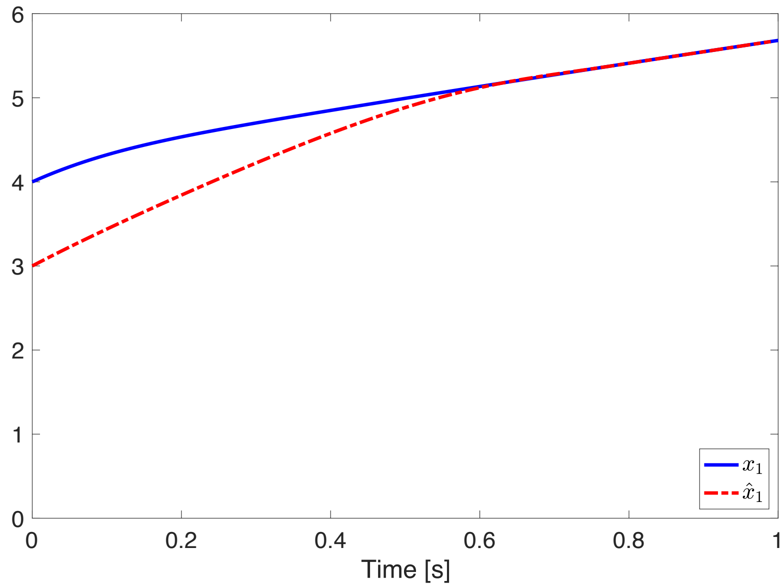

Figure 5.6: Behaviour of x_1 and its estimate for $T = 1s$ and $m = 1$

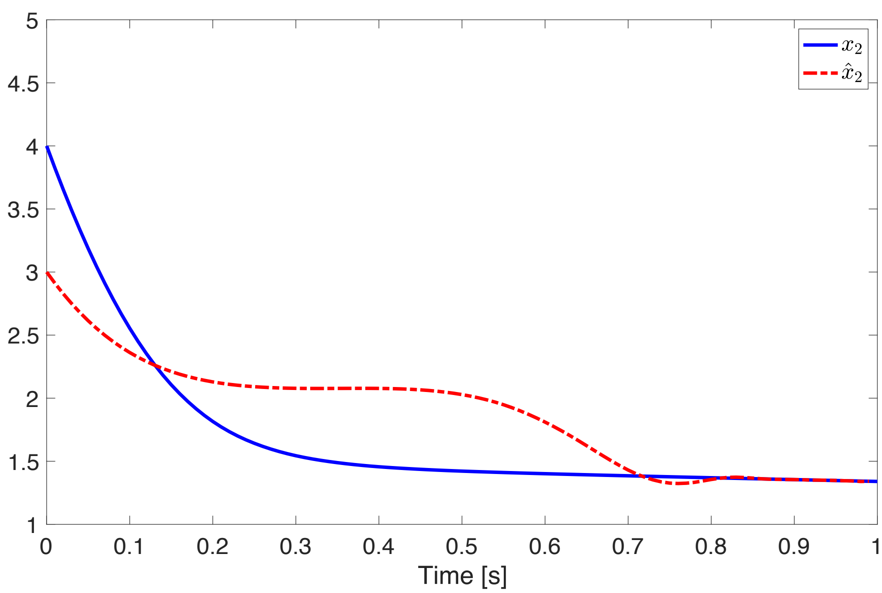

Figure 5.7: Behaviour of x_2 and its estimate for $T = 1s$ and $m = 1$

Comparison between the proposed prescribed-time observer and the standard high-gain observer:

We compare the proposed prescribed-time high-gain observer to the standard high-gain (SHG) observer at different prescribed times $T = 0.01s$, $T = 0.05s$ and $T = 0.08s$. We adjusted both observer's parameters to obtain the same time response around $T = 0.08s$.

The standard high-gain observer observer (2.16) can be constructed as follows

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) + L_1(x_1(t) - \hat{x}_1(t)) \\ \dot{\hat{x}}_2(t) = \phi(\hat{x}_1, \hat{x}_2) + L_2(x_1(t) - \hat{x}_1(t)) \end{cases} \tag{5.27}$$

with the observer's gain is found equal to $L = [159 \quad 12432]^T$.

Denote by $\hat{x}_{HG}(t) = [\hat{x}_{1,HG}, \hat{x}_{2,HG}]^T$ the state estimates for the system (4.23) by using the standard high-gain observer. Let $\hat{x}_{HG}(0) = [3, 3]^T$. The curves of the states x_i and their estimates $\hat{x}_i$ are illustrated in Figures 5.8-5.9. We notice that the proposed prescribed-time observer considerably reduces the peaking phenomenon compared to the standard high-gain observer, which illustrates the proposed observer's fixed-time and time-varying gain functions properties.

