PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

UNIVERSITY OF MOULOUD MAMMERI OF TIZI-OUZOU

FACULTY OF CONSTRUCTION ENGINEERING

DEPARTMENT OF CIVIL ENGINEERING

Course handout

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Title

DYNAMICS OF STRUCTURES

- Year 2025 -

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INTRODUCTION

In the design and evaluation of modern civil engineering structures, the consideration of dynamic effects has become not only important but essential. Structures today are increasingly exposed to various types of dynamic loads—ranging from machinery-induced vibrations to impulsive actions, and, most critically, earthquakes. Understanding how these structures respond under such conditions is the primary objective of structural dynamics.

This handout is specifically prepared for Master 1 professional students aiming to develop both a strong theoretical foundation and practical competency in dynamic analysis. The course begins by introducing the general concepts of dynamic behavior in structures, such as inertia, damping, stiffness, and the fundamental role of differential equations in modeling motion. From there, it moves methodically into the detailed study of **Single Degree of Freedom (SDOF)** systems, which form the building blocks for understanding more complex behavior.

The analysis of SDOF systems spans various loading scenarios:

- **Free vibration**, where no external force is applied;
- **Harmonic and periodic forced vibration**, which models continuous excitations such as rotating machinery or rhythmic ground motion;
- **Impulsive loading**, which represents short-duration high-intensity forces; and
- **Seismic excitation**, where ground motion induces a base excitation requiring careful response analysis.

With these foundations, the course transitions to the dynamic behavior of **Multi-Degree of Freedom (MDOF)** systems. These more realistic models account for distributed mass and stiffness, which are essential in the analysis of multi-story buildings and other complex structures. Modal analysis is introduced as a powerful technique to decouple and solve the equations of motion. The concepts of **modal superposition** and **spectral analysis** provide the basis for efficient and accurate computation of dynamic responses, particularly for seismic applications.

Each chapter builds logically upon the last, ensuring students develop both conceptual clarity and analytical skill. Real-world examples, simplified modeling approaches, and a strong link to structural engineering practice make the material especially relevant for future professionals in design offices or those preparing for further specialization in earthquake engineering or structural reliability.

By the end of this course, students will be able to:

- Develop dynamic models for real structural systems;
- Analyze their response under various types of loading;
- Understand the principles behind modern dynamic analysis tools;
- Interpret results and make informed decisions in the design process.

This handout serves both as a teaching aid and a long-term reference for professional practice in the field.

LIST OF SYMBOLS

CHAPTER 1

T	period of vibration
M	Mass
v	displacement
$\dot{v}$	velocity
$\ddot{v}$	acceleration
$v(x)$	deformation of the structure
b_n	coefficients of the trigonometric series
$\Psi_n(x)$	function compatible with the boundary conditions of the system
Z_n	generalized coordinates of the system

CHAPTER 2

$F(t)$	dynamic loading which varies with time t
$v(t)$	displacement of the mass with respect to its base corresponding to $F(t)$
K	system stiffness
C	system viscous damping coefficient
δv	unit displacement (*translation or rotation*)
$F(\delta v)$	force which makes it possible to induce the unitary displacement δv
$F_E(t)$	elastic force
$F_A(t)$	damping force
$\dot{v}(t)$	relative velocity of the system at time t
$F_I(t)$	inertial force
$\ddot{v}(t)$	acceleration at time t
$A,\ B$	constants
$v(0)$	initial displacement
$\dot{v}(0)$	initial velocity
ρ	amplitude of the response
θ	phase shift
ω	natural pulsation of vibration
f	natural frequency of vibration
ξ	critical damping factor
ξ_{cr}	critical damping
C_{cr}	critical damping coefficient
ω_D	pseudo-pulsation in damped oscillations
δ	logarithmic decrement

CHAPTER 3

$p(t)$	harmonic loading
p_0	amplitude of harmonic loading
$\bar{\omega}$	pulsation of harmonic loading
$v_h(t)$	homogeneous solution
$v_p(t)$	particular solution
β	ratio of the loading pulsation and the natural pulsation
D	dynamic displacement amplification factor
$R(t)$	dynamic response factor
TR	Transmissibilty

CHAPTER 4

$p(t)$	any periodic function
T_p	Period of the function
a_n , b_n	Fourier coefficients, constants

CHAPTER 5

α	impulse of force
$\Delta\dot{v}$	change in velocity
$\delta(t)$	Dirac delta function
$h(t)$	unit impulse
$\Delta\tau$	impulse duration

CHAPTER 6

$\ddot{v}_g(t)$	ground acceleration
$p_{eff}(t)$	effective force
V_0	base shear
M_0	base moment
r	any response parameter
$SD(\omega_D,\xi)$	displacement spectrum
$SV(\omega_D,\xi)$	velocity spetrum
$SA(\omega_D,\xi)$	acceleration spectrum
$D(t)$	relative displacement history of the mass
$A(t)$	absolute pseudo-acceleration

$V(t)$	relative pseudo-velocity history
IS(ξ)	spectral intensity
v_{max}	maximum displacement
V_{max}	maximum deformation energy
T_{0max}	maximum base shear

CHAPTER 7

$[M]$	global mass matrix (usually diagonal)
$[C]$	global damping matrix
$[K]$	global stiffness matrix corresponding to the DOF
$\{F(t)\}$	dynamic loading vector
$\{v(t)\}$	diplacements vector
$\{\dot{v}(t)\}$	velocities vector
$\{\ddot{v}(t)\}$	accelerations vector
$v_g(t)$	base displacement
$\{F_{\text{eff}}(t)\}$	effective forces vector
$\ddot{v}_g(t)$	base absolute acceleration
$\{r\}$	dynamic coupling vector
$[E]$	dynamic stiffness matrix
$\{\phi\}$	vector of eigenmodes
$[\Phi]$	modal matrix of dimension N x N
$[\Lambda]$	diagonal matrix of dimension N x N
$[\tilde{M}]$	diagonal mass matrix whose terms are called generalized mass
$[\tilde{K}]$	diagonal stiffness matrix whose terms are called generalized stiffness
$[\mathrm{I}]$	identity matrix of order N

CHAPTER 8

$\{Y(t)\}$	normal coordinate vector or generalized coordinates
$\{v_i(t)\}$	displacements of the modal component
$\tilde{M}_i$	generalized mass
$\tilde{K}_i$	generalized rigidity
$\tilde{F}_i(t)$	generalized load
$\bar{F}_i(t)$	generalized force per unit generalized mass of mode i

ω_{Di}	damped frequency of mode *i*
$Y_i(0)$	initial displacements of mode *i*
$\dot{Y}_i(0)$	initial velocities of mode *i*

CHAPTER 9

{f}	spatial distribution of loading
$\{x\}^{\mathrm{T}}$	line vector whose components x_j represent the mass height at level *j* measured above the base of the building
$\delta_i^2 / \tilde{M}_i$	effective modal mass
Γ_i	participation factor of mode *i*
$Y_{i,\max}$	modal displacement of mode *i*
$\{v_{i,\max}\}$	d maximum displacement of mode *i* in the geometric coordinate system
$\{F_{Ei,\max}\}$	vector of maximum elastic forces of mode *i*
r_i^{CA}	upper limit of the maximum response of degree of freedom *i*
v_i^{CA}	upper limit of the maximum response of degree of freedom *i*

CHAPTER

1

GENERAL
CONCEPTS

1.1 INTRODUCTION

A phenomenon of dynamic origin is characterized by a load varying both in time and in space, in which the inertia forces, the product of mass by acceleration, play a significant role in the response. In order to realistically evaluate the behavior of a structure subjected to a dynamic load, in other words its dynamic response, dynamic structural analysis techniques are used.

Dynamic structural analysis generates a sequence of time-varying solutions, unlike static structural analysis, which allows a single load to be applied and a single solution to be obtained over time. In this sense, the primary objective of the dynamic of structures is to determine the variations over time of the stresses (*forces*) and deformations (*displacements*) generated by any dynamic loading.

1.2 CHARACTERIZATIONS OF DYNAMIC LOADS

Practically all structures are subject to dynamic loads during their service life. These loads can be classified into deterministic and random loads.

1.2.1 DETERMINISTIC LOADS

If the applied load is perfectly defined by its temporal and spatial variation, i.e. its amplitude, its direction and their points of application are known at all times, the load is qualified as deterministic. There are two classes of deterministic loads: periodic loads and non-periodic loads.

i) Periodic loads

A periodic load is a repetitive load whose loading diagram is reproduced identically after a duration T, called the excitation period. Among the periodic loadings, one will distinguish the harmonic loadings and the anharmonic loadings. A harmonic load is typically that generated by rotating machinery in a building (Fig. 1.1).

The loading can also be periodic without being harmonic, one will qualify it as anharmonic. This type of loading is that generated, for example, by the force of the propeller at the stern of a ship (Fig. 1.2).

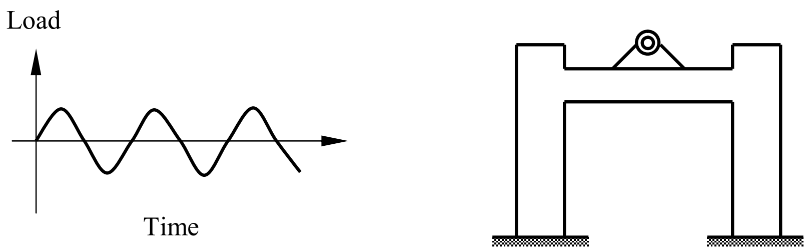


Figure 1.1: Harmonic load.

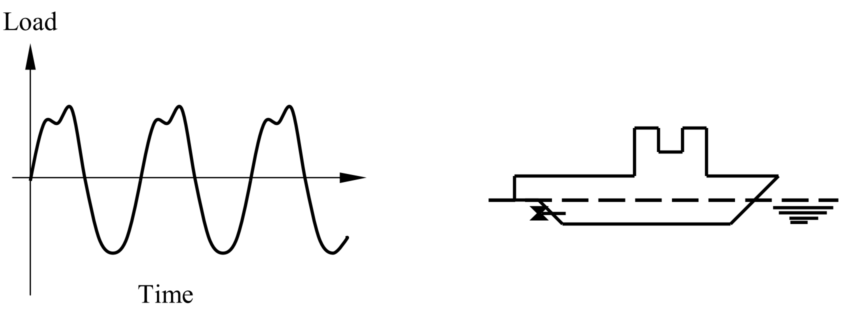


Figure 1.2: Anharmonic periodic load.

ii) Non-periodic loads

A non-periodic load can be short-duration impulsive type or sustained; it does not reproduce identically after a time interval T.

Impulsive loading is characterized by a stress of short duration, such as that induced by the front of a shock wave hitting the structure (Fig. 1.3).

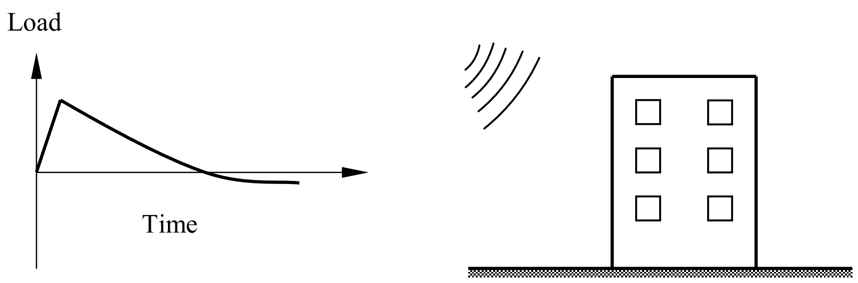


Figure 1.3: Impulsive load.

The sustained load can be defined as the load resulting from a succession of pulses. This is typically the case with seismic solicitation if the ground acceleration is known deterministically (Fig. 1.4).

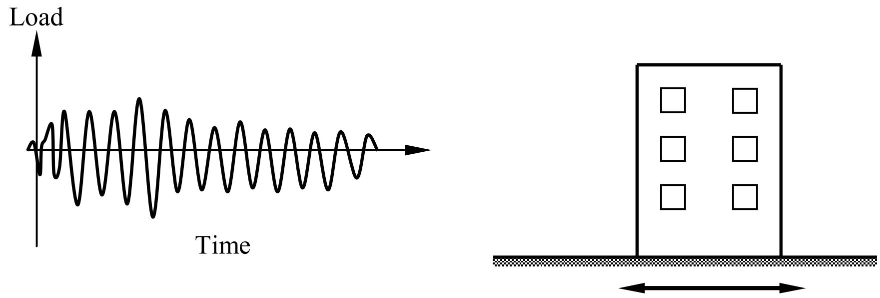


Figure 1.4: Sustained load.

1.2.2 RANDOM LOADS

Many loads affecting civil engineering structures cannot be defined deterministically. They are generally only described statistically (mean, standard deviation, frequency content, etc.). These are generally vibratory movements caused by future earthquakes, wind pressure on a building, rail or road traffic.

The response of the structure to random loadings is the subject of stochastic dynamics. This course will be limited to the analysis of structures subjected to deterministic loads.

1.3 DYNAMIC DEGREE OF FREEDOM (DOF)

The number of degrees of freedom in a dynamical system expresses the smallest number of coordinates necessary to define the position of all the mass particles of the system. In most cases, several motion components are negligible relative to each other, so that a seemingly complicated system can be modeled with only a few degrees of freedom. An important step in the dynamic analysis lies in the choice of the degrees of freedom. Consider, for example, the pendulum shown in figure (1.5) made of a pebble suspended from a ball joint with the help of a bar. In general, the system has an infinite number of dynamic degrees of freedom (DOF) taking into account all the possible deformations of the bar and the pebble. However, the following facts are observed:

- the bar is much lighter than the pebble (its mass is negligible);
- the plane strains of the pebble are very low compared to its rigid body motion;
- the dimensions of the pebble are small compared to the length of the bar (point mass);
- the elongation of the bar is small compared to its motion as a rigid body;

then the system can be reduced to a single DOF: the rotation of the bar relative to the support at the kneecap.

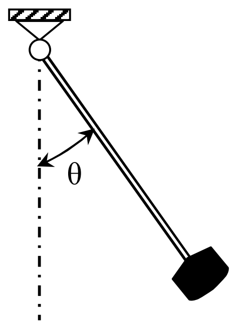


Figure 1.5: Dynamic system: the pendulum.

Figure (1.6) shows a more complex dynamic system in a single frame under lateral loading. Again, the system has an infinite number of DOFs when considering the spatial distribution of mass. But we can reduce the system to a single DOF by making the following assumptions:

- the mass of the frame can be concentrated on the floor;
- the vertical movement of the frame is negligible compared to the lateral movement;
- the columns of the frame retain their lateral rigidity but have no mass.

In most cases, the dynamic analysis of a structure is carried out by considering a restricted number of DOF.

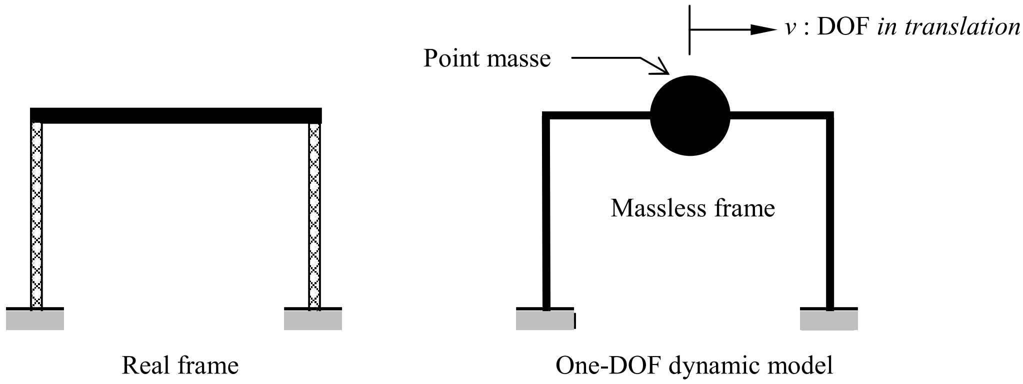


Figure 1.6: Simple frame modeled by a system with one degree of freedom.

1.4 NEWTON'S SECOND LAW

Sir Issac Newton (1642-1727) postulated the laws of motion which form the basis of the dynamic analysis of structures. He introduced the concept of the momentum of a body: the product of its mass, M, and its velocity, V, and posited that if a force, F, acts on a body the rate of change of its momentum is equal to the applied force. So:

$$\mathrm{F}=\frac{\mathrm{d}(MV)}{\mathrm{dt}} \tag{1.1}$$

However, it should be noted that the amount of movement, *MV*, and the applied force, *F*, vary with time.

Equation (1.1) represents Newton's second law and is the cornerstone of the dynamic analysis of structures. We limit ourselves here to systems whose mass is constant. In this case, equation (1.1) can be written:

$$F = M\,\frac{\mathrm{d}(V)}{\mathrm{dt}} = M\,\mathrm{a} \qquad \textbf{(1.2)}$$

This equation represents the common equation: force equals mass multiplied by acceleration, a.

In this course, we use the Newtonian notation stipulating that we denote a derivative with respect to time by a point above the variable in question.

$$\begin{aligned} v &= \text{displacement} \\ \dot{v} &= \frac{dv}{dt} = \text{velocity} \\ \ddot{v} &= \frac{d^2v}{dt^2} = \text{acceleration} \end{aligned} \qquad \textbf{(1.3)}$$

1.5 DYNAMIC MODELING

The modeling of structures under dynamic loading takes on a particular aspect, and this is due to the fact that the inertial forces come from the displacements of the structure, which are themselves connected to the inertial forces. If, moreover, the mass of the system is distributed continuously, the displacements and the accelerations must be calculated at all points, which makes the analysis of a dynamic problem a little more complex.

However, the modeling of a structure can allow important simplifications representing a sufficient simplification from a practical point of view of the exact solution of the problem. These simplifications are illustrated below on a simplified structure representing a beam with a mass density per linear meter, $m(x)$.

1.5.1 CONCENTRATED MASS MODELING

If it is possible to concentrate the mass of the beam in a finite, restricted number of points, called nodes, an important simplification is introduced in the dynamic study because the inertia forces can appear only in these points where the mass is considered concentrated (Fig. 1.7). In this case, it is sufficient to express the displacements and the accelerations at the nodes of the structure. The number of components of displacement necessary for writing completely the field of the inertia forces is called number of dynamic degree of freedom (DOF, to see paragraph 1.3).

Concentrated mass modeling is very useful for systems in which a large part is effectively concentrated in certain locations: this is the case, for example, of buildings where the main mass is located at floor level; the mass of the load-bearing structure (columns, walls) can then be, with a sufficient approximation, distributed at floor level (half at the lower level and half at the upper level).

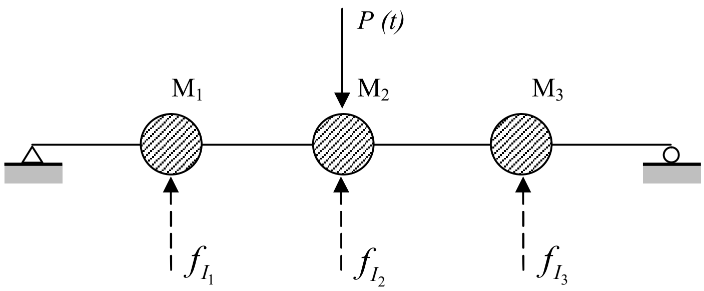


Figure 1.7: Concentrated mass modeling.

1.5.2 GENERALIZED DISPLACEMENTS

This method is used when the mass of the system is distributed almost uniformly everywhere (silo, dam, etc.). In this case, it is obvious that to represent the inertia forces, it will be necessary to hang an infinite number of degrees of freedom. This makes dynamic analysis nearly impossible. There is however an alternative approach allowing to reduce the number of degrees of freedom.

This approach is based on the assumption that the deformation of the structure can be represented by the sum of functions each representing a possible deformation of the system. These functions are called generalized displacements of the structure. Figure (1.8) illustrates this proposition.

In this case, the deformation of the structure can be expressed by an infinite sum of trigonometric series, thus:

$$v(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \textbf{(1.4)}$$

The quantities b_n represent the coefficients of the trigonometric series and can be assimilated to the coordinates of the system. In the example given in fig 1.8, the infinite number of degrees of freedom of the beam is represented by an infinite number of generalized displacements. The advantage of this approach is, by limiting the number of terms of the series to a few terms, to allow a satisfactory representation from a practical point of view of the deformation of the beam.

This concept can be generalized to any function Ψ_n (x) compatible with the boundary conditions of the system. So,

$$v(x) = \sum_{n=1}^{\infty} Z_n \, \psi_n(x) \quad \textbf{(1.5)}$$

For any function Ψ_n (x), the deformation of the beam depends on the amplitudes Z_n which are the generalized coordinates of the system. The number of functions Ψ_n (x) represents the number of degrees of freedom of the system.

As a general rule, a better precision of the dynamic response of a system is obtained with a modeling using generalized displacements than with a modeling in concentrated masses; however, the numerical resolution is more complex.

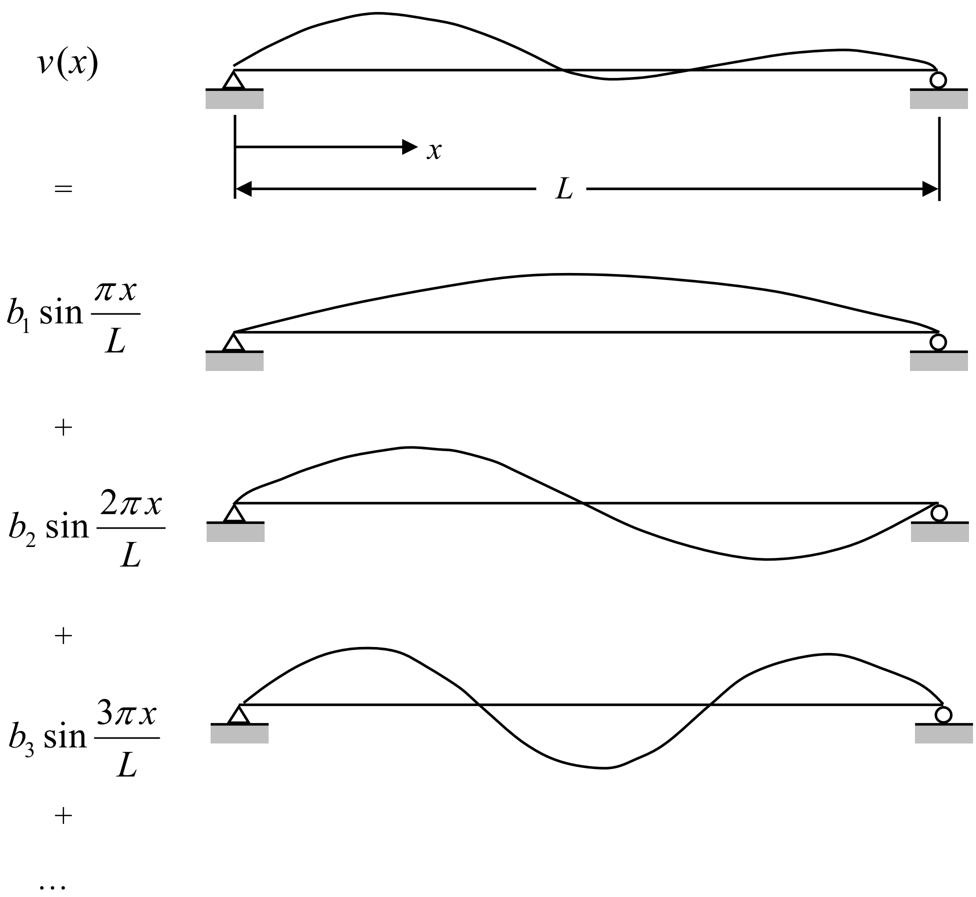


Figure 1.8: Generalized displacements.

1.5.3 FINITE ELEMENT MODELING

This approach combines both the advantages of modeling in concentrated masses and that of modeling using generalized displacements. It is illustrated in figure (1.9). This method is applicable to any type of structure and is particularly powerful from a numerical point of view. Returning to figure 1.9, the structure (beam) is subdivided into an arbitrary number of elements of arbitrary dimensions. The system nodes represent the generalized coordinate system.

The displacement of the structure is expressed as a function of these generalized coordinates using displacement functions similar to the expression given by equation (1.5). These functions are called *interpolation functions* because they define the displacement between the considered nodes.

In principle, the interpolation functions can be defined by any continuous function satisfying the geometric conditions of displacements of the nodes.

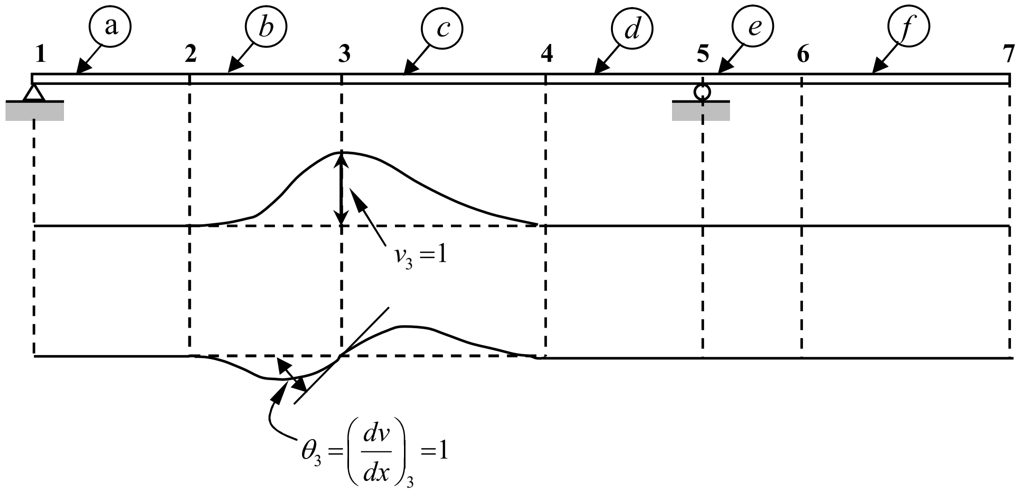


Figure 1.9: Finite element modeling.

The advantages of the procedure lie in the fact:

- that an arbitrary number of nodes can be introduced,
- that the calculations are greatly simplified by choosing identical interpolation functions for all the elements,
- that the dynamic equations of the system are strongly decoupled because the displacement of a node affects only the close nodes.

The finite element method is by far the most powerful method for solving dynamic problems by allowing the displacements of any structure to be expressed using a finite set of coordinates.

1.6 DYNAMIC ANALYSIS STEPS

The main steps of the dynamic analysis are:

1. definition of dynamic load,
2. the idealization of the structure :
   - the definition of important DOFs ;
   - the definition of the properties of the structure (mass, rigidity, damping, elastic limit, etc.),
   - definition of soil properties,
3. dynamic calculation,
4. appreciation and understanding of dynamic behavior and analysis of results,
5. the combination of the dynamic loads with the other (static) loads to obtain the maximum values,
6. dimensioning (or verification) of the structure.

This course focuses mainly on dynamic calculation (step 3). The choice of the method to be used in the dynamic calculation depends on the characteristics of the structure and the dynamic loading (Fig. 1.10).

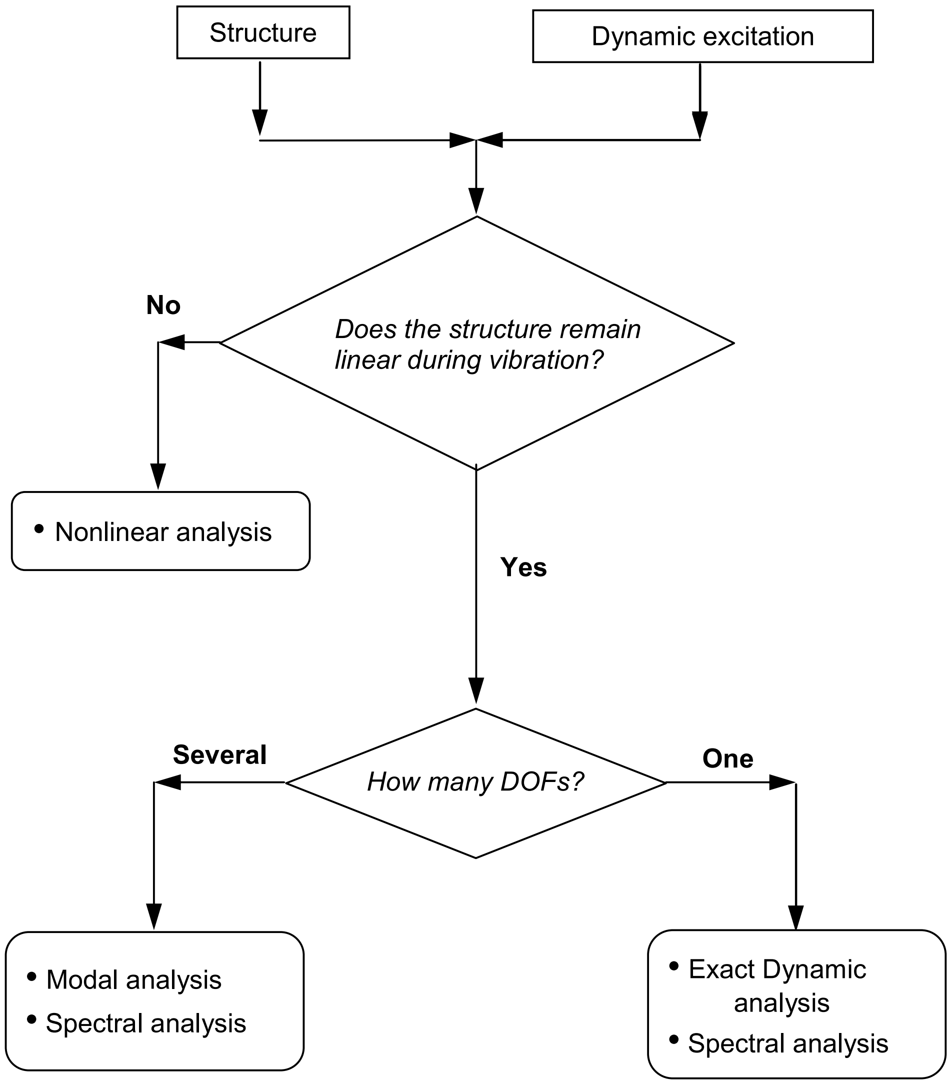


Figure 1.10: Dynamic analysis methods.

CHAPTER

2

DYNAMIC ANALYSIS IN FREE VIBRATION OF STRUCTURES WITH A SINGLE DEGREE OF FREEDOM

2.1 INTRODUCTION

This part presents the dynamic analysis of a linear system (oscillator) comprising a single dynamic degree of freedom (DOF).

The dynamic analysis of a linear system with single DOF is very important for the following reasons:

- It makes it possible to identify certain types of response and thus become familiar with the basic concepts;
- Some structures can be treated as single DOF systems;
- The knowledge of the response of systems with a single DOF is vital in determining the response of systems with multiple DOF;
- Seismic spectral analysis is based on response spectra obtained from single DOF systems excited at their base by accelerations caused by earthquakes.

2.2 MATHEMATICAL MODEL

The representation of a structure as a single DOF system is a delicate task. The precision of the solution (the dynamic response) will depend on the mathematical modeling. In the following, the single DOF system is considered as a system where the inertial forces are all concentrated into a single mass. Figure (2.1) presents the mathematical model of a single DOF system.

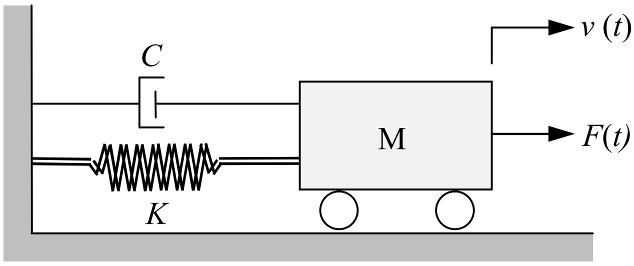


Figure 2.1: Mathematical model of a single DOF system.

where

$F(t)$: dynamic loading which varies with time t.
$v(t)$: displacement of the mass with respect to its base corresponding to $F(t)$ and which also varies with time.
M : mass of the system.
K : rigidity of the system (static force required to obtain a unit displacement). For a building subjected to lateral loads, K represents the lateral stiffness of the structure.
C : system viscous damping coefficient (dynamic force required to obtain unit speed). This viscous damping is introduced to represent the energy dissipated during the vibration of the system. The sources of this energy loss are multiple and are not easy to quantify individually. Viscous damping is used for its mathematical elegance.

2.3 RIGIDITY OF A STRUCTURE

The stiffness of a structure, K (in [N/m]), depends on its geometric dimensions and the modulus of elasticity of the material that composes it, as well as the support conditions. The stiffness is equivalent to the force that must be exerted on the element to induce a unit displacement. It should be noted that this course is limited to the use of linear elastic materials; the stiffness is therefore constant throughout the analyses. The stiffness is equal to [8]:

$$K = \frac{F(\delta v)}{\delta v} \qquad \textbf{(2.1)}$$

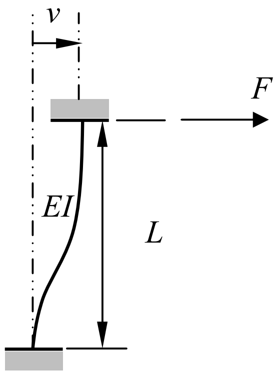
where

δv : displacement (*translation or rotation*) unit.
$F(\delta v)$: force which makes it possible to induce the unitary displacement δv.

2.3.1 EXAMPLES OF LATERAL STIFFNESS COEFFICIENTS

In a building modeled as a single DOF system and subjected to lateral loads (wind and earthquakes), the stiffness coefficient, K, represents the total lateral stiffness of the structure. Several structural elements contribute to this rigidity. To be able to solve practical problems, it is good to review the force-displacement (stiffness) relationship of common structural elements.

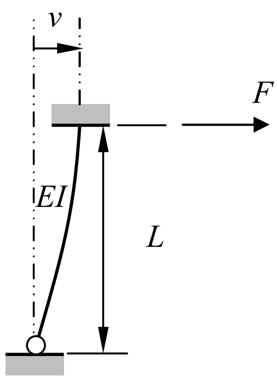
a) Column with fixed supports (Fig. 2.2)



$$k = \frac{F}{v} = \frac{12EI}{L^3} \qquad \textbf{(2.2)}$$

Figure 2.2: Column with fixed supports.

b) Column articulated at its base and embedded in the floor (Fig. 2.3)



$$k = \frac{F}{v} = \frac{3EI}{L^3} \qquad \textbf{(2.3)}$$

Figure 2.3: Column articulated at its base and embedded in the floor.

c) Bi-articulated diagonal bracing (Fig. 2.4)

$$k = \frac{F}{v} = \frac{EA}{L}\cos^2\theta\sin\theta = \frac{EA}{L_D}\cos^2\theta \qquad \textbf{(2.4)}$$

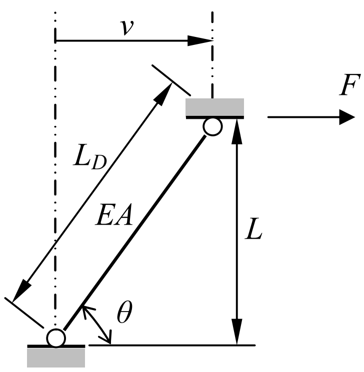


Figure 2.4: Bi-articulated diagonal bracing.

d) Cantilever wall (Fig. 2.5)

$$k = \frac{F}{v} = \frac{3EI}{h^3\left[1 + 0.6\,(1+\nu)\dfrac{d^2}{h^2}\right]} \tag{2.5}$$

where ν is the Poisson's ratio of the material.

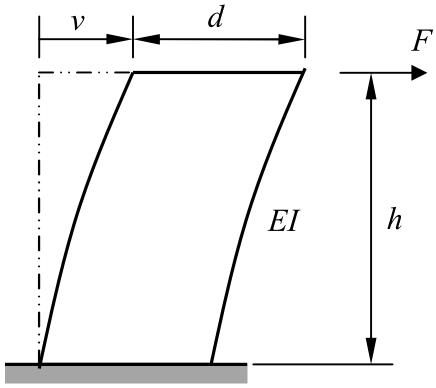
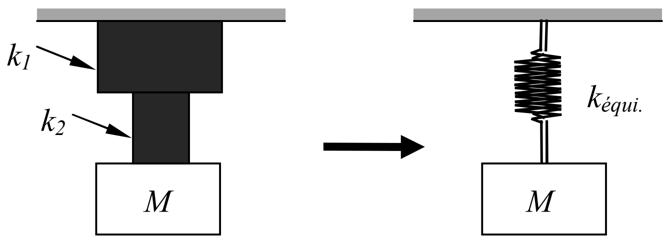


Figure 2.5: Cantilever wall.

2.4 EQUIVALENT RIGIDITY OF AN ASSEMBLED SYSTEM

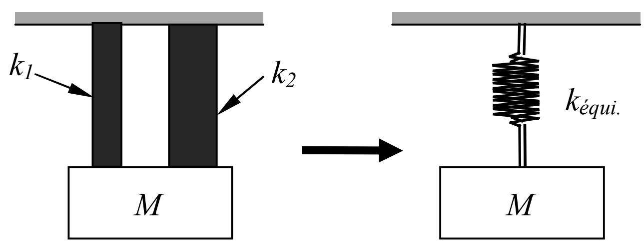
In the case of an assembled system, there are two types of assembly. The rigidities of these two types of assemblies are given in equations (2.6, 2.7).

Serial system (Fig. 2.6)



$$\frac{1}{k_{équi.}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{2.6}$$

Parallel system (Fig. 2.6)



$$k_{équi.} = k_1 + k_2 \tag{2.7}$$

Figure 2.6: Serial and parallel systems.

2.5 EQUATIONS OF MOTION

The time response of certain mechanical or structural systems under dynamic loads can be completely defined by the time response following a single displacement coordinate: these are systems with one dynamic degree of freedom or elementary system, see figure (2.1) of this chapter.

In what follows, we will formulate the equations governing the equilibrium of the system with a single DOF at any time t. Two types of dynamic excitation will be considered: (i) any dynamic loading $F(t)$ and (ii) a support movement (excitation at the base, seismic problem).

2.5.1 ANY DYNAMIC LOADING

Consider the free-body diagram for a single DOF system, as shown in Figure (2.7). The forces acting on the mass at any instant t are the external force, $F(t)$, the elastic force, $F_E(t)$, and the damping force, $F_A(t)$.

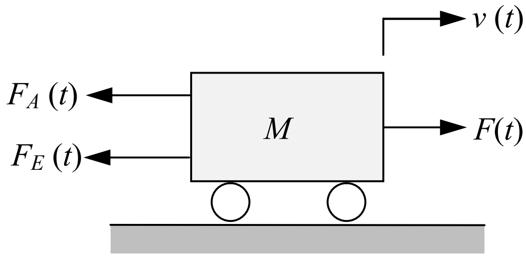


Figure 2.7: Free-body diagram of a single DOF system.

Applying Newton's second law, we get:

$$F(t) - F_A(t) - F_E(t) = M\ddot{v}(t) \quad \textbf{(2.8)}$$

For a linear structure, the elastic force has the expression:

$$F_E(t) = Kv(t) \quad \textbf{(2.9)}$$

The damping force is given by the following formula:

$$F_A(t) = C\dot{v}(t) \quad \textbf{(2.10)}$$

where $\dot{v}(t)$ is the relative velocity of the system at time t.

The force associated with the mass M undergoing acceleration at time t is called the inertia force, and has the expression:

$$F_I(t) = M\ddot{v}(t) \quad \textbf{(2.11)}$$

Substituting equations (2.9) and (2.10) into (2.8), we get:

$$F(t) - C\dot{v}(t) - Kv(t) = M\ddot{v}(t) \quad \textbf{(2.12)}$$

or

$$M\ddot{v}(t) + C\dot{v}(t) + Kv(t) = F(t) \quad \textbf{(2.13)}$$

The relation (2.13) is the equation of motion (dynamic equilibrium equation) of a system with a single DOF. This equation represents the basic relation that we propose to solve for any dynamic load, $F(t)$. Note that equation (2.13) assumes that the properties of the system M, C, K do not change during the response (vibration) of the system, i.e. the system is linear. This means, for example, that the elastic limit of the material is never exceeded and therefore no plasticization is allowed. Mathematically speaking, equation (2.13) is *a non-homogeneous differential equation of the second order, linear and with constant coefficients*.

2.5.2 BASE MOVEMENT (*seismic problem*)

In the case of a structure subjected to an earthquake, the dynamic solicitation does not result from an explicit force applied to the system but from an implicit inertia force caused by the movement of the base. To derive the equation of motion for the seismic problem, consider a single DOF structure subjected to a displacement of its base, $v_g(t)$, (Fig. 2.8). The displacement of the base of the structure induces a displacement of the mass, concentrated at the height of the roof, relatively to the base, $v(t)$, which represents the DOF of the structure. Thus the force generated by the rigidity of the system is proportional to $v(t)$, the force generated by the viscous damping of the system is proportional to $\dot{v}(t)$, but the inertia force generated by the mass of the system is proportional to the total system acceleration, $\ddot{v}_g(t) + \ddot{v}(t)$.

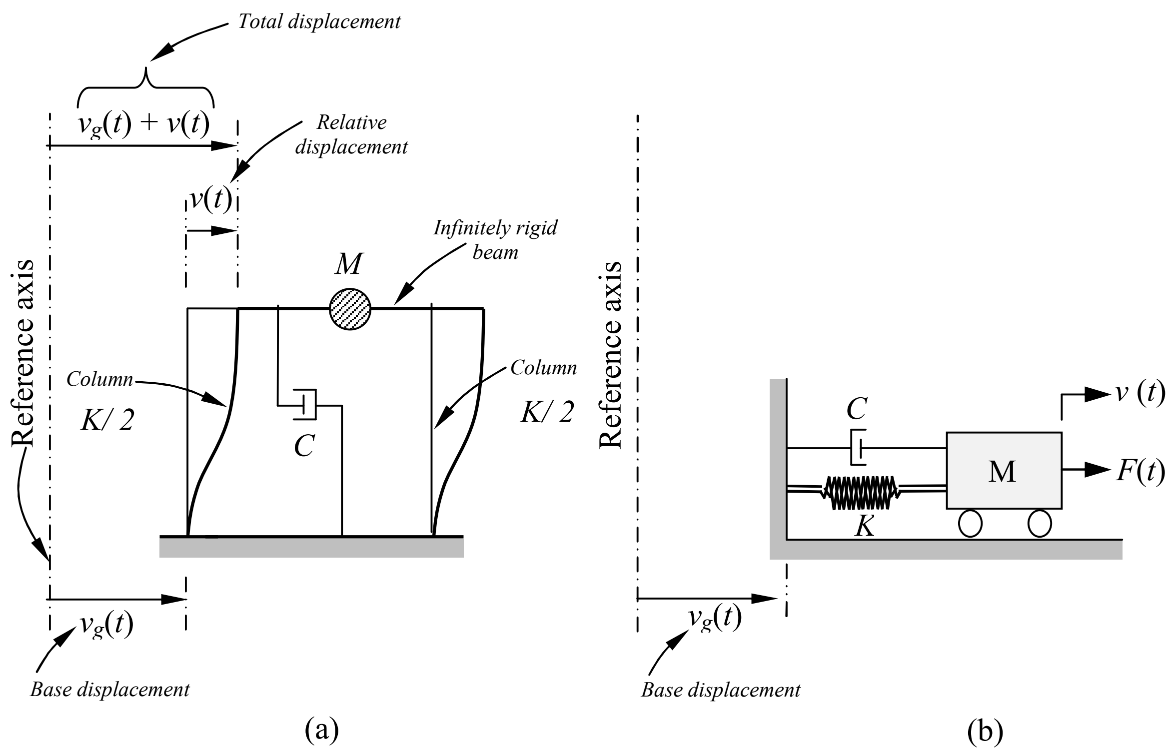


Figure 2.8: Seismic problem: (a) linear-elastic structure loaded by base motion, and (b) equivalent mass-spring-damper model.

