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Présentée par :

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Sujet

ON THE EXISTENCE, UNIQUENESS AND ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF THE NON-LINEAR KIRCHHOFF MODEL IN EXPANDING DOMAINS

Devant le jury d'examen composé de :

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Abstract

In this thesis we study the global and decay solution for large size data of nonlinear hyperbolic-parabolic equation of Kirchhoff type

$$u_{tt} + \mu u_t - \tilde{M}\Big(\int_{\Omega_t} |\nabla u|^2 dx\Big)\Delta u = 0 \text{ in } \Omega_t$$

where $\Omega_t = \{x \in \mathbb{R}^n : \ x = y\sigma(t), \quad y \in \Omega\}$ with Ω being a bounded open domain in $\mathbb{R}^n$, μ is a positive constant and $\sigma(t)$ is a given suitable increasing positive function unbounded from above. The real function $\tilde{M}$ is such that $\tilde{M}(\lambda) > 0$ and $\tilde{M}'(\lambda) \geq 0$ for every $\lambda \in [0, \infty[$.

Keywords : local existence, global existence, asymptotic behaviour, degenerate hyperbolic equation.

Résumé

Cette thèse porte principalement sur l'étude de la solution globale et son comportement asymptotique de l'équation non linéaire hyperbolique-parabolique de Kirchhoff de type

$$u_{tt} + \mu u_t - \tilde{M}\Big(\int_{\Omega_t} |\nabla u|^2 dx\Big)\Delta u = 0 \text{ dans } \Omega_t$$

où $\Omega_t = \{x \in \mathbb{R}^n : \ x = y\sigma(t), \quad y \in \Omega\}$ avec Ω un domaine ouvert et borné de $\mathbb{R}^n$, μ est une constante positive et $\sigma(t)$ une fonction positive, croissante et non bornée supérieurement. La fonction réelle $\tilde{M}$ est telle que $\tilde{M}(\lambda) > 0$ et $\tilde{M}'(\lambda) \geq 0$ pour $\lambda \in [0, \infty[$.

Mots clés : existence locale, existence globale, comportement asymptotique, équation hyperbolique dégénérée.

Contents

Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$ which, without loss of generality, can be assumed to contain the origin, with boundary Γ of class C^2 and $\sigma : [0,\infty[\longrightarrow \mathbb{R}$ a positive continuously differentiable increasing function, unbounded from above. Let us consider the family of bounded increasing subdomains $\{\Omega_t\}_{0\le t<\infty}$ of $\mathbb{R}^n$ given by

$$\Omega_t = h_t(\Omega), \quad \Omega_0 = h_0(\Omega), \quad h_t : y \in \Omega \longmapsto x = \sigma(t)y$$

whose boundaries are denoted by Γ_t, and $\hat{Q}$ the non-cylindrical domain of $\mathbb{R}^{n+1}$

$$\hat{Q} = \bigcup_{0\le t<\infty} \Omega_t \times \{t\}\,,$$

with lateral boundary

$$\hat{\Gamma} = \bigcup_{0\le t<\infty} \Gamma_t \times \{t\}\,.$$

We consider the following mixed problem related to a nonlinear equation of Kirchhoff type

$$u_{tt} + \mu u_t - \tilde{M}\Big(\int_{\Omega_t} |\nabla u|^2 dx\Big)\Delta u = 0 \quad \text{in} \quad \hat{Q}\,, \tag{0.1}$$

$$u|_{\hat{\Gamma}} = 0\,, \tag{0.2}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \tag{0.3}$$

where the given function $\tilde{M}$ satisfies the following conditions

$$\tilde{M} \in C^2([0,\infty[), \quad \tilde{M}(\lambda) \ge m_0 > 0, \quad \tilde{M}'(\lambda) \ge 0 \quad \forall \lambda \in [0,\infty[. \tag{0.4}$$

Here we want to solve the problem (0.1)-(0.3) globally in time regardless of size of the initial data $(u_0, u_1) \in H^2(\Omega_0) \times H^1(\Omega_0)$ provided the expansion of moving domains Ω_t is fairly slow.

In the literature, the equation (0.1) is called of hyperbolic-parabolic type. This class of equations has been studied by several authors, for instance Lar'Kin [26] and Bensoussan et al. [6]. Bisognin proved in [9] the existence of local solution of (1.1) in both bounded and unbounded domains of $\mathbb{R}^n$.
Whenever $\mu = 0$, there is a large number of papers involving the Kirchhoff-Carrier operator

$$Lu = u_{tt} - \Big(1 + \tilde{M}\Big(\int_\Omega |\nabla u|^2 dx\Big)\Big)\Delta u.$$

We recall that in the case $n = 1$ with $M(\lambda) = a\lambda + b$ and a, $b > 0$, the equation $Lu = 0$ was proposed by Kirchhoff [25] in his book of Mathematical Physics in 1883, to describe the oscillations of an elastic stretched string. This equation was studied by some others authors like , Carrier [13], Bernstein [7], Dickey [17, 18], Menzala [33]. The result of local existence for $Lu = 0$ was obtained by some of the authors quoted above with initial data taken in usual Sobolev spaces and for both Dirichlet and periodic boundary conditions. The first result on global solvability for the Kirchhoff equation was established by Bernstein [7] in dimension $n = 1$ for analytic initial data. This result was extended later by Pohozaev [38], Arosio-Spagnolo [1], Kajatani-Yamaguti [24] in dimension $n \geq 2$. Throughout the years, these results on the global solvability for analytical initial data were extended and refined later by several authors (see for instance, Nishihara [36], Ghisi-Gobbino [21]),

The global solvability for large non-analytic initial data has been till now a deep open problem. Several results on the global solvability for small non-analytical (mainly of class $\mathcal{C}^\infty$ with compact support, Gevrey class, or Sobolev spaces) initial data are well established in the literature (see for instance,[10, 15, 16, 22, 32, 40, 41, 42]). We also mention that, for non analytical initial data, Pohozaev [39] and Menzala-Pereira [34] for instance, have obtained some global existence results, using non physical functions $M(r)$ behaving like $(\alpha r + \beta)^{-2}$, α and β being positive constants.

In order to obtain a global solution for $Lu = f$ several authors (see for instance Nishihara [37]), have introduced damping terms like $-\Delta u_t$ or $\Delta^2 u$ which allow to get strong estimates in order to control the nonlinear term proving in this way the global existence result. Another class of dissipative mechanisms was considered by Ikehata and Okazawa in [23], where the authors studied the following equation of a stretched string with "frictional" damping

$$u_{tt} - (1 + \tilde{M}(\int_\Omega |\nabla u(x,t)|^2 dx))\Delta u + \mu u_t = 0$$

and showed the existence of global strong solutions, provided μ (a parameter depending on the initial data) is large enough. Other authors have considered a model with a nonlinear damping term $g(u_t)$ replacing the term of μu_t.