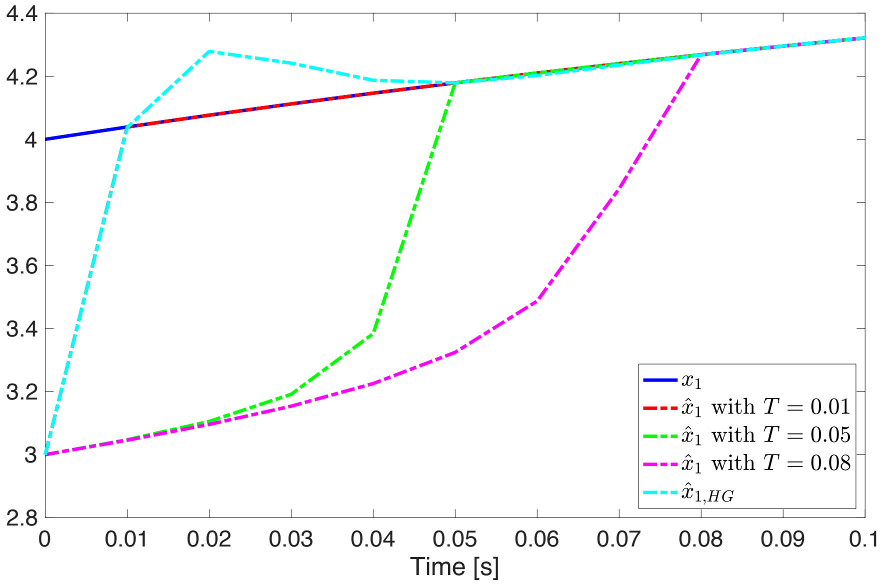

Figure 5.8: Behaviour of x_1 and its estimates by using the proposed prescribed-time observer and the standard high gain observer

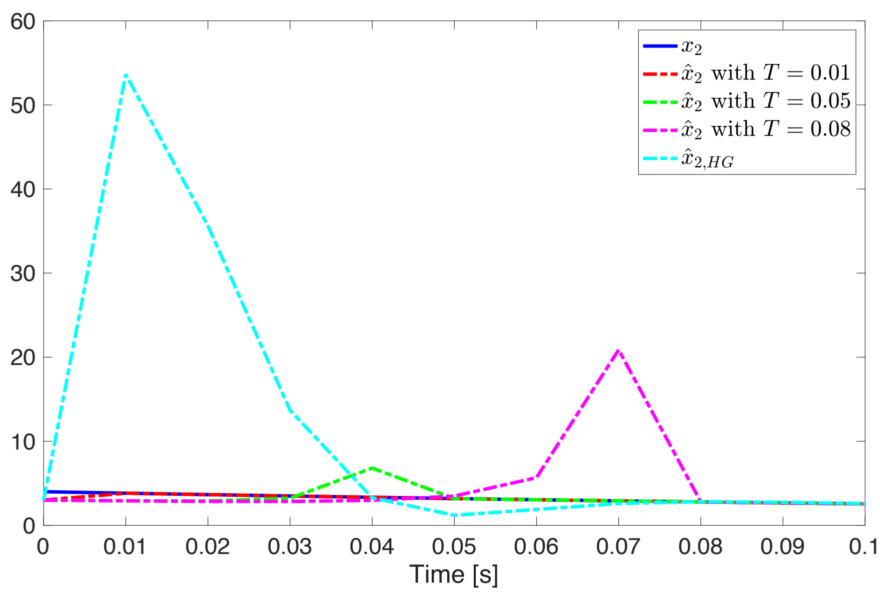

Figure 5.9: Behaviour of x_2 and its estimates by using the proposed prescribed-time observer and the standard high gain observer

5.6 Conclusion

In this chapter, we show the effectiveness of proposed HG/LMI observer and the prescribed-time high-gain observer for water level estimation in the Coupled Tanks system. To deal with the HG/LMI design, we considered that the measured output is delayed and time-varying. The proposed state observer tolerates a smaller tuning parameter allowing a maximum bound of the delay as high as possible while ensuring exponential convergence. We showed the superiority of the proposed approach and how the standard high gain observer fails to reconstruct the water levels in the tanks in the presence of larger values of the time-delay.