Applying Newton's second law, we get:

$$-C\dot{v}(t)-Kv(t)=M\left[\ddot{v}_g(t)+\ddot{v}(t)\right] \qquad \textbf{(2.14)}$$

or

$$M\ddot{v}(t)+C\dot{v}(t)+Kv(t)=-M\ddot{v}_g(t) \qquad \textbf{(2.15)}$$

we pose

$$F_{eff}(t)=-M\ddot{v}_g(t) \qquad \textbf{(2.16)}$$

we will have

$$M\ddot{v}(t)+C\dot{v}(t)+Kv(t)=F_{\text{eff}}(t) \qquad \textbf{(2.17)}$$

where $F_{\text{eff}}(t)$ is an *effective seismic force* equivalent to the effect of an earthquake on the structure. In other words, the displacements $v(t)$ of the structure subjected to an acceleration of the ground $\ddot{v}_g(t)$ will be identical to the displacements of the same structure, resting on a stationary support (fixed base), subjected to an external force, equal to the mass multiplied by the acceleration of the ground, acting in the opposite direction to this same acceleration as indicated by the negative sign (Fig. 2.9). Equation (2.17) is the most popular form of use for the following reasons:

1) Earthquakes are generally measured by the acceleration of the ground (accelerogram) and therefore no modification of the signal is necessary before its use;
2) The stresses depend on the relative displacements $v(t)$ which are directly calculated in this formulation;
3) The damage can be directly assessed because it is a function of the relative displacements.

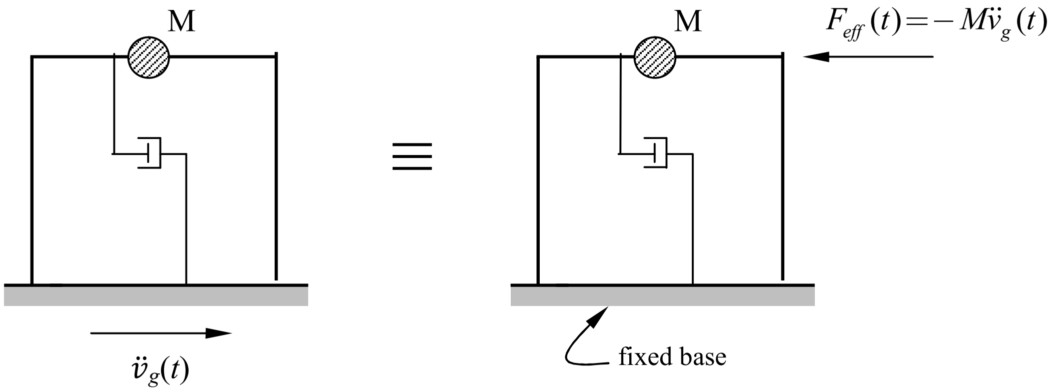


Figure 2.9: Equivalence of the seismic problem.

2.6 FREE VIBRATION

The simplest dynamic response of a single DOF system occurs when the system is in free vibration. The free vibration describes the behavior of the system under the action of the initial conditions (displacement and velocity) without external dynamic excitation. In other words, the free response is the solution of the homogeneous differential equation where the second member of equation (2.13) is equal to zero, i.e. $F(t) = 0$. The response in free vibration is very important to determine a fundamental characteristic of the system: the natural period of vibration.

2.6.1 UNDAMPED FREE VIBRATION OF A STRUCTURE

We speak of undamped oscillations when the damping is zero, i.e. C=0. The equation of motion for a system with an undamped DOF in free vibration is simply written:

$$M\ddot{v}(t) + Kv(t) = 0 \tag{2.18}$$

which can also be written:

$$\ddot{v}(t) + \omega^2 v(t) = 0 \tag{2.19}$$

where

$$\omega^2 = \frac{K}{M} \tag{2.20}$$

The general (homogeneous) solution of equation (2.19) is as follows:

$$v(t) = A\sin\omega t + B\cos\omega t \tag{2.21}$$

in which the constants A and B can be expressed as a function of the initial conditions, i.e. the displacement and the velocity at the instant $t = 0$ when the free vibrations of the system begin. We immediately see that $v(0) = B$ and $\dot{v}(0) = A\omega$. The general solution (2.19) is then written:

$$v(t) = \frac{\dot{v}(0)}{\omega}\sin\omega t + v(0)\cos\omega t \tag{2.22}$$

or equivalently

$$v(t) = \rho\cos(\omega t - \theta) \tag{2.23}$$

ρ designates the amplitude of the response, it is given by the resultant

$$\rho = \sqrt{\left[v(0)^2\right] + \left[\frac{\dot{v}(0)}{\omega}\right]^2} \tag{2.24}$$

and θ the phase shift

$$\theta = \tan^{-1}\frac{\dot{v}(0)}{\omega v(0)} \tag{2.25}$$

Figure (2.10) represents graphically the general solution corresponding to a simple harmonic motion.

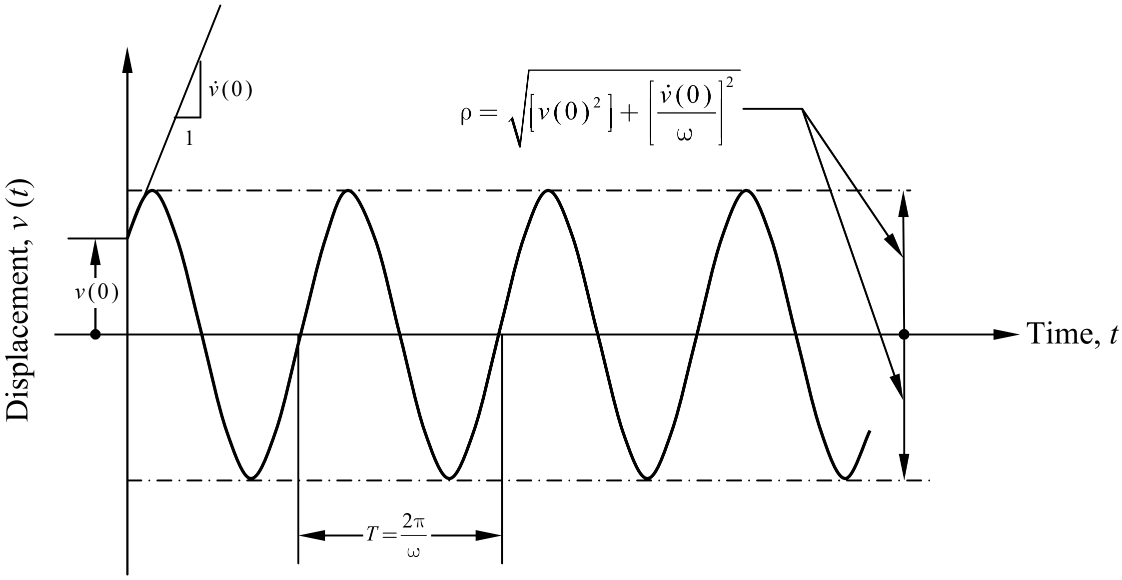


Figure 2.10: Free vibration of a single DOF system without damping.

The result illustrates several fundamental concepts:

- The amplitude of the vibration is constant so the vibration would continue indefinitely. It is physically impossible; therefore, the concept of damping must be applied.
- The time required T (in seconds) to complete one cycle of free vibration is called the natural period of the system.
- The quantity ω is called the angular frequency of vibration or the natural pulsation of the system and is measured in radians per second.
- La The natural frequency f of the system is defined as the reciprocal of the period and is measured in cycles per second or Hertz.

These fundamental properties T, ω and f depend only on the mass and the stiffness of the system and are related by:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

$$T = 2\pi\sqrt{\frac{M}{K}}\,;\ \omega = \sqrt{\frac{K}{M}}\,;\ f = \frac{1}{2\pi}\sqrt{\frac{K}{M}} \tag{2.26}$$

2.6.2 DAMPED FREE VIBRATION OF A STRUCTURE

In reality, a structure in free vibration has a damping which tends to decrease the amplitude of the vibration with time. There are several types of damping:

- Viscous damping, where the force is proportional to the velocity;
- Frictional damping, where the force is constant;
- Internal damping, where the force is proportional to the amplitude of the displacement.

In structural analysis, viscous damping is always used because of its mathematical elegance. In this case, the equation of motion for a damped linear system with a single DOF in free vibration is the following:

$$M\ddot{v}(t) + C\dot{v}(t) + Kv(t) = 0 \tag{2.27}$$

or

$$\ddot{v}(t) + \frac{C}{M}\dot{v}(t) + \frac{K}{M}v(t) = 0 \tag{2.28}$$

if we define:

$$\omega = \sqrt{\frac{K}{M}} \;=\; \text{angular frequency}$$

$$\xi = \frac{C}{2\omega M} \;=\; \text{critical damping factor} \geq 0 \tag{2.29}$$

we obtain:

$$\ddot{v}(t) + 2\xi\omega\dot{v}(t) + \omega^2 v(t) = 0 \tag{2.30}$$

The general (homogeneous) solution of equation (2.30) takes the following form:

$$v(t) = Ge^{st} \tag{2.31}$$

by substituting equation (2.31) into equation (2.30), we get:

$$Ce^{st}\,(s^2 + 2\xi\omega\, s + \omega^2) = 0 \tag{2.32}$$

which is a quadratic equation in s with the following roots?

$$s_1 = -\xi\omega + \omega\sqrt{\xi^2 - 1}, \qquad s_2 = -\xi\omega - \omega\sqrt{\xi^2 - 1} \tag{2.33}$$

According to equation (2.33), three types of motions are possible depending on the amount of damping present in the system, or depending on the value $\xi^2 - 1$ under the radical:

1) *over-damped* system when $\xi > 1$;
2) *critically damped* system when $\xi = 1$;
3) *under-damped* system when $\xi < 1$.

2.6.2.1 Over-damped structure

The system is said to be over-damped when the damping factor is greater than unity, $\xi > 1$. In this case, the roots of equation (2.33) are negative and real.

$$s_1 = -\xi\omega + \omega_s, \qquad s_2 = -\xi\omega - \omega_s \tag{2.34}$$

in which

$$\omega_s = \omega\sqrt{\xi^2 - 1} \tag{2.35}$$

this gives the general solution:

$$v(t) = e^{-\xi\omega t}\,(G_1 e^{\omega_s t} + G_2 e^{-\omega_s t}) \tag{2.36}$$

Figure (2.11) shows graphically the solution. It can be seen that no vibration is possible. Moreover, this case has no practical importance in the dynamic analysis of civil engineering structures.

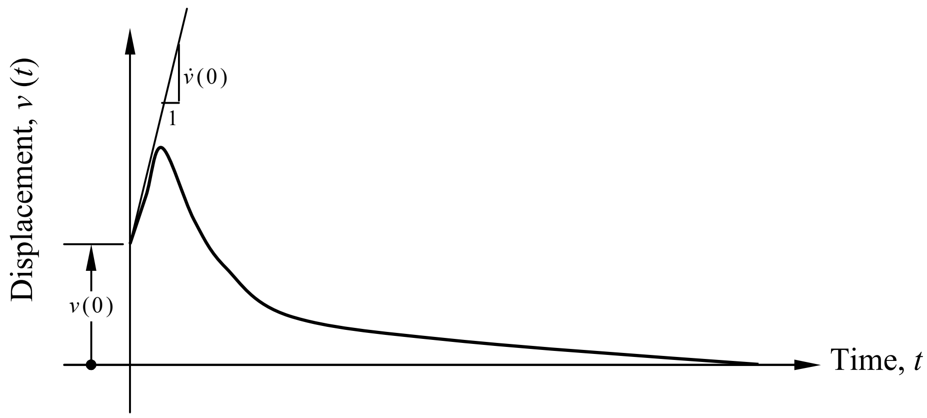


Figure 2.11: Free vibration response of an over- damped system.

2.6.2.2 Critically damped structure

The system is said to have critical damping when $\xi = 1$. In this case, the two roots are equal and the general solution becomes:

$$v(t) = (G_1 + G_2 t)e^{-\omega t} \tag{2.37}$$

with the initial conditions $v\,(0)$ and $\dot{v}\,(0)$ at time $t = 0$, we get:

$$v(t) = \left[v(0)(1 + \omega t) + \dot{v}(0)t\right]e^{-\omega t} \tag{2.38}$$

The solution is presented graphically in figure (2.12). Again, note that the free vibration of a critically damped system has no oscillations. One can notice that the critical damping corresponds to the smallest value of the damping for which the response in free vibration does not comprise oscillations: it presents the limit between an oscillatory response and a monotonous response (without vibration).

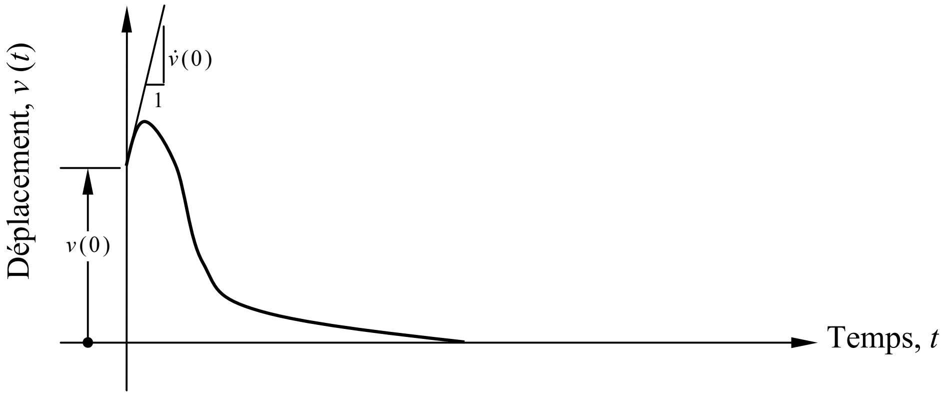


Figure 2.12: Free vibration response of a critically damped system.

We define for this case the critical damping

$$\xi_{cr} = \frac{C}{C_{cr}} \tag{2.39}$$

and the critical damping coefficient

$$C_{cr} = 2M\omega \tag{2.40}$$

The case of critical damping has no practical value. Civil engineering structures are always considered under-damped systems with viscous damping less than 20% critical ($\xi < 0.2$).

2.6.2.3 Under-damped structure

The system is said to be under-damped when the damping factor is less than unity, $\xi < 1$. In this case, the roots of equation (2.33) are conjugate and complex.

$$s_1 = -\xi\omega + i\omega_D\,, \qquad s_2 = -\xi\omega - i\omega_D \tag{2.41}$$

in which

$$i = \sqrt{-1} \qquad \omega_D = \omega\sqrt{1-\xi^2} \tag{2.42}$$

where ω_D is the pseudo angular frequency in damped oscillations or damped pulsation. The general solution is given by equation (2.43).

$$v(t) = G_1 e^{-\xi\omega t + i\omega_D t} + G_2 e^{-\xi\omega t - i\omega_D t} \tag{2.43}$$

which can be written in the following form:

$$v(t)=e^{-\xi\omega t}\,(G_1 e^{i\omega_D t}+G_2 e^{-i\omega_D t}) \qquad \textbf{(2.44)}$$

By using Euler's relation, the solution is written more simply:

$$v(t)=e^{-\xi\omega t}\,(A\sin\omega_D t+B\cos\omega_D t) \qquad \textbf{(2.45)}$$

where A and B are real. The velocity is obtained by derivation, which gives:

$$\dot{v}(t)=e^{-\xi\omega t}\left[(-\xi\omega A-\omega_D B)\sin\omega_D t+(-\xi\omega B+\omega_D A)\cos\omega_D t\right] \qquad \textbf{(2.46)}$$

For the initial conditions $v(0)$ and $\dot{v}(0)$, we deduce the constants from equation (2.45), which gives:

$$A=\frac{\dot{v}(0)+\xi\omega v(0)}{\omega_D}, \qquad B=v(0) \qquad \textbf{(2.47)}$$

Taking into account A and B, the displacement $v(t)$ is expressed as follows:

$$v(t)=e^{-\xi\omega t}\left[\frac{\dot{v}(0)+\xi\omega v(0)}{\omega_D}\sin\omega_D t+v(0)\cos\omega_D t\right] \qquad \textbf{(2.48)}$$

This response can also be written in the following form:

$$v(t)=\rho e^{-\xi\omega t}\cos(\omega_D t-\theta) \qquad \textbf{(2.49)}$$

where

$$\rho=\sqrt{\left[\frac{\dot{v}(0)+\xi\omega v(0)}{\omega_D}\right]^2+\left[v(0)\right]^2}$$

$$\theta=\tan^{-1}\frac{\dot{v}(0)+\xi\omega v(0)}{\omega_D v(0)} \qquad \textbf{(2.50)}$$

The graph of function (2.49) is shown in figure (2.13). This graph is a damped sinusoid between the two envelopes $\pm\rho e^{-\xi\omega t}$. The response is not periodic because of the exponentially decaying term $e^{-\xi\omega t}$. It is said to be an under-damped oscillatory response whose pseudo-period or *damped period* is:

$$T_D=\frac{2\pi}{\omega_D}>T \qquad \textbf{(2.51)}$$

The pseudo-pulsation of the under-damped system is:

$$\omega_D=\omega\sqrt{1-\xi^2}<\omega \qquad \text{pour}\ \xi<1 \qquad \textbf{(2.52)}$$

and the pseudo-frequency of the under-damped system has the expression:

$$f_D = \frac{1}{T_D} = \frac{\omega_D}{2\pi} < f \qquad pour \quad \xi < 1 \qquad \textbf{(2.53)}$$

Damping has the effect of decreasing the pulsation and therefore increasing the period of oscillations of a system compared to the corresponding undamped system.

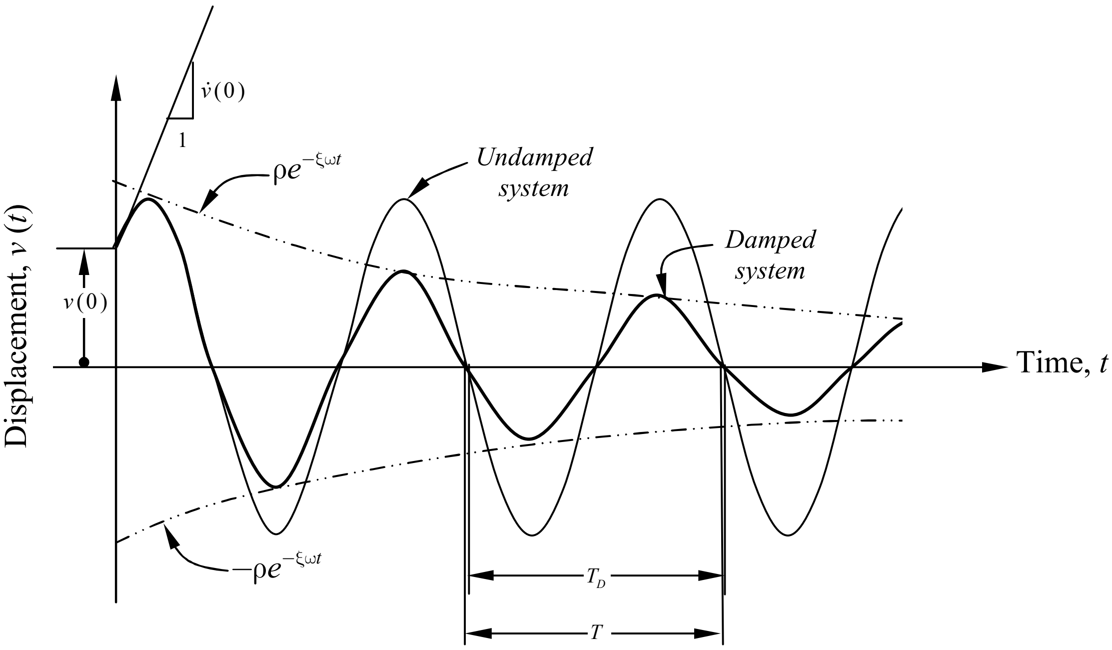


Figure 2.13: Free vibration response of an under-damped system.

Equation (2.42) tells us that the graph of ω_D / ω as a function of ξ is a circle of radius 1. Figure (2.14) shows the graph as a function of ξ where the range of common values of the damping factors of civil engineering structures is shaded and corresponds to $0 < \xi < 0.20$. For this range of values of ξ, the ratio ω_D / ω is practically constant and equal to 1. Indeed, the term ξ^2 is negligible compared to 1 and the pseudo-pulsation ω_D can be confused with the natural pulsation ω without appreciable error. For example, for $\xi = 0.10$, $\omega_D = 0.995\omega$ and $T_D = 2\pi / \omega_D = T / \sqrt{1-\xi^2} = 1.005T$.

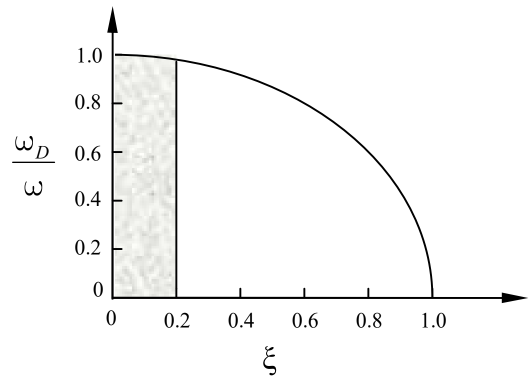


Figure 2.14: Effect of damping on the natural pulsation of vibration.

2.6.2.3.1 Logarithmic decrement

The actual damping characteristics of structures are very complex and difficult to determine. However, it is common practice to express the damping of these systems by means of equivalent viscous damping factors ξ which exhibit comparable decay characteristics in the case of free vibrations. It is therefore now appropriate to study in more detail the influence of the viscous damping factor ξ on the response in free vibration, figure (2.13).

Consider two successive peaks, of the same sign, of the response represented in figure (2.13): v_n et v_{n+1}. From equation (2.49), we derive the ratio of the amplitudes of these two peaks:

$$\frac{v_n}{v_{n+1}} = e^{2\pi\xi\frac{\omega}{\omega_D}} \quad \textbf{(2.54)}$$

By taking the natural logarithm of the two members of this equation, we obtain the *logarithmic decrement* δ:

$$\delta = \ln\frac{v_n}{v_{n+1}} = 2\pi\xi\frac{\omega}{\omega_D} \quad \textbf{(2.55)}$$

with the equation (2.52), we obtain:

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \quad \textbf{(2.56)}$$

For small dampings, this last equation can be reduced to:

$$\delta = 2\pi\xi \quad \textbf{(2.57)}$$

We can develop equation (2.54) in series:

$$\frac{v_n}{v_{n+1}} = e^{\delta} = e^{2\pi\xi} = 1 + 2\pi\xi + \frac{(2\pi\xi)^2}{2} + \cdots \quad \textbf{(2.58)}$$

For low values of ξ, the precision obtained by keeping only the first two terms of the series is sufficient, in which case, we obtain:

$$\xi = \frac{v_n - v_{n+1}}{2\pi v_{n+1}} \quad \textbf{(2.59)}$$

For poorly damped systems, better accuracy can be obtained by using peaks separated by several cycles, *m* cycles for example, in this case:

$$\delta = \ln\frac{v_n}{v_{n+1}} = 2m\pi\xi\frac{\omega}{\omega_D} \quad \textbf{(2.60)}$$

and for extremely low dampings, this last equation leads to the following approximate formulation:

$$\xi = \frac{v_n - v_{n+1}}{2\pi m v_{n+1}} \quad \textbf{(2.61)}$$

EXERCISES

2.1.

Consider the mass-spring systems shown in figure (2.15).

1/ Calculate the natural period of vibration $T(s)$ corresponding to each system.

2/ For case (c), determine the response $v(t)$ at $t = 2$ s.

With : $\boldsymbol{k_1 = k_2 = 2\ 10^5}$ **N / m**, $\boldsymbol{k_3 = 3\ 10^5}$ **N / m.**

$\boldsymbol{M}$ **= 2000 kg.**

Initial conditions: $\boldsymbol{v(t = 0s) = 3}$ **cm** et $\boldsymbol{\dot{v}(t = 0s) = 20}$ **cm / s**

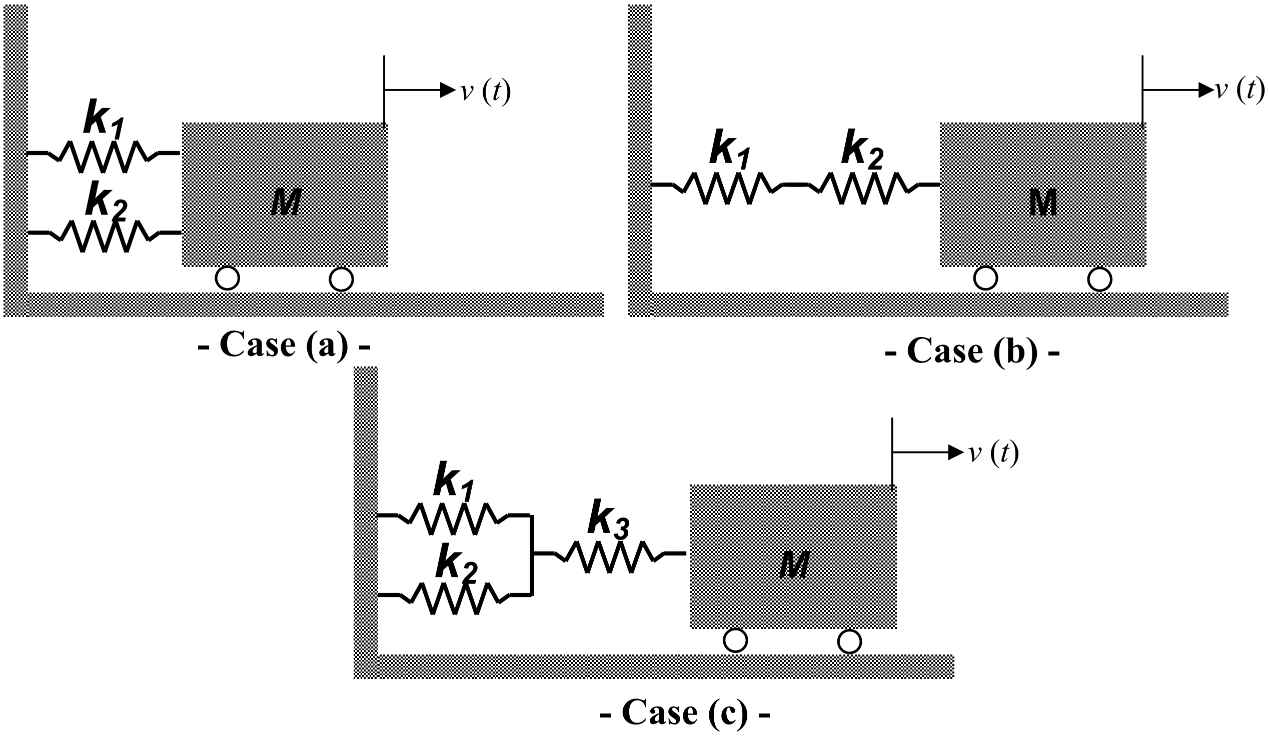


Figure 2.15

2.2.

Consider the steel structure shown in figure (2.16). Its natural period of vibration is **0.6 sec**. When a plate having a mass of **200 kg** is fixed on the floor surface of the steel structure, the natural period of vibration of the latter is lengthened, and it is equal to **0.75 sec**.

- Determine the mass and stiffness of the steel structure.

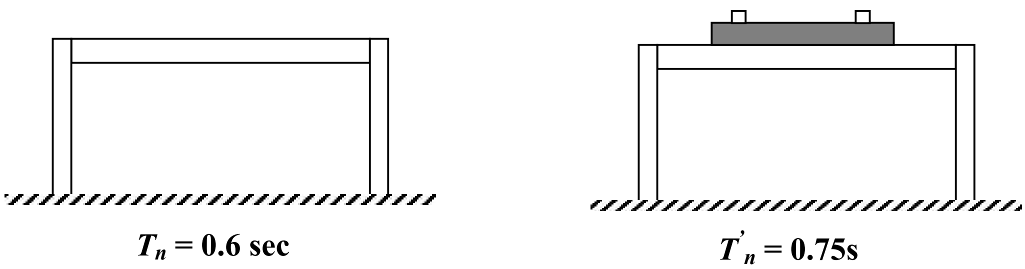


Figure 2.16

2.3.

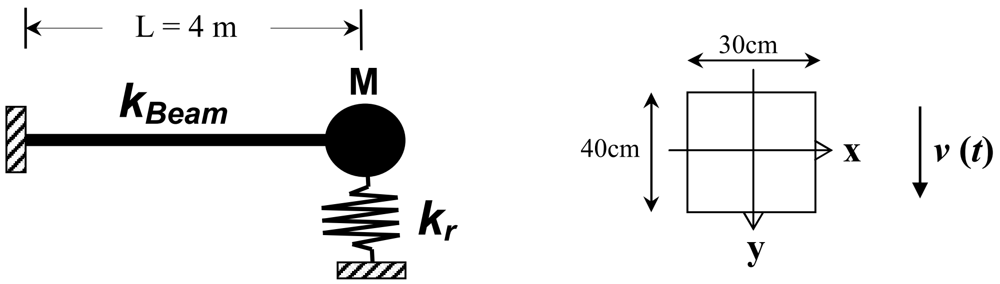
Consider the dynamical system shown in figure (2.17).

1/ Calculate the natural period of vibration $T(s)$ of the system.

2/ Determine the response $v(t)$ at $t = 2$ s.

With: $\boldsymbol{k_r = 2\ 10^5}$ **N / m,** $\boldsymbol{M = 1000}$ **kg,** $\boldsymbol{E = 30000}$ **N/mm²**.

Initial conditions : $\boldsymbol{v(t = 0s) = 2}$ **cm** et $\boldsymbol{\dot{v}(t = 0s) = 15}$ **cm / s**



Cross section of the beam

Figure 2.17

2.4.

Consider the frame structure shown in figure (2.18). It was determined that a static force **F=3500 kN** applied horizontally at the level of the beam produces an horizontal displacement **d = 10 cm**.

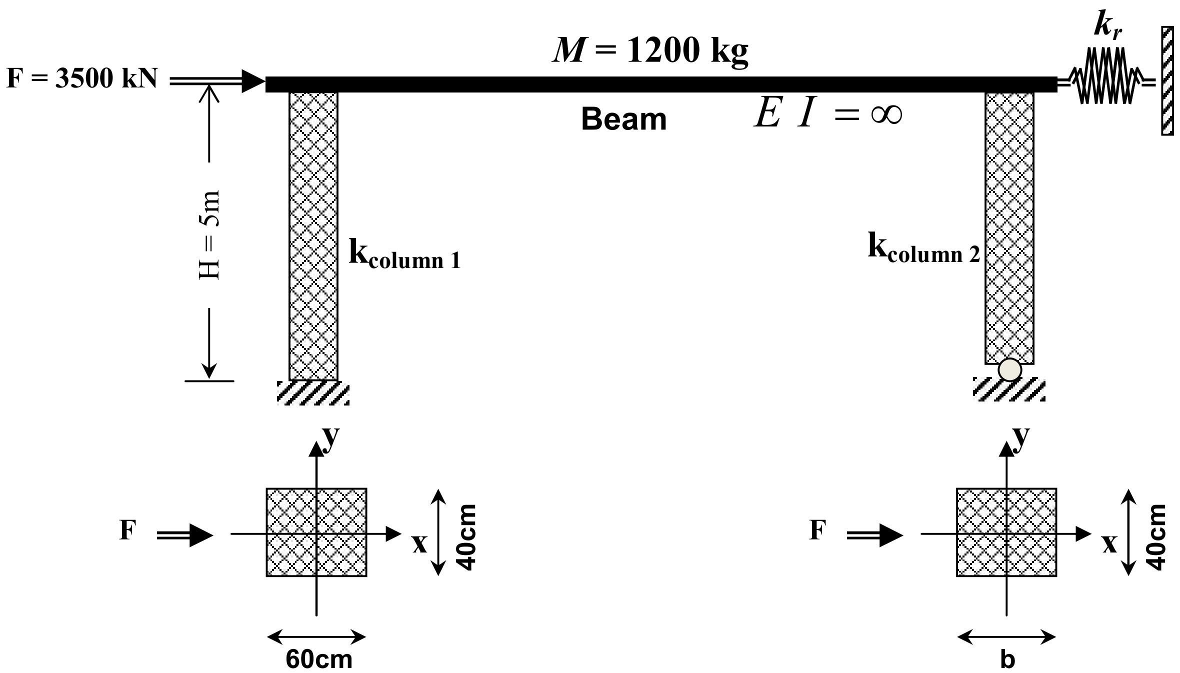
1/ Determine the dimension '**b**' of the column N°2.

2/ Calculate the natural pulsation ω (rad/s) of the structure.

3/ Determine the undamped free vibration response of the structure, $v(t)$ at $t = 1.5$s.

With $\boldsymbol{M = 1200}$ **Kg**, $\boldsymbol{E = 3\ 10^{10}}$ **N/m²** , $\boldsymbol{K_r = 5\ 10^5}$ **N/m**

Initial conditions: v **(t=0s) = 2 cm** and $\boldsymbol{\dot{v}}$**(t=0s) = 13 cm/s**



Cross section of column 1

Cross section of column 2

Figure 2.18

2.5.

A single-storey building is idealized by a rigid beam supported by massless columns, see figure (2.19). In order to determine the dynamic characteristics of this structure, a free vibration test is carried out in which the roof (the rigid beam) is moved laterally by a hydraulic jack and then released. During the pushing action of the cylinder, it can be seen that a force of **90 kN** is necessary to move the beam by **0.5cm**. After the force applied by the actuator has been canceled instantaneously, the maximum displacement on return (after one cycle) is only **0.4cm**, for a displacement cycle duration equal to $\boldsymbol{T}$ **= 1.40s**. Determine:

1/ The effective mass of the beam (M).
2/ The angular frequency of motion (ω).
3/ The damping factor (ξ) and the damping constant (C).
4/ The pseudo frequency of the system (ω_D).

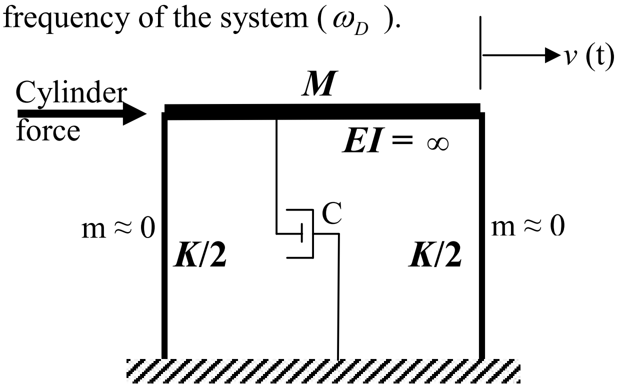


Figure 2.19

2.6.

Consider the dynamical system shown in figure (2.20).

1/ Calculate the natural period of vibration $T(s)$ of the system.
2/ Determine the response $v(t)$ at $t = 2$ s. With : $\boldsymbol{k_r = 2\ 10^5}$ **N / m,** $\boldsymbol{M}$ **= 1500 kg,**
$\boldsymbol{E = 310^{10}}$ **N/m²**. Initial conditions: $\boldsymbol{v\ (t=0s) = 2}$ **cm** and $\boldsymbol{\dot{v}\ (t=0s) = 10}$ **cm / s**

The stiffness of the diagonal bracing: $\boldsymbol{K_{dia.} = (EA / L_D).\cos^2\theta}$, with $\boldsymbol{A}$ **: area of the cross section of the diagonal.**

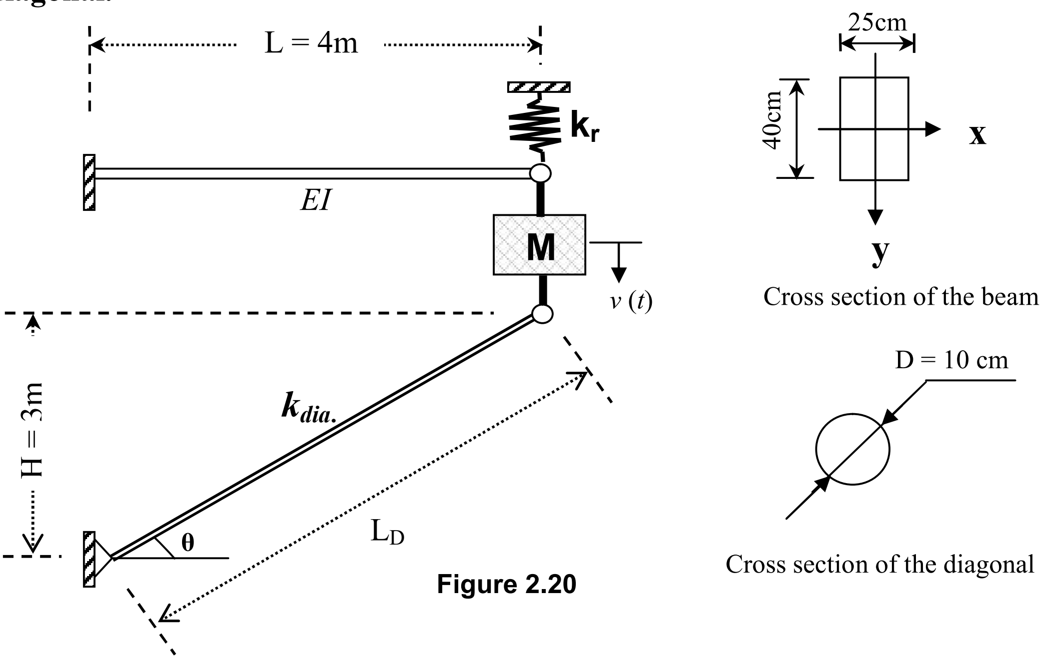


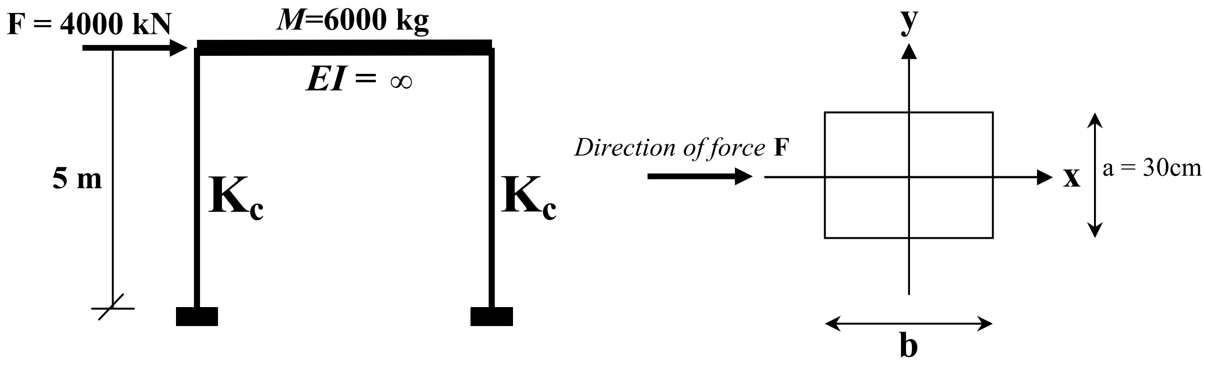
Figure 2.20

2.7.

Consider the structure shown in figure (2.21). The section of the columns is rectangular with dimension **'axb'**. The equivalent mass of the structure is $\boldsymbol{M}$ **= 6000Kg**. By an experiment, it was determined that a static force **F = 4000 kN** applied horizontally at the level of the rigid beam produces a horizontal displacement **d = 30 cm**.

Young's modulus is equal to **E = 30000 MPa**. Determine for this structure:

- The dimension **'b'** of the column, as well as the natural period of vibration $T(s)$.



Cross section of the columns

To reduce the horizontal displacement of the rigid beam, a column of square dimension (a = **40 cm**) is incorporated into the structure via a *massless and infinitely rigid bar element*, see figure below. Determine:

- The new displacement of the rigid beam.
- The reduction rate of the period $T(s)$ of the structure.

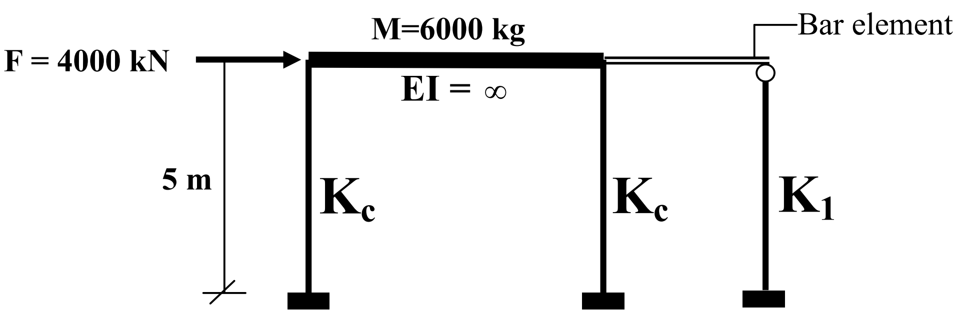


Figure 2.21

CHAPTER

3

DYNAMICS ANALYSIS IN HARMONIC FORCED VIBRATION OF STRUCTURES WITH A SINGLE DEGREE OF FREEDOM

3.1 INTRODUCTION

This part presents the dynamic analysis of a linear system comprising a single DOF subjected to a harmonic loading $p(t)$ of amplitude p_0 and angular frequency $\bar{\omega}$, as shown in figure (3.1).

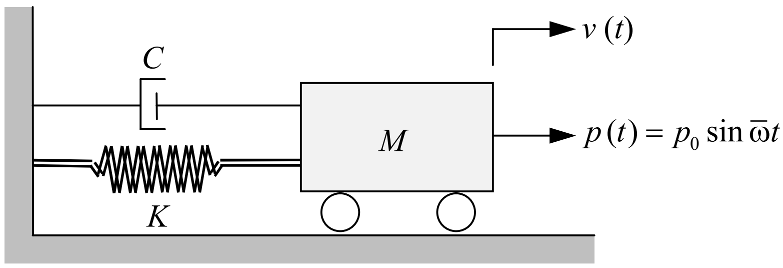


Figure 3.1: Elementary system with a single DOF subjected to a harmonic loading.

The equation of motion is of the following form:

$$M\ddot{v}(t) + C\dot{v}(t) + Kv(t) = p_0 \sin \bar{\omega} t \qquad \textbf{(3.1)}$$

The solution $v(t)$ of equation (3.1) is the sum of the homogeneous solution $v_h(t)$ of the homogeneous equation with $p(t) = 0$ and of the particular solution $v_p(t)$ of the non homogeneous equation , therefore related to the load $p(t)$, so :

$$v(t) = v_h(t) + v_p(t) \qquad \textbf{(3.2)}$$

In this chapter, we will study the solutions of equation (3.1) for undamped and damped systems.

3.2 UNDAMPED FORCED STRUCTURE

The forced vibration is undamped when $C = 0$. The equation of motion (3.1) becomes:

$$M\ddot{v}(t) + Kv(t) = p_0 \sin \overline{\omega} t \qquad \textbf{(3.3)}$$

The homogeneous solution (without second member) of this equation is the response in free vibration of the equation (2.21):

$$v_h(t) = A \sin \omega t + B \cos \omega t \qquad \textbf{(3.4)}$$

Let's look for a particular solution of the following form:

$$v_p(t) = G \sin \overline{\omega} t \qquad \textbf{(3.5)}$$

We therefore assume that the motion is harmonic, of the same angular frequency as the external harmonic loading and in phase with it. Moreover, we assume that the natural pulsation is not present in the expression of the particular solution. We must find the amplitude G so that equation (3.3) is satisfied for all values of $\overline{\omega}$ and t. After deriving equation (3.5) twice, we get the expression for the acceleration:

$$\ddot{v}_p(t) = -\overline{\omega}^2 G \sin \overline{\omega} t \qquad \textbf{(3.6)}$$

Substituting equations (3.6) and (3.5) into equation (3.3), we find:

$$-M\overline{\omega}^2 G \sin \overline{\omega} t + KG \sin \overline{\omega} t = p_0 \sin \overline{\omega} t \qquad \textbf{(3.7)}$$

Dividing back and forth by $sin\overline{\omega}t$ and by K, and noticing that $K / M = \omega^2$, we get:

$$G\left(1 - \frac{\overline{\omega}^2}{\omega^2}\right) = \frac{p_0}{K} \qquad \textbf{(3.8)}$$

The amplitude of the response then becomes:

$$G = \frac{p_0}{K}\frac{1}{1-\beta^2} \qquad \textbf{(3.9)}$$

where β is the ratio of the loading pulsation and the natural (proper) pulsation of free vibration, or else ratio of the pulsations:

$$\beta = \frac{\overline{\omega}}{\omega} \qquad \textbf{(3.10)}$$

Equation (3.9) defines G such that equation (3.3) is satisfied for all values of $\overline{\omega}$ and t. Figure (3.2) shows the variation of G as a function of β. We notice that G suddenly changes from infinitely large positive values to infinitely large negative values, passing through $\overline{\omega} = \omega$: this is the phenomenon of amplitude resonance. When $\beta < 1$ or $\overline{\omega} < \omega$, the amplitude G is positive, indicating that $v_p(t)$ and $p(t)$ have the same algebraic sign, i.e., when the force in figure (3.1) acts to the right, the system also moves to the right. It is said that the displacement is in phase with the applied loading. On the other hand, when $\beta > 1$ or $\overline{\omega} > \omega$, the amplitude G is negative, indicating that $v_p(t)$ and $p(t)$ have opposite algebraic signs. This means that when the force of figure (3.1) acts to the right, the system moves to the left. It is said that the displacement is in phase opposition with the applied loading.