The problem (0.1)-(0.3) was studied in [2, 3] globally in time in dimension two provided the initial data are small and with non homogeneous Dirichlet boundary condition. In the literature, several works have been devoted to evolution problems in non-cylindrical domains (see [4, 5, 8, 11, 14, 20, 30]). For instance, the heat equation, the Navier-Stokes equation and the wave equation have been studied in non-cylindrical domains. The proof of the existence of both local and global solutions in most of those articles is based on suitable change of variables which allows to transform the problem in another problem in a cylindrical domain. Other methods have developed to solve evolution problems in non cylindrical domains. For instance, Cannarsa et al. developed in [11] a method which consists in transforming the problem into a non autonomous initial boundary problem in the Lebesgue space $L^2(\Omega)$, involving a family of unbounded operators with variable coefficients.

As it is well known, the result about local existence of solutions was proved in cylindrical and non-cylindrical domains by many authors cited in the reference. Our principal attention in this paper is devoted to the global existence of solutions and their asymptotic behaviour. We follow here the change of variable method described above. As announced above, this problem has already been studied in [2, 3] in the two-dimensional space case.

Our goal here is to extend the results in the articles [2, 3] in higher dimensional space and for opportunely large initial data. We succeeded to do so under the further assumption that the expansion of the domains Ω_t is slow and that the size of the initial domain Ω_0 is small.

To this aim, we will first study our problem in the cylinder $Q = \Omega\times]0,\infty[$. The domains Q and $\hat{Q}$ are related by the diffeomorphism $\tau : \hat{Q} \longrightarrow Q$ defined by

$$\tau(x,t) := (y,t) = (\frac{x}{\sigma(t)}, t) \quad \text{for} \quad (x,t) \in \hat{Q}. \tag{0.5}$$

Whose inverse $\tau^{-1} : Q \longrightarrow \hat{Q}$ is given by

$$\tau^{-1}(y,t) := (x,t) = (y\sigma(t), t). \tag{0.6}$$

If we set

$$v(y,t) := u \circ \tau^{-1}(y,t) = u(y\sigma(t), t), \tag{0.7}$$

then the initial boundary value problem (0.1)-(0.3) becomes

$$v_{tt} + \mu v_t - \frac{1}{\sigma^2}\tilde{M}\Big(\int_\Omega |\sigma^{\frac{n-2}{2}}\nabla v|^2 dy\Big)\Delta v = \tilde{F}(t,v)\,, \tag{0.8}$$

$$v|_{\partial\Omega} = 0, \qquad v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1, \tag{0.9}$$

where

$$\tilde{F}(t,v) := -\big(\frac{\sigma'}{\sigma}\big)^2 \sum_{i,j=1}^{n} \partial_{y_i}(y_i y_j \partial_{y_j} v) + a_1(t,y)\cdot\nabla v_t + a_2(t)\cdot\nabla v, \tag{0.10}$$

$$a_1(t,y) := 2\frac{\sigma'}{\sigma}y, \quad a_2(t,y) := \sigma^{-2}y(\sigma\sigma'' + \mu\sigma\sigma' + (n-1)\sigma'^2). \tag{0.11}$$

Remark 0.0.1. *Note that the initial data* (v_0, v_1) *is determined by the given couple (0.3)* (u_0, u_1) *and depends of course on the initial position* $\sigma(0)$ *and the initial velocity* $\sigma'(0)$, *thus (see (2.21)) on* σ_0 *and* σ_1. *But considering subsequent assumption (see (3.1)) on* σ_0 *and* σ_1, *the only dependency of* (v_0, v_1) *in terms of* σ_0 *is meaningful. To emphasize this dependency, when required it will be noted* $(v^0_{\sigma_0}, v^1_{\sigma_0})$ *instead of* (v_0, v_1).

Indeed, given (u_0, u_1), the couple of initial data (v_0, v_1) is determined using equations

$$x \in \Omega_0 = \sigma(0)\Omega, \quad u_0(x) = u(\sigma(0)y, 0) = v_0(y), \quad y \in \Omega \tag{0.12}$$

and (see (2.3) and (2.21))

$$u_1(x) = v_1(y) - \alpha\frac{\sigma_1}{\sigma_0}y\cdot\nabla v_0(y), \quad v_1(y) = v_t(y,t)|_{t=0} \tag{0.13}$$

We set

$$M(s) := \tilde{M}(s) - \frac{m_0}{2}, \tag{0.14}$$

$$a_{ij}(t,y) := \frac{m_0}{2\sigma^2}\delta_{ij} - \big(\frac{\sigma'}{\sigma}\big)^2 y_i y_j \quad \big(i,j = 1,n\big)\,. \tag{0.15}$$

According to (2.10) and (0.4), it follows that

$$M(\lambda) \ge \frac{m_0}{2}, \quad M \in C^2([0,\infty[), \quad M'(\lambda) \ge 0, \qquad \forall\lambda \in [0,\infty[\,. \tag{0.16}$$

Given (2.10)-(2.11), the problem (2.4) and (2.5) is rewritten as

$$v_{tt} + \mu v_t - \frac{1}{\sigma^2}M\big(\int_\Omega |\sigma^{\frac{n-2}{2}}\nabla v|^2 dy\big)\Delta v = F(t,v)\,, \tag{0.17}$$

$$v|_{\partial\Omega} = 0\,, \tag{0.18}$$

$$v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1, \tag{0.19}$$

with

$$F(t,v) = A(t)v + a_1(t,y)\cdot\nabla v_t + a_2(t,y)\cdot\nabla v, \tag{0.20}$$

where

$$A(t) = \sum_{i,j=1}^{n} \partial_{y_i}(a_{ij}(t,y)\partial_{y_j}v).$$

We set

$$a(t,u,v) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(t,y)(\partial_{y_i}u)(\partial_{y_j}v)dy \tag{0.21}$$

$$a'(t,u,v) = \sum_{i,j=1}^{n} \int_{\Omega} a'_{ij}(t,y)(\partial_{y_i}u)(\partial_{y_j}v)dy. \tag{0.22}$$

To study (2.13)-(2.15) we need some hypotheses on the function σ. Let us first recall that the function σ is positive, increasing and unbounded from above. Moreover, we assume that

$$\sigma \in C^3([0,\infty[), \quad \sigma(0) > 0, \quad 0 \leq \sigma'(t) \leq \frac{1}{d}\sqrt{\frac{m_0}{2}} \quad \forall t > 0 \tag{0.23}$$

where $d = \operatorname{diam}(\Omega)$. The second condition (2.19) implies that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq 0 \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}. \tag{0.24}$$

In order to avoid tedious abstract computations, we work throughout the paper with a typical family of functions σ which satisfy (2.19), that is

$$\sigma(t) = (\sigma_0 + \sigma_1 t)^{\alpha}, \quad 0 < \alpha < \frac{1}{2}. \tag{0.25}$$

where σ_0 and σ_1 are positive constants chosen so that (2.19) is satisfied. Note that this assumption means that $\hat{Q}$ is increasing in the sense that, if $t > t'$ then Ω_t contains $\Omega_{t'}$.