Furthermore, the prescribed-time high-gain observer successfully reconstructs the water levels at the desired time freely chosen by the user. A comparison to the standard high-gain observer is provided and showed how the prescribed-time high-gain observer considerably reduces the peaking phenomenon compared to the standard high-gain observer.

CHAPTER 6

CONCLUSION AND PERSPECTIVES

This chapter summarises the work established in this thesis and presents some future work perspectives.

6.1 Brief summary

In this thesis, we considered the problem of observer design for a class of nonlinear systems nonlinear systems in general and nonlinear systems with delayed output in particular.

First, we extended the HG/LMI observer design introduced in (Zemouche et al., 2019) for time-varying delayed output measurements. Such an observer led to a considerably higher allowable maximum bound on the delay with a small tuning parameter compared to the standard high-gain methodology. Moreover, extensions to systems with nonlinear outputs and to systems with sampled measurements are established. Two numerical examples are provided to illustrate the performance of the proposed HG/LMI observer design procedure with a comparison to the standard high-gain and to the high-gain observer proposed by (Van Assche et al., 2011).

Second, a prescribed-time observer is designed for a class of nonlinear systems. The objective was to design a high-gain observer that converges within a predefined time T independent on the system's parameters chosen a prior. The proposed observer has the structure of a high-gain observer where the gain depends on scaling function. A numerical example is provided to illustrate the performance of the proposed observer design procedure.

Finally, we illustrated the performance of the proposed approaches in estimating the water levels in the coupled-tanks system. We demonstrated the superiority of the HG/LMI approach and how the standard high gain observer fails to reconstruct the water levels in the tanks in the presence of larger values of the time-delay. Furthermore, the prescribed-time high-gain observer successfully reconstructs the water levels at the prescribed-time. A comparison to the standard high-gain observer is provided and showed how the prescribed-time high-gain observer considerably reduces the peaking phenomenon compared to the standard high-gain observer.

6.2 Future work

The work carried out in this thesis paves the way for new future research directions. As a potential perspective, we can mention few ideas that can be developed:

- Improving the result by exploring new ideas on high-gain observers, namely the introduction of specific nonlinear transformations to decrease the value of the tuning parameter. This will allow the application of the observer to industrial real-world applications.
- Extension of prescribed-time high-gain observer to nonlinear systems with delayed output measurements, and trying to provide less restrictive conditions by using Lyapunov-Krasovskii method.
- Experimental validation of the proposed observers on the Coupled Tanks plant and other real-world applications, and developing efficient observer-based controllers to solve control problems, namely output feedback stabilization, reference trajectory tracking.
- An interesting direction to explore is to introduce learning techniques to find the best values of the design variables related to the observer's parameter, Lipschitz constant

and decision variables of LMIs.

APPENDIX A

USEFUL INEQUALITIES AND DEFINITIONS

A.1 Useful inequalities

In this section, we recall some useful inequalities exploited in the proof of some results established in this thesis.

Jensen's Inequality *(Gu, 2000)*

For any constant symmetric and positive definite matrix $M \in \mathbb{R}^{n\times n}$, scalars t_1, t_2 and vector function $v : [t_1, t_2] \to \mathbb{R}^n$, then the following inequality holds:

$$\left(\int_{t_1}^{t_2} v(\beta)d\beta\right)^T M \left(\int_{t_1}^{t_2} v(\beta)d\beta\right) \leqslant (t_2 - t_1)\left(\int_{t_1}^{t_2} v^\top(\beta)Mv(\beta)d\beta\right).$$

Young's Inequality *(Nguyen and Trinh, 2016)*

Let X and Y be two matrices of appropriate dimensions. Then, for every invertible matrix S and scalar $\mu > 0$, we have

$$X^\top Y + Y^\top X \leqslant \mu X^\top S X + \frac{1}{\mu} Y^\top S^{-1} Y.$$

Schur's Lemma *(Boyd et al., 1993)*

Let A, B and C be three matrices of appropriate dimensions such as $A = A^T$ and $C = C^T$.

Then,

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \leq 0$$

if and only if

$$C < 0 \quad \textit{and} \quad A - BC^{-1}B^T < 0,$$

or equivalent

$$A < 0 \quad \textit{and} \quad C - B^T A^{-1} B < 0.$$

A.2 Useful definitions

In this section, we will recall some definitions which are necessary for the mathematical developments given in this thesis.