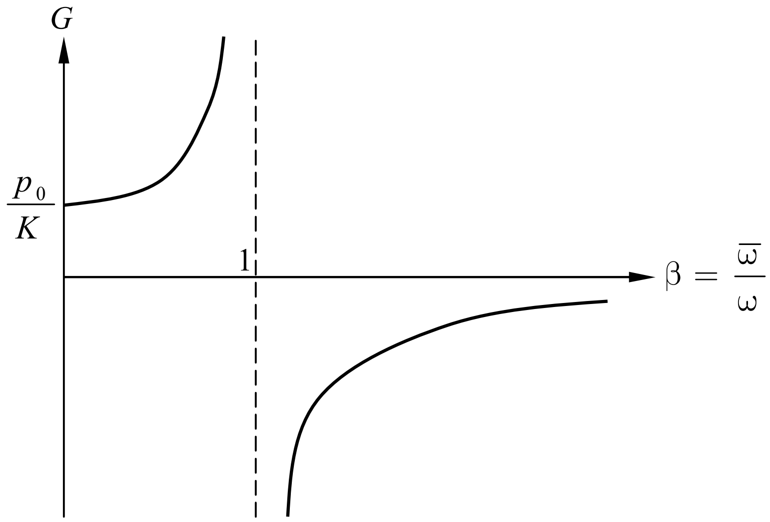


Figure 3.2: Harmonic forced vibration amplitude as a function of the pulsation $\overline{\omega}$ for zero damping. The negative sign of the amplitude for $\overline{\omega} > \omega$ corresponds to a delay of π of the displacement with respect to the exciting load.

After substituting equation (3.9) into equation (3.5), the particular solution is written as follows:

$$v_p(t) = \frac{p_0}{K} \frac{1}{1-\beta^2} \sin \overline{\omega} t \qquad \textbf{(3.11)}$$

The general solution corresponding to a harmonic excitation applied to an undamped system is then given by the sum of the homogeneous solution and the particular solution. So:

$$v(t) = A \sin \omega t + B \cos \omega t + \frac{p_0}{K} \frac{1}{1-\beta^2} \sin \overline{\omega} t \qquad \textbf{(3.12)}$$

and the velocity is obtained by derivation from equation (3.12). Which give:

$$\dot{v}(t) = A\omega \cos \omega t - B\omega \sin \omega t + \overline{\omega} \frac{p_0}{K} \frac{1}{1-\beta^2} \cos \overline{\omega} t \qquad \textbf{(3.13)}$$

For initial conditions at $t = 0$ at rest, i.e., $v(0) = 0$ and $\dot{v}(0) = 0$, it is easy to show that the two constants A and B have the value:

$$A = -\frac{p_0 \beta}{K} \frac{1}{1-\beta^2}, \qquad B = 0 \qquad \textbf{(3.14)}$$

The complete solution then becomes:

$$v(t) = \frac{p_0}{K} \frac{1}{1-\beta^2} \left(\sin \overline{\omega} t - \beta \sin \omega t \right), \qquad \beta \neq 1 \qquad \textbf{(3.15)}$$

Where $P_0 / K = v_{st}$, static displacement that would have been produced by the force p_0 applied statically;

$1/\left(1-\beta^2\right)$ = amplification factor representing the dynamic amplification effect of the harmonic load;

$\sin\bar{\omega}t$ = component of the total displacement having the frequency of the applied load, also called harmonic forced vibration, directly related to the applied load;

$\beta\sin\omega t$ = component of the total displacement having the natural frequency of the system, also called free vibration which depends on the initial conditions.

Define the *dynamic response factor*, $R(t)$, as the ratio of dynamic displacement to maximum static displacement. Thus :

$$R(t) = \frac{v(t)}{v_{st}} = \frac{v(t)}{p_0 / K} \tag{3.16}$$

In the case of the undamped system, for initial conditions at rest, we have:

$$R(t) = \frac{1}{1-\beta^2}(\sin\bar{\omega}t - \beta\sin\omega t) \tag{3.17}$$

When $\beta = 1$, there is *amplification resonance*, i.e., $R(t) \to \infty$ and $v(t) \to \infty$. It does, however, take infinite time to reach infinite displacement. In practice, the phenomenon of resonance causes plasticization of the material before reaching extreme displacements. For $\beta = 1$, $R(t) = 0/0$. However, the displacement $v(t)$ can be calculated using L'Hospital's rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{3.18}$$

So, after replacing $\bar{\omega}$ by $\omega\beta$ in (3.17), we get:

$$\lim_{\beta\to 1} R(t) = \lim_{\beta\to 1}\left[\frac{\omega t\cos\beta\omega t - \sin\omega t}{-2\beta}\right] = \frac{1}{2}(\sin\omega t - \omega t\cos\omega t) \tag{3.19}$$

Figure (3.3) shows the graph of the function $R(t)$ at resonance. The period of the function is $2\pi/\omega$. The term $1/2\sin\omega t$ is small and can be overlooked. The amplitude of $R(t)$ increases linearly. The time to peak is obtained by setting the derivative of $R(t)$ in (3.19) to zero. We have:

$$\frac{1}{2}\omega^2 t\sin\omega t = 0 \tag{3.20}$$

This equation is satisfied for:

$$\omega t = n\pi, \qquad n = 0,1,2,\ldots \tag{3.21}$$

The difference between two successive peaks of the same sign is:

$$R\left(\frac{2\pi}{\omega}-\frac{n\pi}{\omega}\right)-R\left(\frac{n\pi}{\omega}\right)=-\pi\cos n\pi=\pm\pi \qquad \textbf{(3.22)}$$

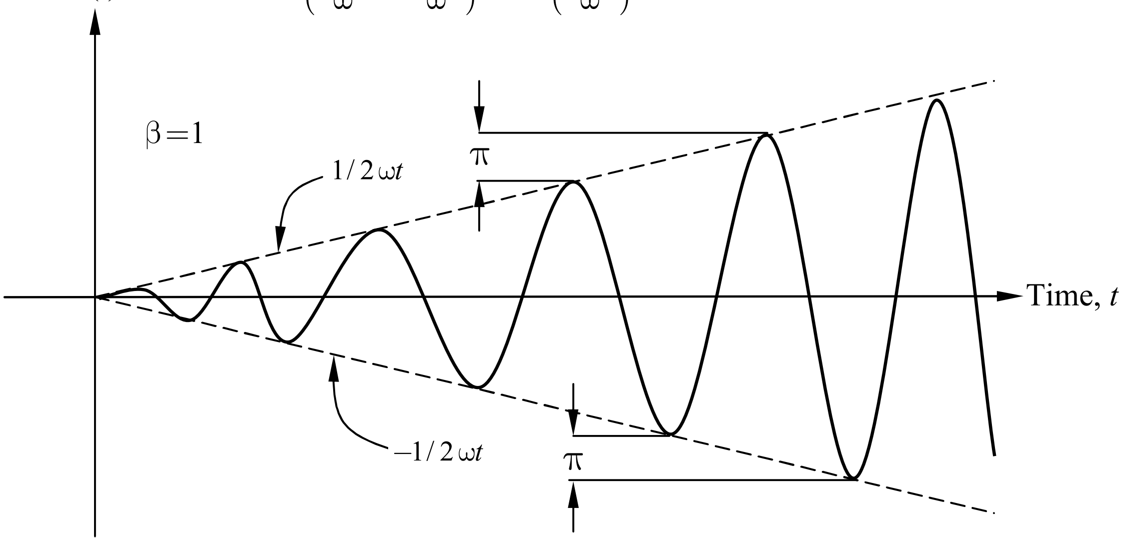


Figure 3.3: Variation of the dynamic response factor as a function of time at resonance.

3.3 DAMPED FORCED STRUCTURE

Taking into account equation (3.1), and noting that $C/M=2\xi\omega$, the equation of motion of the system in damped forced vibration is the following:

$$\ddot{v}(t)+2\xi\omega\dot{v}(t)+\omega^2 v(t)=\frac{p_0}{M}\sin\overline{\omega}\,t \qquad \textbf{(3.23)}$$

The homogeneous solution of this equation gives the response in free vibrations of the damped system; it is given by equation (2.45) (assuming, as is almost always the case in practice, that the structure is under-damped):

$$v_h(t)=e^{-\xi\omega t}\left(A\sin\omega_D t+B\cos\omega_D t\right) \qquad \textbf{(3.24)}$$

Let's look for a particular solution of the following form:

$$v_p(t)=G_1\sin\overline{\omega}t+G_2\cos\overline{\omega}t \qquad \textbf{(3.25)}$$

where the second term is necessary because a damped system is generally not in phase with the applied loading. The velocity is obtained by derivation of the equation (3.25):

$$\dot{v}_p(t)=G_1\overline{\omega}\cos\overline{\omega}t-G_2\overline{\omega}\sin\overline{\omega}t \qquad \textbf{(3.26)}$$

and the acceleration is expressed by:

$$\ddot{v}_p(t)=-G_1\overline{\omega}^2\sin\overline{\omega}t-G_2\overline{\omega}^2\cos\overline{\omega}t \qquad \textbf{(3.27)}$$

Substituting equations (3.25) to (3.27) in equation (3.23) and separating the sine and cosine terms, we get:

$$\left[-G_1\overline{\omega}^2 - G_2\overline{\omega}(2\xi\omega) + G_1\omega^2\right]\sin\overline{\omega}t = \frac{p_0}{M}\sin\overline{\omega}t$$

$$\left[-G_2\overline{\omega}^2 + G_1\overline{\omega}(2\xi\omega) + G_2\omega^2\right]\cos\overline{\omega}t = 0 \qquad \textbf{(3.28)}$$

These two functions must be satisfied individually because the sine and cosine functions do not cancel each other simultaneously. Dividing by ω^2, we get:

$$\left[G_1(1-\beta^2) - G_2(2\xi\beta)\right] = \frac{p_0}{M}$$

$$\left[G_2(1-\beta^2) + G_1(2\xi\beta)\right] = 0 \qquad \textbf{(3.29)}$$

from which we deduce the constants G_1 and G_2:

$$G_1 = \frac{p_0}{K}\frac{1-\beta^2}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

$$G_2 = \frac{p_0}{K}\frac{-2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2} \qquad \textbf{(3.30)}$$

The complete solution is the sum of the homogeneous solution (3.24) and the particular solution (3.25) and is given:

$$v(t) = e^{-\xi\omega t}\left(A\sin\omega_D t + B\cos\omega_D t\right) + \frac{p_0}{K}\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\left((1-\beta^2)\sin\overline{\omega}t - 2\xi\beta\cos\overline{\omega}t\right) \qquad \textbf{(3.31)}$$

where the first term represents the transient response which quickly disappears due to the term $e^{-\xi\omega t}$ and the second term represents the permanent response (Fig. 3.4). The constants A and B of the transient part of the total response can be evaluated for given initial conditions. Let $v(0)$ be the initial displacement and $\dot{v}(0)$ the initial velocity, the constants A and B are worth:

$$A = \frac{p_0}{K}\frac{\omega}{\omega_D}\left(\frac{2\xi^2\beta - \beta(1-\beta^2)}{(1-\beta^2)^2 + (2\xi\beta)^2}\right) + \frac{\dot{v}(0) + v(0)\omega\xi}{\omega_D}$$

$$B = \frac{p_0}{K}\frac{2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2} + v(0) \qquad \textbf{(3.32)}$$

Figure (3.4) shows the response graph of the under-damped system subjected to harmonic loading. The transient response is damped over time, all the more rapidly as the percentage of critical damping is high and the response tends towards the permanent solution. This response then takes place with a period $T = 2\pi / \bar{\omega}$ equal to that of the applied load.

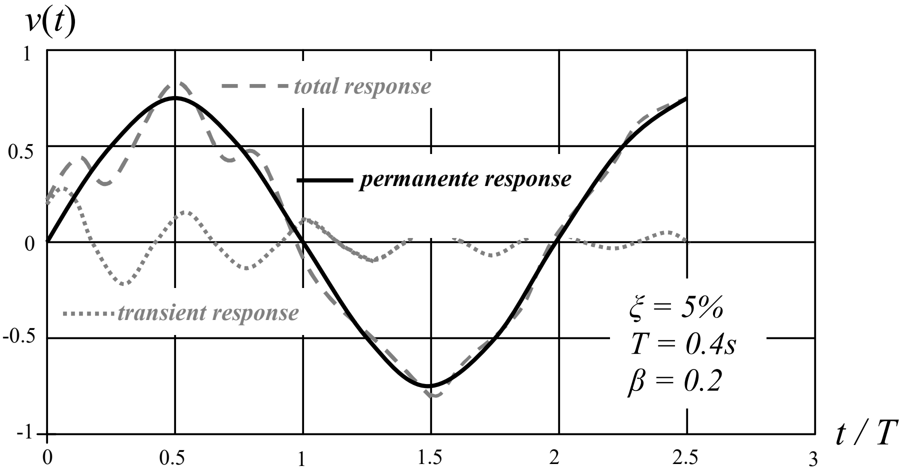


Figure 3.4: Response of the damped system to a harmonic load.

For even a weakly damped system (ξ = 5%), as soon as the duration becomes greater than twice the oscillator's proper vibration period $T = 2\pi / \omega$, the contribution of the transient response can be neglected. One will be interested then, in the rest of this chapter, only with the permanent part of the response after disappearance of the transient terms due to the damping of the system. The permanent response is then written:

$$v_p(t) = \frac{p_0}{K} \frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2} \left((1-\beta^2)\sin\bar{\omega}t - 2\xi\beta\cos\bar{\omega}t \right) \quad \textbf{(3.33)}$$

Figure (3.5) illustrates the position of the two components of the permanent response on the Argand diagram. The resultant of these two vectors is the amplitude ρ of the motion in harmonic steady state. Thus:

$$\rho = \sqrt{G_1^2 + G_2^2} = \frac{p_0}{K} \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \quad \textbf{(3.34)}$$

and the phase angle θ is the delay of the response with respect to the applied force, and is given by:

$$\theta = \tan^{-1} \frac{2\xi\beta}{1-\beta^2} \quad \textbf{(3.35)}$$

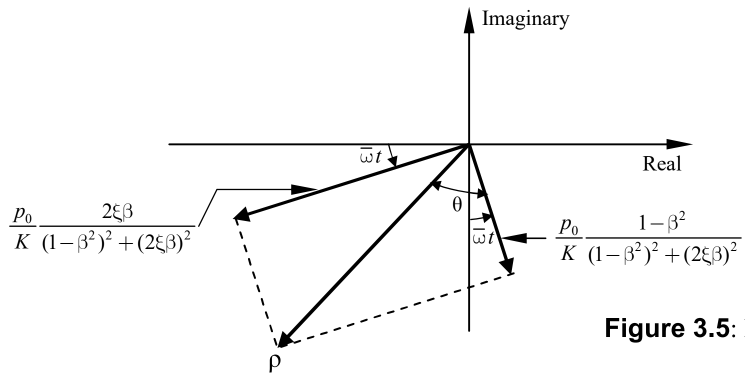


Figure 3.5: Harmonic steady state amplitude.

where the phase shift is between 0 and 180°. The permanent component of the response can therefore be expressed by:

$$v(t) = \rho \sin(\bar{\omega} t - \theta) \quad \textbf{(3.36)}$$

The *dynamic displacement amplification factor* $D = \rho / v_{st}$ is the ratio of the amplitude of the motion ρ given by equation (3.34) to the displacement $v_{st} = p_0 / K$ that would be produced by applying the force p_0 statically. So:

$$D = \frac{\rho}{p_0 / K} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \quad \textbf{(3.37)}$$

This dynamic amplification factor is shown in figure (3.6) as a function of the pulsation ratio β and the damping factor ξ.

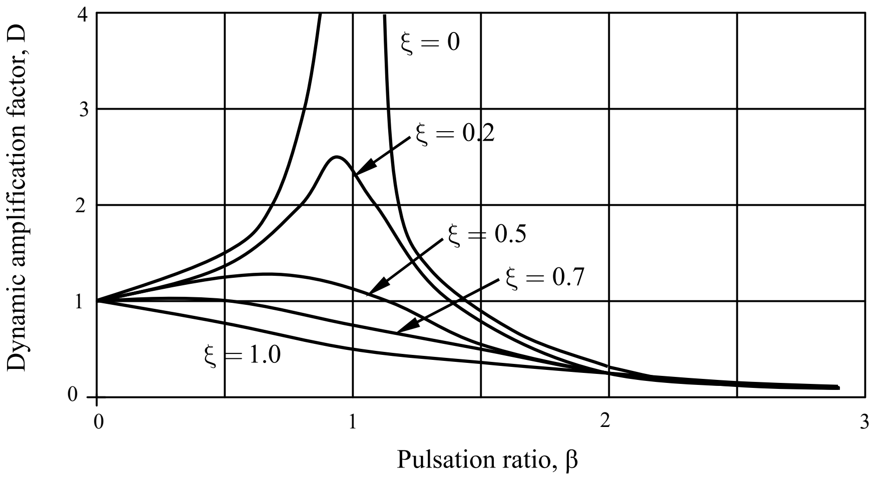


Figure 3.6: Variation of dynamic amplification factor, D, as a function of pulsation ratio, β, and damping factor, ξ.

As for the undamped system, the *dynamic response factor R*(*t*) is defined as the ratio of the dynamic displacement to the static displacement caused by the load p_0. So:

$$R(t) = D \sin(\bar{\omega} t - \theta) \quad \textbf{(3.38)}$$

For the amplification to be maximum, the denominator of equation (3.37) must be minimum, so:

$$\frac{d}{d\beta}\left(\left(1-\beta^2\right)^2 + (2\xi\beta)^2\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{-4\beta(1-\beta^2) + 4\beta(2\xi^2)}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}\right) = 0 \quad \textbf{(3.39)}$$

from where

$$\beta(-1 + \beta^2 + 2\xi^2) = 0 \quad \textbf{(3.40)}$$

A solution corresponds to $\beta = 0$ therefore $\bar{\omega} = 0$. This point corresponds to the maximum static displacement $v_{st} = p_0 / K$ and is a maximum of *D* for $\xi > \sqrt{2}/2 = 0.7$. The nonzero solution gives the maximum of *D*, that is:

$$\beta = \sqrt{1-2\xi^2} \tag{3.41}$$

and the corresponding pulsation is:

$$\omega_{rd} = \omega\sqrt{1-2\xi^2} \tag{3.42}$$

where the subscript rd indicates that it is the *pulsation at the resonance of the displacement*. The *maximum dynamic amplification factor*, D_{max}, is given by substituting the β value given by equation (3.41) into equation (3.37). So:

$$D_{\max} = \frac{1}{2\xi\sqrt{1-\xi^2}} \tag{3.43}$$

There is *amplitude resonance* when the absolute value of the amplitude of the displacement is maximum, so when $R_d = (R_d)_{max}$. If $\beta = 1$, equation (3.37) gives:

$$D_{\beta=1} = \frac{1}{2\xi} \tag{3.44}$$

For low damping values, i.e. $\xi << 1$, D_{max} is substantially equal to $D_{\beta=1}$.

3.4 RESONANCE

To understand the phenomenon of resonance, it is necessary to return to the general equation expressing the total response of the system under a harmonic load, in which both the permanent term and the transient term intervene. For $\beta = 1$ and therefore for $\overline{\omega} = \omega$, the equation of motion (3.31) can be written as follows:

$$v(t) = e^{-\xi\omega t}\,(A\sin\omega_D t + B\cos\omega_D t) - \frac{p_0}{K}\frac{\cos\omega t}{2\xi} \tag{3.45}$$

The transient response depends only on the initial conditions imposed on the system and is not maintained by the excitation. The forced response depends solely on the external force. So, for initial conditions at rest, i.e. $v(0) = 0$ and $\dot{v}(0) = 0$, the constants A and B become:

$$A = \frac{p_0}{K}\frac{\omega}{2\omega_D} = \frac{p_0}{K}\frac{1}{\sqrt{1-\xi^2}}, \qquad B = \frac{p_0}{K}\frac{1}{2\xi} \tag{3.46}$$

The response $v(t)$ becomes:

$$v(t) = \frac{1}{2\xi}\frac{p_0}{K}\left[e^{-\xi\omega t}\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t + \cos\omega_D t\right) - \cos\omega t\right] \tag{3.47}$$

With respect to damping, the sine term of this equation will contribute very little to the magnitude of the response (ξ is very small, for practical purposes); moreover the damped pulsation is almost equal to the undamped pulsation. Thus the answer (3.47) is written:

$$v(t) = \frac{1}{2\xi}\frac{p_0}{K}\left(e^{-\xi\omega t} - 1\right)\cos\omega t \qquad \textbf{(3.48)}$$

The dynamic response factor, $R(t)$, is therefore written:

$$R(t) = \frac{v(t)}{p_0 / K} = \frac{1}{2\xi}\left(e^{-\xi\omega t} - 1\right)\cos\omega t \qquad \textbf{(3.49)}$$

Figure (3.7) shows the establishment phase at resonance of a damped system. It is clear from this figure that the number of cycles to reach the maximum amplitude depends on the degree of damping of the system. It should be noted that, for the damping factors common in structures, it takes relatively few cycles to reach the maximum amplitude.

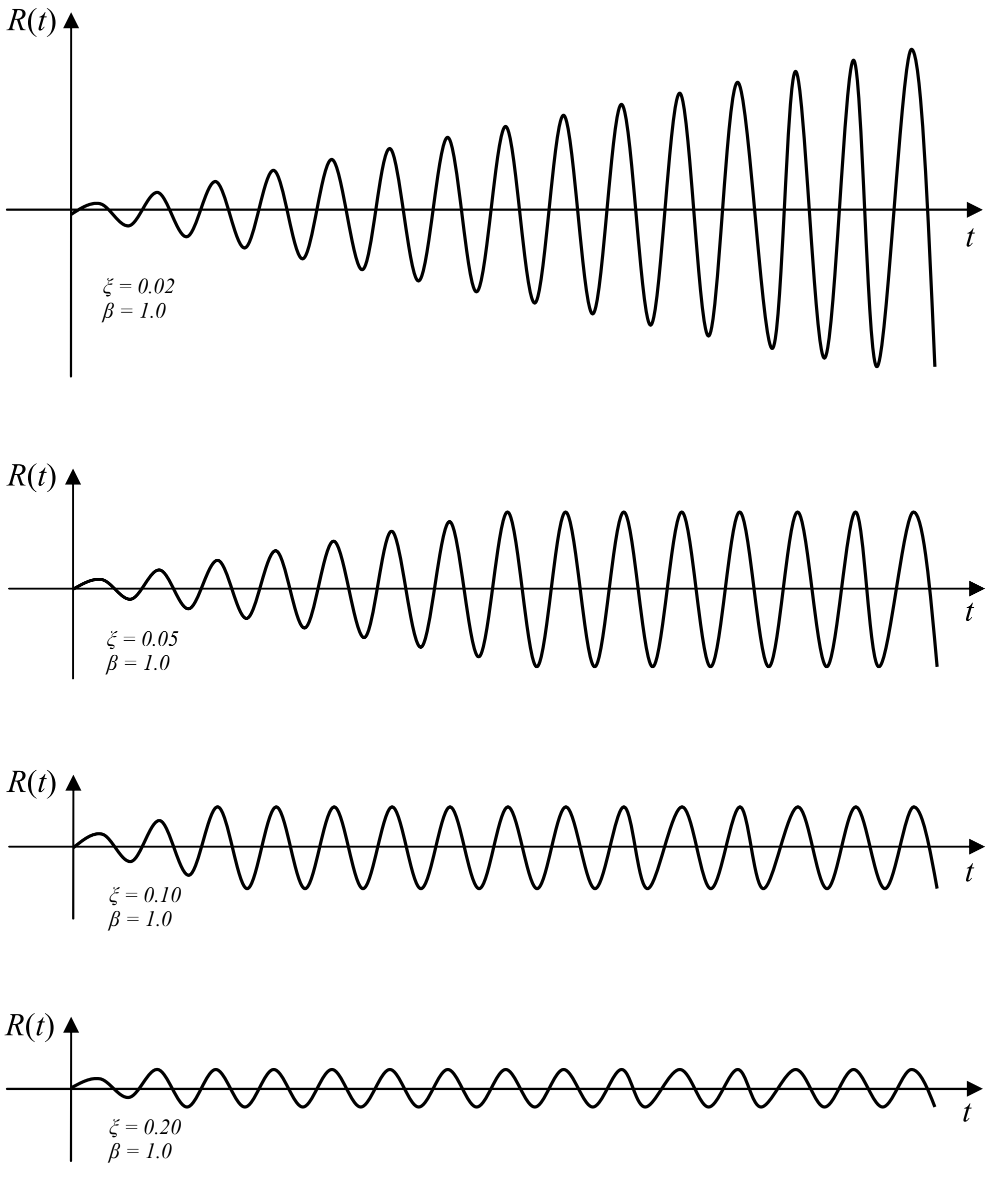


Figure 3.7: Establishment phase at resonance ($\beta = 1$) as a function of the damping factor, ξ, for initial conditions at rest.

3.5 VIBRATION MEASUREMENT DEVICES

Figure (3.8) presents a schematic experimental setup for measuring the dynamic motion of a system. A measuring device (1) is fixed to the structure of which it is desired to measure the vibrations (displacement, velocity or acceleration). This measuring device emits an electrical signal which passes through a charge amplifier (2) before being recorded in binary form for subsequent processing and display (3). In general, two types of devices are used, *amplitude sensors* or *seismographs* and *acceleration sensors* or *accelerometers*, which are seismic measuring devices.

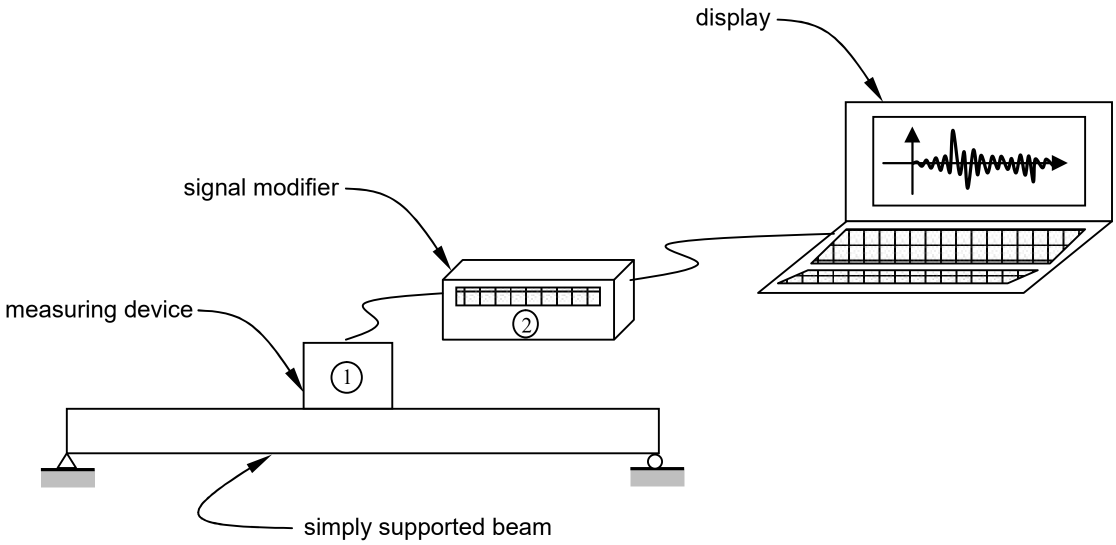


Figure 3.8: Experimental setup for measuring the movement of a structure.

Figure (3.9) shows the basic elements of a seismic measuring device. The device consists of a mass *M* supported by a spring of rigidity *K* and a viscous damping *C*. The case protecting the device is fixed to the surface whose movement is to be studied; the response is measured by means of the movement *v*(t) of the mass relative to the casing. The equation of motion of the system is given by:

$$M\ddot{v}(t) + C\dot{v}(t) + Kv(t) = p_{eff}(t) \tag{3.50}$$

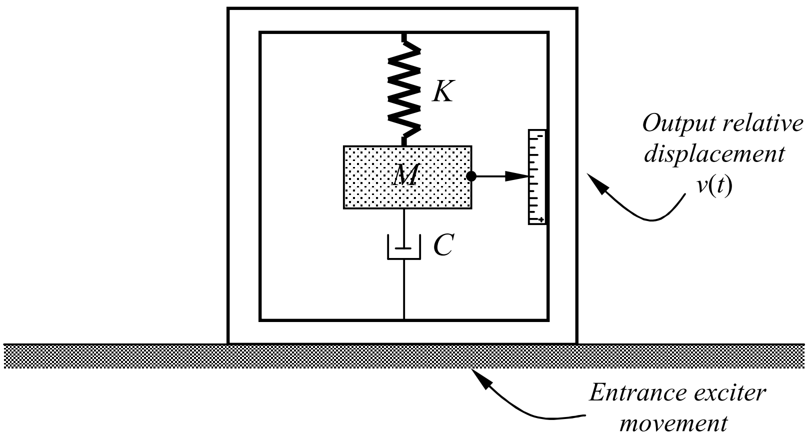


Figure 3.9: Schematic diagram of a vibration measuring device.

3.5.1 DISPLACEMENT SENSOR OR SEISMOGRAPH

We consider an instrument whose base is subject to the following harmonic displacement:

$$v_g(t) = v_{g0} \sin \overline{\omega} t \tag{3.51}$$

The base acceleration is given by:

$$\ddot{v}_g(t) = -\overline{\omega}^2 v_{g0} \sin \overline{\omega} t \tag{3.52}$$

and the effective loading becomes:

$$p_{eff}(t) = M \overline{\omega}^2 v_{g0} \sin \overline{\omega} t \tag{3.53}$$

So we have:

$$M\ddot{v}(t) + C\dot{v}(t) + Kv(t) = M \overline{\omega}^2 v_{g0} \sin \overline{\omega} t = p_0 \sin \overline{\omega} t \tag{3.54}$$

The amplitude of motion can be calculated from equation (3.34) and the definition of the dynamic amplification factor *D* in equation (3.37). So:

$$\rho = \frac{M \overline{\omega}^2 v_{g0}}{K} D = v_{g0} \beta^2 D \tag{3.55}$$

Figure (3.10) shows the variation of $\beta^2 D$ as a function of β and ξ. To serve as a displacement sensor, the device must have a proportional response to the movement of the base. From the curves, it can be seen that the factor $\beta^2 D$ is essentially constant and substantially equal to unity for pulsation ratios $\beta > 1$ if the damping factor is equal to $\xi = 0.5$. Thus, the response of a suitably damped device to high frequency movements is essentially proportional to the amplitude of the displacement of the support; it will therefore be used to measure displacements during such movements. In this case, the range of use of the instrument can be extended by reducing its natural pulsation, i.e. by decreasing the stiffness of the spring or by increasing the mass.

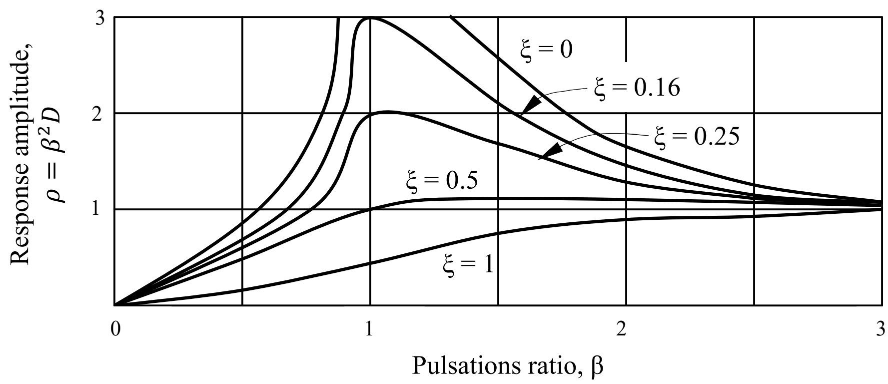


Figure 3.10: Response of a seismograph to a harmonic displacement of the base.

3.5.2 ACCELERATION SENSOR OR ACCELEROMETER

We consider the response of an instrument whose base is subjected to a harmonically varying acceleration according to the following equation:

$$\ddot{v}_g(t) = \ddot{v}_{g0} \sin \overline{\omega} t \tag{3.56}$$

the effective load of the mass is:

$$p_{eff}(t) = -M\,\ddot{v}_{g0} \sin \overline{\omega} t = p_0 \sin \overline{\omega} t \tag{3.57}$$

According to equations (3.34) and (3.37), the amplitude of the permanent response of the system is:

$$\rho = \frac{M\,\ddot{v}_{go}}{K} D \tag{3.58}$$

We see in figure (3.6), which represents the dynamic amplification factor D as a function of the ratio of the pulsations β and the damping factor ξ, that for a damping factor $\xi = 0.7$, the value of D is almost constant for $0 < \beta < 0.6$. From equation (3.58) it is therefore clear that the response indicated by the instrument will be directly proportional to the magnitude of the external acceleration for frequencies up to about 60% of the natural frequency of the 'instrument. Thus, this type of device, if suitably damped, will make a good accelerometer for relatively low frequencies; its field of application can be extended by increasing the stiffness of the spring.

3.6 ISOLATION OF THE VIBRATIONS

In practice, one often encounters the problem of reducing the dynamic forces transmitted by a machine to the structure which supports it or to its immediate environment. In this case, we say that we have a *force isolation problem*. The solution is to insert a suspension consisting of springs and dampers between the machine and the supporting structure or the foundation so as to reduce the force transmitted by the vibration of the machine. The inverse problem is also very common and consists in reducing the vibrations transmitted by the structure to the machine it supports. In this case, we face a *problem of isolation of movements*. The solution to this problem is also to insert a suspension between the moving supporting structure and the machine. We treat these two problems separately in the following to come to the conclusion that they are identical in their solution.

3.6.1 OSCILLATORY VERTICAL FORCE

Consider a vertical oscillatory force $p(t)$ produced by a rotating machine due to a certain unbalance and whose expression is:

$$p(t) = p_0 \sin \overline{\omega} t \tag{3.59}$$

If the machine is mounted on a support (suspension) with a single degree of freedom of the spring-damper type (Fig. 3.11 a), the steady-state response of the system under the action of this exciting force is given by:

$$v(t) = \frac{p_0}{K} D \sin\left(\bar{\omega} t - \theta\right) \qquad \textbf{(3.60)}$$

where D is the dynamic amplification factor given by equation (3.37).

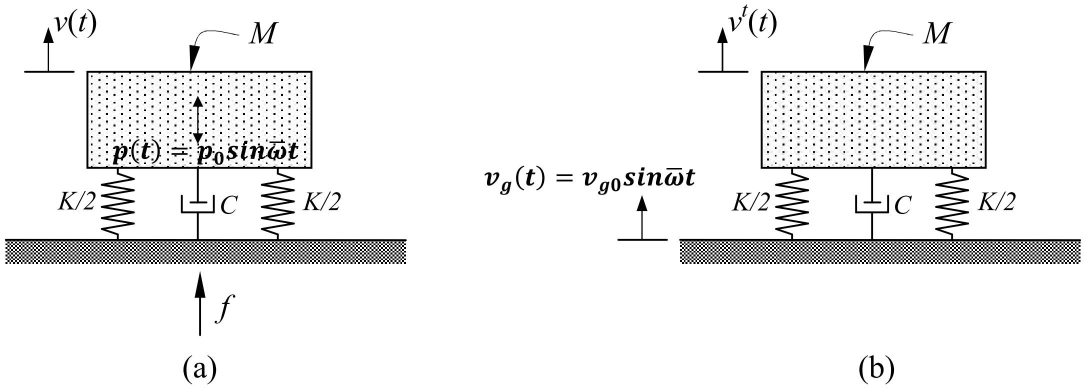


Figure 3.11: Isolation of vibration from (a) applied load and (b) base movement.

The force generated by the machine is transmitted to the support by the springs and the damper as shown in figure (3.11 a). Thus the force transmitted to the base by the spring is:

$$f_E = K v(t) = p_0 D \sin(\bar{\omega} t - \theta) \qquad \textbf{(3.61)}$$

The velocity of movement relative to the base is:

$$\dot{v}(t) = \frac{p_0}{K} D \bar{\omega} \cos(\bar{\omega} t - \theta) \qquad \textbf{(3.62)}$$

so the damping force is:

$$f_A = C\dot{v}(t) = \frac{C p_0 D \bar{\omega}}{K} \cos(\bar{\omega} t - \theta) = 2\xi\beta p_0 D \cos(\bar{\omega} t - \theta) \qquad \textbf{(3.63)}$$

As this force has a phase shift of 90° with that of the spring, it is obvious that the amplitude of the force at the base *f* will be:

$$f_{\max} = \sqrt{f_{E,\max}^2 + f_{A,\max}^2} = p_0 D \sqrt{1 + (2\xi\beta)^2} \qquad \textbf{(3.64)}$$

The ratio of the maximum force at the base and the amplitude of the applied force, which we will call the *transmissibility* of the support, is given by:

$$TR = \frac{f_{\max}}{p_0} = D\sqrt{1 + (2\xi\beta)^2} \qquad \textbf{(3.65)}$$

Figure (3.12) shows the curves of the transmissibility of a system as a function of the pulsation ratio β for various values of the damping factor ξ. Note that all the curves pass through the ordinate point 1 to $\beta = \sqrt{2}$. There is an amplification of the amplitude of the excitation force for $0 < \beta \leq \sqrt{2}$. There is a reduction in the amplitude of the excitation force – the transmissibility is less than 1– for $\beta > \sqrt{2}$.

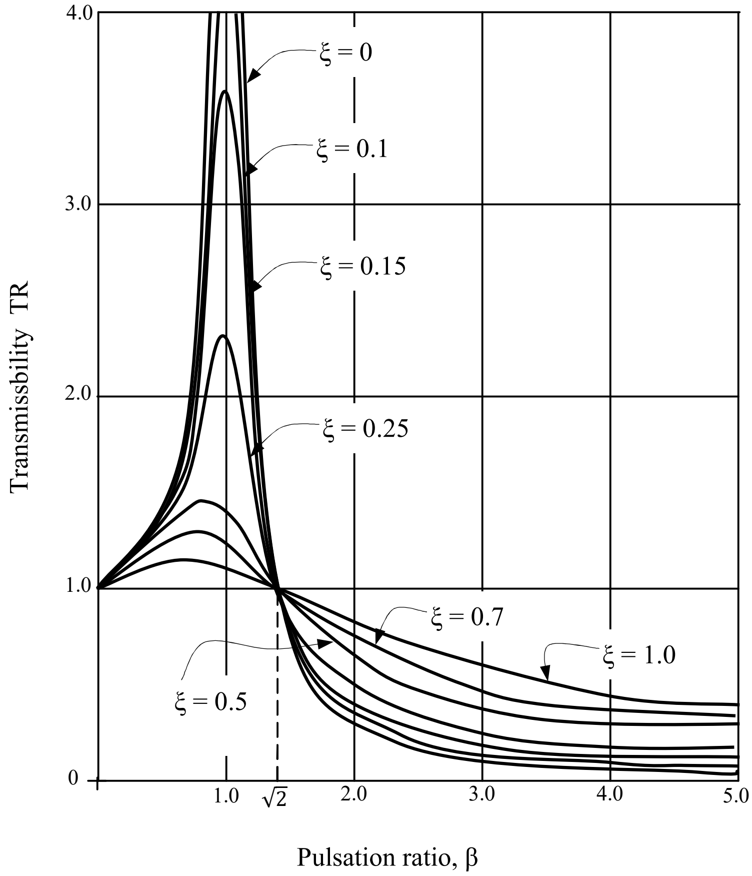


Figure 3.12: Vibration transmissibility ratio (excitation by loading or by displacement).

We also note that, in this interval, the transmissibility decreases with a decrease in damping and reaches its minimum for ξ = 0. At start-up, a rotating machine must go through resonance before reaching its operating frequency in steady state which must be in the range where the transmissibility is less than 1. In theory, zero damping would lead to the lowest transmitted force. In practice, however, some damping must be provided so as to keep the system response within reasonable limits when switching to the resonant frequency when starting the machine.

3.6.2 HARMONIC MOVEMENT OF THE BASE

Now consider the motion isolation problem. The mass *M* to be isolated is carried by a spring-damping system on a slab subjected to vertical harmonic movements as shown in figure (3.11b). The displacement of the mass with respect to the base, as indicated by equation (3.55), is then given by:

$$v(t) = v_{g0}\beta^2 D \sin(\overline{\omega} t - \theta) \qquad \textbf{(3.66)}$$

But by vectorially adding the motion of the base, we can show that the total motion of the mass is given by:

$$v^t(t) = v_{g0}\sqrt{1 + (2\xi\beta)^2}\, D \sin(\overline{\omega} t - \theta) \qquad \textbf{(3.67)}$$

If we define the transmissibility in this case as the ratio of the amplitude of the movement of the mass and the amplitude of the movement of the base, we notice that the expression of this transmissibility is identical to that of the equation (3.65), i.e. for a harmonic vertical force. This can be expressed in the following form:

$$TR = \frac{v^t_{\max}}{v_{g0}} = D\sqrt{1 + (2\xi\beta)^2} \qquad \textbf{(3.68)}$$

Figure (3.12) defines the effectiveness of vibration isolation systems, for the two types of problems with a single degree of freedom that have been considered. Isolators are made with a wide variety of materials and configurations. The type of isolators depends on the dynamic conditions under which it must operate, while the ambient temperature conditions determine the material that will be used. In high temperature conditions, metal springs combined with friction dampers are preferable. However, these isolators can cause problems under high dynamic loads and performance is highly variable due to assembly tolerances. This explains why these isolators are rarely used in delicate applications. Today, the vast majority of isolators are made with natural or synthetic elastomers or polymers. The development of a wide range of elastomeric compounds has made it possible to use this type of insulator in almost all environmental conditions. It is also very easy to vary the stiffness for a type of elastomer. Natural rubber is the most used elastomer for the manufacture of isolators.

EXERCISES

3.1.

Consider the dynamical system shown in figure (3.13). The equivalent mass of the structure is **M=1000Kg**. The damping factor ξ **=7%**. The system is subjected to an external harmonic load equal to:

$$\mathbf{F(t) = 4600 \sin(6t) \quad (Newtons)}$$

With a Young's modulus equal to $\mathbf{E = 2.10^6\ N/m^2}$.

1/ Determine the amplitude of the permanent response.
2/ Determine the maximum stress in the columns.

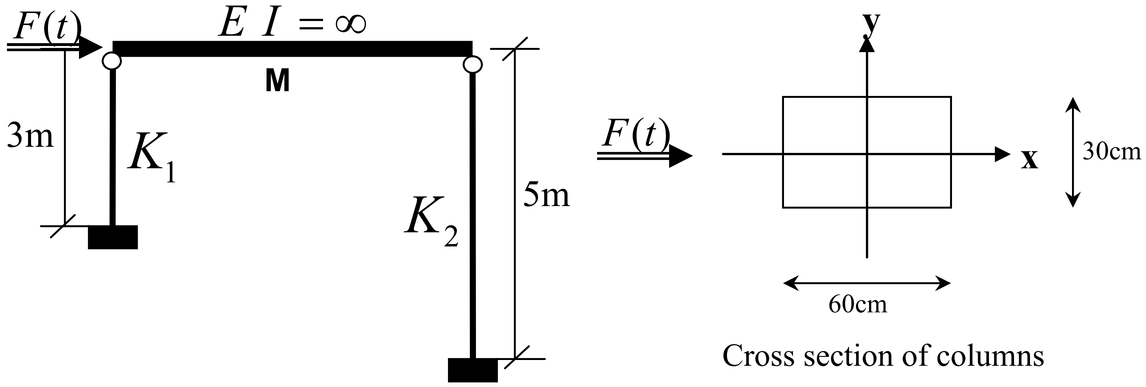


Figure 3.13

3.2.