Structure of the thesis

This thesis is organised as follows. We first derive (for the reader's convenience the Kirchhof equation (1.5) and we collect all the preliminaries results that are used in the thesis. Chapter 2 is devoted to the existence and uniqueness of the local solution following the standard approach that combines the Galerkin approximation method with the a priori estimates and compactness arguments. Chapter 3 is dedicated to the global solution in time. In section 3.1, to adapt to a growing domain, delicate a priori estimates of the local solution had to be established in order to extend the local solution and get the result of global existence for the initial boundary value problem (2.13)-(2.15), and consequently for the problem (0.1)-(0.3). Theses estimates (see lemmas 3.1.1-3.1.6) are obtained by carefully choosing test functions for the equation (2.13), which are products of the unknown function v (or some of its time derivatives) with suitable powers of the function σ describing the expansion of the domain (see (3.11), (3.31), (3.38), (3.42) and (3.1)). Section 3.2 is devoted to the existence of the global solution and its asymptotic behaviour with initial data opportunely large enough.

Chapter 1

Preliminaries

In this chapter, we recall some essential notions and concepts from abstract analysis that will be used throughout this thesis, as well as some results on L^p-spaces ($1 \leq p \leq \infty$) and Sobolev spaces. Let $x = (x_1, x_2, \cdots, x_n)$ be a generic point in an open subset Ω of $\mathbb{R}^n$. Given a vector $a = (a_1, a_2, \cdots, a_n)$ in $\mathbb{R}^n$, we denote by $\|a\|^2 = \sum_{i=1}^{n} a_i^2$ the Euclidean norm of the vector a. Let $u : \Omega \longrightarrow \mathbb{R}$ be a function. We recall that $\partial_{x_i} u$ and $\partial_{x_i}^2 u$ is the first and second partial derivative of u with respect to the ith variable x_i and the gradient and Laplacian of u are $\nabla u = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})^t$ and $\Delta u = \sum_{i=1}^{n} \partial_{x_i}^2 u$ respectively

$C^0(\Omega)$ denotes the space of continuous functions from Ω to $\mathbb{R}$. For every $k \geq 1$ integer, $C^k(\Omega)$ is the space of continuous functions together with their partial derivatives up to order k and $C_c^k(\Omega)$ is the space of function of $C^k(\Omega)$ having a compact support contained in Ω. We also recall that $C^k(\overline{\Omega})$ denotes the space of the restrictions to $\overline{\Omega}$ of functions in $C^k(\mathbb{R}^n)$. We recall also that $C_c^\infty(\Omega) = \mathcal{D}(\Omega)$ is the space of infinitely differentiable functions with compact supports in Ω and is also called the space of test functions.

1.0.1 Strong and Weak convergence

Definition 1.0.1. *Let $(V, \|\cdot\|)$ be a normed space over $\mathbb{C}$ or $\mathbb{R}$.*

- *(Strong convergence). A sequence $\{u_n\}_{n\geq 1}$ of V is said to be strongly convergent (or convergent in the norm) if there is an $u \in V$ such that*

$$\lim_{n\to\infty} \|u_n - u\| = 0$$

- *(Weak convergence). We say $\{u_n\}_{n\geq 1}$ converges weakly to $u \in V$, written*

$u_n \rightharpoonup u$ *if*

$$< f, u_n > \longrightarrow < f, u >$$

For each bounded linear functional $f \in V'$ (dual of V).

It is easy to check that if $u_n \longrightarrow u$, then $u_n \rightharpoonup u$. It is also true that any weakly convergent sequence is bounded. In addition, if $u_n \rightharpoonup u$, then

$$\|u\| \leq \underset{n \to \infty}{\text{liminf}} \|u_n\|$$

- (Weak Star Convergence). *A sequence $\{f_n\}_{n \geq 1}$ of V' is weak star convergent to $f \in V'$ if $f_n(u)$ converges to $f(u)$ for all $u \in V$.*

Note that, the weak star limit of $\{f_n\}_{n \geq 1}$ is unique

Theorem 1.0.1. ((Weak compactness) *Let V be a reflexive Banach space and suppose the sequence $\{u_n\}_{n \geq 1}$ of V is bounded. Then there exists a subsequence $\{u_{k_n}\}_{n \geq 1}$ of $\{u_n\}_{n \geq 1}$ and $u \in V$ such that*

$$u_{k_n} \longrightarrow u.$$

In other words, bounded sequences in a reflexive Banach space are weakly precompact. In particular, a bounded sequence in a Hilbert space contains a weakly convergent subsequence.

1.0.2 Compact Operators

Now we introduce a class of operators having interesting spectral properties even infinite dimensional spaces and which play an important role in the study of ordinary and partial differential equations as well as integral equations.

Definition 1.0.2. *Let $(V, \|\cdot\|_V)$ and Let $(W, \|\cdot\|_W)$ be two Banach spaces. An operator $T \in \mathcal{L}(V, W)$ called a compact operator if $T(B_V)$ has compact closure in W. Equivalently, T is compact if for every norm-bounded sequence $\{u_n\}_{n \geq 1}$ of V, $\{T(u_n)\}_{n \geq 1}$ has a strongly convergent subsequence in W. We denote the set of compact operators from V to W by $\mathcal{K}(V, W)$ and write $\mathcal{K}(V) = \mathcal{K}(V, V)$.*

Theorem 1.0.2. ((Spectral theorem for Compact, Self adjoint Operator) *Let $A : H \longrightarrow H$ be a compact, self adjoint operator on a Hilbert space H. There is an orthonormal basis of H consisting of eigenvectors of A. The non-zero eigenvalues of A form a finite or countably infinite set $\{\lambda_n\}_{n \geq 1}$ of real numbers, and*

$$A = \sum_n \lambda_n P_n$$

where P_n is the orthogonal projection onto the finite dimensional eigenspace of eigenvectors with eigen- value λ_n. If the number of non-zero eigenvalues is countably infinite, then the above series converges to A in the operator norm.