Definition A.2.1 *(Khalil, 2002) A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $a(0) = 0$.*

Definition A.2.2 *(Khalil, 2002) A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if:*

- *For each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r.*
- *For each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

Definition A.2.3 *(Holloway and Krstic, 2019; Chen et al., 2020a) The system $\dot{x} = f(x, t), (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ with $f(t - t_0, 0) = 0$ is said to be fixed-time, globally asymptotically stable in time T (FT-GAS) over the interval $I_T := [t_0, t_0 + T)$ is there exists a class $\mathcal{KL}$ function β such that, for any initial state $x(0) \in \mathbb{R}^n$, the system states are well-defined on I_T and satisfy:*

$$\|x(t)\| \leqslant \beta(\|x_0(t)\|, \mu(t - t_0, T) - 1), \tag{A.1}$$

where the function $\mu(t - t_0, T)$ is defined in (4.4).

Definition A.2.4 *(Chen et al., 2020a) The system $\dot{x} = f(x,t)$ is said to be globally convergent to zero in any prescribed-finite time T, if for any initial state $x(0) \in \mathbb{R}^n$, the system states are well defined on $t \in [t_0, t_0 + T)$ and satisfy*

$$\lim_{t \to t_0 + T} \|x(t)\| = 0. \tag{A.2}$$

Lemma A.2.1 *(Su et al., 2013) Suppose the matrix $D \in \mathbb{R}^{n \times n}$ is defined as $D = diag\{1, \ldots, n\}$ and $P \in \mathbb{R}^{n \times n}$ is a positive definite matrix. Then, a positive constant λ can be found to satisfy:*

$$PD + DP \geqslant \lambda I.$$

APPENDIX B

OBSERVABLE CANONICAL FORM

We consider nonlinear single-input single-output system of the form

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t), \\ y(t) = h(x(t)), \end{cases} \tag{B.1}$$

where $x(t) \in \Omega \in \mathbb{R}^n$ is the n-dimensional state vector defined on $\Omega \in \mathbb{R}^n$, u is the input, and $y(t)$ the measured output. $f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth real valued vector fields and scalar function, respectively. Moreover, we assume that Ω is positively invariant set for the dynamics of system (B.1).

The drift-observability map $z = \Psi(x)$ of system (B.1) is defined as

$$\Psi(x) = \left[h(x), \quad , L_f h(x), \ldots, L_f^{n-1} h(x)\right]^\top \tag{B.2}$$

Definition B.0.1 *(Cacace et al., 2016; Gauthier et al., 1992) The system is said to be globally drift-observable if the function $z = \Psi(x)$ is a diffeomorphism in all $\mathbb{R}^n$. A system is said to be globally uniformly Lipschitz drift-observable (GULDO) if it is globally drift-observable and the maps Ψ and Ψ^{-1} are uniformly Lipschitz.*

When the system is globally drift-observable, the map $z = \Psi(x)$ defines a global of coordinates, and the Jacobian

$$J(x) = \frac{\mathrm{d}\Psi(\mathrm{x})}{\mathrm{dx}} \tag{B.3}$$

is nonsingular for all $x \in \mathbb{R}^n$.

Definition B.0.2 *(Gauthier et al., 1992; Cacace et al., 2016) The triple $(f(x); G(x); h(x))$ is said to have observation relative degree r in a set $\Omega \in \mathbb{R}$ if*

$$L_g L_f^k h(z) = 0, k = 1, \dots, r-2, \forall z \in \Omega \tag{B.4}$$

$$L_g L_f^{r-1} h(z) \neq 0 \quad \textit{for some} \quad z \in \Omega \tag{B.5}$$

If $\Omega = \mathbb{R}$, the triple is said to have observation relative degree r.

If the system (B.1) is globally drift-observable and the observation relative degree in $\mathbb{R}^n$ is n, the following function is defined

$$p(x, u) = \left(L_f^n h(z) + L_g L_f^{n-1} h(z)\right)_{x=\Psi^{-1}(z)} \tag{B.6}$$

and, the representation in x-coordinates is

$$\begin{cases} z(t) = Az(t) + Bp(z; u(t)); \\ y(t) = Cz(t); \end{cases} \tag{B.7}$$

The following hypotheses are needed for the construction of an observer for system (B.1):

- The nonlinear system described by the triple $(f(x); G(x); h(x))$ is GULDO.
- The function $p(z, u$ defined in (B.6) is globally uniformly Lipschitz with respect to z and the Lipschitz coefficient γ_p is a (non decreasing) function of $\|u\|$,i.e.,$\forall z_1, z_2 \in \mathbb{R}^n$

$$\|p(z_1, u) - p(z_2, u)\| \leq \gamma_p(\|u\|)\|z_1 - z_2\| \tag{B.8}$$

- The triple $(f(x); G(x); h(x))$ has uniform observation degree at least equal to n.

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