An alternative machine, in which the total mass is **300 kg**, is supported by four isolators resting on a floor. The total stiffness of the isolators is $\mathbf{1500 \times 10^3}$ **N/m**. In operation, the machine generates a vertical harmonic force whose amplitude is **550 N** at a velocity of **40 Hz**. Assuming damping ξ **=0.20**, it is required to verify that the amplitude of the movement due to the harmonic force does not exceed the allowable amplitude of **0.04mm** and that the force transmitted to the floor does not exceed the allowable force of **85 N**.

3.3.

A reciprocating machine with a mass of **10000 kg** produces vertical harmonic forces with amplitude of **2500 N** at an operating velocity of **40 Hz**. In order to limit the vibrations induced in the building where this machine is to be installed, provision is made for stand on four springs: one at each corner of its rectangular base.

- Determine the stiffness of each of the four springs required to limit the total harmonic force transmitted to the building to **400 N**.

3.4.

A fragile instrument is carried by a structure whose ambient disturbances induce vibrations with a frequency of **24 Hz** and amplitude of **0.25mm**. We want the amplitude of the isolated instrument, with a mass **M= 1 ton**, not to exceed **0.05 mm**.

1/ Determine the pulsation of the bearing device (Fig. 3.14). The damping is equal to zero.
2/ What is the stiffness (K_1) of the springs?
3/ A damper is added to the device. What is the stiffness (K_1) of the springs if **ξ=20%**.
4/ Conclude with respect to the contribution of the damper to the support device.

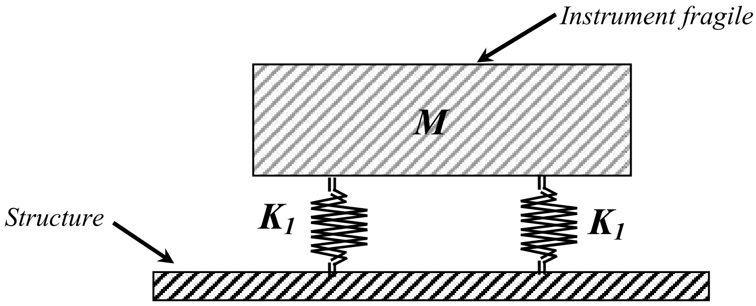


Figure 3.14

3.5.

A vehicle of mass **M** travels at constant speed on a pavement whose surface is longitudinally deformed, see figure 3.15.

- Determine the total vertical amplitude experienced by the vehicle for the following damping factors:

$\xi = 0$ et $\xi = 0.5$

With: **L = 20 m, V = 30 m/s, M = 3 t, *K* = 30 kN/m, S_0 = 4 cm.**

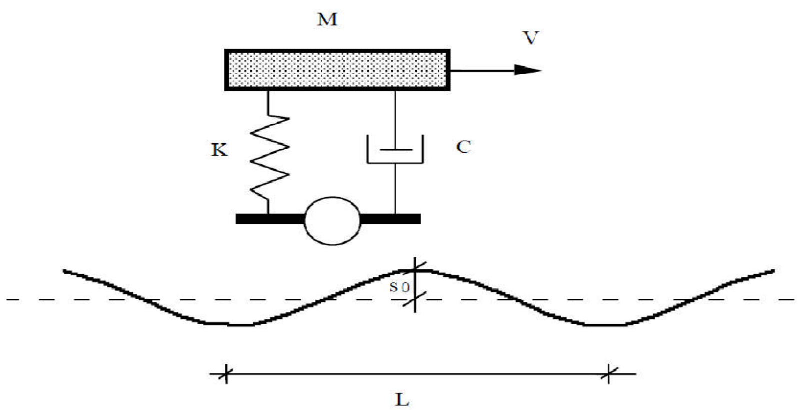


Figure 3.15

3.6.

Consider the dual structure shown in figure (3.16). The equivalent mass of the structure is **M = 5000Kg**. The damping ratio $\xi = 10\%$. The beam of this structure is subjected to an external harmonic load equal to:

$$\mathbf{F\ (\mathit{t}) = 800\ sin\ (7\mathit{t})} \qquad \textbf{(Newtons)}$$

With a Young's modulus $\mathbf{E = 2\ 10^6\ N/m2.}$
and a Poisson's ratio of the material **ν = 0.2**

1/ Determine, in the case of resonance, the amplitude of the permanent response.
2/ Determine the maximum stress in column 2.

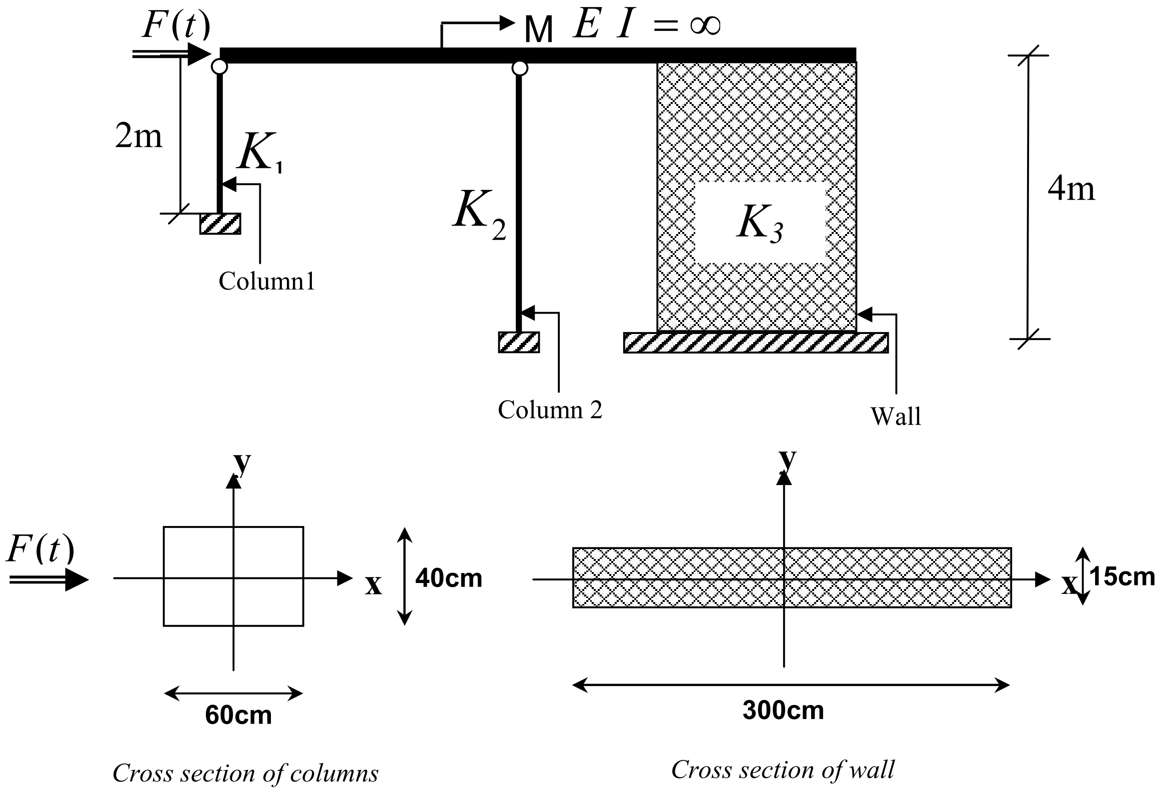


Figure 3.16

3.7.

Consider the structure shown in figure (3.17). The floor, infinitely rigid to lateral force, with mass $\mathbf{M = 3.5\ 10^5\ kg}$, is supported by columns of negligible mass; their stiffness to lateral force is $\mathbf{K = 7\ 10^6\ N/m.}$ A small mass $\mathbf{m_1}$, negligible compared to that of the floor, which can roll without friction on the latter, is attached to a horizontal spring with stiffness $\mathbf{k_1 = 1250\ N/m}$. This spring is attached to a support fixed to the floor and itself infinitely rigid to lateral force. The floor is set into free oscillation by releasing it instantly (canceling the jack action) at time $\boldsymbol{t} = \mathbf{0\ s.}$
With: $\boldsymbol{v}\ (\boldsymbol{t} = \mathbf{0\ s}) = \mathbf{v_0 = 2\ cm.}$

- Determine the expression for the relative displacement of the small mass $\mathbf{m_1}$ during free vibrations of the floor. Take $\mathbf{m_1 = 50}$ **kg.**

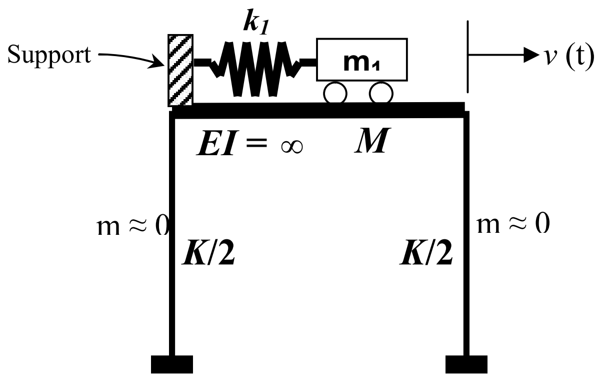


Figure 3.17

CHAPTER

4

DYNAMIC ANALYSIS IN PERIODIC FORCED VIBRATION OF STRUCTURES WITH A SINGLE DEGREE OF FREEDOM

4.1 INTRODUCTION

A periodic load is a load whose variation during a given period T_p is repeated indefinitely (Fig. 4.1). These loads are caused by, among other things, the steps of a person on a floor or crossing a pedestrian bridge, the hydraulic forces generated by the rotation of the propellers of a ship, the alternating vortices caused by the winds on the very slender structures, waves on an offshore drilling rig, etc.

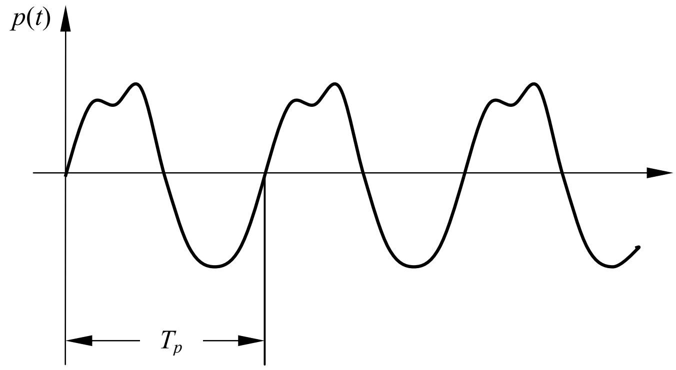


Figure 4.1: Any periodic loading.

In this chapter, we will study the response of elementary linear systems subjected to any periodic force. All the periodic loading functions encountered in structural dynamics can be developed as a series of trigonometric functions called the *Fourier series*. The response of an elementary system to a periodic load therefore amounts to the summation of the responses of the system to forces of different amplitudes but all varying according to harmonic functions. Summation restricts the application of this method to linear systems.

4.2 REPRESENTATION OF A PERIODIC FUNCTION IN A FOURIER SERIES IN TRIGONOMERTIC FORM

Consider any periodic function $p(t)$, as shown in figure (4.1). The period of the function is T_p. We have:

$$p(t + nT_p) = p(t), \qquad n = \pm 1, \pm 2, \pm 3, \ldots \tag{4.1}$$

Let $\overline{\omega}$ denote the pulsation of the force $p(t)$. We can write the following:

$$\overline{\omega} = \frac{2\pi}{T_p} \tag{4.2}$$

The periodic function $p(t)$ can be developed into a series of trigonometric functions called the Fourier series. Thus,

$$p(t) = \sum_{n=0}^{\infty} \left(a_n \cos \frac{2\pi n}{T_p} t + b_n \sin \frac{2\pi n}{T_p} t \right) \tag{4.3}$$

where a_n and b_n are constant coefficients. Taking into account equation (4.2), we can write the previous series in the following form:

$$p(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\overline{\omega} t + \sum_{n=1}^{\infty} b_n \sin n\overline{\omega} t \tag{4.4}$$

The pulsation $\overline{\omega}$ is called a fundamental pulsation, and the pulsations $n\overline{\omega}$ are called harmonics for $n \geq 2$. For the determination of the constants a_n and b_n, let us recall the following relations for m and n integers:

$$\int_{-T_p/2}^{T_p/2} \cos n\overline{\omega} t \cos m\overline{\omega} t \, dt = 0, \qquad m \neq n \tag{4.5}$$

$$\int_{-T_p/2}^{T_p/2} \cos^2 n\overline{\omega} t \, dt = \frac{T_p}{2}, \qquad n \neq 0 \tag{4.6}$$

$$\int_{-T_p/2}^{T_p/2} \sin n\overline{\omega} t \sin m\overline{\omega} t \, dt = 0, \qquad m \neq n \tag{4.7}$$

$$\int_{-T_p/2}^{T_p/2} \sin^2 n\overline{\omega} t \, dt = \frac{T_p}{2}, \qquad n \neq 0 \tag{4.8}$$

$$\int_{-T_p/2}^{T_p/2} \cos n\overline{\omega} t \sin m\overline{\omega} t \, dt = 0 \tag{4.9}$$

in which the two functions are said to be orthogonal to each other when the integral vanishes. We also have:

$$\int_{-T_p/2}^{T_p/2} \cos n\varpi t\, dt = 0, \qquad n \neq 0 \tag{4.10}$$

$$\int_{-T_p/2}^{T_p/2} \sin n\varpi t\, dt = 0 \tag{4.11}$$

We assume that the series can be integrated term by term in the interval of one period T_p, either from $t = -T_p/2$ to $T_p/2$ or $t = 0$ to $t = T_p$ and that the function $p(t)$ is always integrable. We will determine the coefficient a_0 by integrating the series (4.4) in the interval of one period. So:

$$\int_0^{T_p} p(t)\, dt = a_0 \int_0^{T_p} dt + a_1 \int_0^{T_p} \cos \varpi t\, dt + \cdots + a_n \int_0^{T_p} \cos n\varpi t\, dt + \cdots + b_1 \int_0^{T_p} \sin \varpi t\, dt + \cdots + b_n \int_0^{T_p} \sin n\varpi t\, dt + \cdots \tag{4.12}$$

All the terms on the right-hand side are zero, according to equations (4.10) and (4.11), except the first, which is equal to $a_0 T_p$. We are then left with:

$$a_0 = \frac{1}{T_p} \int_0^{T_p} p(t)\, dt \tag{4.13}$$

We will determine the coefficients a_n (n = 1, 2, 3,…) by multiplying the two sides of equation (4.4) by $cosn\varpi t$ and by integrating each term in the interval of one period. We have:

$$\int_0^{T_p} p(t) \cos n\varpi t\, dt = a_0 \int_0^{T_p} \cos n\varpi t\, dt + a_1 \int_0^{T_p} \cos \varpi t \cos n\varpi t\, dt + \cdots + a_n \int_0^{T_p} \cos^2 n\varpi t\, dt + \cdots + b_n \int_0^{T_p} \sin \varpi t \cos n\varpi t\, dt + \cdots + b_n \int_0^{T_p} \sin n\varpi t \cos n\varpi t\, dt + \cdots \tag{4.14}$$

Taking into account equations (4.5), (4.9) and (4.10), all the terms of the second member in cosine cancel each other out except for one in $\cos^2 n\varpi t$ which, by virtue of equation (4.6), is worth $T_p/2$. We therefore obtain:

$$a_n = \frac{2}{T_p} \int_0^{T_p} p(t) \cos n\varpi t\, dt \tag{4.15}$$

We obtain the coefficients b_n (n = 1, 2, 3,…) in a similar way by multiplying the two members of the equation (4.4) by $sin n\overline{\omega}t$ and by integrating each term in the interval of one period. We have:

$$\int_0^{T_p} p(t)\sin n\overline{\omega}t\,dt = a_0\int_0^{T_p}\sin n\overline{\omega}t\,dt + a_1\int_0^{T_p}\cos\overline{\omega}t\sin n\overline{\omega}t\,dt + \cdots + a_n\int_0^{T_p}\cos n\overline{\omega}t\sin n\overline{\omega}t\,dt + \cdots + b_n\int_0^{T_p}\sin\overline{\omega}t\sin n\overline{\omega}t\,dt + \cdots + b_n\int_0^{T_p}\sin^2 n\overline{\omega}t\,dt + \cdots \quad \textbf{(4.16)}$$

As before, taking into account equations (4.7), (4.9) and (4.11), all the terms of the second member of the equation thus obtained cancel out except for one in $\sin^2 n\overline{\omega}t$ which, by virtue of l equation (4.8), is worth $T_p/2$. We therefore obtain:

$$b_n = \frac{2}{T_p}\int_0^{T_p} p(t)\sin n\overline{\omega}t\,dt \quad \textbf{(4.17)}$$

The Fourier coefficients (4.13), (4.15) and (4.17) are known as *Euler's formulas* or *Euler-Fourier formulas*. From (4.13), we see that the coefficient a_0 is the average value of $p(t)$. One can thus represent the dynamic loading by the sum of a constant load a_0, representing the average load, and of a series of harmonic loads, of frequencies $n\overline{\omega}$ and amplitude a_n and b_n, representing the variation of the loading compared to the average value. By combining the cosines and sines of the same frequency, equation (4.4) can be put in the following form:

$$p(t) = p_0 + \sum_{n=1}^{\infty} p_n\cos(n\overline{\omega}t - \delta_n) \quad \textbf{(4.18)}$$

where $p_0 = a_0$ and p_n is the amplitude of the n^e harmonic, it is given as follows:

$$p_n = \sqrt{a_n^2 + b_n^2} \quad \textbf{(4.19)}$$

and δ_n is the phase angel, given as follows:

$$\delta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \quad \textbf{(4.20)}$$

Consider a function $p(t)$ defined in the interval [$-Tp/2$, $Tp/2$]. When $p(-t) = p(t)$, we say that the function $p(t)$ is *even* (it is symmetric with respect to the y-axis). In this case, the b_n constants are zero and the series includes only the cosine terms. The function $p(t)$ is *odd* if $p(-t) = -p(t)$ (it is symmetric with respect to the origin), then the constants a_n are zero and the series includes only the sine terms.

4.3 RESPONSE TO A PERIODIC LOAD EXPRESSED IN FOURIER SERIES

The response of a linear elementary system to a periodic loading can be obtained by superimposing the responses of the system to each harmonic component of the development in Fourier series of the loading. The response of an elementary system to a constant load p_0 is expressed as follows:

$$v(t) = \frac{p_0}{K} \tag{4.21}$$

We have developed, in chapter 3, expressions giving the permanent response of a system to harmonic loads. The permanent response of an undamped system – this is a misnomer because the permanent response does not, strictly speaking, exist for an undamped system – to a sine function, $p(t) = p_0 \sin \overline{\omega} t$, is given by the equation (3.15):

$$v(t) = \frac{p_0}{K}\frac{1}{1-\beta^2}\left(\sin \overline{\omega}\, t - \beta \sin \omega\, t\right), \qquad \beta \neq 1 \tag{4.22}$$

where $\beta = \overline{\omega}/\omega$ and ω is the natural pulse of the system. According to equation (3.33), the permanent response of a damped system to a sine function is:

$$v(t) = \frac{p_0}{K}\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\left((1-\beta^2)\sin \overline{\omega} t - 2\xi\beta \cos \overline{\omega} t\right) \tag{4.23}$$

For a cosine function, $p(t) = p_0 cos\overline{\omega}t$, the response of an undamped system is expressed by:

$$v(t) = \frac{p_0}{K}\frac{1}{1-\beta^2}\left(\cos \overline{\omega}\, t - \cos \omega\, t\right), \qquad \beta \neq 1 \tag{4.24}$$

and the permanent response of a damped system to a sine function is:

$$v(t) = \frac{p_0}{K}\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\left(2\xi\beta \sin \overline{\omega} t - (1-\beta^2)\cos \overline{\omega} t\right) \tag{4.25}$$

The Fourier series decomposition makes it possible to calculate the steady state of an undamped system excited by a periodic force $p(t)$. Let's pose

$$\beta_n = \frac{n\overline{\omega}}{\omega} = n\beta \tag{4.26}$$

Taking into account equations (4.21), (4.22) and (4.24), the permanent response of an undamped system to a loading decomposed in Fourier series according to (4.4) is given by the following sums:

$$v(t) = \frac{a_0}{K} + \sum_{n=1}^{\infty} \frac{1}{K}\frac{1}{1-\beta_n^2}\left(a_n \cos n\overline{\omega}t + b_n \sin n\overline{\omega}t - a_n \cos \omega t - b_n\beta_n \sin \omega t\right) \tag{4.27}$$

In a similar way taking into account the equations (4.21), (4.23) and (4.25), the permanent response of a damped system to a loading decomposed in Fourier series is given by the following sums:

$$v(t)=\frac{a_0}{K}+\sum_{n=1}^{\infty}\frac{1}{K}\frac{1}{(1-\beta_n^2)^2+(2\xi\beta_n)^2}\left[\left(a_n(1-\beta_n^2)-b_n 2\xi\beta_n\right)\cos n\bar{\omega}t+\left(a_n 2\xi\beta_n+b_n(1-\beta_n^2)\right)\sin n\bar{\omega}t\right]$$

(4.28)

Expressing the loading by (4.18), the response of the damped system is expressed as follows:

$$v(t)=\frac{p_0}{K}+\sum_{n=1}^{\infty}\frac{p_n}{K}\frac{1}{\sqrt{(1-\beta_n^2)^2+(2\xi\beta_n)^2}}\cos(n\bar{\omega}t-\delta_n-\theta_n) \quad \textbf{(4.29)}$$

where the phase angle of the response with respect to the n^e harmonic of the loading, θ_n, is given by equation (3.35):

$$\theta_n=\tan^{-1}\left(\frac{2\xi\beta_n}{1-\beta_n^2}\right) \quad \textbf{(4.30)}$$

Recall that θ_n varies between 0 and π. Neglecting the transient terms (functions of the natural frequency), the response of the undamped system is expressed as follows:

$$v(t)=\frac{p_0}{K}+\sum_{n=1}^{\infty}\frac{p_n}{K}\frac{1}{1-\beta_n^2}\cos(n\bar{\omega}t-\delta_n), \qquad \beta\neq 1 \quad \textbf{(4.31)}$$

Equations (4.29) and (4.31) are preferable to equations (4.28) and (4.27), respectively, because they clearly show the magnitudes of each harmonic of the loading and the phase angles of each harmonic of the loading and displacement.

EXERCISES

4.1.

Expand in Fourier series the slot loading function shown in figure (4.2).

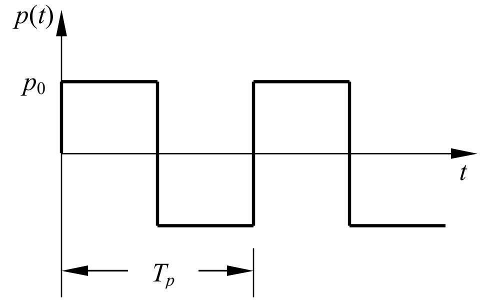


Figure 4.2

4.2.

Calculate the response of a damped elementary system to a step loading shown in figure (4.3) with $\boldsymbol{T_p}$ **= 1s** and $\boldsymbol{p_0}$ **= 500 kN**. The system has the following properties: ***K*=10 000 kN/m**, ***M* = 10 000 kg** and **ξ = 0.05**.

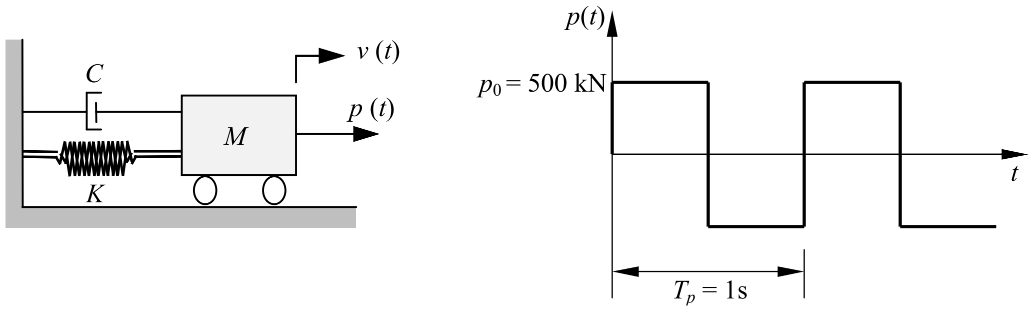


Figure 4.3

4.3.

First develop, in Fourier series, the loading function shown in figure (4.4.b) which represents the rhythmic jumps of a person on a floor. The function consists of the positive part of a sinusoidal function. Then calculate the system response, shown in figure (4.4.a) in the undamped and damped cases. Take $\frac{\boldsymbol{T_p}}{\boldsymbol{T}} = \frac{\boldsymbol{4}}{\boldsymbol{3}}$

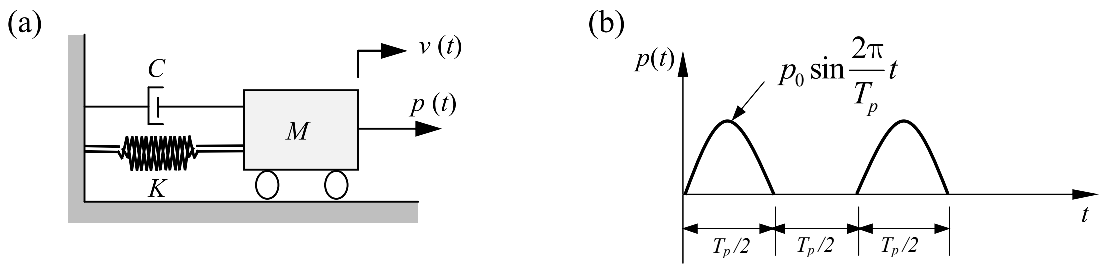


Figure 4.4

CHAPTER

5

DYNAMIC ANALYSIS IN IMPULSIVE FORCED VIBRATION OF STRUCTURES WITH A SINGLE DEGREE OF FREEDOM

5.1 INTRODUCTION

An impulsive load consists of a single pulse; it is usually very intense and short-lived (Fig. 5.1). These impulsive loads, for example shocks, are of great importance in the design of certain structures. The damping is much less important for the maximum response than in the case of periodic loads. The maximum response to an impulsive load will be reached in a very short time, before the damping forces have time to absorb substantial energy. For this reason, we consider in this chapter only the undamped response.

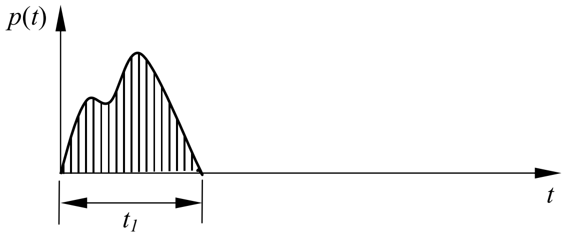


Figure 5.1: Any impulsive charging.

5.2 SINUSOIDAL IMPULSE

Consider a half-cycle pulse of sinusoidal loading. The force function is shown in figure (5.2) and is written:

$$p(t) = \begin{cases} p_0 \sin \overline{\omega} t & \text{pour} \quad t \leq t_1 \\ 0 & \text{pour} \quad t > t_1 \end{cases} \tag{5.1}$$

The response will be divided into two phases, one corresponding to the interval during which the load acts (forced system), and the other to the free vibration phase (free system).

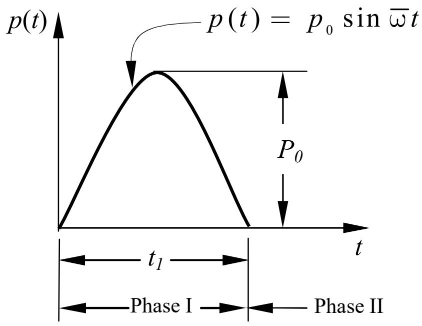


Figure 5.2: Sinusoidal impulse.

PHASE I: the loading is harmonic and the initial conditions correspond to rest. The response includes the transient part and the permanent part and is therefore given by equation (3.15):

$$v(t) = \frac{p_0}{K}\frac{1}{1-\beta^2}\left(\sin \overline{\omega}\, t - \beta \sin \omega\, t\right) \qquad 0 \le t \le t_1 \qquad \textbf{(5.2)}$$

PHASE II: from $t = t_1$, we are in free mode with the initial conditions $v(t_1)$ and $\dot{v}(t_1)$ at the end of phase I. The response is given by equation (2.22):

$$v(t) = \frac{\dot{v}(t_1)}{\omega}\sin \omega (t - t_1) + v(t_1) \cos\, \omega (t - t_1) \qquad t \ge t_1 \qquad \textbf{(5.3)}$$

Figure (5.3) shows the variation of $v(t)/\left(p_0/K\right)$, for $t_1/T = 3/4$, as a function of t/T as well as the ratio of the quasi-static displacement to the maximum static displacement $v_{st}(t)/v_{st0} = \left[p(t)/K\right]/\left(p_0/K\right)$, dotted in the figure. The maximum response that results from impulsive loading depends on the relationship between the duration of the load and the system's own vibration period t_1/T . The time $t_{\max}$ when the maximum response occurs during phase I can be obtained by setting the time derivative of equation (5.2) to zero; we obtain:

$$\frac{dv(t)}{dt} = \frac{p_0}{K}\frac{1}{1-\beta^2}\left(\overline{\omega} \cos \overline{\omega}\, t - \overline{\omega} \cos \omega\, t\right) = 0 \qquad \textbf{(5.4)}$$

from where

$$\cos \overline{\omega} t = \cos \omega t \qquad pour \quad \overline{\omega} t \le \pi \qquad \textbf{(5.5)}$$

and

$$\overline{\omega} t_{\max} = 2\pi n \pm \omega t_{\max} \qquad n = 0,\ 1,\ 2,\ldots \qquad \textbf{(5.6)}$$

The smallest value of $t_{\max}$, other than 0, is obtained for $n = 1$ and using the negative sign. The maximum response will occur at time:

$$t_{\max} = \frac{2\pi}{\overline{\omega} + \omega} = \frac{2\pi}{\overline{\omega}\left(1 + \dfrac{1}{\beta}\right)} = \frac{2t_1}{1 + \dfrac{1}{\beta}} \le t_1 \qquad \textbf{(5.7)}$$

from where

$$\beta \leq 1 \tag{5.8}$$

which implies, taking into account $\bar{\omega} = \pi / t_1$,

$$\frac{t_1}{T} \geq \frac{1}{2} \tag{5.9}$$

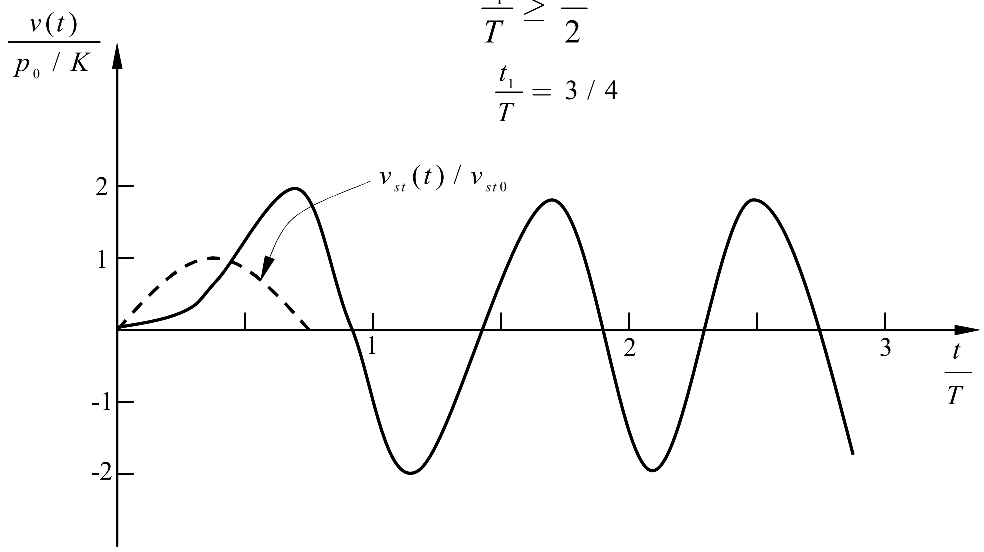


Figure 5.3: Response to a sinusoidal impulse for $t_1 = 0.75T$.

We can calculate v_{max} by introducing $t_{\max}$ in equation (5.2), we obtain:

$$v_{\max} = \frac{p_0}{K}\frac{1}{1-\beta^2}\left(\sin\frac{2\pi\beta}{1+\beta} - \beta\sin\frac{2\pi\beta}{1+\beta}\right), \qquad \frac{t_1}{T} \geq \frac{1}{2} \tag{5.10}$$

The maximum displacement is given by equation (5.10) only if $t_{\max} \leq t_1$, so if $\beta \leq 1$ and $\bar{\omega} \leq \omega$. The dynamic amplification factor $D = v_{\max} / v_{st0}$ is expressed by:

$$D = \frac{v_{\max}}{v_{st0}} = \frac{1}{1-\beta^2}\left(\sin\frac{2\pi\beta}{1+\beta} - \beta\sin\frac{2\pi\beta}{1+\beta}\right), \qquad \frac{t_1}{T} \geq \frac{1}{2} \tag{5.11}$$

The maximum response occurs during the free regime (phase II), if $\beta > 1$, either $t_1 / T < 1/2$. The initial conditions are therefore, considering that $t = t_1$ and $\bar{\omega} t_1 = \pi$,

$$v(t_1) = \frac{p_0}{K}\frac{1}{1-\beta^2}(-\beta\sin\frac{\pi}{\beta}) \tag{5.12}$$

$$\dot{v}(t_1) = \frac{p_0}{K}\frac{1}{1-\beta^2}\left(-1-\cos\frac{\pi}{\beta}\right) \tag{5.13}$$

The amplitude of the movement can be obtained by calculating the resultant of the two orthogonal components of the movement, that is:

$$v_{\max} = \sqrt{\left(v(t_1)\right)^2 + \left(\frac{\dot{v}(t_1)}{\omega}\right)^2} = \frac{p_0}{K}\frac{\beta}{1-\beta^2}\sqrt{2\left(1+\cos\frac{\pi}{\beta}\right)} \tag{5.14}$$

which, by considering the trigonometric identity $\sqrt{2(1+\cos 2\alpha)} = 2|\cos\alpha|$, reduces to the following expression:

$$v_{max} = \frac{p_0}{K}\frac{2\beta}{1-\beta^2}\cos\frac{\pi}{2\beta}, \qquad \frac{t_1}{T} < \frac{1}{2} \tag{5.15}$$

The dynamic amplification factor D is expressed:

$$D = \frac{v_{max}}{v_{st0}} = \frac{2\beta}{1-\beta^2}\cos\frac{\pi}{2\beta}, \qquad \frac{t_1}{T} < \frac{1}{2} \tag{5.16}$$

5.3 STEP IMPULSE

The step impulse is shown in figure (5.4) and is described by the following force function:

$$p(t) = \begin{cases} p_0 & \text{pour } t \le t_1 \\ 0 & \text{pour } t > t_1 \end{cases} \tag{5.17}$$

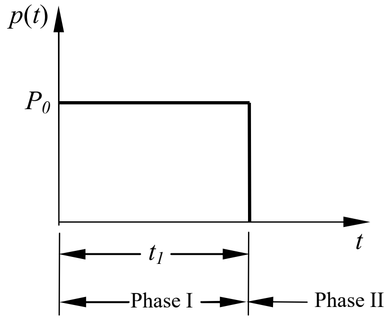


Figure 5.4: Step impulse.

With initial conditions at rest, i.e., $v(0) = \dot{v}(0) = 0$, the two phases of the analysis are:

PHASE I : Forced vibration for $t \le t_1$, the loading is a step force. The particular solution for a step loading is simply the corresponding static displacement:

$$v(t) = \frac{p_0}{K} \tag{5.18}$$

We draw from this result the general solution, with free vibration constants (A and B) calculated to satisfy the initial conditions at rest:

$$v(t) = \frac{p_0}{K}(1-\cos\omega t), \qquad t \le t_1 \tag{5.19}$$

PHASE II : For $t > t_1$, we are in a free regime with the initial conditions $v(t_1)$ and $\dot{v}(t_1)$ at the end of phase I. The response is given by equation (2.22):

$$v(t) = \frac{\dot{v}(t_1)}{\omega}\sin\omega(t-t_1) + v(t_1)\cos\,\omega(t-t_1), \qquad t > t_1 \tag{5.20}$$

The displacement and velocity at time t_1 are given by equation (5.19):

$$v(t_1) = \frac{p_0}{K}(1 - \cos\omega t_1) \qquad and \qquad \dot{v}(t_1) = \frac{p_0}{K}\omega \sin\omega t_1 \quad \textbf{(5.21)}$$

which, after substitution in equation (5.20), gives us:

$$v(t) = \frac{p_0}{K}\left[(1 - \cos\omega t_1)\cos\omega(t - t_1) + \sin\omega t_1 \sin\omega(t - t_1)\right], \quad t > t_1 \quad \textbf{(5.22)}$$

which we simplify to:

$$v(t) = \frac{p_0}{K}(\cos\omega(t - t_1) - \cos\omega t), \qquad t > t_1 \quad \textbf{(5.23)}$$

The variation of the dynamic response factor $R(t) = v(t)/(p_0/K)$ as a function of t/T is shown in figure (5.5) for an elementary system, whose natural period is *T*, subjected to a rectangular pulse p_0 whose duration is $t_1 = 1.75T$. During phase I for which $t \le t_1$, the system oscillates around the shifted position $v_{st0} = p_0 / K$ according to its natural frequency of vibration ω . After the impulse $(t > t_1)$, the system vibrates around the initial position of equilibrium, always according to its natural frequency of vibration ω .

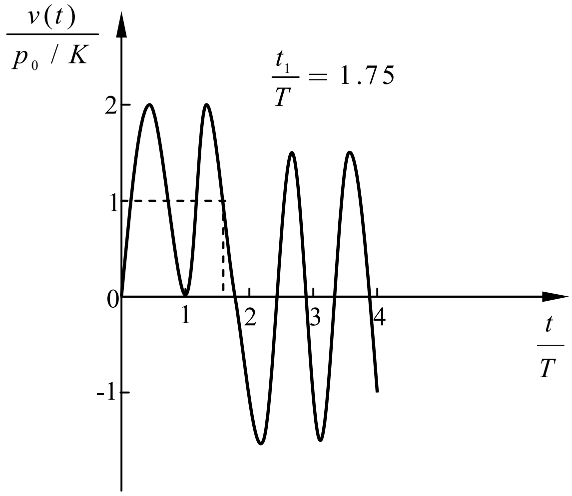


Figure 5.5: Response to a step impulse for $t_1 = 1.75T$.

The maximum of the response will be obtained in phase I or phase II depending on the value of the ratio t_1 / T . Suppose first that the maximum is obtained at time $t_{max} < t_1$. We obtain t_{max} by equalizing to zero the derivative of equation (5.19) with respect to time. Thus,

$$\frac{dv(t)}{dt} = \frac{p_0}{K}(\omega \sin\omega t) = 0 \quad \textbf{(5.24)}$$

either for

$$\omega t_{max} = n\pi \qquad n = 1,3,5,... \quad \textbf{(5.25)}$$

where *n* is odd because the displacement is a minimum for *n* even. The maximum displacement only occurs in phase I if the smallest value of t_{max}, obtained by taking *n* = 1, is less than t_1 , thus

$$t_{max} = \frac{\pi}{\omega} = \frac{T}{2} < t_1 \quad \textbf{(5.26)}$$

Several peaks of the same amplitude – two peaks in the case of figure (5.5) for $t_1 / T = 1.75$ – can occur during phase I depending on the ratio t_1 / T. The first peak is obtained by replacing t by t_{max} given by equation (5.26) in equation (5.19) to give:

$$v_{max} = 2\frac{p_0}{K} \quad \textbf{(5.27)}$$

When the maximum of the response occurs during phase I, the dynamic amplification factor is expressed as follows:

$$D = \frac{v_{max}}{v_{st0}} = 2 \quad \textbf{(5.28)}$$

If $t_1 / T < 0.5$, v_{max} occurs during phase II of the response. t_{max} is obtained in this case by equalizing to zero the derivative of equation (5.23) with respect to time. Thus,

$$\frac{dv(t)}{dt} = \frac{p_0}{K}\omega(-\sin\omega(t - t_1) + \sin\omega t) = 0 \quad \textbf{(5.29)}$$

which is satisfied for

$$\tan\omega t_{max} = -\cot\frac{\omega t_1}{2} \quad \textbf{(5.30)}$$

either for

$$t_{max} = (2n+1)\frac{\pi}{2\omega} + \frac{t_1}{2} = (2n+1)\frac{T}{4} + \frac{t_1}{2}, \qquad n = 0,1,2,\ldots \quad \textbf{(5.31)}$$

The maximum response is obtained by substituting the value of t_{max} obtained in equation (5.31) for t in equation (5.23). So:

$$v_{max} = 2\frac{p_0}{K}\sin\frac{\omega t_1}{2} = 2\frac{p_0}{K}\sin\frac{\pi t_1}{T} \quad \textbf{(5.32)}$$

When the maximum of the response occurs in phase II, the dynamic amplification factor is expressed as follows:

$$D = \frac{v_{max}}{v_{st0}} = 2\sin\frac{\pi t_1}{T} \quad \textbf{(5.33)}$$

5.4 LINEARLY DEGENERATE IMPULSE

Let us consider an elementary system initially at rest and subjected to a dynamic loading whose value is p_0 at $t = 0$ and which decreases linearly to zero during a time interval t_1, see figure (5.6). This type of charge can represent in a simplified way the effect of an explosion. The force function is described as follows:

$$p(t)=\begin{cases} p_0\left(1-\dfrac{t}{t_1}\right) & \text{pour } t\le t_1 \\ 0 & \text{pour } t>t_1 \end{cases} \tag{5.34}$$

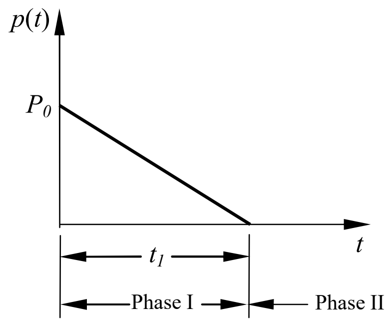


Figure 5.6: Linearly degenerate impulse.

Phase I : In this phase, the particular solution corresponding to this loading is easily written as follows:

$$v(t)=\frac{p_0}{K}\left(1-\frac{t}{t_1}\right) \tag{5.35}$$

One supposes null initial conditions; we calculate the constants of the free vibration and we obtain:

$$v(t)=\frac{p_0}{K}\left(\frac{\sin\omega t}{\omega t_1}-\cos\omega t-\frac{t}{t_1}+1\right) \tag{5.37}$$

Phase II : We use the equation (2.22) which describes the response in free vibration, with as initial conditions the displacement and the speed at $t=t_1$. So:

$$v(t_1)=\frac{p_0}{K}\left(\frac{\sin\omega t_1}{\omega t_1}-\cos\omega t_1\right) \tag{5.38}$$

and

$$\dot v(t_1)=\frac{p_0\omega}{K}\left(\frac{\cos\omega t_1}{\omega t_1}+\sin\omega t_1-\frac{1}{\omega t_1}\right) \tag{5.39}$$

from where

$$v(t)=\frac{p_0}{K}\left(\frac{\cos\omega t_1}{\omega t_1}+\sin\omega t_1-\frac{1}{\omega t_1}\right)\sin\omega(t-t_1) + \frac{p_0}{K}\left(\frac{\sin\omega t_1}{\omega t_1}-\cos\omega t_1\right)\cos(t-t_1) \tag{5.40}$$

The response is shown in figure (5.7) for $t_1 / T = 1.5$. The maximum is obtained before t_1 and is equal to 1.7 for a report $t_1 / T = 1.5$. The response after t_1 is harmonic around zero.