We will be having an abstract framework with the following structure: Let $(H, (\cdot,\cdot)_H)$ be a real Hilbert space with $\|\cdot\|_H$ being the norm induced by the scalar product $(\cdot,\cdot)_H$. Let $A : H \longrightarrow H$ be a linear operator with domain $V = D(A)$ dense in H.
We assume that the operator A is self-adjoint and (strictly) positive. This implies that there exists $\alpha > 0$ such that $(Au, u) \leq \alpha\|u\|_H^2$. Hence, we obtain a scalar product on V defined by

$$(u, v)_A := (Au, v)_H \quad \forall u, v \in V$$

with norm induced

$$\|u\|_A := \sqrt{(Au, u)_H} \quad \forall u \in V$$

equivalent on V to the norm $\|\cdot\|_H$. Hence, $(V, (\cdot,\cdot)_A)$ is a Hilbert space. Under these assumptions, we get the so-called Hilbert triple: $V = D(A) \subset H \subset V'$ with H identified to its dual H'.
The following spectral theorem holds

Theorem 1.0.3. *Assume that the operator A is such that the space V is compactly embedded in H. Then, under the assumptions above, there exists a sequence $\{\lambda_n\}_{n\geq1}$ of eigenvalues of the operator A suchthat*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \quad \textit{with} \;\; \lambda_n \to +\infty$$

and an orthonormal system of eigenvectors $\{w_n\}_{n\geq1}$ with Span$(w_n : n \geq 1)$ dense in H.

Proof. the idea is to first prove the operator A is invertible and that the inverse operator A^{-1} is compact. First of all let $f \in H$. We seek a unique $u \in D(A)$ such that $A(u) = f$. Let us define $F \in V'$ by $< F, v >_{V'\times V} := (f, v)_H$ forall $v \in V$. Then, by the Riesz's representation theorem, it follows that there exists a unique $u \in V$ such that

$$< F, v >_{V'\times V} = (A(u), v)_H \iff (f, v)_H = (A(u), v)_H \quad \forall v \in V$$

which, since V is assumed dense in H, gives $Au = f$. Also, from Cauchy-Schwartz inequality and the coercivity inequality we get

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_H$$

which implies that the operator A is invertible and its inverse A^{-1} is continuous. Hence, every bounded subset B of H is mapped to $A^{-1}(B)$ a bounded subset of V . Now since, V is compactly embedded in H, we have that $A^{-1}(B)$ is a relatively compact subset of H. Therefore, A^{-1} is a compact operator.

1.0.3 L^p-spaces for real valued functions

We assume that the open subset Ω is equipped with the Lebesgue measure that we denote by dx. For $1 \leq p < \infty$ we denote the space of classes of functions $L^p(\Omega)$ by

$$L^p(\Omega) := \{\text{classes of measurable functions } u : \Omega \longrightarrow \mathbb{R} \text{ such that } \int_\Omega |u(x)|^p \, dx < \infty\}$$

equipped with the norm

$$\|u\|_p := \Big(\int_\Omega |u(x)|^p \; dx \Big)^{\frac{1}{p}}.$$

We say that a measurable function $u : \Omega \longrightarrow \mathbb{R}$ is essentially bounded if there exists $\alpha > 0$ such that $|u(x)| \leq \alpha$ for a.e. $x \in \Omega$ and the space $L^\infty(\Omega)$ is defined as

$$L^\infty(\Omega) := \{\text{classes of measurable functions } u : \Omega \longrightarrow \mathbb{R} \text{ essentially bounded}\}$$

equipped with the norm

$$\|u\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| =: \inf\{\alpha \geq: \; |u(x)| \leq \alpha \text{ for a.e. } x \in \Omega\}$$

Also, the space $L^p_{\text{loc}}(\Omega)$ is defined by

$$L^p_{\text{loc}}(\Omega) := \{\text{classes } u : \Omega \longrightarrow \mathbb{R} \text{ such that } u1_K \in L^p(\Omega) \forall K \subset \Omega \text{ compact}\}$$

where 1_K denotes the characteristic function of the set K.

We recall that $(L^p(\Omega), \| \cdot \|_p)$ $(1 \leq p \leq \infty)$ is a Banach space and $L^2(\Omega)$ is a Hilbert space equipped with the scalar product

$$(u, v) := \int_\Omega u(x)v(x) \; dx.$$

1.0.4 Sobolev spaces for real valued functions

Sobolev spaces are natural generalizations of the Lebesgue spaces L^p. Simply stated, if $\Omega \subseteq \mathbb{R}^N$ is an open set, $1 \leq p \leq \infty$ and k is a positive integer, the Sobolev space $W^{k,p}(\Omega)$ consists of functions (or equivalence classes of functions) in $L^p(\Omega)$ whose partial derivatives up to order k are in $L^p(\Omega)$. We will start by giving a more precise definition of what we mean by 'partial derivatives', since these are not defined in the classical sense. Throughout these section, dx refers to Lebesgue measure on $\mathbb{R}^N$ and we assume that all functions are real-valued. We will also identify elements of L^p with functions. Before discussing weak derivative, we introduce some notation. Let $C_c(\Omega)$ denote the class of continuous real-valued functions on Ω with compact support and denote $C_c^\infty(\Omega)$ as the subset of smooth functions. we also, use the definition of $L^p_{\text{loc}}(\Omega)$ in above section and note that $L^p_{\text{loc}}(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$ whenever $1 \leq p \leq \infty$: If u and φ are C^1 functions on Ω and φ has compact support, then

$$\int_\Omega \varphi \partial_{x_k} u \; dx = -\int_\Omega u \partial_{x_k} \varphi \; dx$$

This can e.g. be proved using Fubini's theorem and the usual integration by parts formula in one dimension. We take this as our definition of weak derivatives.

Definition 1.0.3. *Let* $u \in L^1_{\text{loc}}(\Omega)$. *We say that f is weakly partially differentiable with respect to* x_k *if there exists a function* $v \in L^1_{\text{loc}}(\Omega)$ *such that*

$$\int_\Omega u \partial_{x_k} \varphi \; dx = -\int_\Omega v \varphi \; dx \quad \text{for every} \quad \varphi \in C_c^\infty(\Omega).$$

If u is weakly partially differentiable with respect to x_k, we call v the weak partial derivative of u with respect to x_k and denote it by $\partial_{x_k} u$. More generally, if α is a multi-index, we say that $v \in L^1_{\text{loc}}(\Omega)$ is the α^{th} weak derivative of u if

$$\int_\Omega u D^\alpha \varphi \; dx = (-1)^{|\alpha|} \int_\Omega v \varphi \; dx \quad \text{for every} \quad \varphi \in C_c^\infty(\Omega).$$

where $D^\alpha = \frac{\partial^{\alpha_1}}{\partial^{\alpha_1}_{x_1}} \cdots \frac{\partial^{\alpha_n}}{\partial^{\alpha_n}_{x_n}} \varphi$. For the above definition to make sense then the weak derivative v must be unique up to a set of measure zero. This is a consequence of the following:

Lemma 1.0.1. (Uniqueness of weak derivative) *Let* $v, \bar{v} \in L^1_{\text{loc}}(\Omega)$ *be such that*

$$\int_\Omega u D^\alpha \varphi \; dx = (-1)^{|\alpha|} \int_\Omega \bar{v} \varphi \; dx = (-1)^{|\alpha|} \int_\Omega v \varphi \; dx \quad \textit{for every} \quad \varphi \in C_c^\infty(\Omega).$$

Then

$$\int_\Omega (v-\bar{v})\varphi\, dx = 0 \quad \text{for every} \ \ \varphi \in C_c^\infty(\Omega)$$

whence $v=\bar{v}$ a.e.

We remark that a weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero. Note that, if u is k-times continuously differentiable on Ω then, for each α with $|\alpha| \le k$, the classical partial derivative $D^\alpha u$ is also the α^{th} weak derivative of u. For this reason, we use the notation $D^\alpha u$ also for the weak derivative of u

The Sobolev space are indispensable tools in the study of boundary value problems. We are now going to define the Sobolev space.

Definition 1.0.4. *Let k be be a non-negative integer, $k \in \mathbb{N}$ and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(\Omega)$ is defined as the set of all functions $u \in L^1_{\text{loc}}(\Omega)$ such that for each multi-index α with $|\alpha| \le k$ the α^{th}-partial derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$.*

The norm in the space $W^{k,p}(\Omega)$ is defined as

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \Big(\displaystyle\sum_{|\alpha|\le k} \int_\Omega \big|D^\alpha u\big|^p dx\Big)^{\frac{1}{p}} & 1 \le p < \infty \\ \displaystyle\sum_{|\alpha|\le k} \operatorname*{esssup}_\Omega \big|D^\alpha u\big| & p=\infty \end{cases}$$

We note $W^{0,p}(\Omega) = L^p(\Omega)$ and when $p=2$, we write $W^{k,2}(\Omega) := H^k(\Omega)$. It is not difficult to see that $W^{k,p}(\Omega)$ is a normed space. Moreover, we have the following results, which summarize the basic properties of Sobolev spaces.

Theorem 1.0.4. *Let be Ω an open bounded set in $\mathbb{R}^N$, and $1 \le p \le \infty$. Then*
i) $W^{k,p}(\Omega)$ is a Banach space,
ii) $W^{k,p}(\Omega)$ is reflexive if $1 < p < \infty$,
iii) $W^{k,p}(\Omega)$ is separable if $1 \le p < \infty$.

Corollary 1.0.1. *The Sobolev space $H^k(\Omega)$ is a separable Hilbert space with the inner product*

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha|\le k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx \quad \forall u,v \in H^k(\Omega).$$

The closure of the space $C_c\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}}$ gives a closed subspace of $W^{k,p}(\Omega)$, denoted $W_0^{k,p}(\Omega)$. When $p=2$, we use the notation $H_0^k(\Omega)\equiv W_0^{k,2}(\Omega)$. It follows from the above theorem and corollary that $W_0^{k,p}(\Omega)$ is a Banach space and $H_0^k(\Omega)$ the Hilbert space. It can be shown that the seminorm $|u|_{W^{k,p}}=\sum_{|\alpha|=k}\|D^\alpha u\|_{L^2}$ is the norm on $W_0^{k,p}(\Omega)$ and there exists a positive constant c_Ω such that

$$|u|_{W^{k,p}}\leq\|u\|_{W^{k,p}}\leq c_\Omega|u|_{W^{k,p}}$$

Theorem 1.0.5. *Assume that Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator $T: W^{1,p}(\Omega)\longrightarrow L^p(\partial\Omega)$, called the trace operator, such that $Tu=u|_{\partial\Omega}$ if $u\in W^{1,p}(\Omega)\cap C^0(\bar{\Omega})$ and $\|Tu\|_{L^p(\partial\Omega)}\leq C\|u\|_{W^{1,p}(\Omega)}$ for each $u\in W^{1,p}(\Omega)$ with the constant C that depends on p and Ω only.*

We state that Tu is the trace of u on $\partial\Omega$. Now, we need to examine what it means for a function to have zero trace.

Theorem 1.0.6. *Assume that Ω is bounded and $\partial\Omega$, and $u\in W^{1,p}(\Omega)$. Then $u\in W_0^{1,p}(\Omega)$ if and only if $Tu=0$ on $\partial\Omega$)*

Theorem 1.0.7. (Gagliardo-Nirenberg-Sobolev inequality.) *Let $1\leq p<N$. Then there exists a constant C, depending only on p, N and Ω, such that*

$$\|u\|_{L^{p^*}(\Omega)}\leq C\|Du\|_{L^p(\Omega)}\quad \textit{where}\ \ p^*=\frac{Np}{N-p}$$

forall $u\in C_c^1(\Omega)$.

The idea is to show that if $u\in W^{k,p}(\Omega)$ then, in addition to $u\in L^p(\Omega)$, $W^{k,p}(\Omega)$ is also embedding into other L^q-spaces.

Theorem 1.0.8. *Let $\Omega\subset\mathbb{R}^N$ be bounded domain with smooth boundary and $1\leq p<N$. Then $W^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$ and there exists a constant $C>0$, dependent only on p, N and Ω, such that*

$$\|u\|_{L^{p^*}(\Omega)}\leq C\|u\|_{W^{1,p}(\Omega)}\quad \textit{for every}\ \ u\in W^{1,p}(\Omega)$$

Theorem 1.0.9. (Rellich-Kondrachov Compactness Theorem) *Assume Ω is a bounded open subset of $\mathbb{R}^N$, $\partial\Omega$ is C^1 and $1\leq p<N$. Then $W^{1,p}(\Omega)\subset L^q(\Omega)$ for every $1\leq q<p^*$.*

Remark 1.0.2. *It follows that $W_0^{1,p}(\Omega)$ is also compactly embedded into $L^q(\Omega)$ for every $1\leq q<p^*$. The embedding is not compact when $q=p^*$ (for $N\geq 3$)*

Theorem 1.0.10. (Poincare's Inequality) *Let Ω be a bounded, connected, open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ is C^1. Asumme $1 \leq p \leq \infty$ then there exist a constant C, depending only on p, N and Ω, such that*

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \quad \textit{where} \;\; \bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)\, dx$$

for each $u \in W^{1,p}(\Omega)$.

1.0.5 Functions taking values in a Banach space

Bochner spaces

In this section, we deal with functions defined on a bounded interval on $(0,T)$ with $0 < T \leq \infty$ and taking values in a Banach space $(V, \|\cdot\|_V)$ with $< \cdot, \cdot >_{V' \times V}$ denoting the duality brackets.

Definition 1.0.5. *i) A function $u : (0,T) \longrightarrow V$ is called simple if it takes only a finite number of values $v_i \in V$ and $E_i = u^{-1}(v_i)$ is Lebesgue measurable and in that case, the integral of the simple function u is defined as*

$$\int_0^T u(t)dt = \sum_{\text{finite}} meas(E_i)v_i$$

ii) We say that $u : (0,T) \longrightarrow V$ is strongly measurable if u is a pointwise limit (in the norm on V) of a sequence $\{u_n\}_n$ of simple functions, i.e.,
$\|u_n(t) - u(t)\|_V \to 0$ for a.e. $t \in (0,T)$
iii) We say that $u : (0,T) \longrightarrow V$ is weakly measurable if

$$t \in (0,T) \longrightarrow < f, u(t) > V \times V'$$

is measurable forall $f \in V'$.