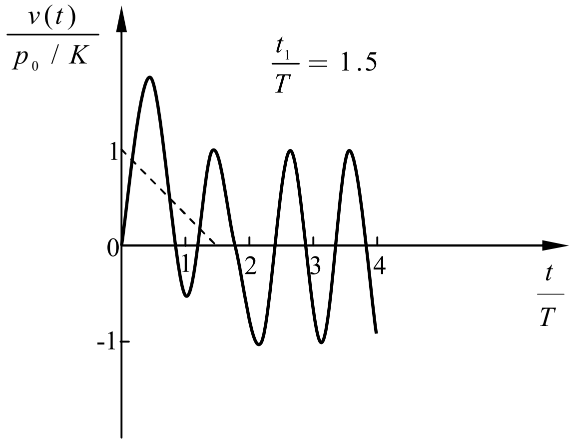


Figure 5.7: Response to a linearly degenerate impulse for $t_1 = 1.5T$.

5.5 APPROXIMATE CALCULATION OF THE RESPONSE TO AN IMPULSIVE LOADING

A convenient method to approximate the maximum response to a very short-duration impulse loading consists in expressing the variation in momentum of a mass *M*. Consider the impulse load $p(t)$ as shown in figure (5.8) and whose duration of application, t_1, is very short.

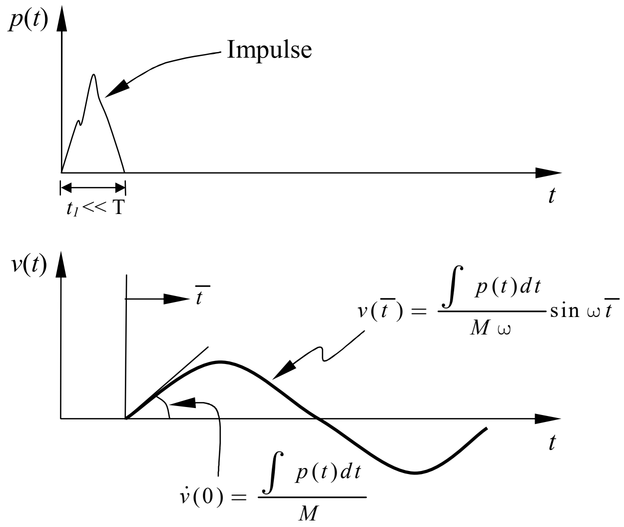


Figure 5.8: Simplified impulse response.

Let's define the impulse of force, α, by the area under the loading curve. So:

$$\alpha = \int_0^{t_1} p(t)dt \tag{5.41}$$

If the impulsive force application duration is very short compared to the vibration period T of the elementary system, i.e. $t_1 \ll T$, we can assume that the impulsive force intensity is very large compared to others forces present. The elastic and damping forces can therefore be neglected. There is no noticeable change in displacement during the time the load is applied, only a change in velocity, $\Delta \dot{v}$. According to Newton's 2nd law which states that the change in the momentum of a mass is equal to the impulse of the resultant force, we have:

$$M \, \Delta \dot{v} = \int_0^{t_1} p(t) dt \tag{5.42}$$

from where

$$\Delta \dot{v} = \frac{1}{M} \int_0^{t_1} p(t) dt \tag{5.43}$$

The free-running response, after, t_1, is described by equation (2.22). Introducing the initial conditions $v(0) = v(t_1)$ and $\dot{v}(0) = \dot{v}(t_1) = \Delta \dot{v}$ into (2.22), taking into account (5.43), gives us the approximate impulse response for an undamped system. So:

$$v(\overline{t}) \approx \frac{1}{M\omega} \left(\int_0^{t_1} p(t) dt \right) \sin \omega \overline{t} \tag{5.44}$$

where $\overline{t} = t - t_1$. For the purposes of further study, in particular the principle of obtaining the response spectrum in seismic calculation, which will be presented in chapter 6, we introduce here and in the following sections, in addition to the undamped response, the case of the damped response.

Introducing the initial conditions in (2.48) similarly gives us the approximate impulse response for a damped system.

$$v(\overline{t}) \approx \frac{1}{M\omega_D} \left(\int_0^{t_1} p(t) dt \right) e^{-\xi\omega\overline{t}} \sin \omega_D \overline{t} \tag{5.45}$$

5.6 DIRAC IMPULSE OR DELTA FUNCTION

Consider a force function whose duration $t_1 \to 0$ while the integral remains finite and equal to unity, i.e. $I = \int_0^{t_1} p(t) \mathrm{dt} = 1$. Such a force has a finite value. This function does not exist in practice but it is widely used in signal theory in relation to signal transformations and in structural dynamics in relation to the response to an arbitrary loading. We say that we have a unit impulse or a Dirac delta function or more simply a delta function. One of the ways to define this function is to start from a rectangular pulse $\Delta_\varepsilon(t)$ given by:

$$\Delta_\varepsilon(t) = \begin{cases} \dfrac{1}{2\varepsilon} & \text{pour } |t| \leq \varepsilon \\ 0 & \text{pour } |t| > \varepsilon \end{cases} \tag{5.46}$$

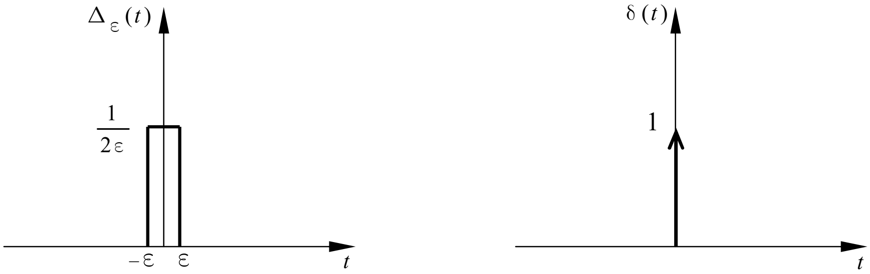


Figure 5.9: Unit impulse.

If we continuously decrease the duration ε, the impulse $\Delta_\varepsilon(t)$ becomes narrower and higher while the area remains equal to 1. In the limit, when $\varepsilon \to 0$, we obtain the Dirac delta function $\delta(t)$:

$$\delta(t) = \lim_{\varepsilon \to 0} \Delta_\varepsilon(t) \tag{5.47}$$

which is conventionally represented by a vertical arrow of length one as shown in figure (5.9). We deduce from this definition that the Dirac function takes the following values:

$$\begin{aligned} \delta(t) &= 0 \qquad pour\ t \neq 0 \\ \delta(t) &= \infty \qquad pour\ t = 0 \end{aligned} \tag{5.48}$$

and that its integral on the real axis equals unity, thus:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{5.49}$$

from which we deduce the fundamental relationship known as the following filtering property:

$$\int_{-\infty}^{\infty} f(t)\delta(t) dt = f(0) \tag{5.50}$$

where $f(t)$ is an arbitrary function without discontinuity at $t = 0$.

We will use the Dirac delta function in the form $\delta(t-\tau)$ acting at τ as shown in figure (5.10). It takes the following values:

$$\begin{aligned} \delta(t-\tau) &= 0 \qquad pour\ t \neq \tau \\ \delta(t-\tau) &= \infty \qquad pour\ t = \tau \end{aligned} \tag{5.51}$$

and its integral is:

$$\int_{-\infty}^{\infty} \delta(t-\tau) dt = 1 \tag{5.52}$$

In this case, the filtering integral becomes:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau) dt = f(\tau) \tag{5.53}$$

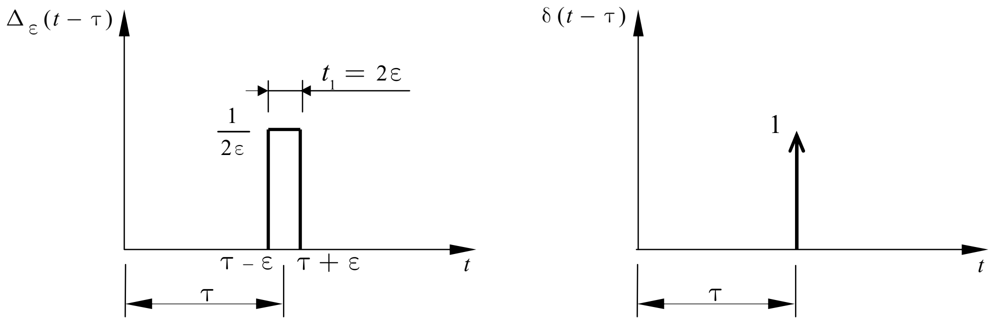


Figure 5.10: Unit impulse.

5.7 RESPONSE TO A DIRAC IMPULSE

If the force function is a delta function whose duration is $t_1 = 2\varepsilon \to 0$, we have $\int_{-\infty}^{\infty} p(t)\mathrm{dt} = 1$. Equation (5.44), expressing the approximate response to any pulse of finite duration t_1, becomes an exact response which is written:

$$v(t) = h(t) = \frac{1}{M\omega}\sin \omega t \tag{5.53}$$

where $h(t)$ is, by definition, the impulse response function (IRF). For a damped system, the IRF is obtained from equation (5.45).

$$h(t) = \frac{1}{M\omega_D} e^{-\xi\omega t}\sin \omega_D t \tag{5.54}$$

By considering equations (5.41), (5.44) and (5.45), the response of an elementary system to any impulse can be written as follows:

$$u(\overline{t}) \approx \alpha h(\overline{t}) \tag{5.55}$$

5.8 DUHAMEL'S INTEGRAL

The dynamic response of a linear elastic system to any loading can be described by its impulse response, i.e. the unit impulse response $h(t)$ caused by a loading which is a function of delta $p(t) = \delta(t)$. To demonstrate this, consider the arbitrary load shown in figure (5.11). Let us approximate the force function by a sequence of impulsive loads of duration $\Delta\tau$. The unit impulse response $\delta(t-\tau)$, applied at $t = \tau$ is the impulse response $h(t-\tau)$ given by equation (5.53) or (5.54). So at a later time t (i.e. $t \geq \tau$), the contribution to the total response of an impulse of magnitude $p(\tau)\Delta\tau$, applied at $t = \tau$, is the following quantity:

$$\Delta v(t) \approx p(\tau)\Delta\tau h(t-\tau) \tag{5.56}$$

By virtue of the linearity of the system, its response $v(t)$ to the approximate function is obtained by calculating the response to each of the rectangular pulses and adding them.

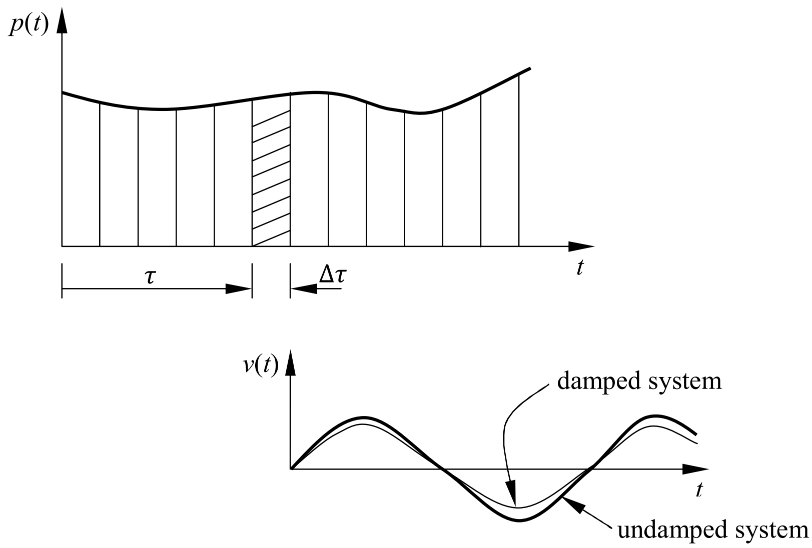


Figure 5.11: Any continuous force function approximated by a sequence of pulses of duration Δτ and response of the system to a pulse at t=τ.

As Δτ gets smaller, the system response approaches the exact response. In the limit, when $\Delta\tau \to 0$, we can replace the rectangle at $t=\tau$ by a delta function $p(\tau)\delta(t-\tau)$ and, according to equation (5.53), we can write:

$$p(t) = \int_0^\infty p(\tau)\delta(t-\tau)d\tau \quad \textbf{(5.57)}$$

where we assume the load function is defined for $t \geq 0$. The differential contribution of the delta function to the response is:

$$dv(t) = p(\tau)h(t-\tau)d\tau \quad \textbf{(5.58)}$$

where $h(t-\tau)$ is the unit impulse response at $t-\tau$. The total response is obtained by the sum of the differential responses. So :

$$v(t) = \int_0^t p(\tau)h(t-\tau)d\tau \quad \textbf{(5.59)}$$

For an undamped system and taking into account equation (5.53), equation (5.59) can be written as follows:

$$v(t) = \frac{1}{M\omega}\int_0^t p(\tau)\sin\omega(t-\tau)d\tau \quad \textbf{(5.60)}$$

where the number on the right of equation (5.60) is the Duhamel integral. Equation (5.60) can be used to calculate the forced response of a system to any dynamic excitation. In the above, we assumed initial conditions at rest. For nonzero initial conditions at $t=0$, the total response is the sum of the free regime due to the initial conditions and the forced regime due to the excitation; we therefore have for an undamped system:

$$v(t) = \frac{\dot{v}(0)}{\omega}\sin\omega t + v(0)\cos\omega t + \frac{1}{M\omega}\int_0^t p(\tau)\sin\omega(t-\tau)d\tau \quad \textbf{(5.61)}$$

Similar to an undamped system and taking into account the equation (5.54), equation (5.60) applied to a damped system is expressed as:

$$v(t) = \frac{1}{M\omega_D}\int_0^t p(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\,d\tau \qquad \textbf{(5.62)}$$

in which the right hand side is the Duhamel integral for a damped system. Equation (5.62) can be used to calculate the forced response of a system to any dynamic excitation. For non-zero initial conditions at $t=0$, the total response is the sum of the free regime due to the initial conditions and the forced regime due to the excitation; we therefore have for a damped system:

$$v(t) = e^{-\xi\omega t}\left[\frac{\dot{v}(0)+v(0)\xi\omega}{\omega_D}\sin\omega_D t + v(0)\cos\omega_D t\right] + \frac{1}{M\omega_D}\int_0^t p(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\,d\tau \qquad \textbf{(5.63)}$$

The Duhamel integral is a general method for calculating the response of a linear elementary system to any dynamic loading. One can obtain the exact response of a linear elementary system for simple loading functions for which the integral can be calculated exactly. In general, however, a numerical integration of the Duhamel integral is required for any loading functions. We emphasize, however, that the Duhamel integral is based on the principle of superposition and is therefore only applicable to linear systems.

5.9 SHOCK RESPONSE SPECTRUM

In general, we are interested in the maximum value of the response of an elementary oscillator for a dimensioning. According to equations (5.11) and (5.16), the maximum response depends only on t_1 and β for a sinusoidal pulse and therefore on t_1/T. It can be shown that this is true for other impulsive loads. In practice, we use a graph of the dynamic amplification factor D as a function of t_1/T for different forms of impulse loads. Such a graph is called *a displacement response spectrum*. Figure (5.12) shows the response spectra for four types of impulsive loads.

The curves can be used to determine the maximum response of a structure under the action of a sudden acceleration $\ddot{v}_g(t)$ applied to the base. The maximum impulse force is $p_{0eff} = -M\ddot{v}_{g0}$, where $\ddot{v}_{g0}$ is the maximum acceleration. The dynamic amplification factor becomes:

$$D = \left|\frac{v_{max}}{M\ddot{v}_{g0}/K}\right| \qquad \textbf{(5.64)}$$

For an undamped system, $F_{I,max} = F_{E,max}$, *or* $M\ddot{v}^t_{max} = Kv_{max}$. Equation (5.64) becomes:

$$D = \left| \frac{\ddot{v}^{t}_{\max}}{\ddot{v}_{g0}} \right| \quad \textbf{(5.65)}$$

where $\ddot{v}^{t}_{\max}$ is the absolute maximum acceleration. Thus, the spectrum of *D* presented in figure (5.12) can as well be used to determine the relative maximum displacement of the mass of an elementary system under the action of a shock as the absolute maximum acceleration under the action of an impulsive acceleration applied to the support. In the latter case, the graph *D* as a function of the period is called the shock response spectrum. Although these spectra were developed for undamped elementary systems, they can also be used for damped systems because, for damping values commonly encountered in practice, damping has little effect on the maximum response for impulsive loads.

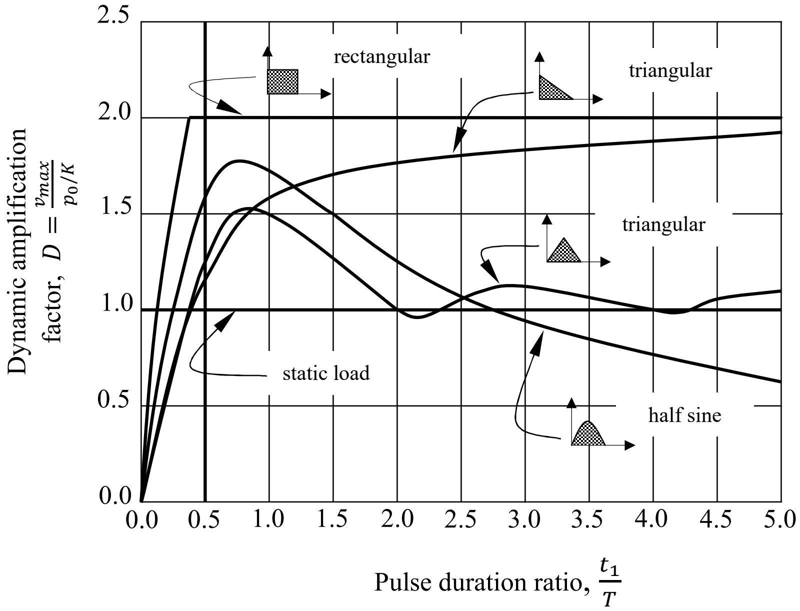


Figure 5.12: Displacement response spectrum or shock spectrum for four types of impulse loads.

EXERCISES

5.1.

A dynamic system with a single degree of freedom shown in figure (5.13 a) is subjected to a load whose variation over time is shown in figure (5.13 b).

- Calculate the displacement $v(t)$ at $t = 0.1$**s** and $t = 0.3$**s**.

The initial conditions are zero.

With: $E = 3\ 10^{10}$ **N/m²**, $M = 15000$ **kg**, $k_R = 2\ 10^5$ **N/m**

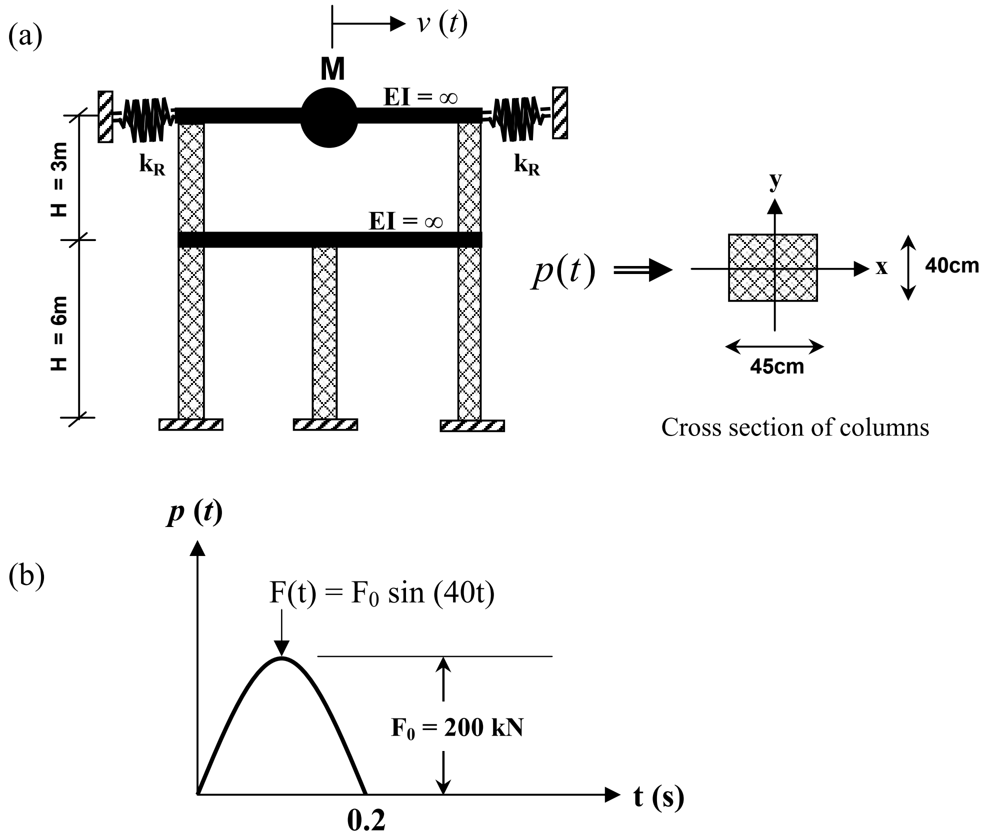


Figure 5.13

5.2.

Find the expression for the time t_{max} where the maximum displacement v_{max} of an undamped elementary system occurs under the action of a triangular impulse as shown in figure (5.6). Find the expression for the dynamic amplification factor D corresponding to this pulse.

5.3.

A one-storey frame shown in figure (5.14 a) is subjected to a load whose variation over time is shown in figure (5.14 b).

- Calculate the displacement $v(t)$ at $t = 0.2$**s**.

The initial conditions are zero and Young's modulus $E = 3\ 10^{10}$ **N/m²**.

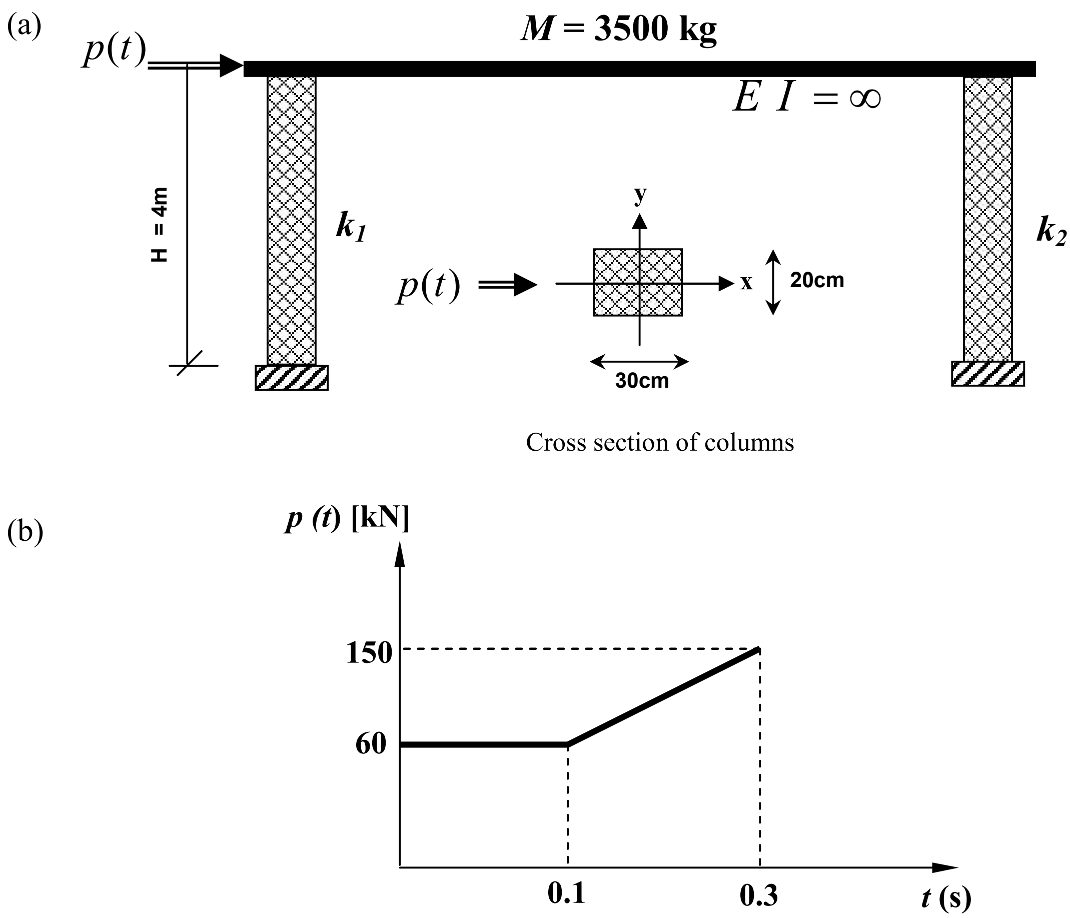


Figure 5.14

5.4.

Determine analytically using the Duhamel integral the displacement at the top of the column as a function of time when the structure is subjected to an explosive load, see figure (5.15). Young's modulus $\boldsymbol{E} = \mathbf{3\ 10^{10}\ N/m^2}$.

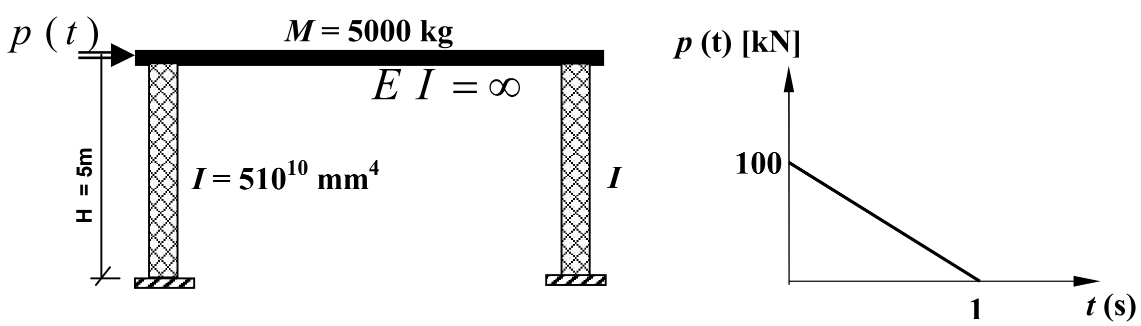


Figure 5.15

5.5.

The one-storey building shown in figure (5.16) is subjected to a load from an explosion, the effect of which can be represented by a lateral force applied at the level of the roof and the variation of which with time is indicated. The total lateral stiffness of the building is **2500 kN/m** and the total mass of the roof is **17500 kg**. Calculate the maximum displacement of the roof and the maximum shear at the base of the building using the shock spectrum shown in figure (5.12).

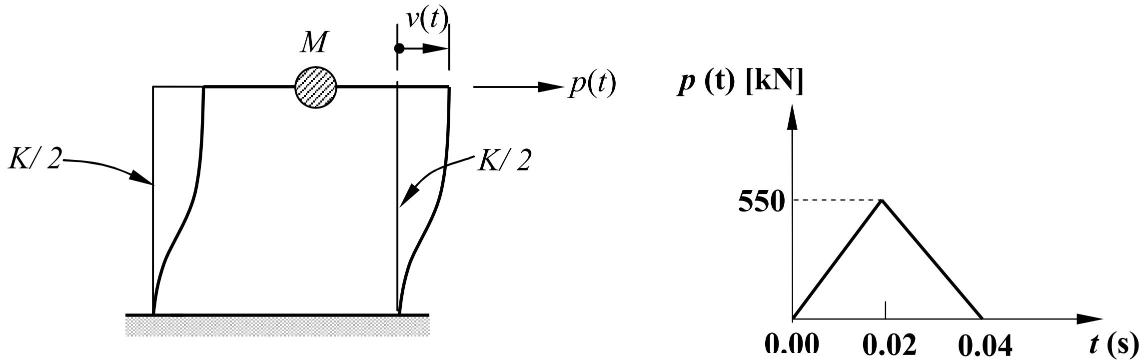


Figure 5.16

CHAPTER

6

RESPONSE OF STRUCTURES WITH A SINGLE DEGREE OF FREEDOM TO AN EARTHQUAKE

6.1 INTRODUCTION

The most widespread application of structural dynamics in civil engineering is in the study of the response of structures to earthquakes. The main reason is that earthquakes generate significant inertia forces for the vast majority of buildings and civil engineering structures, with the exception of very flexible structures such as very large suspension bridges, very slender pylons and very tall skyscrapers for which the wind becomes the dominant mode of dynamic excitation. An earthquake gives rise to a translational movement of the ground in three directions which is transmitted to the structures through their foundation. The simplest instrument for measuring the intensity of ground motion is the accelerograph, which records ground acceleration over time in three orthogonal directions, two horizontal and one vertical. These recordings are called accelerograms.

The distinctive character of earthquakes is that there are no forces applied directly to the structures as for the other types of dynamic loading but rather movements imposed on the supports. However, we saw in section 2.5.2 of chapter 2 that the effect of these movements was similar to the application of an effective force at the level of the mass.

6.2 TEMPORAL REPONSE OF THE SYSTEM

The response to an earthquake of a one-storey structure, comparable to an elementary system, can be obtained from the Duhamel integral. Noting that the effective force due to an earthquake is given by the product of the mass, *M*, by the acceleration of the ground, $\ddot{v}_g(t)$, i.e. $p_{eff}(t) = -M\ddot{v}_g(t)$. Thus, according to the Duhamel integral, we obtain:

$$v(t) = -\frac{1}{\omega_D}\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\mathrm{d}\tau \qquad \textbf{(6.1)}$$

The relative velocity history is obtained by derivation of equation (6.1). So:

$$\dot{v}(t) = -\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\cos\omega_D(t-\tau)d\tau + \frac{\xi\omega}{\omega_D}\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)d\tau \qquad \textbf{(6.2)}$$

To find the expression for the total acceleration, let us write the equation of motion of an elementary system with a motion of the base:

$$M\ddot{v}^t(t) + C\dot{v}(t) + Kv(t) = 0 \qquad \textbf{(6.3)}$$

where $\ddot{v}^t(t)$ is the total acceleration of the mass *M*. This equation can be put in the following form:

$$\ddot{v}^t(t) = -2\xi\omega\dot{v}(t) - \omega^2 v(t) \qquad \textbf{(6.4)}$$

The total acceleration history is obtained by substituting equations (6.1) and (6.2) into equation (6.4). So:

$$\ddot{v}^t(t) = \frac{\omega(1-2\xi^2)}{\sqrt{1-\xi^2}}\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\mathrm{d}\tau + 2\xi\omega\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\cos\omega_D(t-\tau)\mathrm{d}\tau \qquad \textbf{(6.5)}$$

Figure (6.1) presents the response *v*(*t*) of four different elementary systems to the north-south component of the 1940 El Centro earthquake. The masses and rigidities of the elementary systems are such that their period of vibration *T* are 0.5 s, 1.0 s, 2.0 s and 3.0 s while having the same damping factor $\xi = 0.02$. Examination of figure (6.1) shows that the amplitude, duration and shape of the response strongly depend on the natural period of the elementary system.

It can be seen that the maximum displacement does not necessarily occur at the time of the maximum acceleration and that the maximum displacement is different for each system and that, apparently, it increases with an increase in the natural period of vibration of the system. However, one should not make a general rule of this observation which is only valid for a certain range of periods.

6.3 CALCULATION OF EFFORTS

Having determined the time response, the shear and moment at the base of the building can be calculated as a function of the elastic force, F_E, i.e. the force which, if applied statically, would have caused the displacement *v*(*t*). The elastic force, F_E, is given by:

$$F_E(t) = Kv(t) \qquad \textbf{(6.6)}$$

Equation (6.6) can also be expressed as a function of the mass *M*. Thus:

$$F_E(t) = M\omega^2 v(t) \qquad \textbf{(6.7)}$$

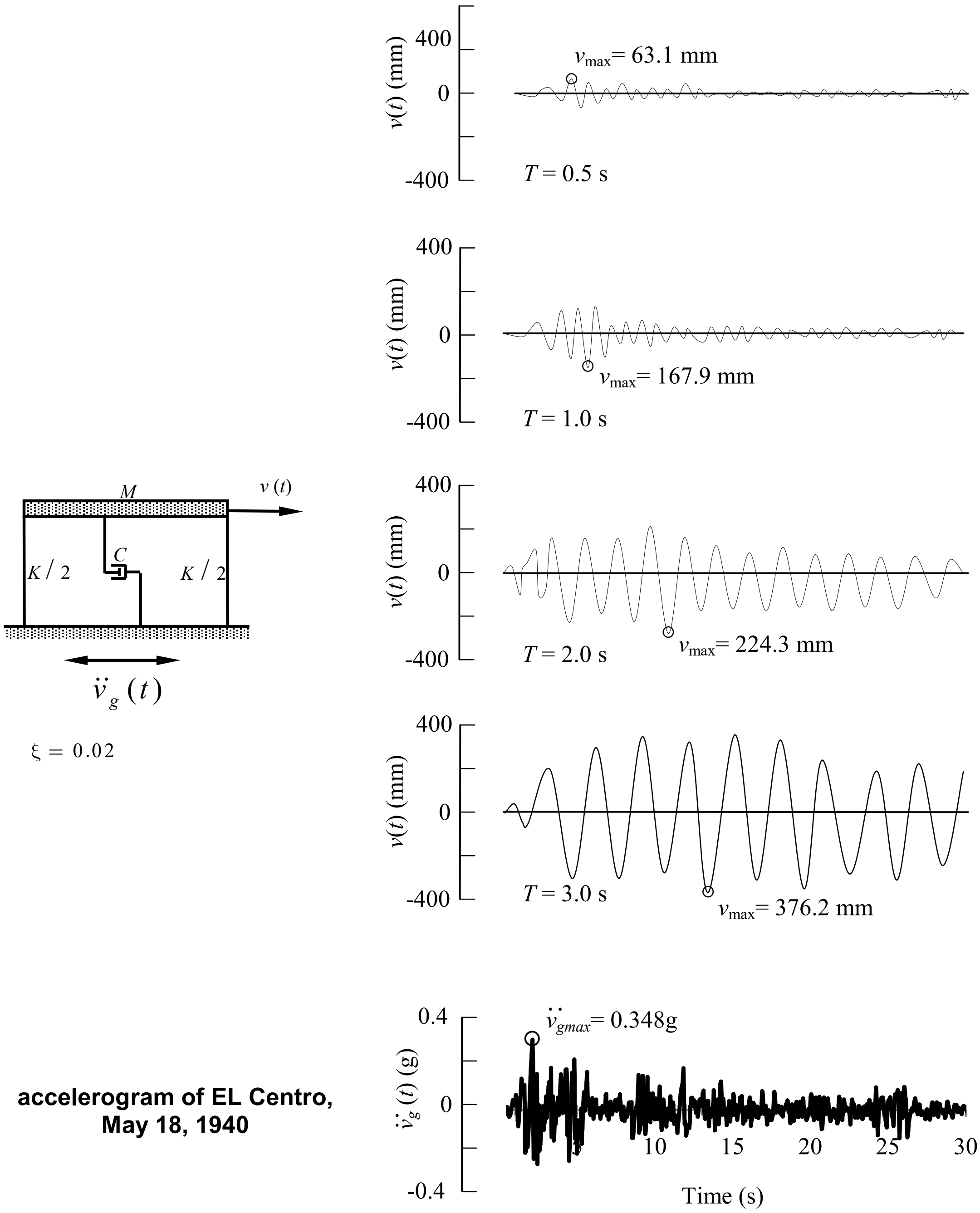


Figure 6.1: Responses of four structures with a single degree of freedom having natural periods T = 0.5s, 1s, 2s and 3s and a damping factor ξ=0.02 submitted to the first thirty seconds of the N-S component of the El Centro accelerogram.

The shear at the base V_0 and the moment M_0 at the base of the structure can be determined by static analysis of the structure subjected to an equivalent lateral force F_E (t). We obtain:

$$V_0(t) = F_E(t) \qquad \textbf{(6.8)}$$
$$M_0(t) = hF_E(t) \qquad \textbf{(6.9)}$$

where h is the height of the mass.

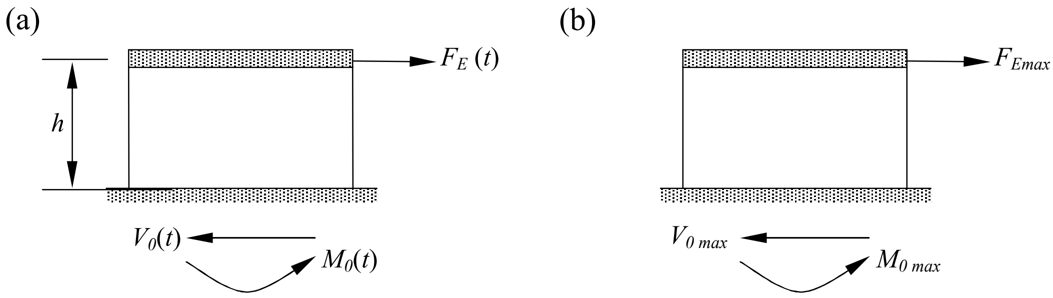


Figure 6.2: Elastic force, overturning moment and shear at the base caused by an earthquake: (a) value at time t and (b) maximum values.

Equations (6.8) and (6.9) can also be expressed in the following form:

$$V_0(t) = M\omega^2 v(t) \quad \textbf{(6.10)}$$
$$M_0(t) = hV_0(t) \quad \textbf{(6.11)}$$

6.4 RESPONSE SPECTRUM

The concept of response spectrum was first used by M.A. Biot to characterize the effect of earthquakes on structure. G.W. Housner propelled its use in earthquake engineering. The importance of this concept comes from the fact that, during a dimensioning, one is mainly interested only in the maximum value of the response of a structure to an earthquake. We have for any response parameter *r:*

$$r_{\max} = \max_t \left(\left| r(t) \right| \right) \quad \textbf{(6.12)}$$

where the max index designates the maximum value of the response over time. The curve of the maximum value of any response parameter (displacement, velocity or acceleration) as a function of the natural period (or frequency) of an elementary system is called a response spectrum. Let us define the following response spectra:

$$\text{displacement spectrum SD}(\omega_D, \xi) = \max_t \left| v(t, \omega_D, \xi) \right| \quad \textbf{(6.13)}$$

$$\text{velocity spectrum SV}(\omega_D, \xi) = \max_t \left| \dot{v}(t, \omega_D, \xi) \right| \quad \textbf{(6.14)}$$

$$\text{absolute acceleration spectrum SA}(\omega_D, \xi) = \max_t \left| \ddot{v}(t, \omega_D, \xi) \right| \quad \textbf{(6.15)}$$

For a given earthquake, the response spectra are therefore functions of frequency and damping. They are presented in the form of graphs for a given damping factor ξ over a wide frequency band. For a natural frequency (natural pulsation) and a given damping factor, the value of the maximum relative displacement is obtained from equation (6.1) to give us:

$$\text{SD}(\omega_D, \xi) = \max_t \left| -\frac{1}{\omega_D} \int_0^t \ddot{v}_g(\tau) e^{-\xi\omega(t-\tau)} \sin(\omega_D(t-\tau)) d\tau \right| = \max_t \left| D(t) \right| \quad \textbf{(6.16)}$$

where $D(t)$ is the relative displecement history of the mass given by equation (6.1). So:

$$D(t) = -\frac{1}{\omega_D}\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\mathrm{d}\tau \qquad \textbf{(6.17)}$$

Similarly, for a given frequency and damping factor, the relative speed is obtained by calculating the absolute maximum of the speed given by equation (6.2). So:

$$SV(\omega_D,\xi) = \max_t\left|-\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\cos\omega_D(t-\tau)\mathrm{d}\tau + \frac{\xi\omega}{\omega_D}\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\mathrm{d}\tau\right| \qquad \textbf{(6.18)}$$

and the maximum absolute acceleration is obtained from equation (6.5). So:

$$SA(\omega_D,\xi) = \max_t\left|\frac{\omega(1-2\xi^2)}{\sqrt{1-\xi^2}}\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\mathrm{d}\tau + 2\xi\omega\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\cos\omega_D(t-\tau)\mathrm{d}\tau\right|$$

(6.19)

Response spectra are determined by numerically evaluating the integrals of equations (6.16) to (6.19) for a given earthquake and damping value and a large number of natural frequencies (periods). The displacement spectrum evaluation process is shown in figure (6.3).

Considerable simplification can be achieved by noting that $\omega \approx \omega_D$ for the small values of damping encountered in practice, i.e. for $\xi < 0.2$, and assuming that the terms in ξ and ξ^2 are small. We obtain:

$$SV(\omega_D,\xi) \approx \max_t\left|-\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\cos\omega(t-\tau)\mathrm{d}\tau\right| \qquad \textbf{(6.20)}$$

This expression can be further simplified by replacing the cosine term of equation (6.20) with a sine term, which has the same maximum value although at a different time, to obtain the following final relation:

$$S_v(\omega_D,\xi) \approx \max_t\left|-\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\sin\omega(t-\tau)\mathrm{d}\tau\right| = \max_t\left|V(t)\right|$$

(6.21)

where S_v is used to designate the pseudo-velocity spectrum and distinguish it from the exact velocity spectrum called *SV* and $V(t)$ is the relative pseudo-velocity history. So:

$$V(t) = -\int_0^t \ddot{v}_g(\tau)e^{-\xi\omega(t-\tau)}\sin\omega(t-\tau)\mathrm{d}\tau \qquad \textbf{(6.22)}$$

We obtain in a similar way the pseudo-spectrum of acceleration. So:

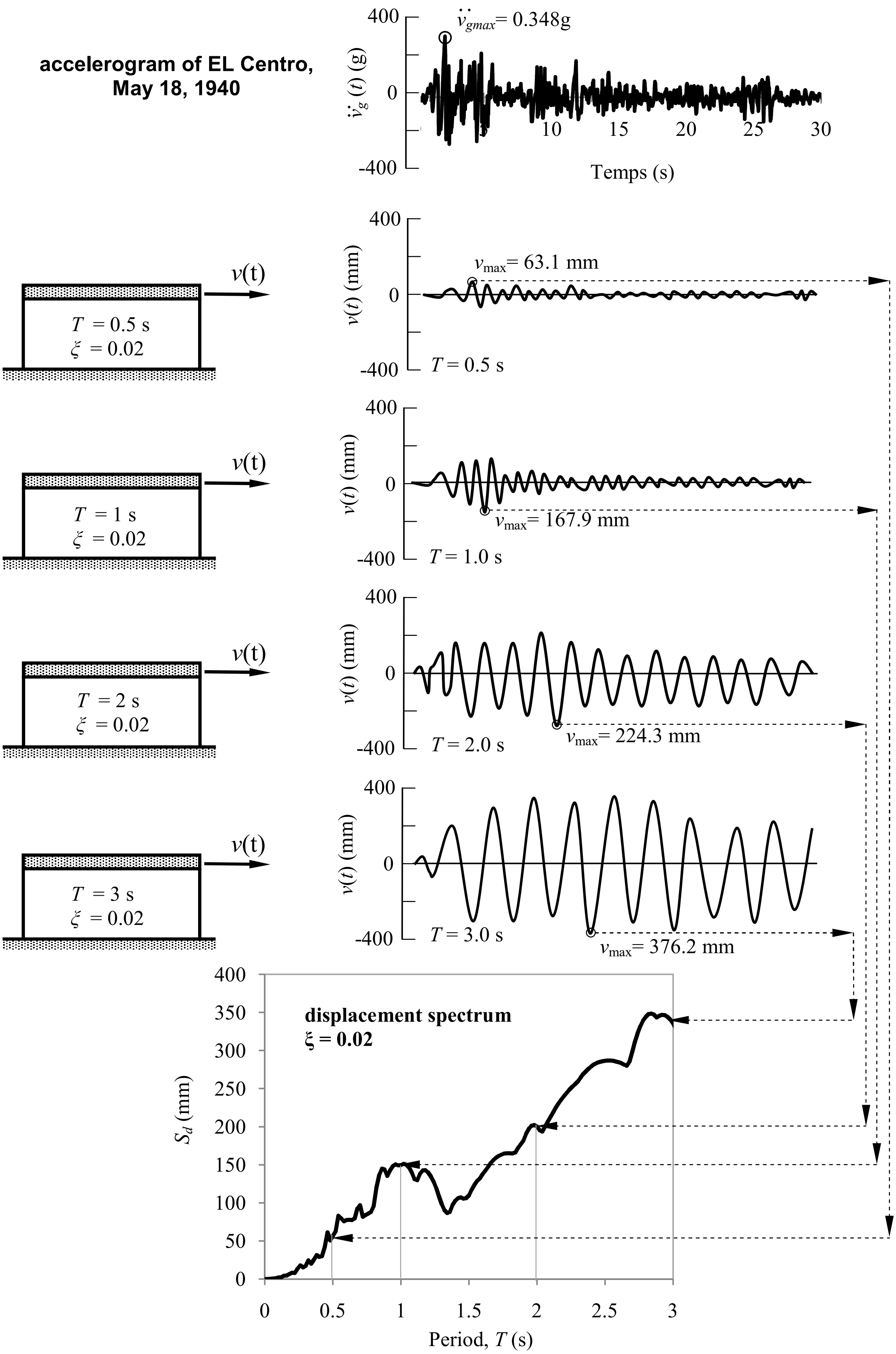


Figure 6.3: Illustration of the calculation of the displacement spectrum for the accelerogram of El Centro.

$$S_a(\omega_D, \xi) = \max_t \left| \int_0^t \ddot{v}_g(\tau) e^{-\xi\omega(t-\tau)} \omega \sin \omega(t-\tau) d\tau \right| = \max_t |A(t)|$$ **(6.23)**

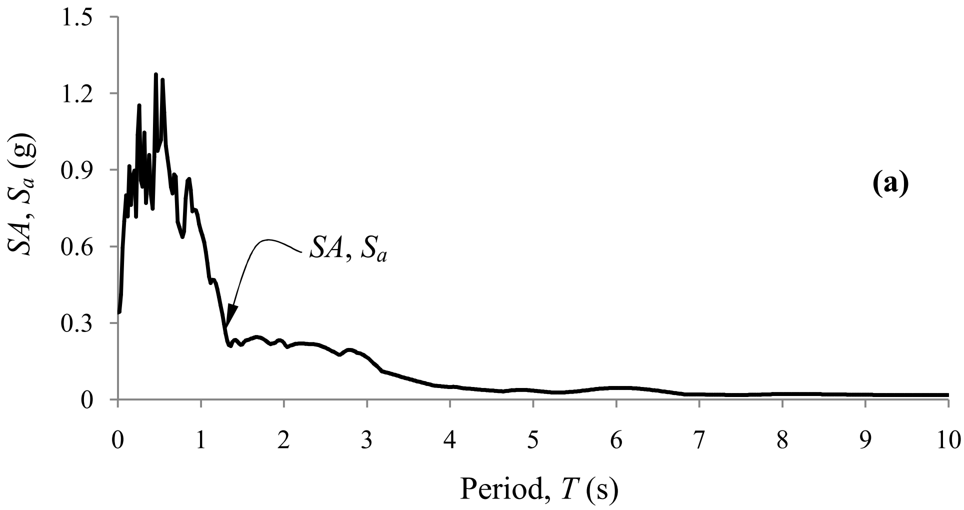
where the history of the absolute pseudo-acceleration is defined by the following relation:

$$A(t) = \int_0^t \ddot{v}_g(\tau) e^{-\xi\omega(t-\tau)} \omega \sin \omega(t-\tau) d\tau = -\omega V(t) = -\omega^2 D(t)$$ **(6.24)**

By comparing equations (6.17), (6.21) and (6.23) and setting S_d = SD, we see that:

$$S_v = \omega S_d \quad \text{et} \quad S_a = \omega^2 S_d = \omega S_v$$ **(6.25)**

The displacement, velocity and acceleration response spectra for the El Centro accelerogram are shown in figure (6.4) for a damping factor $\xi = 0.02$. It can be seen that the difference between SA and S_a is negligible while this difference is significant between SV and S_v for vibration periods greater than 3s. For a sinusoidally vibrating elementary oscillator with a maximum displacement S_d, the maximum speed and the maximum acceleration are equal to S_v and S_a, respectively. For an earthquake, it can be shown that $S_v \approx$ SV and $S_a \approx$ SA over the usual range of periods and dampings. Acceleration being, as we have seen, more precise than speed. When $T \rightarrow \infty$ (infinitely flexible oscillator), SD tends towards the maximum displacement of the ground (P.G.D: Peak Ground Displacement), see figure (6.5), SV tends towards the maximum speed and S_a tends towards zero. However, $S_v = 2\pi S_d / T$ tends to zero when $T \rightarrow \infty$. So, for long periods, the S_v pseudo-velocity spectrum can be significantly different from the SV velocity spectrum. It should be noted, moreover, that for a null damping, $S_a \approx$ SA and $S_v \neq$ SV since the expression of S_v contains a term in sine whereas the expression SV contains a term in cosine. When $T = 0$ (infinitely rigid oscillator), the relative displacement (SD) of the mass with respect to the ground is zero (Fig. 6.5). The value of the pseudo-acceleration (S_a) read on the spectrum for $T = 0$ is therefore equal to the maximum value of the absolute acceleration of the ground (P.G.A: Peak Ground Acceleration).