We recall the following theorem by Pettis [9]

Theorem 1.0.11. *If the Banach space $(V, \|\|_V)$ is separable, then u is strongly measurable if u is weakly measurable*

Definition 1.0.6. *(Bochner Integral) A function $u : (0,T) \longrightarrow V$ is Bochner integrable if there exists a sequence of simple functions $u_n u : (0,T) \longrightarrow V$ such that the following two conditions are met:*

i) $\lim\limits_{n\to\infty} \|u_n(t) - u(t)\| \to 0$ *for a.e.* $t \in (0, T)$

ii) $\lim\limits_{n\to\infty} \displaystyle\int_0^T \|u_n(t) - u(t)\|\ dt = 0.$

If u *is Bochner integrable, then the limit*

$$\int_0^T u(t)dt = \lim_{n\to\infty} \int_0^T u_n(t)dt$$

called the Bochner integral of u*, exists in* V *and is independent of the sequence* $\{u_n\}$*.*

Theorem 1.0.12. *Let* $u : (0, T) \longrightarrow V$ *be Bochner integrable function. Then*

i) $\Big\| \displaystyle\int_0^T u(t)dt \Big\|_V \leq \displaystyle\int_0^T \|u(t)\|_V dt$

ii) Let $(W, \|\|_W)$ *another Banach space and* $L : V \longrightarrow W$ *be a linear continuous mapping, then the function* $Lu : (0, T) \longrightarrow Lu(t) \in W$ *is also Bochner inegrable and*

$$L\Big(\int_0^T u(t)dt\Big) = \int_0^T Lu(t)dt$$

In particular, for every $f \in V'$ *the function* $< f, u(t) >_{V\times V'}$ *is integrable and*

$$\Big\langle f, \int_0^T u(t)dt \Big\rangle_{V\times V'} = \int_0^T < f, u(t) >_{V\times V'} dt$$

The following theorem by Bochner gives a characterization of Bochner integrability of a function $u : (0, T) \longrightarrow V$ in terms of the summability of the real valued function $t \in (0, T) \longrightarrow \|u(t)\|_V$.

Theorem 1.0.13. (Bochner)) *A function* $u : (0, T) \longrightarrow V$ *is Bochner integrable if and only if it is strongly measurable and* $\int_0^T \|u(t)\|dt < \infty$*.*

Remark 1.0.3. *As in the case of the Lebesgue integral for real valued functions, we also regard here functions that are equal a.e. as equivalent and identified them. They have the same Bochner integral.*

The dominated convergence theorem also holds for Bochner inegrals.

Theorem 1.0.14. *Let* $u_n : (0, T) \longrightarrow V$ *be a sequence of Bochner integrable functions such that* $\lim_{n\to\infty} \|u_n(t) - u(t)\|_V = 0$ *for a.e.* $t \in (0, T)$ *and there exists a summable function* $g : (0, T) \longrightarrow \mathbb{R}$ *such that*

$$\|u_n(t)\|_V \leq g(t) \quad \text{for a.e. } t \in (0, T) \text{ and every } n \in \mathbb{N}$$

Then $u : (0, T) \longrightarrow V$ is Bochner integrabe and

$$\int_0^T u(t)dt = \lim_{n\to\infty} \int_0^T u_n(t)dt \quad \textit{and} \quad \lim_{n\to\infty} \int_0^T \|u_n(t) - u(t)\|_V dt = 0$$

We now introduce the L^p-spaces of V-valued functions that are used for the study of evolution problems.

Definition 1.0.7. *Let $1 \leq p < \infty$, the spaces $L^p(0, T, V)$ consists of all classes of strongly measurables functions $u : (0, T) \longrightarrow V$ such that $\int_0^T \|u(t)\|_V^p dt < \infty$ and is equipped with the norm*

$$\|u\|_{L^p(0,T,V)} = \Big(\int_0^T \|u(t)\|_V^p dt \Big)^{\frac{1}{p}} \quad \forall u \in L^p(0, T, V)$$

We collect in the next theorems some key properties of the L^p-spaces of V-valued functions

Theorem 1.0.15. *i) If $(V, \|\|_V)$ is Banach space then the normed space $L^p(0, T, V)$ equipped with the norm $\| \cdot \|_{L^p(0,T,V)}$ is also a Banach space for all $p \in [1, +\infty]$*
ii) The vector subspace of simple functions

$$S := \{u = \sum_{finite} v_i 1_{E_i}, v_i \in V \text{ and } E_i \text{ measurable subsets of } (0, T)\}$$

is dense in $(L^p(0, T, V)$. Moreover, by mollification with respect to t also the vector subspace

$$S_c := \{u = \sum_{finite} v_i\varphi_i, v_i \in V \text{ and } \varphi \in C_c^\infty(0, T)\}$$

is dense in $(L^p(0, T, V)$.
iii) Let $p \in [1, +\infty[$ and $(V, \|\|_V)$ be reflexive Banach space with dual space V', then the dual space of $L^p(0, T, V)$ is isomorphe to $L^{p'}(0, T, V')$ for $1/p+1/p' = 1$ and the and the duality brackets $< \cdot, \cdot >$ between $L^{p'}(0, T, V')$ and $L^p(0, T, V)$ is defined by

$$< f, u >:= \int_0^T < f(t), u(t) >_{V' \times V} \quad \forall f \in L^{p'}(0, T, V') \textit{ and } \forall u \in L^p(0, T, V).$$

We are now in a position to introduce Sobolev spaces of V -valued functions.

1.0.6 Sobolev spaces of V-valued functions

We will first start as in the case of real valued functions, with the concept of weak-differentiability. Since we deal with V -valued functions with V having the strong topology (induced by the norm $\|\cdot\|_V$ and some weak topology), the notion of weak differentiability in this case might be confusing.

Definition 1.0.8. *(Strong continuity - strong differentiability) We say that $u:(0,T)\longrightarrow V$ is is strongly continuous at $t\in(0,T)$ if*

$$\lim_{s\to t} u(s)=u(t) \quad \textit{in } (V,\|\cdot\|_V)$$

and, we say that it is strong differentiable at $t\in(0,T)$ if

$$\lim_{s\to t}\frac{u(s)-u(t)}{s-t} \textit{ exist in } \ (V,\|\cdot\|_V)$$

and the vector

$$u'(t):=\lim_{s\to t}\frac{u(s)-u(t)}{s-t}$$

is called the strong derivative of u at $t\in(0,T)$.

Moreover, a function $u:(0,T)\longrightarrow V$ is continuously differentiable in $(0,T)$ if u is strongly differentiable at every $t\in(0,T)$ and its strong derivative $u':(0,T)\longrightarrow V$ is a strongly continuous.

As in the case of real valued functions, the above-mentioned notion of strong differentiability excludes classes of functions which naturally arise from the study of various physical phenomena. Therefore, the notion of distributional or weak derivative has been introduced in order to accommodate such

Definition 1.0.9. *(Weak differentiability of V-valued functions) A function $u\in L^1_{\text{loc}}(0,T,V)$ is weakly differentiable with weak derivative $v=D_tu\in L^1_{\text{loc}}(0,T,V)$ if the following identity holds*

$$\int_0^T u(t)\varphi'(t)dt=-\int_0^T v(t)\varphi(t)dt \quad \forall\varphi\in C_c^\infty(0,T) \tag{1.2}$$

where the integrals in (1.2) are Bochner integrals.
In general, for every integer $j\geq 1$, if there exists a function $v_j\in L^1_{\text{loc}}(0,T,V)$ such that

$$\int_0^T u(t)\varphi^{(j)}(t)dt=(-1)^j\int_0^T v_j(t)\varphi(t)dt \quad \forall\varphi\in C_c^\infty(0,T)$$

Then, we say that the function v_j *is the weak derivative of the function* u *of order* j *and is written* $v_j = D_t^{(j)}u$.

Theorem 1.0.16. (Lebesgue differentiation) *Let* $(V, \|\cdot\|_V)$ *be a Banach space and* $u \in L^1(0,T,V)$. *Then*

$$u(t) = \lim_{h\to 0}\int_t^{t+h} u(s)ds \quad \text{for a.e. } t \in (0,T)$$

Corollary 1.0.2. *Let* $u \in L^1_{\text{loc}}(0,T,V)$ *be such that*

$$\int_0^T u(t)\varphi(t)\ dt = 0 \quad \forall \varphi \in C_c^\infty(0,T)$$

then $u(t) = 0$ *for* a.e. $t \in (0,T)$.

Corollary 1.0.3. *If the function* $u : (0,T) \longrightarrow V$ *is differentiable and* $D_t u = 0$ *then* u *is constant a.e. in* $(0,T)$.

As a consequence of the Lebesgue differentiation theorem 1.0.16 and the corollaries 1.0.2 and 1.0.3 we have the characterization of a weakly differentiable function as the Bochner integral of a Bochner integrable function.

Theorem 1.0.17. *Let* $(V, \|\cdot\|_V)$ *be a Banach space and* $u \in L^1(0,T,V)$ *Then* u *is weakly differentiable with Bochner integrable derivative* $D_t u = v \in L^1(0,T,V)$ *if and only if*

$$u(t) = v_0 + \int_0^t v(s)ds$$

In that case, the function u *is strongly differentiable a.e. and its strong derivative coincides with its weak derivative.*

We also have in the following theorem, a characterization of weak derivative of a V-valued function in terms of weak derivatives of the real-valued functions obtained by duality.

Theorem 1.0.18. *Let* $(V, \|\cdot\|_V)$ *be a Banach space with dual space* V'. *The function* $u \in L^1(0,T,V)$ *is weakly differentiable with weak derivative* $D_t u = v \in L^1(0,T,V)$ *if and only if for every* $f \in V'$, *the function* $< f, u >: (0,T) \longrightarrow \mathbb{R}$ *is weakly differentiable with weak derivative*

$$\frac{d}{dt} < f, u > = < f, v >$$

that is

$$\int_0^T < f, u(t) > \varphi'(t)dt = -\int_0^T < f, v(t) > \varphi(t)dt \quad \forall \varphi \in C_c^\infty(0,T)$$

We are now in a position to define Sobolev spaces of V -valued functions also called *Bochner Sobolev spaces.*

Definition 1.0.10. *Let $(V, \|\cdot\|_V)$ be a Banach space, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. The V-valued Sobolev space $W^{m,p}(0,T,V)$ consists of all (equivalent classes of) strongly measurable functions $u : (0,T) \longrightarrow V$ whose weak derivatives $D_t^{(j)}u$ of order $0 \leq j \leq m$ belong to belong to $L^p(0,T,V)$. For $1 \leq p < \infty$, the space $W^{m,p}(0,T,V)$ is a Banach space when equipped with the norm*

$$\|u\|_{W^{m,p}(0,T,V)} := \Big(\sum_{j=0}^m \int_0^T \|D_t^{(j)}u\|_V^p dt\Big)^{\frac{1}{p}} = \Big(\sum_{j=0}^m \|D_t^{(j)}u\|_{L^p(0,T,V)}^p dt\Big)^{\frac{1}{p}}.$$

For $p = +\infty$, the space $W^{m,\infty}(0,T,V)$ is a Banach space when equipped with the norm

$$\|u\|_{W^{m,\infty}(0,T,V)} := \sup_{1 \leq j \leq m} \|D_t^{(j)}u\|_{L^\infty(0,T,V)}.$$

When $p = 2$ and $V = H$ a Hilbert space with scalar product $(\cdot,\cdot)_H$ the space $W^{m,2}(0,T,V) = \mathrm{H}^{m,2}(0,T,H)$ is a Hilbert space with scalar product defined by

$$(u,v)_{\mathrm{H}^{m,2}(0,T,H)} = \sum_{j=0}^m \int_0^T (D_t^{(j)}u, D_t^{(j)}v)_H dt \quad \forall u, v \in \mathrm{H}^{m,2}(0,T,H).$$

We also have the following embedding theorem known for real valued Sobolev spaces.

Theorem 1.0.19. *Let $1 \leq p < \infty$. Then $W^{1,p}(0,T,V) \hookrightarrow C^0(0,T,V)$ (The space $W^{1,p}(0,T,V)$ is continuously embedded in $C^0(0,T,V)$)), i.e. there exists a constant $C = C(p,V) > 0$ such that*

$$\sup_{0 \leq t \leq T} \|u(t)\|_V \leq C\|u\|_{W^{1,p}(0,T,V)} \quad \forall u \in W^{1,p}(0,T,V).$$

Theorem 1.0.20. (Aubin-Lions) *Let X, V, Y be Banach spaces with the embeddings: $X \hookrightarrow V$ compact and $V \hookrightarrow Y$ continuous. For $1 \leq p, q \leq \infty$ let*

$$W := \{u \in L^p(0,T,X) : \ u' \in L^p(0,T,Y)\}.$$

i) If $p < \infty$ then the embedding of W into $L^p(0,T,X)$ is compact
ii) If $p = +\infty$ and $q > 1$, then the embedding of W into $L^p(0,T,X)$ is compact.