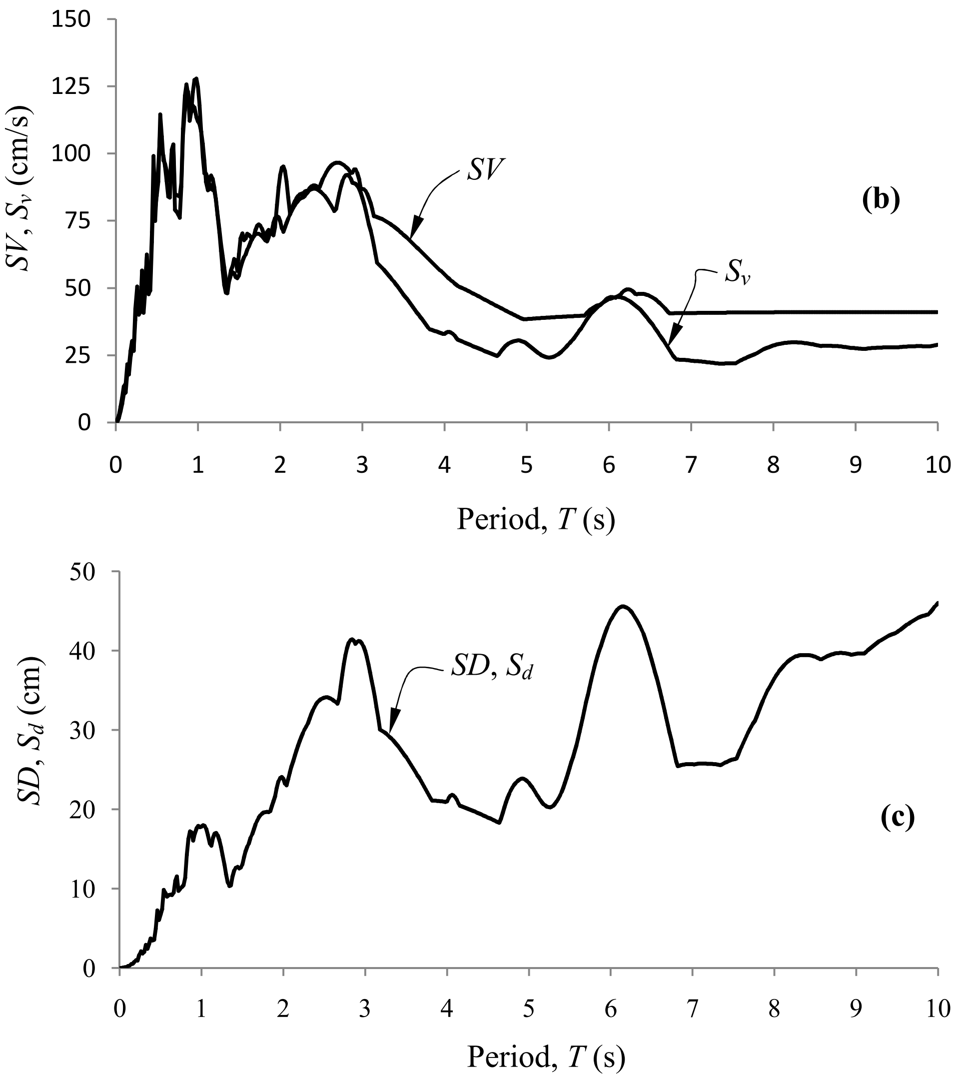


Figure 6.4: Spectra and pseudo-spectra: (a) acceleration, (b) velocity and (c) displacement of the El Centro accelerogram for ξ = 2%.

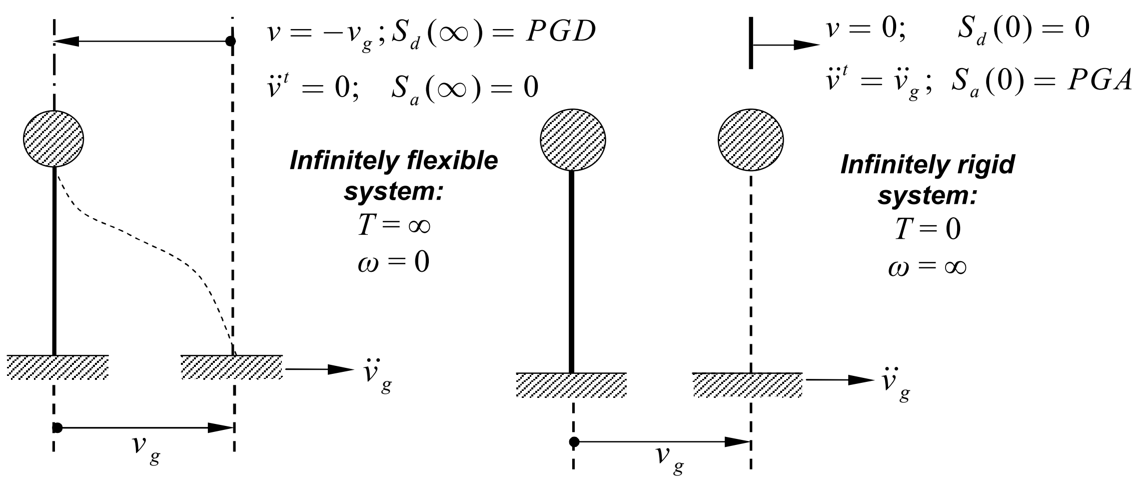


Figure 6.5: Response of infinitely flexible and infinitely rigid single degree of freedom systems to an earthquake.

6.5 DESIGN SPECTRUM

When it comes to determining the response spectrum to be taken into account for the calculation of constructions at a given site, it is of course excluded to use a single accelerogram, even if by chance it was recorded in the vicinity of the site. Indeed, the accelerogram of the earthquake against which we want to protect ourselves is not a priori predictable. It is therefore necessary to determine a calculation spectrum which will be the envelope of a set of spectra corresponding to accelergrams recorded in comparable sites from the point of view of the nature of the ground.

The accelerograms used result from earthquakes of different magnitudes and the response spectra deduced from them are not directly comparable. It is therefore necessary to first apply an affinity to each spectrum so that they all have the same value of spectral intensity. The spectra thus obtained are then called "normalized spectra".

Housner defined the spectral intensity as being the area of the velocity spectrum between the *T* axis and the abscissa lines 0.10 and 2.5 seconds (Fig. 6.6); this range of periods covering most of the cases encountered. Thus, the spectral intensity, *IS*, which measures the severity of the effects of ground motions on elastic structures is defined as follows:

$$\mathrm{IS}(\xi) = \int_{0.1s}^{2.5s} S_v(\xi, T)\,\mathrm{d}T \tag{6.27}$$

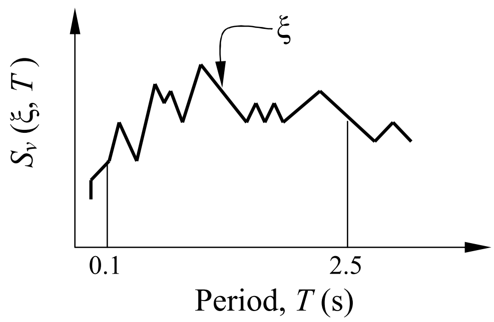


Figure 6.6: Measure of the intensity of an earthquake according to Housner.

One can also normalize the spectra by taking into account the maximum acceleration of the ground instead of the spectral intensity.

The calculation spectrum is obtained by making the envelope of the various normalized spectra; it must then be "calibrated" by subjecting it to an affinity to take into account:

- The probable intensity of the earthquake;
- The importance for the local community of the building studied.

Figure (6.7) represents the appearance that such calculation spectra can have. However, it is important to realize that the envelope spectrum is not representative of any seismic movement and it is useless to search in a database of recorded earthquakes for a recording with an identical response spectrum, or even a similar one, of a envelope spectrum.

On the other hand, the definition of an envelope spectrum is very useful for earthquake engineering regulations. These spectra are defined there from statistical processing of the real recording spectra and generally all have identical shapes, see figure (6.8).

- For low periods, the pseudo-acceleration increases linearly up to a maximum amplification value reached for a period T_B;
- The pseudo-acceleration is constant and maximum for periods between T_B and T_C ;
- Between the T_C and T_D periods, the pseudo-velocity is constant, which means that the pseudo-acceleration decreases in $1/T$ (equation 6.26);
- Beyond T_D, the relative displacement is constant, which means that the pseudo-acceleration decreases in $1/T^2$ (equation 6.26).

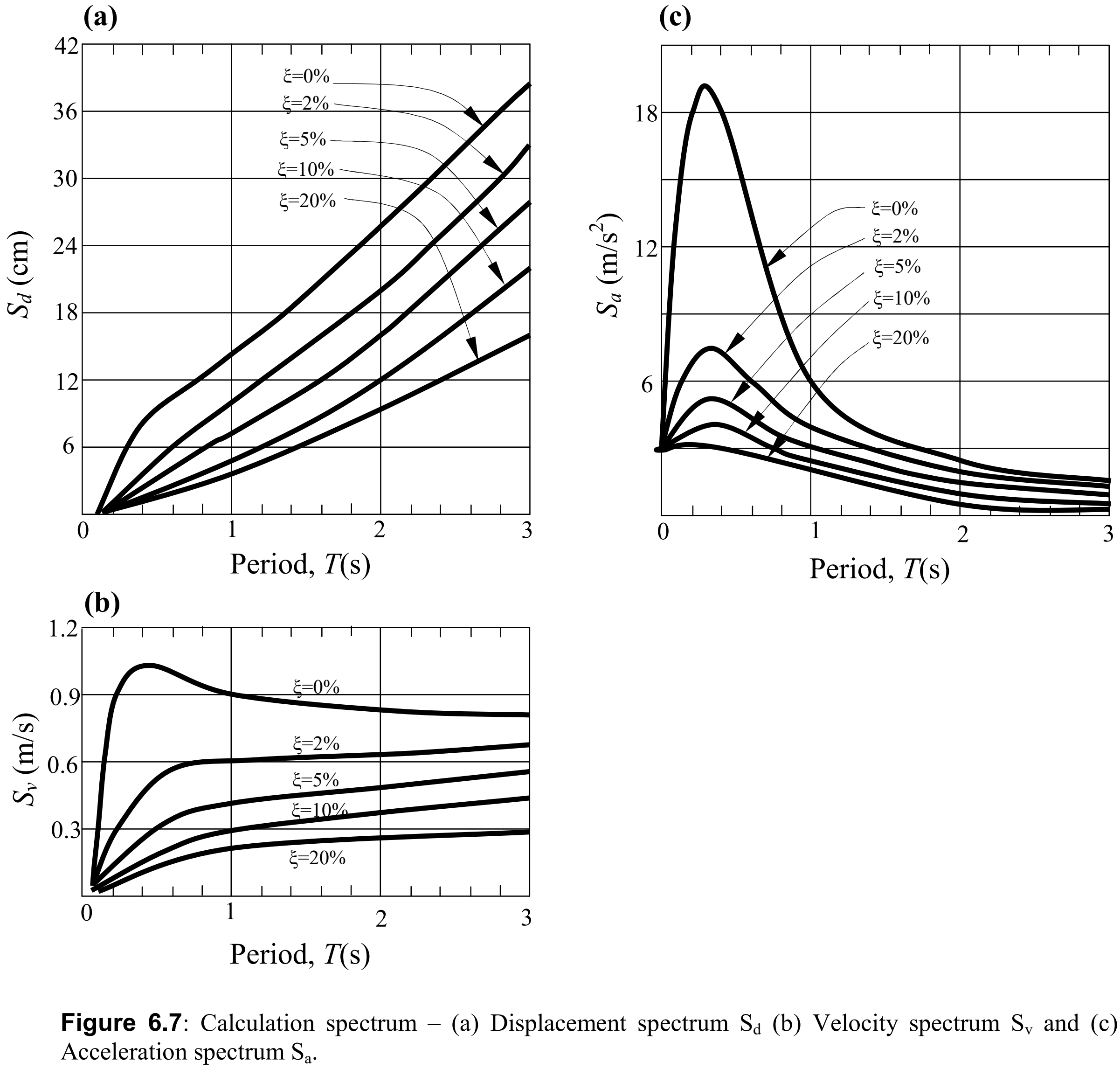


Figure 6.7: Calculation spectrum – (a) Displacement spectrum S_d (b) Velocity spectrum S_v and (c) Acceleration spectrum S_a.

6.6 THE USE OF RESPONSE SPECTRA

Given a structure of mass *M*, stiffness *K* and damping ξ, the use of the response spectra is as follows:

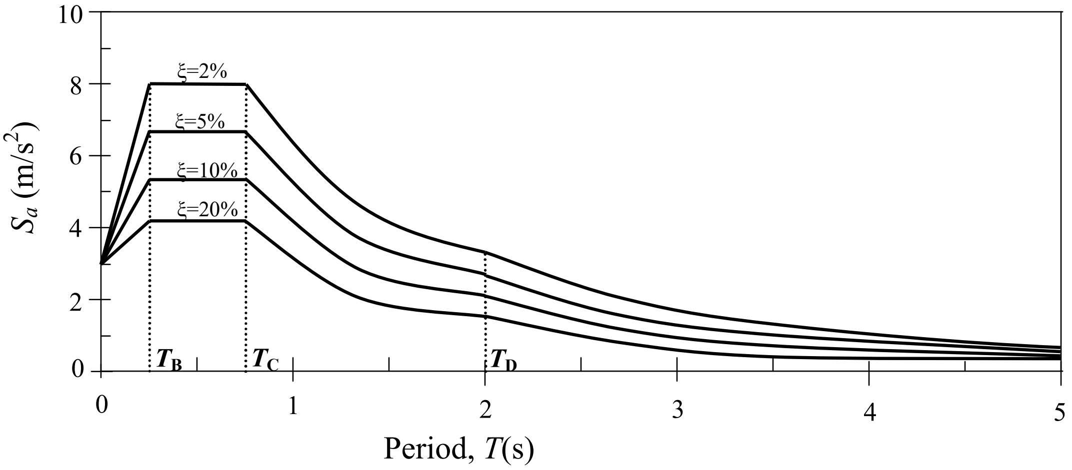


Figure 6.8: Regulatory response spectra.

1) Calculate the period of the structure:

$$T = 2\pi\sqrt{\frac{M}{K}} \quad \textbf{(6.28)}$$

2) Obtain the maximum displacement from T, ξ and the calculation spectrum:

$$v_{max} = S_d \quad \textbf{(6.29)}$$

3) Calculate the maximum deformation energy:

$$V_{max} = \frac{1}{2}Kv_{max}^2 = \frac{1}{2}KS_d^2 = \frac{1}{2}M\left(\frac{S_a}{\omega}\right)^2 \quad \textbf{(6.30)}$$

4) Calculate the base shear :

$$T_{0max} = KS_d = M\omega^2 S_d = MS_a \quad \textbf{(6.31)}$$

from where

$$T_{0max} = \frac{S_a}{g}W \quad \textbf{(6.32)}$$

where W is the weight of the structure, g is the acceleration of gravity. In this equation, S_a / g can be interpreted as the base shear coefficient as given in building codes for the calculation of the equivalent static lateral load.

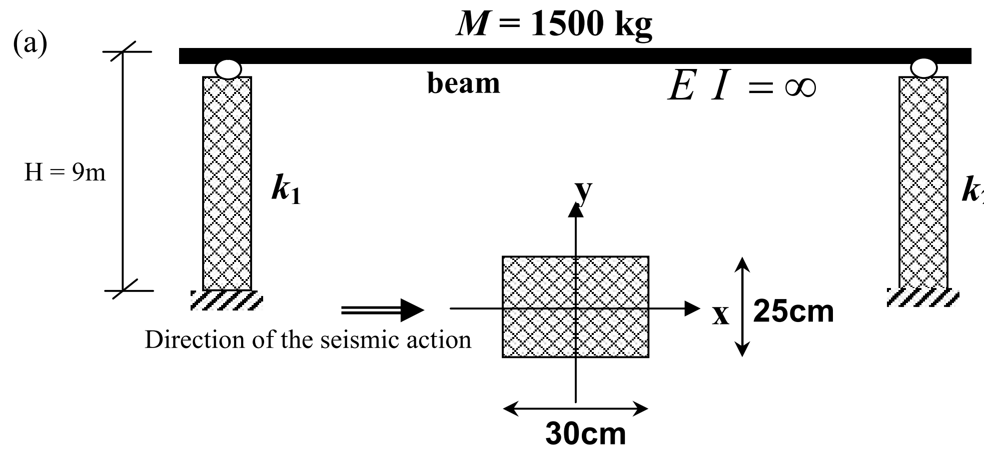
EXERCISES

6.1.

A one-storey frame shown in figure (6.9.a) is subjected to an earthquake characterized by an acceleration response spectrum given in figure (6.9.b)
With a Young's modulus $\mathbf{E = 3\ 10^{10}\ N/m^2}$.

1/ Calculate the maximum seismic force acting on the frame.
2/ Determine the maximum displacement of the frame beam.

(a)



(b)

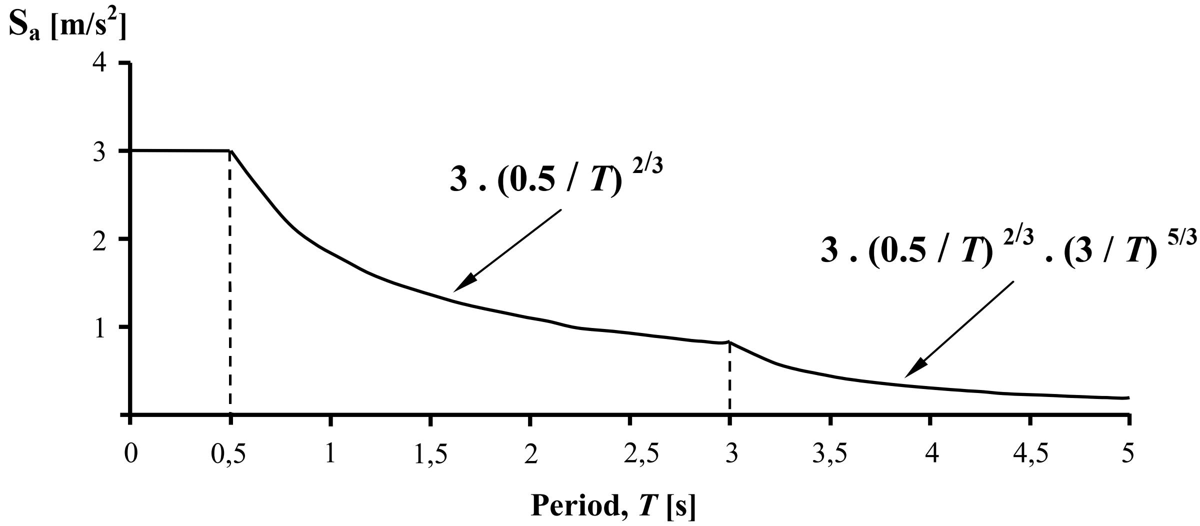


Figure 6.9

6.2.

Consider two neighboring frames separated by a seismic joint whose width must be greater than **d**, see figure (6.10.a). The two frames are subjected to an earthquake characterized by the acceleration response spectrum given in figure (6.10.b)
With a Young's modulus $\mathbf{E = 3\ 10^{10}\ N/m^2}$.

- Determine the width of the seismic joint, **d**, to prevent the collision of the two frames.

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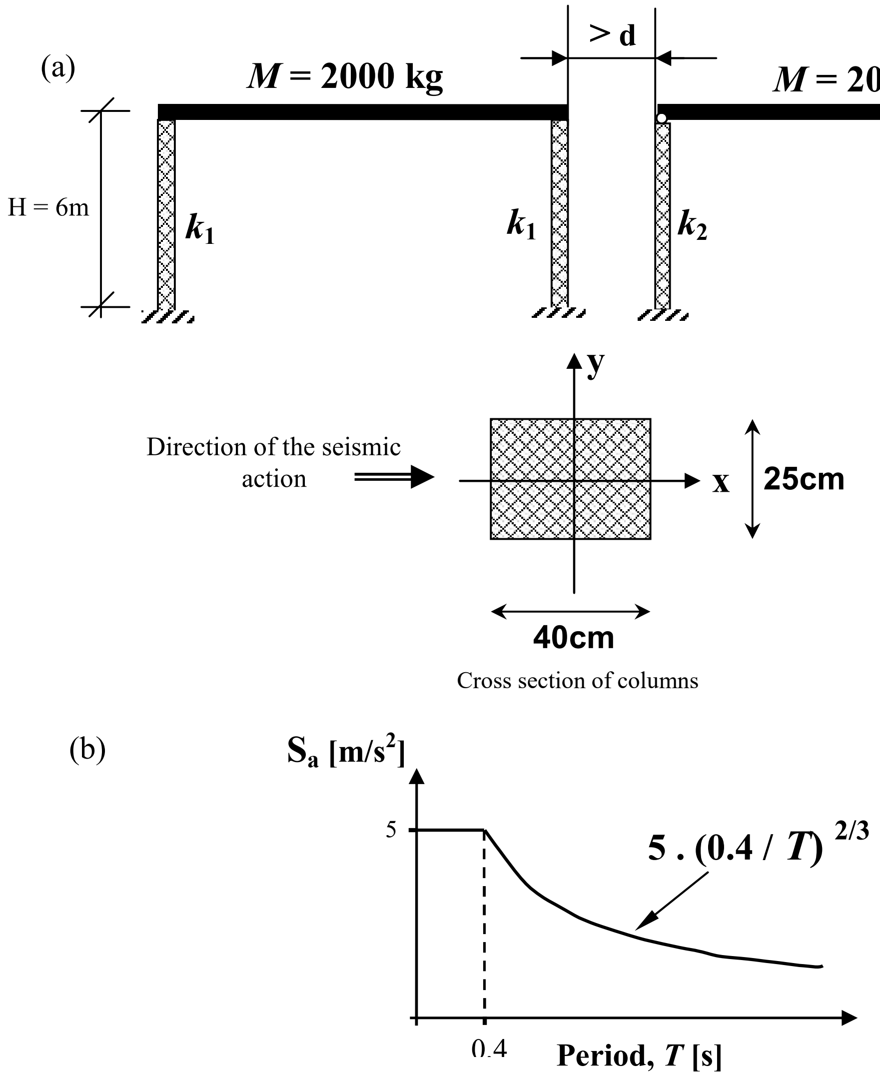


Figure 6.10

6.3.

Consider a simple pendulum of mass (M) and lateral stiffness (K) represented in two different configurations:

- 1st configuration: pendulum with a **fixed base** (embedded), figure (6.11.a);
- 2nd configuration: pendulum with **flexible base** (elastic support: spring), figure (6.11.b).

The pendulum is subjected to an earthquake characterized by an acceleration response spectrum given in figure (6.11.c). If the admissible resistance to the lateral force of the fixed base pendulum is $\mathbf{F_{adm.}}$ **= 4000 N** and its admissible displacement is $\mathbf{X_{adm.}}$**= 2 cm**, study the influence of the elastic support (spring) on these two response parameters.
With: $\boldsymbol{k_r} = \mathbf{2\ 10^5}$ **N / m,** $\boldsymbol{K} = \mathbf{3\ 10^5}$ **N/m,** $\boldsymbol{M}$ **= 1500 kg.**

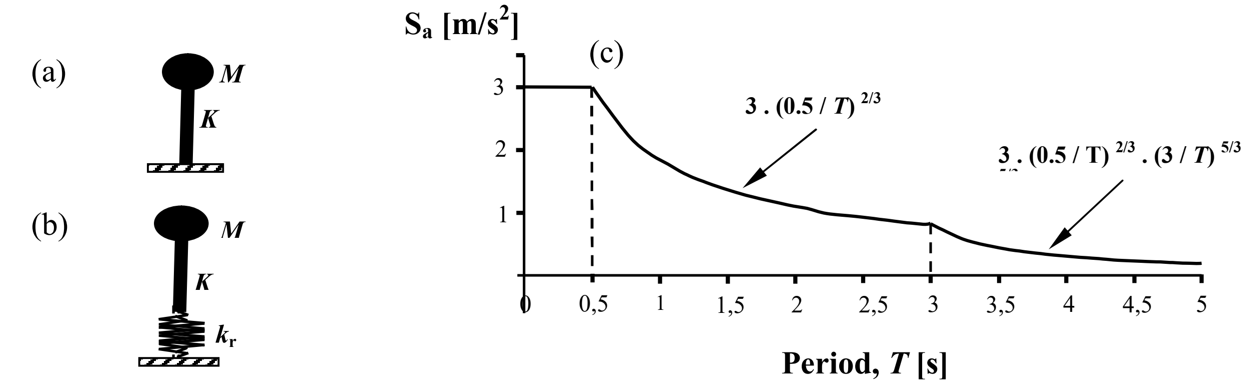


Figure 6.11

6.4.

A one-storey frame shown in figure (6.12) is subjected to an earthquake characterized by the same acceleration response spectrum given in figure (6.11.c).

With a Young's modulus $\mathbf{E = 3\ 10^{10}\ N/m^2}$.

- Determine for this frame the dimension 'a' of the column if the maximum seismic force acting on it is equal to **5000 N**.

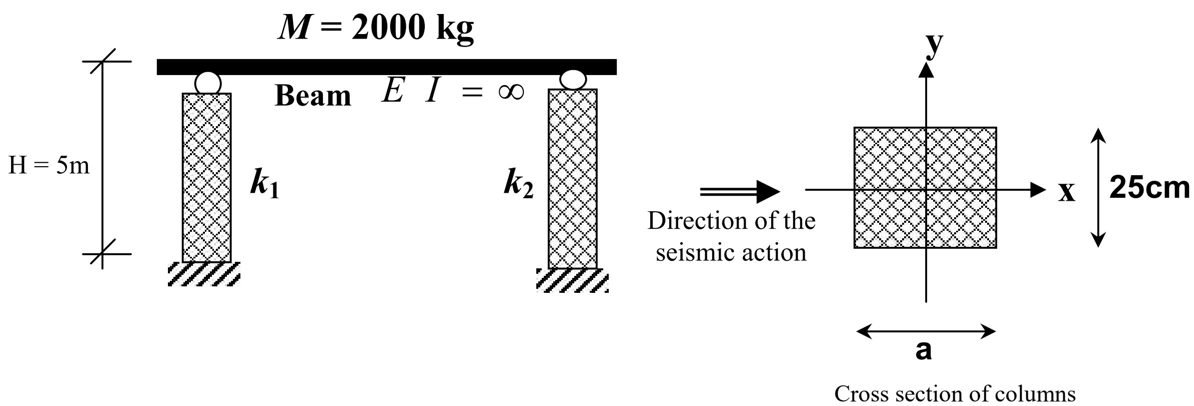


Figure 6.12

CHAPTER

7

DYNAMIC ANALYSIS OF STRUCTURES WITH MULTI DEGREE OF FREEDOM

7.1 INTRODUCTION

In more complicated structures, it is necessary to resort to a system having multi-DOF to obtain a satisfactory dynamic model. This section deals with the dynamic analysis of a linear system having multi-DOF. In the context of so-called deterministic dynamic analysis, it is assumed, on the one hand, that the geometric and mechanical characteristics of the structure (M, K) are perfectly known and, on the other hand, that the forces $F(t)$ applied in each point of the structure are perfectly determined (these are known functions of time). Moreover, this section deals only with concentrated mass systems (*discrete system*); it is assumed that the concentrated masses attract the largest inertial forces and that the system has only a finite number of DOFs.

7.2 EQUATIONS OF MOTION FOR A SYSTEM WITH MULTI DEGREE OF FREEDOM

Recall that the number of degrees of freedom is the number of components of the displacement required to express the inertial forces developing in it. These displacements are evaluated at a number of points in the structure, called nodes where the masses are concentrated. In the most general case, a node has six possible movements (three translations and three rotations) and the number of degrees of freedom of the system is equal to $N = 6p$ where p is the number of nodes.

Figure (7.1) presents two examples of modeling of plane structures falling within this framework. In the case of the frame with two bays (Fig. 7.1.a), the degrees of freedom are constituted by the displacements of the nodes located at the intersection of the columns and the beams, the masses of the structure are concentrated in these nodes. Each node is affected by the mass of the column and floor elements located in its vicinity and symbolized in figure (7.1.a) by the dotted rectangle for the central node. If the axial stiffness of the columns is infinite, the only possible movements of the nodes are the horizontal translation and the rotation around an axis perpendicular to the plane of the figure, that is to say a number of degrees of freedom $2p$ =18. If moreover the floors are considered as infinitely rigid, the kinematics of a floor level is described by a movement of one of its points; we end up with a ″skewer″ model of figure (7.1.b), in which the mass of a level is concentrated at a point, generally the center of mass of the floor. The number of degrees of freedom has been reduced in this case to $2p$ = 6.

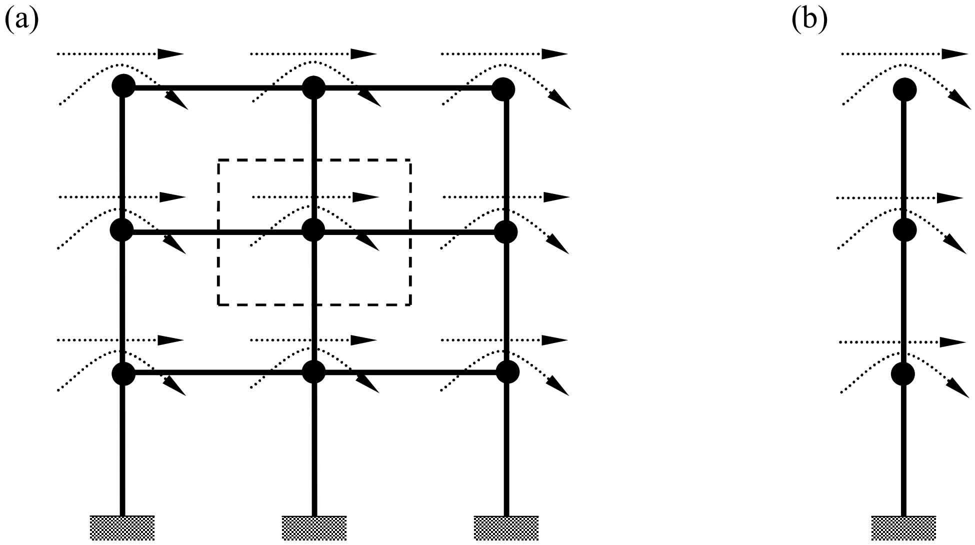


Figure 7.1: Modeling in concentrated masses: (a) plan frame and (b) vertical console.

There are N equations of motion for a system with N dynamic degrees of freedom. These equations can be written in matrix form with N x N order matrices. Like the one-DOF system, the dynamic excitation can take the form of any loading or the form of a base excitation.

Any dynamic loading. Consider an undamped two-DOF modeled system as shown in figure (7.2). The free body diagrams for each mass are shown in figure (7.3). By applying Newton's second law to each mass, we get:

$$\begin{aligned} M_1\ddot{v}_1(t) + K_1 v_1(t) - K_2\left[v_2(t) - v_1(t)\right] &= F_1(t) \\ M_2\ddot{v}_2(t) + K_2\left[v_2(t) - v_1(t)\right] &= F_2(t) \end{aligned} \quad \textbf{(7.1)}$$

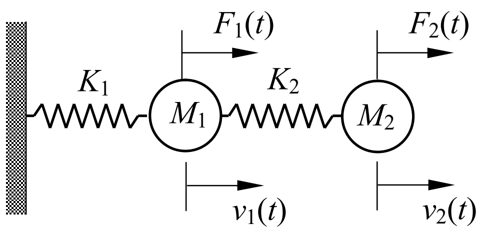


Figure 7.2: Two DOF system.

In matrix form, we write:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{v}_1(t) \\ \ddot{v}_2(t) \end{Bmatrix} + \begin{bmatrix} (K_1 + K_2) & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} v_1(t) \\ v_2(t) \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \tag{7.2}$$

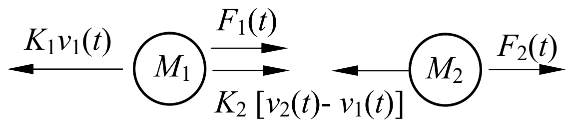


Figure 7.3: Free-body diagram of a two-DOF system.

Equation (7.2) can also be written:

$$[M]\{\ddot{v}(t)\} + [K]\{v(t)\} = \{F(t)\} \tag{7.3}$$

But if we integrate the viscous damping into the model, we introduce the damping matrix [C] and the equation (7.3) becomes:

$$[M]\{\ddot{v}(t)\} + [C]\{\dot{v}(t)\} + [K]\{v(t)\} = \{F(t)\} \tag{7.4}$$

where

[M]	= global mass matrix (usually diagonal)
[C]	= global damping matrix (difficult to evaluate)
[K]	= global stiffness matrix corresponding to the DOF
$\{F(t)\}$	= dynamic loading vector
$\{v(t)\},\{\dot{v}(t)\},\{\ddot{v}(t)\}$	= vectors of displacements, velocities and accelerations of DOF relative to the base

Relation (7.4) represents a system of linear differential equations governing the motion of a system with multi-DOFs. This system of equations describes the basic fundamental relations that we propose to solve for any dynamic loading vector, $\{F(t)\}$. It is important to understand that equation (7.4) assumes that the properties of the system ([M], [K] and [C]) do not change during the system response, i.e. the system is linear. The elastic limit of the elements of the system will never be exceeded and therefore no plasticization is allowed.

Excitation at the base (seismic problem). To deduce the system of equations of motion, consider the same two-DOF system subject to the base displacement, $v_g(t)$, figure (7.4). By applying Newton's second law to each mass, we get:

$$\begin{aligned} M_1\left[\ddot{v}_1(t) + \ddot{v}_g(t)\right] + K_1 v_1(t) - K_2\left[v_2(t) - v_1(t)\right] = 0 \\ M_2\left[\ddot{v}_2(t) + \ddot{v}_g(t)\right] + K_2\left[v_2(t) - v_1(t)\right] = 0 \end{aligned} \tag{7.5}$$

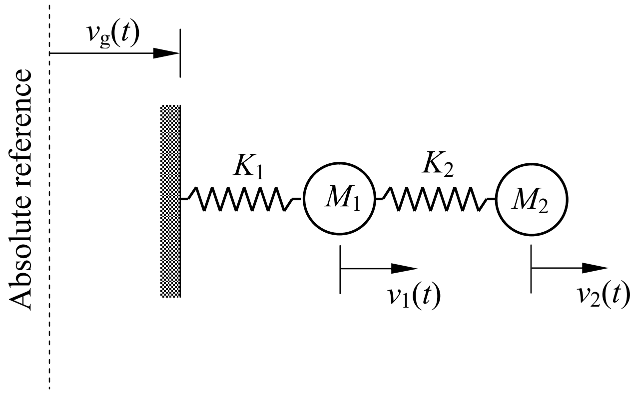


Figure 7.4: System with two DOF subjected to a displacement of its base.

In matrix form, we write:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{v}_1(t) \\ \ddot{v}_2(t) \end{Bmatrix} + \begin{bmatrix} (K_1 + K_2) & -K_1 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} v_1(t) \\ v_2(t) \end{Bmatrix} = - \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{v}_g(t) \tag{7.6}$$

or:

$$[M]\{\ddot{v}(t)\} + [K]\{v(t)\} = -[M]\{r\}\ddot{v}_g(t) = \{F_{\text{eff}}(t)\} \tag{7.7}$$

If one introduces the viscous damping in the model, one obtains:

$$[M]\{\ddot{v}(t)\} + [C]\{\dot{v}(t)\} + [K]\{v(t)\} = \{F_{\text{eff}}(t)\} \tag{7.8}$$

where

$[M]$	= global mass matrix (usually diagonal)
$[C]$	= global damping matrix (difficult to evaluate)
$[K]$	= global stiffness matrix corresponding to the DOF
$\{F_{\text{eff}}(t)\}$	= effective load vector
$\{v(t)\},\{\dot{v}(t)\},\{\ddot{v}(t)\}$	= vectors of displacements, velocities and accelerations of DOF relative to the base
$\ddot{v}_g(t)$	= absolute base acceleration
$\{r\}$	= dynamic coupling vector

The dynamic coupling vector, $\{r\}$, relates the direction of motion at the base with the direction of each DOF as the structure moves as a rigid body. In general, this vector is composed of 1 and 0. For example, the dynamic coupling vector for the DDLs of figure (7.5) is:

$$\{r\} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{7.9}$$

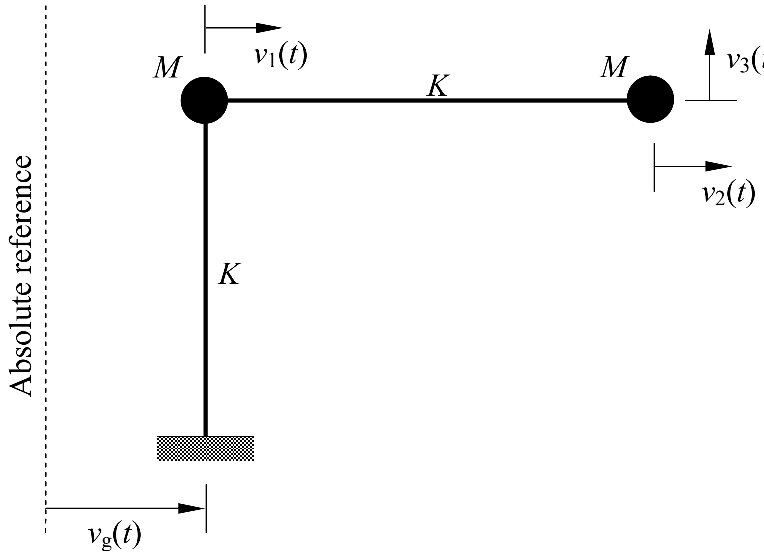


Figure 7.5: System with three DOF subjected to a horizontal movement of its base.

Note that equation (7.8) is identical to equation (7.4) except for the dynamic load vector, $\{F(t)\}$, replaced by an effective load vector : $-[M]\{r\}\ddot{v}_g(t)$.

7.3 DEVELOPMENT OF STIFFNESS, MASS AND DAMPING MATRICES

7.3.1 STIFFNESS MATRIX

One considers a structure with p nodes comprising on the whole N degrees of freedom numbered from 1 to N. In the general case, there exist six degrees of freedom per node; the number N therefore has the value: $N = 6p$, as mentioned in paragraph 7.2.

We call :

- $F_{Ej}(t)$: the elastic force, or the moment, applied to the structure in the direction of the degree of freedom j,
- $v_j(t)$: the displacement (translation or rotation) according to the degree of freedom j.

The force $F_{Ej}(t)$ is counted positive if it causes a positive displacement $v_j(t)$, see figure (7.6).

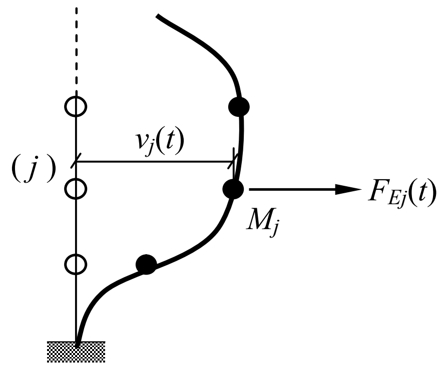


Figure 7.6: Deformed shape of an N DOF system under lateral load.

The set of elastic forces constitutes the force-vector $\{F_E(t)\}$ with n components:

$$\{F_E(t)\} = \begin{Bmatrix} F_{E1}(t) \\ F_{E2}(t) \\ . \\ . \\ . \\ F_{Ej}(t) \\ . \\ . \\ . \\ F_{EN}(t) \end{Bmatrix} \tag{7.10}$$

We consider the vector of displacements in the same way:

$$\{v(t)\} = \begin{Bmatrix} v_1(t) \\ v_2(t) \\ . \\ . \\ . \\ v_j(t) \\ . \\ . \\ . \\ v_N(t) \end{Bmatrix} \tag{7.11}$$

The structure is supposed elastic; thus, there exist linear relations between the elastic forces and displacements. So:

$$\begin{aligned} F_{E1}(t) &= K_{11}v_1(t) + K_{12}v_2(t) + \cdots + K_{1j}v_j(t) + \cdots + K_{1N}v_N(t) \\ F_{E2}(t) &= K_{21}v_1(t) + K_{22}v_2(t) + \cdots + K_{2j}v_j(t) + \cdots + K_{2N}v_N(t) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_{Ei}(t) &= K_{i1}v_1(t) + K_{i2}v_2(t) + \cdots + K_{ij}v_j(t) + \cdots + K_{iN}v_N(t) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_{EN}(t) &= K_{N1}v_1(t) + K_{N2}v_2(t) + \cdots + K_{Nj}v_j(t) + \cdots + K_{NN}v_N(t) \end{aligned} \tag{7.12}$$

Expression (7.12) can be put in the following form:

$$\{F_E(t)\} = [K]\{v(t)\} \tag{7.13}$$

where $\{F_E(t)\}$ is the elastic force vector.

$[K]$ is the N x N dimensional stiffness matrix:

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1j} & \cdots & K_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{i1} & K_{i2} & \cdots & K_{ij} & \cdots & K_{iN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{N1} & K_{N2} & \cdots & K_{Ni} & \cdots & K_{NN} \end{bmatrix} \quad \textbf{(7.14)}$$

In the simple cases, one will be able to calculate the coefficients of influence K_{ij} by evaluating the forces which it is necessary applied to the nodes of structure so that all displacements, except one, are null. Indeed, if all displacements are null, except displacement v_j to which one imposes a unit displacement, the equations (7.12) become:

$$\begin{aligned} F_{E1} &= K_{1j} \\ F_{E2} &= K_{2j} \\ &\cdots\cdots\cdots \\ F_{Ei} &= K_{ij} \\ &\cdots\cdots\cdots \\ F_{EN} &= K_{Nj} \end{aligned} \quad \textbf{(7.15)}$$

It is noted that the column of rank j of the stiffness matrix is identical to the vector force corresponding to this case of imposed displacement. We can therefore calculate the coefficients of this matrix by evaluating the forces at the nodes for N cases of imposed displacements, see figure (7.7).

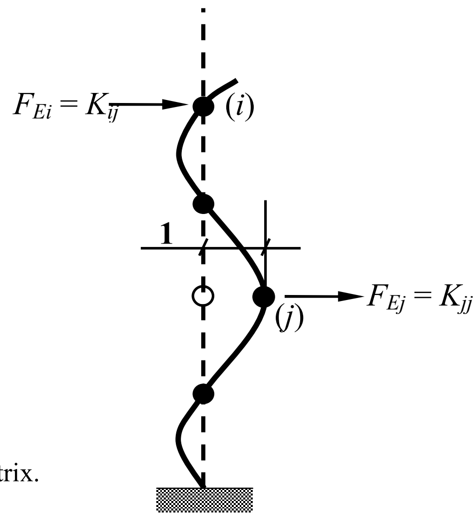


Figure 7.7: Calculation of coefficients of the stiffness matrix.

7.3.2 MASS MATRIX

It is assumed that all the mass of the structure is concentrated at the nodes; with each node, one thus associates in general three masses and three mass moments of inertia corresponding to the six degrees of freedom.

We notice :

- M_j the mass relative to the degree of freedom j,

- $\ddot{v}_j(t)$ the acceleration of the mass in the direction of the degree of freedom *j*.

The masses being concentrated, the acceleration of only one of them imposes inertial force only on itself: the modeling in concentrated masses does not introduce any coupling between the degrees of freedom, which does not would not be the case if there were masses distributed between two nodes; the mass influence coefficient is therefore $M_{ii} = M_i$.

When all the masses are concentrated, the inertial forces have the following values:

$$\begin{aligned} F_{I1}(t) &= M_1\ddot{v}_1(t) \\ F_{I2}(t) &= M_2\ddot{v}_2(t) \\ &\ldots\ldots\ldots\ldots \\ F_{Ii}(t) &= M_i\ddot{v}_i(t) \\ &\ldots\ldots\ldots\ldots \\ F_{IN}(t) &= M_N\ddot{v}_N(t) \end{aligned} \tag{7.16}$$

These relations are written as follows:

$$\{F_I(t)\} = [M]\{\ddot{v}(t)\} \tag{7.17}$$

$\ddot{v}(t)$ represents the vector of accelerations:

$$\{\ddot{v}(t)\} = \begin{Bmatrix} \ddot{v}_1(t) \\ \ddot{v}_2(t) \\ . \\ . \\ . \\ \ddot{v}_j(t) \\ . \\ . \\ . \\ \ddot{v}_N(t) \end{Bmatrix} \tag{7.18}$$

[*M*] represents the matrix of the masses, diagonal and of dimension *N* x *N*:

$$[M] = \begin{bmatrix} M_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & M_2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & M_3 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & M_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & M_N \end{bmatrix} \tag{7.19}$$

7.3.3 DAMPING MATRIX

It is impossible to evaluate the damping forces that apply to a structure. Given its mathematical elegance, we adopt the simple model of viscous damping. According to the latter, the coefficients of the damping matrix, $\left[C\right]$, are generally defined according to modal damping. As we will see later, one chooses the form of the matrix of damping in order to obtain the solution of the system.

7.4 UNDAMPED FREE VIBRATIONS

We assume here that the discrete system with *N* degrees of freedom is not subjected to any external force, that is to say $\left\{F(t)\right\}=\left\{0\right\}$, and that the damping is null. The solution of the following system of differential equations

$$\left[M\right]\left\{\ddot{v}(t)\right\}+\left[K\right]\left\{v(t)\right\}=\left\{0\right\} \qquad \textbf{(7.20)}$$

is the undamped regime that describes the response of the system over time when *arbitrary initial conditions* are imposed on it. An example is shown in figure (7.8).

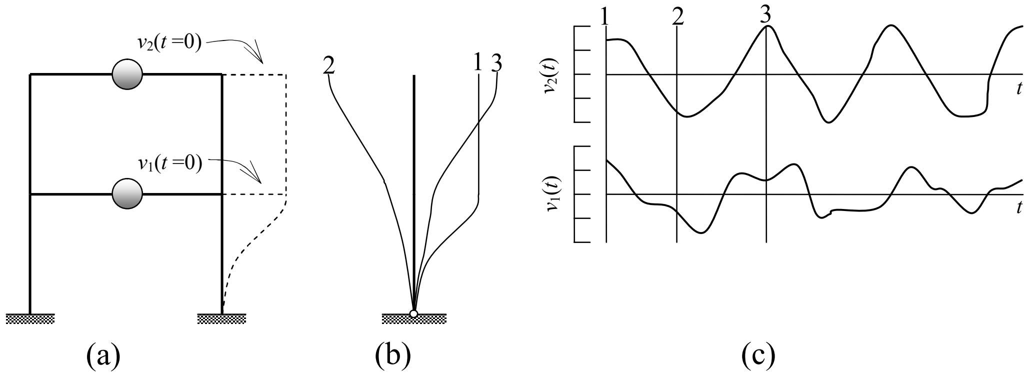


Figure 7.8 : Undamped free vibration of a two-storey building after any initial release: (a) idealized structure with two degrees of freedom, (b) initial and time displacements, and (c) displacement histories.

7.4.1 PHYSICAL SIGNIFICANCE OF FREQUENCIES AND EIGENMODES

Considering the two-storey structure shown in figure (7.8.a) there is no form of damping. If we impose arbitrary displacements on the masses concentrated at floor level, the structure will oscillate around its equilibrium position. A mathematical solution of this problem can be obtained by solving equation (7.20) numerically. The response to initial displacements corresponding to displacement configuration 1, shown in figure (7.8.b), is illustrated in figure (7.8.c) in the form of a graph of displacements at levels 1 and 2 as a function of time. We observe that the movement of each mass of the system is no longer a simple harmonic movement as in the case of the undamped free vibration of an elementary oscillator. Note, moreover, that the shape of the deformation of the structure changes with time. On the other hand, if the free vibration of the structure is initiated by appropriate displacements and velocities imposed on the masses, the structure will oscillate according to a simple harmonic movement without change in the pace imposed on the deformation. Two characteristic curves of the deformation exist for a system with two degrees of freedom. The first corresponds to displacement configuration 1 shown in figure (7.9.b) imposed on $t = 0$. It can be seen that the responses of the two degrees of freedom shown in figure (7.9.c) are harmonic and in phase and have the same vibration period T_1. The degree of freedom 2 undergoes the maximum displacement and the ratio of displacements $v_2(t)/v_1(t)$ remains constant. There is only one point, at the base, whose displacement is null during all the duration of the response. This point is called a *node*.

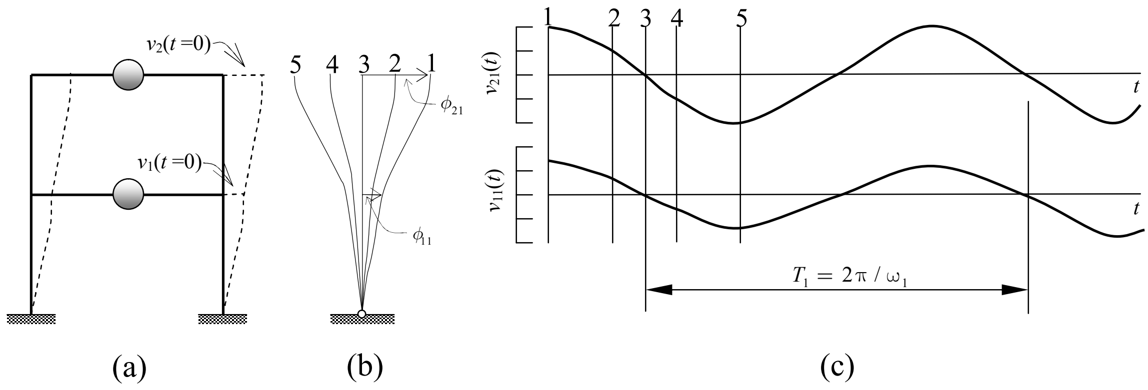


Figure 7.9 : Undamped free vibration of an idealized two-storey building in the first vibration mode: (a) idealized structure with two degrees of freedom, (b) displacements 1 imposed at t = 0 and displacements 2 to 5 at t > 0, and (c) displacement histories.

The second characteristic curve corresponds to the configuration of displacements 1 shown in figure (7.10.b) imposed at $t = 0$. It can be seen that the responses of the two degrees of freedom shown in figure (7.10.c) are once again harmonic, but in phase opposition and have the same vibration period T_2. The degree of freedom 1 undergoes this time the maximum displacement and the ratio of displacements $v_2(t)/v_1(t)$ remains once again constant. There exist in this case, two nodes whose displacements remain null all the time.

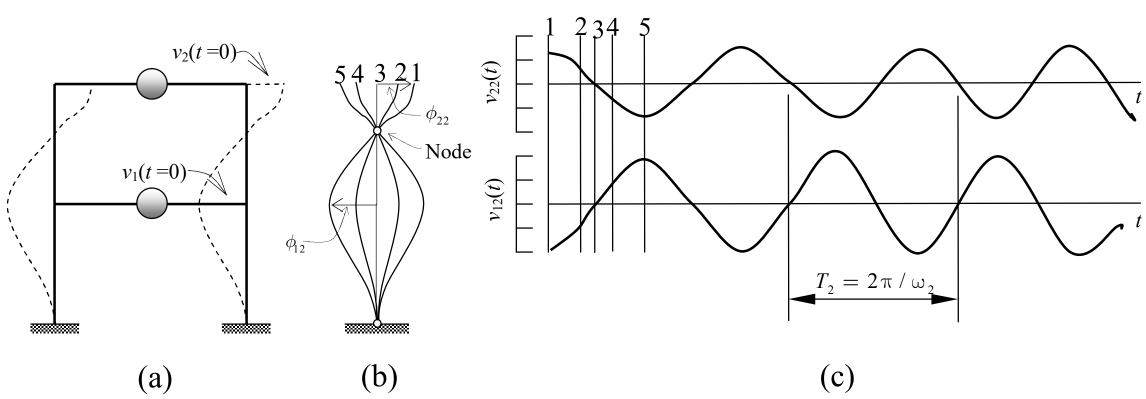


Figure 7.10 : Undamped free vibration of an idealized two-storey building in the second vibration mode: (a) idealized structure with two degrees of freedom, (b) displacements 1 imposed at t = 0 and displacements 2 to 5 at t > 0, and (c) displacement histories.

A comparison of responses 7.9.c and 7.10.c shows that the period of vibration T_2 is less than T_1. The characteristic deformed shapes are called *eigenmodes* and the vectors containing the displacements of each eigenmode are generally called $\{\phi\}$. All the displacements of the coordinates or degree of freedom corresponding to a mode of vibration reach their maximum at the same instant and also pass through the initial position of equilibrium at the same instant. We can associate with each eigenmode a natural period of vibration T_j and a natural frequency of vibration f_j given by the following relations:

$$T_j = \frac{2\pi}{\omega_j} \ (\text{s}), \qquad f_j = \frac{1}{T_j} = \frac{\omega_j}{2\pi} \ (\text{Hz}) \tag{7.21}$$

The smallest natural pulsation is denoted by ω_1 and consequently the longest natural period is denoted by T_1. In general, there exist N eigenmodes and eigenpulsations for a system with N degrees of freedom with

$$\omega_1 \le \omega_2 \le \cdots \le \omega_j \le \cdots \le \omega_N \tag{7.22}$$

$$T_1 \ge T_2 \ge \cdots \ge T_j \ge \cdots \ge T_N \tag{7.23}$$

The vector $\{\phi_j\}$ associated with the natural pulsation ω_j only defines the shape of the deformation of the oscillating structure in its j^e mode of vibration, i.e. the vector only defines the ratio of the displacements of the floors between them. By convenience, the eigenmodes are normalized: the displacement of the last floor is arbitrarily chosen equal to the unit for example. The quantities ω_1 , T_1 and $\{\phi_1\}$ correspond to the fundamental pulsation, period and mode of vibration of the structure, respectively.

7.4.2 DETERMINATION OF NATURAL VIBRATION FREQUENCIES

From the above, the motion of a discrete system is, under certain conditions, harmonic. Thus, the vector of displacements is expressed in the form:

$$\{v(t)\} = \{\phi\}\sin(\omega t - \theta) \tag{7.24}$$

where $\{\phi\}$ represents the possible deformation modes, which do not change with time and θ is the phase angle. The acceleration vector in free vibration is then given by the second derivative of the previous expression:

$$\{\ddot{v}(t)\} = -\omega^2\{\phi\}\sin(\omega t - \theta) = -\omega^2\{v(t)\} \tag{7.25}$$

By transferring these two expressions (7.24-25) into equation (7.20), we obtain the expression:

$$-\omega^2[M]\{\phi\}\sin(\omega t - \theta) + [K]\{\phi\}\sin(\omega t - \theta) = \{0\} \tag{7.26}$$

This expression must be verified whatever t, therefore for all the values of the term in sine, therefore:

$$[K]\{\phi\} = \omega^2 [M]\{\phi\} \quad \textbf{(7.27)}$$

Equation (7.27) is a generalized eigenvalue problem that can be put in the form of a standard eigenvalue problem. Recall that the standard eigenvalue problem is of the form:

$$[A]\{x\} = \lambda \{x\} \quad \textbf{(7.28)}$$

However, if we premultiply the two members of the equation (7.27) by $[M]^{-1}$, we obtain the following expression:

$$[M]^{-1}[K]\{\phi\} = \omega^2 \{\phi\} \quad \textbf{(7.29)}$$

Which we can rewrite:

$$[E]\{\phi\} = \omega^2 \{\phi\} \quad \textbf{(7.30)}$$

where $[E]$ is an asymmetric matrix called *dynamic stiffness matrix*:

$$[E] = [M]^{-1}[K] \quad \textbf{(7.31)}$$

Equation (7.30) is of the form of equation (7.28). The dynamic stiffness matrix $[E]$ contains all the information about the characteristics of the system, represented by the mass $[M]$ and stiffness $[K]$ matrices. Equation (7.27) can also be written:

$$\left[[K] - \omega^2 [M]\right]\{\phi\} = \left[[K] - \lambda [M]\right]\{\phi\} = \{0\} \quad \textbf{(7.32)}$$

where $\lambda = \omega^2$ is a positive real scalar. The trivial solution $\{\phi_i\} = \{0\}$ of equation (7.32) corresponds to no motion and does not interest us. A non-trivial solution is only possible if the determinant of the coefficient matrix $\left[[K] - \omega^2 [M]\right]$ vanishes. In other words, finite amplitude vibrations are possible only if:

$$\det\left([K] - \omega^2 [M]\right) = \det\left([K] - \lambda [M]\right) = 0 \quad \textbf{(7.33)}$$

If the mass matrix is diagonal, we obtain:

$$\begin{vmatrix} (k_{11} - \lambda M_1) & K_{12} & \cdots & \cdots & K_{1N} \\ K_{21} & (k_{22} - \lambda M_2) & \cdots & \cdots & K_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{N1} & K_{N2} & \cdots & \cdots & (k_{NN} - \lambda M_N) \end{vmatrix} = 0 \quad \textbf{(7.34)}$$

Equation (7.34) is called the *frequency equation* or *the characteristic equation of the system*. By expanding this determinant, we obtain a polynomial equation of degree *N* in $\lambda_j = \omega_j^2$ for a system with *N* degrees of freedom. The *N* roots of this equation $\left(\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots, \lambda_N\right)$ are the eigenpulsations, more frequently called natural frequencies (or eigenfrequencies) of the system which are each associated with a modal vector or eigenmode of vibration of the system. The lowest frequency of the system is called the fundamental frequency of the system; the corresponding (highest) period is called the fundamental period of the system.

7.4.3 DETERMINATION OF THE EIGENMODES OF VIBRATION

In order to complete the solution of the eigenvalue problem, one can calculate *N* forms of vibration corresponding to the *N* eigenmodes of the system. If we substitute the values of ω_j in equation (7.27), we get:

$$\left[[K] - \omega_j^{\,2}[M]\right]\left\{\phi_j\right\} = \{0\} \qquad \textbf{(7.35)}$$

where $\left\{\phi_j\right\}$ is the deformation mode corresponding to the vibration frequency ω_j. As $\det\left[[K]-\omega_j^{\,2}[M]\right]=0$, it follows that the components $\left\{\phi_j\right\}$ are indeterminate. However, the general shape associated with a given frequency can be determined by solving the displacements according to a reference displacement. For this purpose, we can assume that the first element ϕ_{1j} (or the last ϕ_{Nj}) of the vector $\left\{\phi_j\right\}$ has a magnitude of 1. Thus, we can calculate the other elements ϕ_{ij}, where $i = 2, 3, \ldots, N$ and $j = 1, 2, \ldots, N$.

NOTE– The elements of the *j*-mode vector $\left\{\phi_j\right\}$ are named ϕ_{ij} and correspond to the *i* displacements of the *N* degrees of freedom in the *j*-mode of vibration. The use of the two indices allows us to identify the mode of vibration and the displacement of the degree of freedom.

All the information on the eigenmodes of vibration is gathered in the matrix of eigenvectors or eigenmodes of which each column *j* is a mode of vibration $\left\{\phi_j\right\}$. So:

$$[\Phi] = \begin{bmatrix}\phi_1 & \phi_2 & \cdots & \phi_N\end{bmatrix} = \begin{bmatrix}\phi_{11} & \phi_{12} & \cdots & \phi_{1N}\\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN}\end{bmatrix} \qquad \textbf{(7.36)}$$

where $[\Phi]$ is the *modal matrix* of dimension *N* x *N*. Taking account of (7.36), the solution for *N* eigenvalues and the associated eigenvectors can therefore be written:

$$[K][\Phi] = [M][\Phi][\Lambda] \qquad \textbf{(7.37)}$$

in which $[\Lambda]$ is an *N* x *N* dimensional diagonal matrix containing the eigenvalues $\lambda_j = \omega_j^2$. The matrix $[\Lambda]$ is called a *spectral matrix*. Thus,

$$[\Lambda] = \begin{bmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \omega_N^2 \end{bmatrix} \qquad \textbf{(7.38)}$$

7.4.4 ORTHOGONALITY OF THE EIGENMODES OF VIBRATION

The eigenmodes of a structure have the fundamental property of being orthogonal. This property makes it possible to reformulate the coupled equations of motion of a discrete system with *N* unknowns into *N* decoupled equations (see paragraph 7.5.1). Consider two distinct eigenfrequencies ω_i and ω_j and their associated eigenmodes $\{\phi_i\}$ and $\{\phi_j\}$ to demonstrate this property. Let us return to equation (7.27) for frequencies *i* and *j*. We have:

$$[K]\{\phi_i\} = \omega_i^2 [M]\{\phi_i\} \qquad \textbf{(7.39)}$$

$$[K]\{\phi_j\} = \omega_j^2 [M]\{\phi_j\} \qquad \textbf{(7.40)}$$

Premultiply equation (7.39) by $\{\phi_j\}^{\mathrm{T}}$ and equation (7.40) by $\{\phi_i\}^{\mathrm{T}}$, we get:

$$\{\phi_j\}^{\mathrm{T}} [K]\{\phi_i\} = \omega_i^2 \{\phi_j\}^{\mathrm{T}} [M]\{\phi_i\} \qquad \textbf{(7.41)}$$

$$\{\phi_i\}^{\mathrm{T}} [K]\{\phi_j\} = \omega_j^2 \{\phi_i\}^{\mathrm{T}} [M]\{\phi_j\} \qquad \textbf{(7.42)}$$

in which both sides of the equations are scalars. If we transpose both sides of equation (7.42), we get:

$$\{\phi_j\}^{\mathrm{T}} [K]^{\mathrm{T}} \{\phi_i\} = \omega_j^2 \{\phi_j\}^{\mathrm{T}} [M]^{\mathrm{T}} \{\phi_i\} \qquad \textbf{(7.43)}$$

Since $[K]$ and $[M]$ are symmetric, we have $[K]=[K]^{\mathrm{T}}$ and $[M]=[M]^{\mathrm{T}}$. It therefore follows, according to equation (7.43):

$$\{\phi_j\}^{\mathrm{T}} [K]\{\phi_i\} = \omega_j^2 \{\phi_j\}^{\mathrm{T}} [M]\{\phi_i\} \qquad \textbf{(7.44)}$$

From equations (7.41) and (7.44), we derive the following equalities:

$$\omega_i^2 \{\phi_j\}^{\mathrm{T}} [M]\{\phi_i\} = \omega_j^2 \{\phi_j\}^{\mathrm{T}} [M]\{\phi_i\} \qquad \textbf{(7.45)}$$

from where

$$(\omega_j^2 - \omega_i^2) \{\phi_j\}^{\mathrm{T}} [M]\{\phi_i\} = 0 \qquad \textbf{(7.46)}$$

The first member of this last equation is a scalar. With the conditions that $\omega_j \neq \omega_i$, we obtain the first property of orthogonality which is written:

$$\left\{\phi_j\right\}^{\mathrm{T}} \left[M\right]\left\{\phi_i\right\} = 0, \qquad i \neq j \tag{7.47}$$

After substitutions of equation (7.47) in equation (7.44), we obtain the second property of orthogonality:

$$\left\{\phi_j\right\}^{\mathrm{T}} \left[K\right]\left\{\phi_i\right\} = 0, \qquad i \neq j \tag{7.48}$$

NOTE– The properties of orthogonality of the eigen modes imply that the work of the inertia forces and the elastic forces of a mode *i* on displacements according to another mode *j* is null.

The modal vectors are said to be orthogonal with respect to the matrices $\left[M\right]$ and $\left[K\right]$. The important conclusions of the orthogonality properties are the following: if $\left[M\right]$ is an arbitrary mass matrix, symmetric and positive definite (this is the case studied), and $\left[K\right]$ a stiffness matrix symmetric and positive definite (this is the case studied) , given the modal matrix $\left[\Phi\right]$, we have:

$$\left[\tilde{M}\right] = \left[\Phi\right]^{\mathrm{T}} \left[M\right]\left[\Phi\right] \tag{7.49}$$

where $\left[\tilde{M}\right]$ is a diagonal matrix whose terms on the diagonal are:

$$\tilde{M}_i = \left\{\phi_i\right\}^{\mathrm{T}} \left[M\right]\left\{\phi_i\right\} \tag{7.50}$$

where $\tilde{M}_i$ is a scalar quantity called *modal mass* or *generalized mass*. We also have:

$$\left[\tilde{K}\right] = \left[\Phi\right]^{\mathrm{T}} \left[K\right]\left[\Phi\right] \tag{7.51}$$

where $\left[\tilde{K}\right]$ is a diagonal matrix whose terms on the diagonal are:

$$\tilde{K}_i = \left\{\phi_i\right\}^{\mathrm{T}} \left[K\right]\left\{\phi_i\right\} \tag{7.52}$$

where $\tilde{K}_i$ is a scalar quantity called *modal stiffness* or *generalized stiffness*.

7.4.4.1 Normalization of eigenvectors

As the eigenmodes (eigenvectors) do not have absolute values and are defined only with a factor near, one must always normalize them. The most widespread method in numerical dynamics is that which makes the generalized masses unitary $\tilde{M}_i = 1, i = 1, 2, \ldots, N$. It is said that the eigenvectors are *orthonormal* with respect to the mass matrix. The conditions of orthonormality are therefore expressed as:

$$\begin{aligned}\{\phi_i\}^{\mathrm{T}}[M]\{\phi_j\} &= \delta_{ij} \\ \{\phi_i\}^{\mathrm{T}}[K]\{\phi_j\} &= \omega_i^2\delta_{ij}\end{aligned} \qquad i,j = 1,2,\ldots,N \tag{7.53}$$

Where δ_{ij} is the Kronecker symbol. The conditions of orthonormality can be clarified according to the following matrix relations:

$$\begin{aligned}[\Phi]^{\mathrm{T}}[M][\Phi] &= [\mathrm{I}] \\ [\Phi]^{\mathrm{T}}[K][\Phi] &= [\Lambda]\end{aligned} \tag{7.54}$$

where $[\mathrm{I}]$ is the identity matrix of order *N* and $[\Lambda]$ is the spectral matrix, storing the *N* frequencies. It is useful to explicitly show the operations necessary for such normalization. The scalar is calculated as:

$$\{\hat{\phi}_i\}^{\mathrm{T}}[M]\{\hat{\phi}_i\} = \tilde{M}_i \tag{7.55}$$

where $\{\hat{\phi}_i\}$ is any eigenvector. We can express the above equality as:

$$\{\hat{\phi}_i\}^{\mathrm{T}}\tilde{M}_i^{-\frac{1}{2}}[M]\tilde{M}_i^{-\frac{1}{2}}\{\hat{\phi}_i\} = 1 \tag{7.56}$$

from which we see that the $[M]$-orthonormé eigenmode is simply:

$$\{\phi_i\} = \tilde{M}_i^{-\frac{1}{2}}\{\hat{\phi}_i\} \tag{7.57}$$

7.5 DAMPED FREE VIBRATIONS

We assume here that the discrete system with *N* degrees of freedom is not subjected to any external force, i.e. $\{F(t)\} = \{0\}$, and that the damping is non-zero. The solution of the following system of differential equations

$$[M]\{\ddot{v}(t)\} + [C]\{\dot{v}(t)\}[K]\{v(t)\} = \{0\} \tag{7.58}$$

is the damped regime that describes the response of the system over time when *arbitrary initial conditions* are imposed on it. In equation (7.58), $[M]$(*N*x*N*) and $[K]$ (*N*x*N*) are symmetric mass and stiffness matrices and $[C]$(*N*x*N*) is the symmetric damping matrix.

7.5.1 Proportional Damping Matrix

The eigenmodes of vibration of the damped system, determined by solving the eigenvalue problem (see equation 7.37), are not in general orthogonal with respect to the damping matrix $[C]$. We cannot therefore decouple the equations of motion of a damped system. However, under certain conditions imposed on the expression of the coefficients of damping, the properties of orthogonalities of the eigenmodes are applicable to the matrix of damping according to the following relation:

$$\{\phi_j\}^{\mathrm{T}}[C]\{\phi_i\} = 0, \qquad i \neq j \tag{7.59}$$

Lord Rayleigh showed that a damping matrix proportional to the mass matrix and/or the stiffness matrix satisfies the orthogonality conditions. Either:

$$[C] = a_0[M] + a_1[K] \tag{7.60}$$

where a_0 and a_1 are arbitrary coefficients. We have in fact:

$$\{\phi_i\}^{\mathrm{T}}[C][\Phi] = a_0\{\phi_i\}^{\mathrm{T}}[M][\Phi] + a_1\{\phi_i\}^{\mathrm{T}}[K][\Phi] \tag{7.61}$$

from where

$$\tilde{C}_i = a_0\tilde{M}_i + a_1\tilde{K}_i = a_0\tilde{M}_i + a_1\omega_i^2\tilde{M}_i = (a_0 + a_1\omega_i^2)\tilde{M}_i \tag{7.62}$$

The preceding method supposes that two values of modal damping are known for two distinct modes. A modal damping value may be known for more than two vibration modes. Caughey showed that an infinite number of matrices formed from mass and stiffness matrices also satisfy the orthogonality conditions. Let us suppose that a number p of factor of modal damping $\xi_i = 1, i = 1, 2, \ldots, p$ is given for the determination of the matrix $[C]$. The orthogonal damping matrix is expressed in the form of the following series called Caughey:

$$[C] = [M]\sum_{k=0}^{p-1} a_k\left([M]^{-1}[K]\right)^k \tag{7.63}$$

where the coefficients $a_k, k = 0, 1, \ldots, p-1$ are determined from the following p simultaneous equations:

$$\xi_i = \frac{1}{2\omega_i}\sum_{k=0}^{p-1} a_k\omega_i^{2k} \tag{7.64}$$

With this type of damping, it is possible to calculate the damping coefficients so as to obtain a decoupled system having as many damping ratios in as many modes as desired. When $p = 2$, equation (7.63) reduces to the Rayleigh damping presented in equation (7.60), and equation (7.64) gives us:

$$\xi_i = \frac{a_0}{2\omega_i} + \frac{a_1\omega_i}{2} \tag{7.65}$$

if $a_1 = 0$ in equation (7.65), we have damping of the system proportional to the mass. The damping factor for mode i becomes:

$$\xi_i = \frac{a_0}{2\omega_i} \tag{7.66}$$

where we see that the damping factor ξ is inversely proportional to the vibration frequency. Thus, the higher modes of the structure will have very little damping. If $a_0 = 0$ in equation (7.65), we have system damping which is purely proportional to the stiffness. Equation (7.65) gives us:

$$\xi_i = \frac{a_1 \omega_i}{2} \tag{7.67}$$

where we see that the damping factor becomes directly proportional to the frequency. In this case, the higher modes of the structure will be very deadened. Knowing the damping factor corresponding to a certain frequency, the constants a_0 and a_1 can be determined from equations (7.66) and (7.67) for systems whose damping is purely proportional to mass or stiffness, respectively. One can thus deduce the damping factor of the other modes knowing a_0 and a_1. If the two types of damping are present, it is necessary to know the modal damping factors corresponding to two frequencies to determine a_0 and a_1. For damping factors ξ_i and ξ_j corresponding to the modal frequencies ω_i and ω_j respectively, one can write the equation (7.65) in matrix form by expressing the two conditions in the form:

$$\begin{bmatrix} \xi_i \\ \xi_j \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \dfrac{1}{\omega_i} & \omega_i \\ \dfrac{1}{\omega_j} & \omega_j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \tag{7.68}$$

from where

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = 2 \frac{\omega_i \omega_j}{\omega_j^2 - \omega_i^2} \begin{bmatrix} \omega_j & -\omega_i \\ -\dfrac{1}{\omega_j} & \dfrac{1}{\omega_i} \end{bmatrix} \begin{bmatrix} \xi_i \\ \xi_j \end{bmatrix} \tag{7.69}$$

The determination of the factors a_0 and a_1 gives the relation between damping factor and frequency of the form shown in figure (7.11). As can be seen in figure (7.11), this way of constructing a diagonal damping matrix in the eigenmode database gives rise to very high damping for very low and very high frequency modes and lower for intermediate frequencies. Considering the lack of information on the variation of the factor of damping with the frequencies, one supposes, in general, the same rate of damping for the two frequencies of control ω_i and ω_j. It is recommended to choose the fundamental frequency as the first control frequency $\omega_i = \omega_1$ and to choose, as the second control frequency ω_j, one of the highest frequencies among the modes which contribute the most to the dynamic response. This ensures that all intermediate modes will have damping somewhat less than $\xi_1 = \xi_j = \xi$ and that modes with frequency $\omega_k > \omega_j$ will have damping that increases monotonically with frequency above ξ_j. Therefore, the mode responses with very high frequencies will be eliminated by their very high damping.

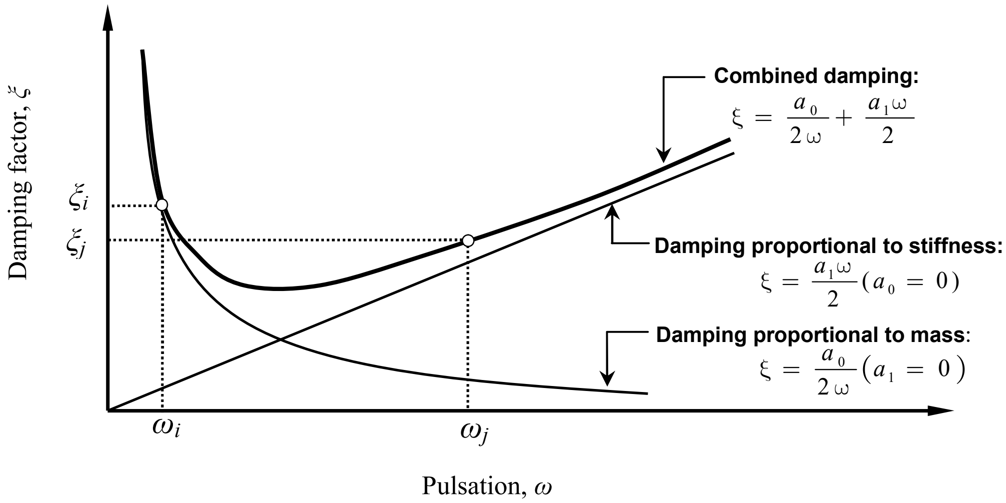


Figure 7.11: Relationship between the damping factor ξ and the natural frequency for Rayleigh type damping.

EXERCISES

7.1.

We ask to calculate the stiffness matrix and the mass matrix for the mass-spring system with two degrees of freedom shown in figure (7.12).

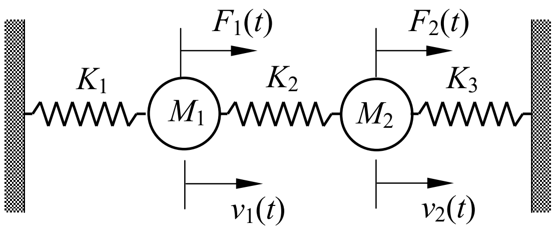


Figure 7.12

7.2.

Consider the structure shown in figure (7.13). The floors (1) and (2) each have a distributed mass of **2000 kg/m²** of surface. These two floors are assumed to be infinitely rigid to the lateral force and are supported by columns of negligible mass.
One considers only the horizontal degrees of freedom in the direction X-X.
Young's modulus is equal to **E = 30000 N/mm²**.

- Set up the stiffness matrix and the mass matrix of the building with two degrees of freedom shown in figure (7.13).

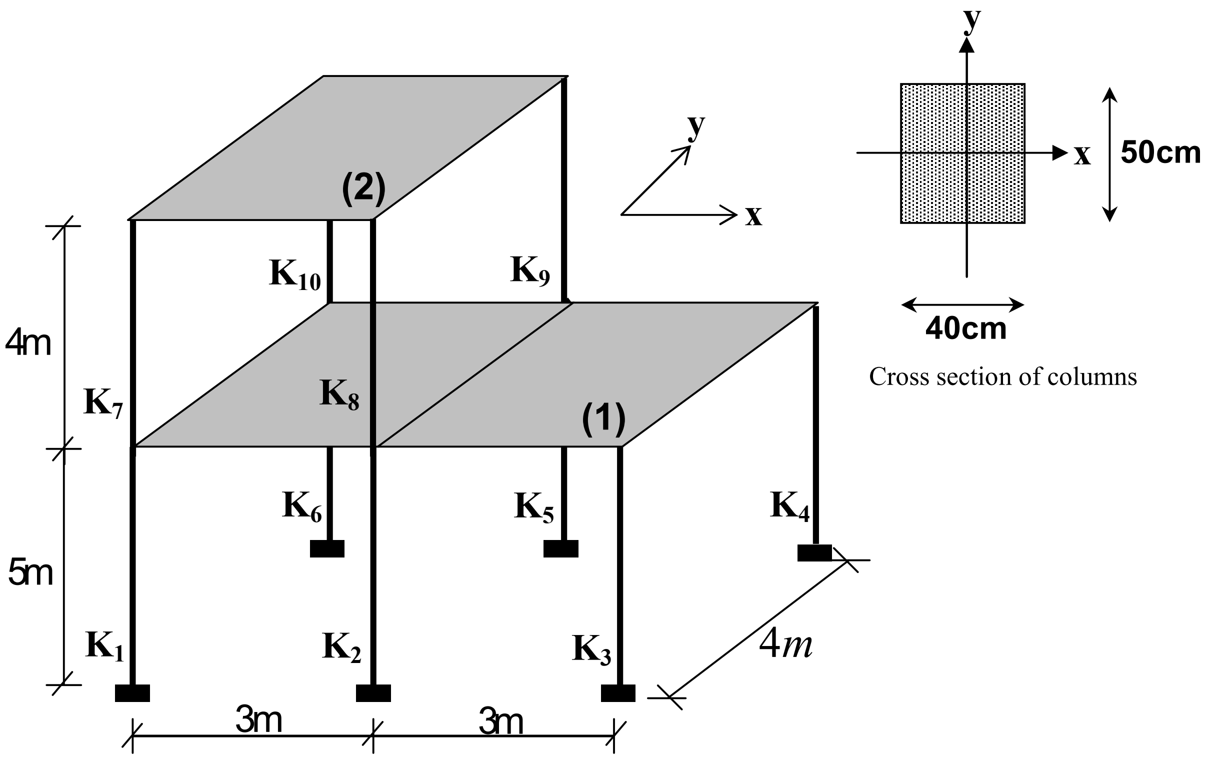


Figure 7.13

7.3.

We ask to calculate the natural frequencies of vibration of the system with two degrees of freedom shown in the figure (7.12).

7.4.

Consider the structure shown in figure (7.14). The floors (1) and (2) each have a mass of **20000 kg** and a total lateral stiffness per floor of $\mathbf{18\times10^6}$ **N/m**.

- Calculate the natural frequencies of vibration of the structure.
- Calculate the eigenmodes of vibration of the structure.
- Calculate the masses and the generalized rigidities which correspond to the eigenmodes of vibration of the structure.
- Standardize, with respect to the mass matrix, the eigenmodes of vibration of the structure.

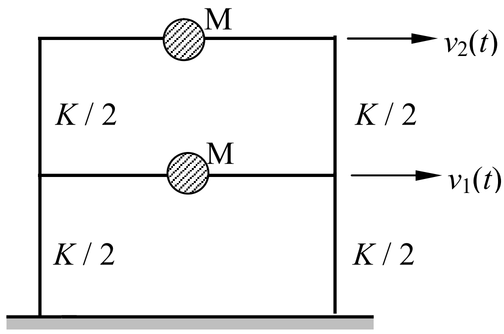


Figure 7.14

7.5.

Calculate the mass and stiffness matrices of the building with three degrees of freedom shown in figure (7.15) for a mass of $\mathbf{80x10^3}$**kg** and a total lateral stiffness per storey of $\mathbf{140x10^6}$ **N/m**. Calculate the frequencies and natural modes of vibration of the building. Assuming Rayleigh damping, determine the damping matrix that provides **5%** and **10%** damping factors at the first and 4th eigenmodes, respectively. What will be the damping factors at the 2^{nd} and 3^{rd} eigenmodes?

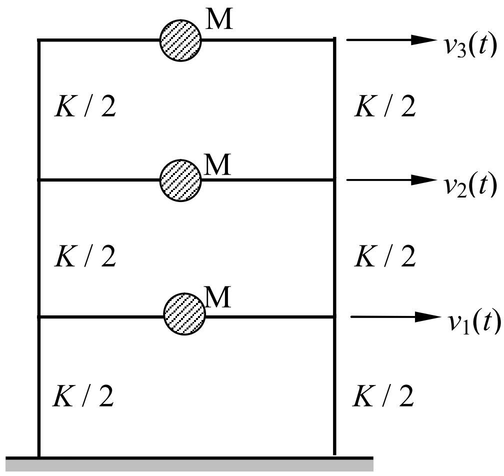


Figure 7.15

CHAPTER

8

DYNAMIC RESPONSE OF STRUCTURES WITH MULTI DEGREE OF FREEDOM TO AN ARBITRARY LOAD BY MODAL SUPERPOSITION

8.1 INTRODUCTION

The response of a discrete system with N degrees of freedom is governed by the following equations of motion:

$$[M]\{\ddot{v}(t)\} + [C]\{\dot{v}(t)\} + [K]\{v(t)\} = \{F(t)\} \quad \textbf{(8.1)}$$

In general, it is a system of N equations coupled by $[M]$, $[C]$ or $[K]$. In other words, these matrices are not necessarily diagonal matrices. If M_{ij} exists for $i \neq j$, the system is said to have a dynamic coupling. If K_{ij} exists for $i \neq j$, the system is said to have a static coupling. These equations can only be solved approximately by numerical integration of the N coupled equations. For linear systems, the properties of orthogonality of the vibration modes allow an important simplification of the equations of motion. This simplification consists in transforming the equations of motion expressed in the geometric coordinate system into equations of motion expressed in a set of so-called normal coordinates (also called generalized coordinates). This transformation decouples the N simultaneous equations into N independent equations analogous to N elementary systems. The exact dynamic response can thus be obtained by calculating the response of each normal coordinate and superimposing them in order to obtain the response in the original geometric coordinate system. This calculation procedure is called the *modal superposition method*.

8.2 NORMAL COORDINATES

The normal modes of vibration define a set of coordinates which are not dynamically linked, i.e. for a system with N degrees of freedom; these modes constitute N independent forms of displacement. The amplitudes of these modal shapes can therefore be used as generalized coordinates serving to express any shape of the deformation. These coordinates are called *modal* or *normal coordinates*.

Any set of N orthogonal vectors can be used as a basis to represent any vector of order N. Any displacement vector $\{v(t)\}$ can therefore be found by superposition of the eigenmodes each multiplied by a certain scalar or modal amplitude. The procedure is illustrated in figure (8.1) for a system with six degrees of freedom.

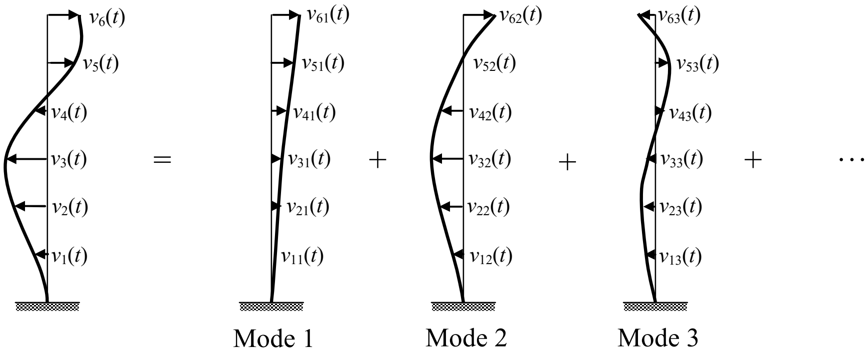


Figure 8.1: Representation of a deformation by a sum of normal components.

The displacements of the modal component $\{v_i(t)\}$ are given by the modal vector $\{\phi_i\}$ multiplied by the modal amplitude $Y_i(t)$; thus:

$$\{v_i(t)\} = \{\phi_i\} Y_i(t) \tag{8.2}$$

The total displacement is given by the sum of the modal components:

$$\{v(t)\} = \{\phi_1\} Y_1(t) + \{\phi_2\} Y_2(t) + \cdots + \{\phi_N\} Y_N(t) \tag{8.3}$$

which we write:

$$\{v(t)\} = \sum_{i=1}^{N} \{\phi_i\} Y_i(t) = [\Phi]\{Y(t)\} \tag{8.4}$$

where $\{Y(t)\}$ is the vector of normal coordinates or generalized coordinates and $[\Phi]$ is the matrix of the eigenmodes. The matrix therefore serves to transform the generalized coordinates $\{Y(t)\}$ into geometric coordinates $\{v(t)\}$.

Conversely, any modal coordinate can be calculated easily by taking into account the orthogonality properties of the eigenmodes. Indeed, if we premultiply equations (8.4) by $\{\phi_i\}^{\mathrm{T}}[M]$, we get:

$$\{\phi_i\}^{\mathrm{T}}[M]\{v(t)\} = \{\phi_i\}^{\mathrm{T}}[M][\Phi]\{Y(t)\} \tag{8.5}$$

but

$$\begin{aligned}\{\phi_i\}^{\mathrm{T}}[M]\{v(t)\} = {} & \{\phi_i\}^{\mathrm{T}}[M]\{\phi_1\}Y_1(t) + \{\phi_i\}^{\mathrm{T}}[M]\{\phi_2\}Y_2(t) + \cdots \\ & + \{\phi_i\}^{\mathrm{T}}[M]\{\phi_i\}Y_i(t) + \cdots + \{\phi_i\}^{\mathrm{T}}[M]\{\phi_N\}Y_N(t)\end{aligned} \tag{8.6}$$

However, all the terms of the preceding series are null except the term corresponding to $\{\phi_i\}$ because of the properties of orthogonality of the eigenmodes compared to the mass matrix. So we have:

$$\{\phi_i\}^{\mathrm{T}}[M]\{v(t)\} = \{\phi_i\}^{\mathrm{T}}[M]\{\phi_i\}Y_i(t) \quad \textbf{(8.7)}$$

from where

$$Y_i(t) = \frac{\{\phi_i\}^{\mathrm{T}}[M]\{v(t)\}}{\{\phi_i\}^{\mathrm{T}}[M]\{\phi_i\}} \quad \textbf{(8.8)}$$

8.2.1 DECOUPLED EQUATIONS OF UNDAMPED MOTION

Consider the equations of motion of an undamped system with *N* degrees of freedom:

$$[M]\{\ddot{v}(t)\} + [K]\{v(t)\} = \{F(t)\} \quad \textbf{(8.9)}$$

We can calculate $\{\ddot{v}(t)\}$ from equation (8.4) noting that $[\Phi]$ does not vary over time. We obtain:

$$\{\ddot{v}(t)\} = [\Phi]\{\ddot{Y}(t)\} \quad \textbf{(8.10)}$$

Inserting equations (8.4) and (8.10) into equation (8.9), we get:

$$[M][\Phi]\{\ddot{Y}(t)\} + [K][\Phi]\{Y(t)\} = \{F(t)\} \quad \textbf{(8.11)}$$

Pre-multiply all the terms by $[\Phi]^{\mathrm{T}}$, we get:

$$[\tilde{M}]\{\ddot{Y}(t)\} + [\tilde{K}]\{Y(t)\} = \{\tilde{F}(t)\} \quad \textbf{(8.12)}$$

where

$$[\tilde{M}] = [\Phi]^{\mathrm{T}}[M][\Phi] \quad \textbf{(8.13)}$$

$$[\tilde{K}] = [\Phi]^{\mathrm{T}}[K][\Phi] \quad \textbf{(8.14)}$$

$$\{\tilde{F}(t)\} = [\Phi]^{\mathrm{T}}\{F(t)\} \quad \textbf{(8.15)}$$

According to the properties of orthogonality of the proper modes compared to the matrices of mass and rigidity, the equation (8.12) can be written:

$$\tilde{M}_i\ddot{Y}_i(t) + \tilde{K}_iY_i(t) = \tilde{F}_i(t), \qquad i = 1,\ 2, \ldots, N \quad \textbf{(8.16)}$$

in which

$$\tilde{M}_i = \{\phi_i\}^{\mathrm{T}} [M] \{\phi_i\} \quad \textbf{(8.17)}$$

$$\tilde{K}_i = \{\phi_i\}^{\mathrm{T}} [K] \{\phi_i\} \quad \textbf{(8.18)}$$

$$\tilde{F}_i(t) = \{\phi_i\}^{\mathrm{T}} \{F(t)\} \quad \textbf{(8.19)}$$

where $\tilde{M}_i$, $\tilde{K}_i$ and $\tilde{F}_i(t)$ are called the *generalized mass*, *generalized stiffness*, and *generalized load* of normal coordinates for mode *i*, respectively. Equation (8.16) is the equation of motion of an elementary oscillator with one degree of freedom for mode *i*. From equation (7.39), we know that $[K]\{\phi_i\} = \omega_i^2 [M]\{\phi_i\}$. Pre-multiply both sides of this equation by $\{\phi_i\}^{\mathrm{T}}$, we get the generalized stiffness for mode *i* as a function of the generalized mass for that mode. So:

$$\{\phi_i\}^{\mathrm{T}} [K] \{\phi_i\} = \omega_i^2 \{\phi_i\}^{\mathrm{T}} [M] \{\phi_i\} \quad \textbf{(8.20)}$$

from where

$$\tilde{K}_i = \omega_i^2 \tilde{M}_i \quad \textbf{(8.21)}$$

Replacing (8.21) in (8.16) and dividing both sides by $\tilde{M}_i$, we get:

$$\ddot{Y}_i(t) + \omega_i^2 Y_i(t) = \frac{\tilde{F}_i(t)}{\tilde{M}_i} = \overline{F}_i(t), \qquad i = 1,\ 2, \ldots, N \quad \textbf{(8.22)}$$

where $\overline{F}_i(t)$ is the generalized force per generalized unit mass of mode *i*. Equation (8.22) is the equation of motion of a system with a single degree of freedom that we solve for $Y_i(t)$ = 1, 2, …, *N*. Knowing $\{Y(t)\}$, we can calculate the response in the system geometric coordinates from equation (8.4).

8.2.2 DECOUPLED EQUATIONS OF DAMPED MOTION

The eigenmodes are not in general orthogonal compared to the matrix of damping [*C*]. We cannot therefore decouple the equations of motion of a damped system. However, under the conditions imposed on the expression of the damping coefficients in chapter 7, the orthogonality properties of the eigenmodes are applicable to the damping matrix. This way of doing things is justified when it is considered that it is difficult, if not impossible, to calculate the damping matrix and that, moreover, its effects are negligible on the value of the eigenfrequencies. The matrix equation of motion of a damped system is:

$$[M]\{\ddot{v}(t)\} + [C]\{\dot{v}(t)\} + [K]\{v(t)\} = \{F(t)\} \quad \textbf{(8.23)}$$

Let us make the change of coordinates given by equation (8.4):

$$[M][\Phi]\{\ddot{Y}(t)\} + [C][\Phi]\{\dot{Y}(t)\} + [K][\Phi]\{Y(t)\} = \{F(t)\} \qquad \textbf{(8.24)}$$

and premultiply all the terms by $\{\phi_i\}^{\mathrm{T}}$, we get:

$$[\tilde{M}]\{\ddot{Y}(t)\} + [\tilde{C}]\{\dot{Y}(t)\} + [\tilde{K}]\{Y(t)\} = \{\tilde{F}(t)\} \qquad \textbf{(8.25)}$$

where

$$[\tilde{M}] = [\Phi]^{\mathrm{T}}[M][\Phi] \qquad \textbf{(8.26)}$$

$$[\tilde{C}] = [\Phi]^{\mathrm{T}}[C][\Phi] \qquad \textbf{(8.27)}$$

$$[\tilde{K}] = [\Phi]^{\mathrm{T}}[K][\Phi] \qquad \textbf{(8.28)}$$

$$\{\tilde{F}(t)\} = [\Phi]^{\mathrm{T}}\{F(t)\} \qquad \textbf{(8.29)}$$

According to the properties of orthogonality of the proper modes related to the matrices of mass and rigidity, the equation (8.25) can be written:

$$\tilde{M}_i \ddot{Y}_i(t) + \tilde{C}\dot{Y}_i(t) + \tilde{K}_i Y_i(t) = \tilde{F}_i(t), \qquad i = 1,\ 2, \ldots, N \qquad \textbf{(8.30)}$$

in which

$$\tilde{M}_i = \{\phi_i\}^{\mathrm{T}}[M]\{\phi_i\} \qquad \textbf{(8.31)}$$

$$\tilde{K}_i = \{\phi_i\}^{\mathrm{T}}[K]\{\phi_i\} = \omega_i^2 \tilde{M}_i \qquad \textbf{(8.32)}$$

$$\tilde{C}_i = \{\phi_i\}^{\mathrm{T}}[C]\{\phi_i\} = 2\xi_i\omega_i\tilde{M}_i \qquad \textbf{(8.33)}$$

$$\tilde{F}_i(t) = \{\phi_i\}^{\mathrm{T}}\{F(t)\} \qquad \textbf{(8.34)}$$

dividing both sides of equation (8.30) by $\tilde{M}_i$, we get:

$$\ddot{Y}_i(t) + 2\xi_i\omega_i\dot{Y}_i(t) + \omega_i^2 Y_i(t) = \frac{\tilde{F}_i(t)}{\tilde{M}_i} = \overline{F}_i(t), \qquad i = 1,\ 2, \ldots, N \qquad \textbf{(8.35)}$$

where $\bar{F}_i(t)$ is the generalized force per generalized unit mass of mode *i*. Equation (8.35) is the equation of motion of a system with one degree of freedom at $Y_i(t)$ = 1, 2, …, *N*. Knowing $\{Y(t)\}$, one can calculate the response in the geometric coordinate system from the equation (8.4).

8.3 MODAL SUPERPOSITION METHOD

The mode superposition method is based on the fact that, for certain types of damping which are reasonable models for structures and are accessible in the form of modal damping factors, the set of *N* coupled equations of motion of a discrete system with *N* degrees of freedom can be changed into a set of *N* decoupled equations by a transformation to the modal or normal coordinates. Each mode responds with its own mode of vibration $\{\phi_i\}$, its own frequency ω_i and its own modal damping ξ_i. Having made the transformation on the basis of the modal coordinates, the total response of the systems with multi degrees of freedom can be obtained by summation of the response of *N* elementary systems, from where the name of modal superposition given to the method.

8.3.1 RESPONSE CALCULATION

The general expression of the response is given for each mode by the integral of Duhamel, which, for a damped system, is written:

$$Y_i(t) = \frac{1}{\tilde{M}_i \omega_{Di}} \int_0^t \tilde{F}_i(\tau) e^{-\xi_i \omega_i (t-\tau)} \sin \omega_{Di}(t-\tau) d\tau \qquad \textbf{(8.36)}$$

where $\omega_{Di} = \omega_i \sqrt{1-\xi_i^2}$ is the damped frequency of mode *i*.

8.3.2 INITIAL CONDITIONS

The Duhamel integral is applicable to a system which is at rest at the origin of the times (*t* = 0). If the initial velocities and displacements are not null, a response in free vibrations must be added for each mode to the expression given by the Duhamel integral. The most general expression of the damped free vibration response is given for each mode by equation (2.48):

$$Y_i(t) = e^{-\xi_i \omega_i t} \left[\frac{\dot{Y}_i(0) + \xi_i \omega_i Y_i(0)}{\omega_{Di}} \sin \omega_{Di} t + Y_i(0) \cos \omega_{Di} t \right] \qquad \textbf{(8.37)}$$

$Y_i(0)$ and $\dot{Y}_i(0)$ represent the initial displacements and velocities for each mode. They can be obtained for each mode as follows, from the initial imposed displacements $\{v(0)\}$ and the initial velocities $\{\dot{v}(0)\}$ expressed in the geometric coordinates from equation (8.8):

$$Y_i(0) = \frac{\{\phi_i\}^{\mathrm{T}} [M] \{v(0)\}}{\{\phi_i\}^{\mathrm{T}} [M] \{\phi_i\}} \tag{8.38}$$

$$\dot{Y}_i(0) = \frac{\{\phi_i\}^{\mathrm{T}} [M] \{\dot{v}(0)\}}{\{\phi_i\}^{\mathrm{T}} [M] \{\phi_i\}} \tag{8.39}$$

8.3.3 TOTAL RESPONSE

One calculates the answer in the system of geometrical coordinates knowing the responses of each mode $Y_i(t)$ according to the following relation:

$$\{v(t)\} = [\Phi]\{Y(t)\} \tag{8.40}$$

The previous equation can be written:

$$\{v(t)\} = \{\phi_1\} Y_1(t) + \{\phi_2\} Y_2(t) + \cdots + \{\phi_N\} Y_N(t) = \sum_{i=1}^{N} \{\phi_i\} Y_i(t) \tag{8.41}$$

where we see that the total response in the geometric coordinate system is the summation of the contributions of each mode, hence the name of modal summation or modal superposition given to the method.

8.3.4 CALCULATION OF ELASTIC FORCES

Although the determination of the displacements completes the calculation of the response, the internal forces and stresses must still be determined. The elastic forces during the response are determined from the following equation:

$$F_E(t) = [K]\{v(t)\} = [K][\Phi]\{Y(t)\} \tag{8.42}$$

which can also be written:

$$F_E(t) = [K]\{\phi_1\} Y_1(t) + [K]\{\phi_2\} Y_2(t) + \cdots + [K]\{\phi_N\} Y_N(t) \tag{8.43}$$

If we substitute $[K]\{\phi_i\} = \omega_i^2 [M]\{\phi_i\}$ in equation (8.43), we obtain:

$$F_E(t) = \omega_1^2 [M]\{\phi_1\} Y_1(t) + \omega_2^2 [M]\{\phi_2\} Y_2(t) + \cdots + \omega_N^2 [M]\{\phi_N\} Y_N(t) \tag{8.44}$$

which can be rewritten:

$$F_E(t) = [M][\Phi]\{\omega_i^2 Y_i(t)\} \quad \textbf{(8.45)}$$

where $\{\omega_i^2 Y_i(t)\}$ represents a vector whose components are modal amplitudes each multiplied by its modal frequency squared. This multiplication of each modal contribution by the square of the modal frequency implies that the higher modes have a greater importance in the determination of the forces than in the determination of the displacements. Consequently, it will be necessary to use more modes to define the forces than to define the displacements with the same precision.

8.4 SUMMARY OF THE MODAL SUPERPOSITION METHOD

The calculation of the response of an elastic structure to an arbitrary loading by the method of modal superposition is summarized in the following stages:

1) Write the coupled equations of motion according to the relative responses:

$$[M]\{\ddot{v}(t)\} + [C]\{\dot{v}(t)\} + [K]\{v(t)\} = \{F(t)\} \quad \textbf{(8.46)}$$

2) Calculate the frequencies and eigenmodes of vibration for the problem of undamped free vibration which is reduced to the following eigenvalue problem:

$$[K][\Phi] = [M][\Phi][\Lambda] \quad \textbf{(8.47)}$$

where $[\Phi]$ is the N x N dimensional *modal matrix* and $[\Lambda]$ is a diagonal matrix of dimension N x N containing the terms ω_i^2 .

3) Normalize the eigenvectors compared to the mass matrix [*M*], i.e.:

$$\tilde{M}_i = \{\phi_i\}^T [M]\{\phi_i\} = 1 \quad \textbf{(8.48)}$$

4) Determine the damping factor ξ_i.

5) Calculate the generalized loadings relating to each mode:

$$\tilde{F}_i(t) = \{\phi_i\}^T \{F(t)\} \quad \textbf{(8.49)}$$

6) Write *N* decoupled equations of motion:

$$\ddot{Y}_i(t) + 2\xi_i\omega_i\dot{Y}_i(t) + \omega_i^2 Y_i(t) = \frac{\tilde{F}_i(t)}{\tilde{M}_i} = \overline{F}_i(t) \quad \textbf{(8.50)}$$

7) Calculate the modal response to the imposed loading. Each equation obtained in step 6 corresponds to a mode of vibration and represents a system with a single degree of freedom which is solved by any method suitable for the type of loading (see chapters 3, 4 and 5 of this present course). In general, the response for each mode can be given by the Duhamel integral (equation 8.36).

8) Calculate the response in the original geometric coordinate system knowing the modal response $Y_i(t)$:

$$\left\{v(t)\right\} = \left[\Phi\right]\left\{Y(t)\right\} = \sum_{i=1}^{N} \{\phi_i\}Y_i(t) \qquad \textbf{(8.51)}$$

9) Determine the elastic forces during the response from the following equations:

$$F_E(t) = \left[K\right]\left\{v(t)\right\} = \left[K\right]\left[\Phi\right]\left\{Y(t)\right\} \qquad \textbf{(8.52)}$$

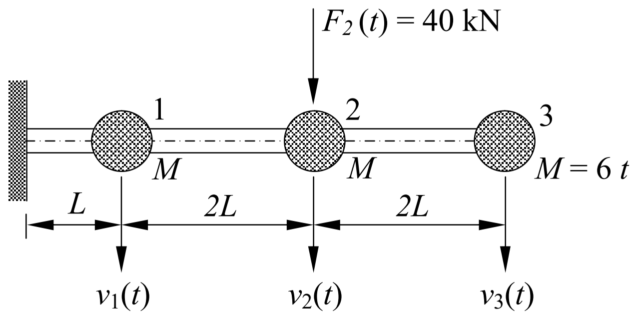
EXERCISES

8.1.

Calculate, by modal superposition, the permanent response of the two degrees of freedom of the two-storey building of exercise 7.4 under the action of the following initial conditions: $\boldsymbol{v_1(0) = 0.02}$ **m** and $\boldsymbol{v_2(0) = 0.02}$ **m**. Consider that the system has no damping.

8.2.

A cantilever beam carrying three equal concentrated masses is shown in figure (8.2); we have also indicated its undamped modes [Φ] and its vibration frequencies ω. Find an expression for the dynamic response of degree of freedom $v_3(t)$ after a **40 kN** step load is applied to degree of freedom 2. Include all modes and consider damping to be zero. Suggestion: use the response of a system with a single degree of freedom at a step load seen in chapter 5.



$$[\Phi] = \begin{bmatrix} 0.054 & 0.283 & 0.957 \\ 0.406 & 0.870 & -0.281 \\ 0.913 & -0.402 & 0.068 \end{bmatrix}; \qquad \{\omega\} = \begin{Bmatrix} 3.61 \\ 24.2 \\ 77.7 \end{Bmatrix} \text{ rad / s}$$

Figure 8.2

8.3.

We now consider the beam of Ex.8.2, but this time the load is permanent and harmonic, applied to degree of freedom 2:

$$F(t) = 15 \sin \overline{\omega} t \text{ (kN), où } \overline{\omega} = 3/4\ \omega_1$$

1/ Find the expression of the motion in steady state of the degree of freedom $v_1(t)$, assuming that the structure is not damped.

2/ Calculer les déplacements de tous les degrés de liberté à l'instant de la réponse maximale.

8.4.

Consider the structure shown in figure (8.3). The floors (1) and (2) are assumed to be infinitely rigid to the lateral force and supported by columns of negligible self-weight in front of the floors. The structure is excited by a force $F_2(t)$ applied at floor level (2).

- Calculate the displacements $\boldsymbol{v_1(t)}$ and $\boldsymbol{v_2(t)}$ at $\boldsymbol{t = 0.2}$**s** by the modal superposition method.

The initial conditions are zero.
Young's modulus is equal to **E = 30000 N/mm^2.**

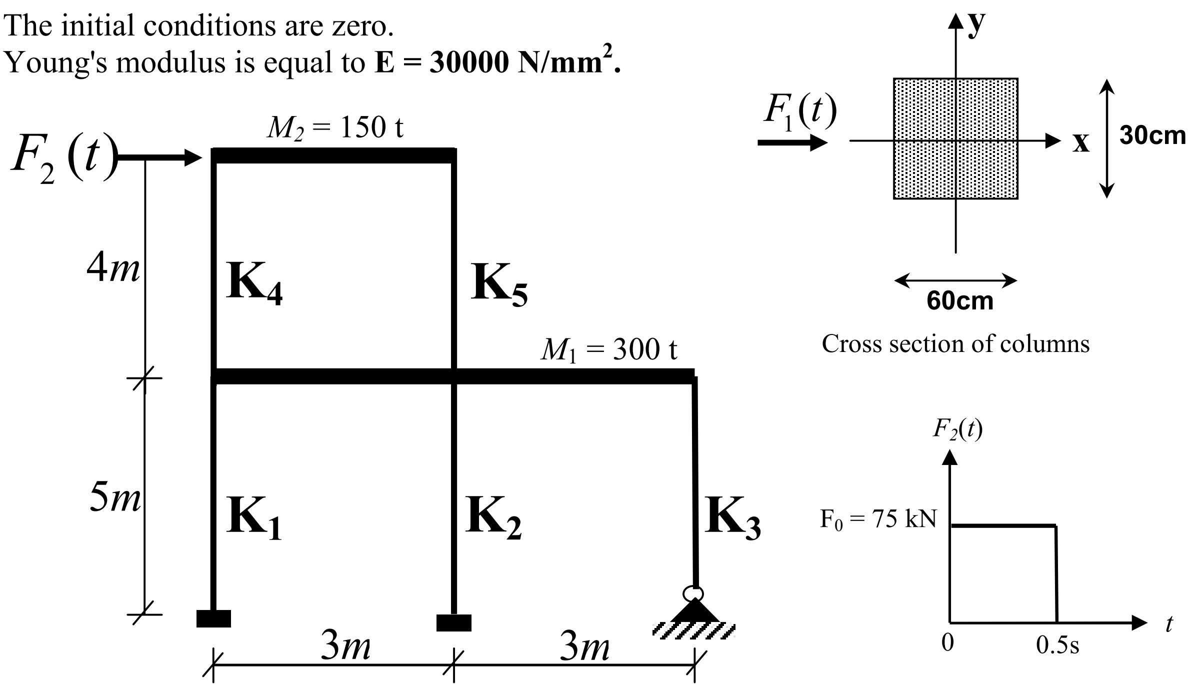
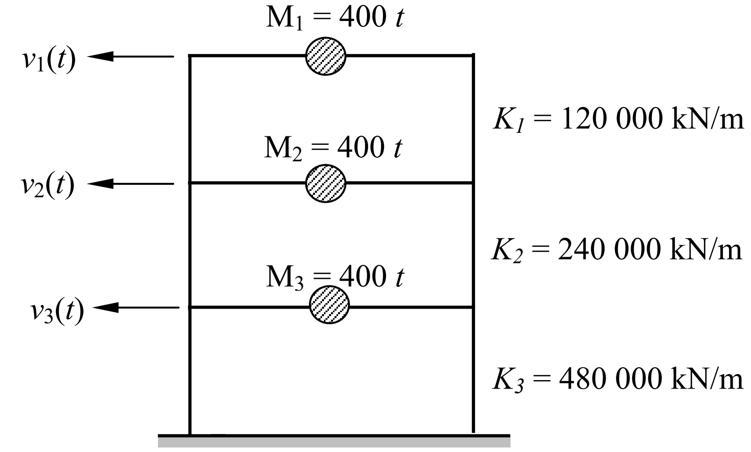


Figure 8.3

8.5.

The mass and stiffness characteristics of a three-storey building, as well as its undamped vibration modes and frequencies, are shown in figure (8.4). The structure is brought into free vibration by moving the floors as follows: $v_1(0) = 0.75$ **cm**, $v_2(0) = -2$ **cm** and $v_3(0) = 0.75$ **cm**, then suddenly releasing them at time $t = 0$. Determine the deformation at time $t = 2\pi/\omega_1$:

1/ Assuming no damping.
2/ Assuming ξ = 10% for each mode.



$$[\Phi] = \begin{bmatrix} 1.000 & 1.000 & 1.00 \\ 0.548 & -1.522 & -6.26 \\ 0.198 & -0.872 & 12.10 \end{bmatrix}; \qquad \{\omega\} = \begin{Bmatrix} 11.62 \\ 27.5 \\ 45.9 \end{Bmatrix} \text{ rad / s}$$

Figure 8.4

CHAPTER

9

SEISMIC RESPONSE OF STRUCTURES WITH MULTI DEGREES OF FREEDOM BY SPECTRAL MODAL SUPERPOSITION

9.1 INTRODUCTION

The previous chapter dealt with the dynamic response of a system with multi degrees of freedom subjected to loads directly applied to the masses composing the system. The movement of the base of the system, such as that caused by an earthquake, represents another form of load whose study constitutes the object of this chapter. The seismic calculation is presented using a method called the spectral modal method. This method uses the concept of design spectra (Chapter 6) to estimate the maximum responses of the structure for a given ground accelerogram at its base.

The ground acceleration at the base of structures is different from the free-field acceleration due to the interaction between the structure and the ground. We only consider in this chapter the response of structures based on rigid ground.

9.2 MODAL SUPERPOSITION

The procedure is similar to that followed in the case of any loading when $F(t)$ is replaced by $F_{\text{eff}}(t)$ given by equation (7.7). The steps to follow are:

1) Write the coupled equations of motion in function of the relative responses:

$$[M]\{\ddot{v}(t)\} + [C]\{\dot{v}(t)\} + [K]\{v(t)\} = \{F_{\text{eff}}(t)\} \quad \textbf{(9.1)}$$

in which $\{F_{\text{eff}}(t)\}$ is the vector of effective forces which is expressed:

$$\{F_{\text{eff}}(t)\} = -[M]\{r\}\ddot{v}_g(t) = -\{\text{f}\}\ddot{v}_g(t) \quad \textbf{(9.2)}$$

where $\{\text{f}\}$ is the spatial distribution of the load equal to:

$$\{\text{f}\} = [M]\{r\} \quad \textbf{(9.3)}$$

2) Calculate the N frequencies and eigenmodes of vibration for the problem of undamped free vibration which reduces to the following eigenvalue problem:

$$[K][\Phi] = [M][\Phi][\Lambda] \qquad \textbf{(9.4)}$$

where $[\Phi]$ is the *modal matrix* of dimension N x N and $[\Lambda]$ is a diagonal matrix of dimension N x N containing the terms ω_i^2.

3) Calculate the mass and the generalized loading relating to each mode:

$$\tilde{M}_i = \{\phi_i\}^{\mathrm{T}}[M]\{\phi_i\} \qquad \textbf{(9.5)}$$

$$\tilde{F}_i(t) = \{\phi_i\}^{\mathrm{T}}\{F_{\mathrm{eff}}(t)\} = -\delta_i \ddot{v}_g(t) \qquad \textbf{(9.6)}$$

where

$$\delta_i = \{\phi_i\}^{\mathrm{T}}\{\mathrm{f}\} = \{\phi_i\}^{\mathrm{T}}[M]\{r\} \qquad \textbf{(9.7)}$$

4) Write the N independent equations of motion:

$$\ddot{Y}_i(t) + 2\xi_i\omega_i\dot{Y}_i(t) + \omega_i^2 Y_i(t) = -\frac{\delta_i}{\tilde{M}_i}\ddot{v}_g(t) \qquad \textbf{(9.8)}$$

5) Calculate the modal response. By analogy with equation (8.36) and taking into account equation (6.17), the response of poorly damped systems is expressed as a function of the Duhamel integral. So :

$$Y_i(t) = \frac{\delta_i}{\tilde{M}_i}D_i(t) \qquad \textbf{(9.9)}$$

in which

$$D_i(t) = -\frac{1}{\omega_i}\int_0^t \ddot{v}_g(\tau)e^{-\xi_i\omega_i(t-\tau)}\sin\omega_i(t-\tau)\mathrm{d}\tau \qquad \textbf{(9.10)}$$

where we assume that $\omega_{Di} = \omega_i$

6) Calculate the response in the geometric coordinate system from the modal responses $Y_i(t)$. Displacements of the modal component $\{v_i(t)\}$ are given by:

$$\{v_i(t)\} = \{\phi_i\}Y_i(t) = \{\phi_i\}\frac{\delta_i}{\tilde{M}_i}D_i(t) \qquad \textbf{(9.11)}$$

The vector of total displacements is the sum of all the modal components:

$$\{v_i(t)\} = [\Phi]\{Y(t)\} = [\Phi]\left\{\frac{\delta_i}{\tilde{M}_i}D_i(t)\right\} \qquad \textbf{(9.12)}$$

where $\left\{\frac{\delta_i}{\tilde{M}_i}D_i(t)\right\}$ is a vector whose components are the displacement response of each mode considered in the analysis.

7) Calculate the elastic forces corresponding to the relative displacements:

$$\{F_E(t)\} = [K]\{v(t)\} = [K][\Phi]\{Y(t)\} \quad \textbf{(9.13)}$$

Substituting equation (9.4) into equation (9.13), we get:

$$\{F_E(t)\} = [M][\Phi][\Lambda]\{Y(t)\} \quad \textbf{(9.14)}$$

which we can still write:

$$\{F_E(t)\} = [M][\Phi]\left\{\frac{\delta_i}{\tilde{M}_i}\omega_i^2 D_i(t)\right\} = -[M][\Phi]\left\{\frac{\delta_i}{\tilde{M}_i}A_i(t)\right\} \quad \textbf{(9.15)}$$

where $A_i(t) = -\omega_i^2 D_i(t)$ and $\left\{\frac{\delta_i}{\tilde{M}_i}A_i(t)\right\}$ is a vector whose components are the *absolute pseudo-acceleration* of each mode considered in the analysis. The vector of elastic forces associated with each mode *i* in equation (9.15) is given by:

$$\{F_{Ei}(t)\} = -[M]\{\phi_i\}\frac{\delta_i}{\tilde{M}_i}A_i(t) \quad \textbf{(9.16)}$$

Let $F_E^{(j)}(t)$ be the elastic force at level *j*, which is the sum of all the elastic forces associated with each mode considered in the analysis, the base shear is given by:

$$V_0(t) = \sum_{j=1}^{n} F_E^{(j)}(t) = \{r\}^{\mathrm{T}}\{F_E(t)\} \quad \textbf{(9.17)}$$

where $\{r\}^{\mathrm{T}}$ is the transpose of the vector of pseudo-static displacements. Substituting equation (9.15) into equation (9.17), we get:

$$V_0(t) = -\{r\}^{\mathrm{T}}[M][\Phi]\left\{\frac{\delta_i}{\tilde{M}_i}A_i(t)\right\} \quad \textbf{(9.18)}$$

For models of simple structures whose degrees of freedom are represented by the horizontal displacements of the floors, the vector $\{r\}^{\mathrm{T}}$ becomes a unit vector $\{1\}$, since for unit ground acceleration in the horizontal direction; all degrees of freedom have unit horizontal acceleration. Taking this into account, equation (9.7) allows us to write:

$$\{r\}^{\mathrm{T}}[M][\Phi] = \{\delta_1 \quad \delta_2 \quad \delta_3 \quad \dots \quad \delta_N\} \quad \textbf{(9.19)}$$

Substituting equation (9.19) into equation (9.18) finally gives us:

$$V_0(t) = -\sum_{i=1}^{n}\frac{\delta_i^2}{\tilde{M}_i}A_i(t) \quad \textbf{(9.20)}$$

where n is the number of floors which, in the case of simple structural models, is also equal to the number of equations N.
The overturning moment at the base of the structure shown in figure (9.1) is given by the following relationship:

$$M_0(t) = \sum_{j=1}^{n} x_j F_s^{(j)}(t) = \{x\}^{\mathrm{T}} \{F_E(t)\} \qquad \textbf{(9.21)}$$

where $\{x\}^{\mathrm{T}}$ is a line vector whose components xj represent the mass height at level j measured above the base of the building. Substituting equation (9.15) into equation (9.21), we get:

$$M_0(t) = \{x\}^{\mathrm{T}} [M][\Phi][\Lambda]\{Y(t)\} = -\{x\}^{\mathrm{T}} [M][\Phi]\left\{\frac{\delta_i}{\tilde{M}_i} A_i(t)\right\} \qquad \textbf{(9.22)}$$

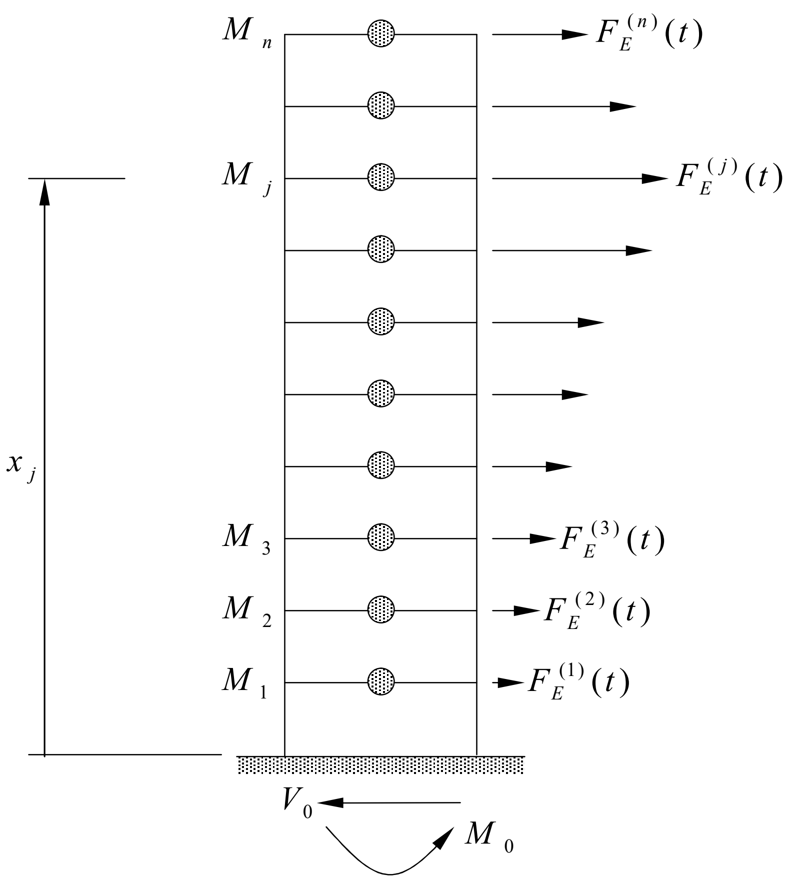


Figure 9.1: Distribution of elastic forces in a discrete system with multi degrees of freedom.

9.3 EFFECTIVE MODAL MASSES

Consider the structure shown in figure (9.1) subjected to horizontal ground movements caused by an earthquake. Let us adopt a diagonal concentrated mass matrix corresponding to a mass M_i per floor i. The total mass M_{tot} is given by:

$$M_{tot} = \{1\}^{T} [M] \{1\} \qquad \textbf{(9.23)}$$

Representing the vector $\{1\}^{T} = \{1 \ ... \ 1\}$ in the modal base according to the following expression:

$$\{1\} = [\Phi]\{Y\} \qquad \textbf{(9.24)}$$

Pre-multiply equation (9.24) by $\{\phi_i\}^{T}[M]$, we get:

$$\{\phi_i\}^{T} [M] \{1\} = \{\phi_i\}^{T} [M][\Phi]\{Y\} = \tilde{M}_i \{Y_i\} \qquad \textbf{(9.25)}$$

from where

$$Y_i = \frac{\delta_i}{\tilde{M}_i} \qquad \textbf{(9.26)}$$

Taking into account the equation (9.26), the equation (9.24) can be written:

$$\{1\} = [\Phi]\left\{\frac{\delta_i}{\tilde{M}_i}\right\} \qquad \textbf{(9.27)}$$

Substituting equation (9.27) into equation (9.23), we get:

$$M_{tot} = \{1\}^{T} [M][\Phi]\left\{\frac{\delta_i}{\tilde{M}_i}\right\} = \{\delta_1 \quad \delta_2 \quad \cdots \quad \delta_n\}\left\{\frac{\delta_i}{\tilde{M}_i}\right\} \qquad \textbf{(9.28)}$$

from where

$$M_{tot} = \sum_{i=1}^{n} \frac{\delta_i^2}{\tilde{M}_i} \qquad \textbf{(9.29)}$$

The factor $\delta_i^2 / \tilde{M}_i$ has the dimensions of a mass and is called the *effective modal mass* of the structure. It represents the part of the total mass responding to the earthquake in each mode i. However, this interpretation is only valid for a building of the type shown in figure (9.1) having masses concentrated along a vertical axis and for an excitation coming from an earthquake. The factor Γ_i is called participation factor of the mode i and indicates how much mode i is excited by the earthquake. It is written:

$$\Gamma_i = \frac{\delta_i}{\tilde{M}_i} = \frac{\{\phi_i\}^{T}\{f\}}{\{\phi_i\}^{T}[M]\{\phi_i\}} \qquad \textbf{(9.30)}$$

This modal participation factor is therefore a major factor in determining the magnitude of the modal response. In most finite element programs, the eigenmodes are normalized so that $\tilde{M}_i = 1$. Therefore, we have:

$$\Gamma_i = \delta_i = \{\phi_i\}^T [M]\{r\} \quad \textbf{(9.31)}$$

9.4 SUPERPOSITION OF SPECTRAL RESPONSES

Modal superposition makes it possible to calculate the complete response of a system with *N* degrees of freedom. However, for dimensioning, we are generally interested in the maximum value of the response and not in the complete temporal response. The modal analysis method, by decoupling the *N* equations of a system with *N* degrees of freedom, makes it possible to model the response of each vibration mode by the response of a system with one degree of freedom. The design spectrum can therefore be used to find the maximum response for each mode of vibration and by combining the maximum responses for each mode according to certain rules; one can obtain the probable maximum response of the discrete system with *N* degrees of freedom. Thus, the modal displacement for the mode *i* is written:

$$Y_{i,\max} = \frac{\delta_i}{\tilde{M}_i} S_d(\xi_i, T_i) = \Gamma_i S_d(\xi_i, T_i) \quad \textbf{(9.32)}$$

The maximum displacement of mode *i* in the geometric coordinate system is therefore given by:

$$\{v_{i,\max}\} = \{\phi_i\} Y_{i,\max} = \{\phi_i\} \frac{\delta_i}{\tilde{M}_i} S_d(\xi_i, T_i) = \{\phi_i\} \Gamma_i S_d(\xi_i, T_i) \quad \textbf{(9.33)}$$

The vector of maximum elastic forces of mode *i* can be obtained from equation (9.15), that is:

$$\{F_{Ei,\max}\} = [M]\{\phi_i\} \frac{\delta_i}{\tilde{M}_i} S_a(\xi_i, T_i) = [M]\{\phi_i\} \Gamma_i S_a(\xi_i, T_i) \quad \textbf{(9.34)}$$

The total response of a structure to an earthquake comes from the superposition of the contributions of the eigenmodes. Also, one can determine the maximum response for a mode *i* directly from the design spectrum. In general, the maximums for each mode do not occur at the same time and their addition gives very safe results. Therefore, they cannot be added directly in order to obtain the maximum total response. This procedure, known as the *absolute sum method* or *the arithmetic combination* or AC method, however, provides the upper limit of the maximum response which, in general, will be greatly overestimated. Therefore, the upper limit of the maximum of a given response quantity is expressed as:

$$r_i^{CA} = \sum_{j=1}^{n} \left| (r_{ij})_{\max} \right| \quad \textbf{(9.35)}$$

Where r_i^{CA} is the upper limit of the maximum response of degree of freedom *i*. According to this method, the upper limit of the maximum displacements will be given by:

$$v_i^{CA} = \sum_{j=1}^{n} \left| \left(v_{ij} \right)_{max} \right| \quad \textbf{(9.36)}$$

Where v_i^{CA} is the upper limit of the maximum response of the degree of freedom *i* according to the CA method. Several methods have been proposed to estimate the probable value of the maximum response from the spectral responses. The most popular of these methods, and indeed the simplest, is to calculate the quadratic mean of the modal responses which we will call the *quadratic combination method* or CQ (*Square Root of the Sum of the Squares* or SRSS), which one writes:

$$r_i^{CQ} = \left(\sum_{j=1}^{n} \left(r_{ij} \right)_{max}^2 \right)^{\frac{1}{2}} \quad \textbf{(9.37)}$$

By the CQ method, the probable maximum total displacements are given by the following expression:

$$v_i^{CQ} = \left(\sum_{j=1}^{n} \left(v_{ij} \right)_{max}^2 \right)^{\frac{1}{2}} \quad \textbf{(9.38)}$$

The CQ method provides a good estimate of the maximum response for systems that exhibit well-separated eigenmodes. It was originally proposed in two-dimensional building analyses; thus, the eigenmodes are not brought closer. However, for structures with very close modes: the case of three-dimensional analyzes where modes in different directions can have very similar natural frequencies, the maximum response calculated by the CQ method can be marred by errors. In the latter case, the *Complete Quadratic Combination* or CQC method is preferable. In the CQC method, the probable maximum responses are obtained from the following relationship:

$$r_i^{CQC} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \left(r_{ki} \right)_{max} \left(r_{kj} \right)_{max} \right)^{\frac{1}{2}} \quad \textbf{(9.39)}$$

where each of the n^2 terms of the right-hand side is the product of the maximum response to the degree of freedom *k* of modes *i* and *j*, and of a correlation coefficient ρ_{ij} between these two modes. ρ_{ij} varies between 0 and 1 and $\rho_{ij} = 1$ for *i* = *j*. Rewrite equation (9.39) in the following form:

$$r_i^{CQC} = \left(\sum_{i=1}^{n} \left(r_{ki} \right)_{max}^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \rho_{ij} \left(r_{ki} \right)_{max} \left(r_{kj} \right)_{max} \right)^{\frac{1}{2}} \quad \textbf{(9.40)}$$

which shows that the CQC method is reduced to the CQ method when the modes are well separated for which $\rho_{ij} \approx 0$. The expression of ρ_{ij} is given by the following relation:

$$\rho_{ij} = \frac{8\sqrt{\xi_i \xi_j}\left(\xi_i + \beta_{ij}\xi_j\right)\beta_{ij}^{3/2}}{\left(1-\beta_{ij}^2\right)^2 + 4\xi_i\xi_j\beta_{ij}\left(1+\beta_{ij}^2\right) + 4\left(\xi_i^2+\xi_j^2\right)\beta_{ij}^2} \quad \textbf{(9.41)}$$

where

$$\beta_{ij} = \frac{\omega_i}{\omega_j} \quad \textbf{(9.42)}$$

Equation (9.41) implies that $\rho_{ij} = \rho_{ji}$ and $\rho_{ij} = 1$ for $i = j$ or for two modes having the same frequencies and the same damping factors. For $\xi_i = \xi_j = \xi$, this equation simplifies to the following form:

$$\rho_{ij} = \frac{8\xi^2\left(1+\beta_{ij}\right)\beta_{ij}^{3/2}}{\left(1-\beta_{ij}^2\right)^2 + 4\xi^2\beta_{ij}\left(1+\beta_{ij}^2\right)^2} \quad \textbf{(9.43)}$$

EXERCISES

9.1.

Calculate, by modal superposition, the response of the two degrees of freedom of the two-storey building given in Exercise 7.4 under the action of the El Centro accelerogram. Assume that the building has modal dampings of 2% for both vibration modes. Use only the first 30 seconds of the accelerogram.

9.2.

Calculate the probable maximum roof displacement and the probable maximum shear at the base of the two-storey building given in Exercise 7.4 by the QC method.

9.3.

Calculate, by modal superposition of the spectral responses, the displacement of two degrees of freedom of the two-storey building given in Exercise 7.4 under the action of the El Centro earthquake. Consider that the building has a constant damping rate ξ = 2% at the two eigenmodes. Figure (9.2) gives the displacement spectrum of the El Centro accelerogram for ξ = 2%.

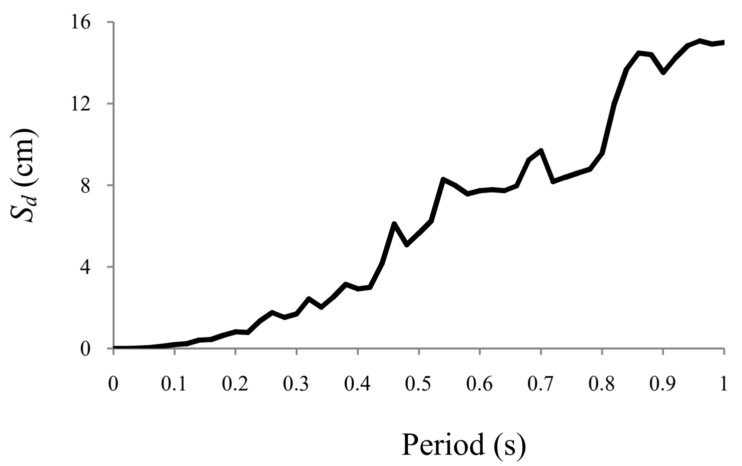


Figure 9.2

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