Theorem 1.0.21. (Ascoli-Arzelà theorem) *Let* (X, d_X) *and* (Y, d_Y) *be two metric spaces with* (X, d_X) *compact and* (Y, d_Y) *complete. Let* $\mathcal{F}$ *be a family of continuous functions from* X *into* Y *. Then* $\mathcal{F}$ *is relatively compact in* $C(X, Y)$ *if and only if*
(i) $\mathcal{F}$ *is an equicontinuous family,*
(ii) for every $x \in X$*, the set* $\mathcal{F}(x) := \{f(x) : f \in \mathcal{F}\}$ *is relatively compact in* Y*.*

1.0.7 The Galerkin approximation method

Several methods for studying nonlinear partial differential equations were developed with the years. We refer the interested reader for instance to the master piece by J.L. Lions [29] in which a large collection of different methods allowing to study various examples coming from applications has been presented. We explain in this section how the Galerkin approximation method works through a simple example.

Let V and H be two Hilbert space with $V \subset H$, V dense in H and continuously embedded in H. We will identify H to its dual and have

$$V \subset H \subset V' \quad \text{with continuous embeddings.}$$

Let $A \in \mathcal{L}(V, V')$. We consider the following linear problem

$$\begin{cases} \dfrac{du}{dt} + Au = f \quad \text{in } (0, T) \\ u(0) = u_0 \end{cases} \tag{1.4}$$

For instance, the case $A = -\Delta$ with Dirichlet boundary conditions corresponds to the spaces $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$). We assume that $u_0 \in H$ and $f \in L^2(0, T, V')$ and thus the problem

$$\begin{cases} D_t u + Au = f \quad \text{in } (0, T) \\ u(0) = u_0 \end{cases} \tag{1.5}$$

where $D_t u$ is the weak derivative of u in the sense of Definition 1.0.18. We are looking for $u \in L^2(0, T, V) \cap C^0(0, T, H)$ such that $D_t u \in L^2(0, T, V')$.

Theorem 1.0.22. *Under the assumptions given above, the problem (1.5) has a unique solution*

$$u \in L^2(0, T, V) \cap C^0(0, T, H), \quad D_t u \in L^2(0, T, V').$$

Proof. Assuming that V is separable, self-adjoint and positive i.e.

$$< Av, v >_{V'\times V} \geq \alpha \|v\|_V^2$$

with some positive constant α. let $\{w_m\}_{n\geq 1}$ be a basis in V such that span$\{w_n : n \in N\}$ is dense in V . For each $m \geq 1$ integer, define the approximate solution u_m of (1.5):

$$(1.6) \qquad \begin{cases} u_m(t) = \displaystyle\sum_{i=1}^{m} g_{i,m}(t) w_i \\ \dfrac{d}{dt}(u_m, w_i) + < Au_m, w_i > = < f, w_i >, \quad i = 1, \cdots, m \\ u_m(0) = u_{0m} \end{cases}$$

where u_{0m} is the projection in H of u_0 onto the subspace span$\{w_1, \cdots, w_m\}$. The functions w_n $(n \geq 1)$ are often chosen as orthonormal eigenfunctions of the operator A:

$$Aw_j = \lambda_j w_j \quad j \geq 1.$$

Notice that such a basis in H exists provided that A is self-adjoint i.e., $(Au, v)_H = (u, Av)_H$ for every $u, v \in V$ and the embedding $V \hookrightarrow H$ is compact.

In the case where $\{w_n\}$ is an orthonormal system of H, the equation (1.6) then becomes

$$(1.7) \quad \begin{cases} u_m(t) = \displaystyle\sum_{i=1}^{m} g_{i,m}(t) w_i \\ \dfrac{d}{dt} g_{i,m}(t)(w_i, w_i) + \lambda_i g_{i,m}(t) < w_i, w_i > = < f, w_i >, \quad i = 1, \cdots, m \\ g_{im}(0) = (u_0, w_i) \end{cases}$$

(1.7) is a system of linear differential equation and hence has a unique global solution $g_m = (g_{1,m}, \cdots, g_{m,m}) \in C(0, T, \mathbb{R}^m)$. Multiply the equation (1.6) by $g_{im}(t)$ and adding up $i = 1, \cdots, m$ we get

$$\frac{d}{dt}\|u_m\|_H^2 + \alpha \|u_m\|_V^2 \leq \frac{1}{\alpha}\|f\|_{V'}^2$$

and, integrating over $(0, t)$, we obtain

$$\|u_m(t)\|_H^2 + \alpha \int_0^t \|u_m(s)\|_V^2 ds \leq \|u_0\|_H^2 + \frac{1}{\alpha}\int_0^t \|f(s)\|_{V'}^2 ds.$$

Hence $\{u_m\}$ is bounded in $L^2(0,T,V)\cap L^\infty(0,T,H)$, we can so extract from $\{u_m\}$ a subsequence $\{u_{k_m}\}$ such that $u_{k_m}\to u$ weakly in $L^2(0,T,V)$ and weakly* in $L^\infty(0,T,H)$

$$\frac{d}{dt}(u,w_i)+<Au,w_i>=<f,w_i>,\quad i=1,\cdots,m.$$

Hence, since $\{w_i\}$ is a basis in V , we get

$$\frac{d}{dt}(u,v)+<Au,v>=<f,v>,\quad \forall v\in V.$$

Using Theorem 1.0.18, we find that $\frac{d}{dt}(u,v)=(D_tu,v)$ and hence

$$D_tu+Au=f$$

Now, since $f\in L^2(0,T,V')$ and $Au\in L^2(0,T,V')$ we get that $D_tu\in L^2(0,T,V')$. Since $L^2(0,T,V')\subset L^1(0,T,V')$, it follows from Theorem 1.0.19 that u coincides a.e. with a continuous function form $[0,T]$ to V'. Hence, we will also pass to the limit in the initial condition to get that $u(0)=u_0$.

Remark Notice that when the mapping A is not linear, the only estimates on u_m are not sufficient to pass to the limit in nonlinear terms in (1.6). We need a strong convergence which will be obtained thanks to a compactness theorem like for example the Aubin-Lions compactness theorem 1.0.20 and the Ascoli-Arzelà compactness theorem 1.0.20 or others which will allow us to pass to the limit in the linear terms.

1.0.8 Regularity of Weak solutions

Let $\Omega\subset\mathbb{R}^N$ be a bounded open set and $u\in H_0^1(\Omega)$ be a weak solution of the elliptic equation

$$Lu=f\quad\text{dans }\Omega \tag{1}$$

where L is the elliptic operator in divergence form

$$Lu=-\sum_{i,j=1}^N\partial_{x_j}(a_{i,j}(x)\partial_{x_i}u)+\sum_{i=1}^N b_i(x)\partial_{x_i}u+c(x)u$$

where $a(x)=(a_{i,j}(x))$ is a uniformly elliptic symmetric matrix, i.e.

$$\exists\alpha>0,\ \forall\xi\in\mathbb{R}^N\text{ tel que p.p }x\in\mathbb{R}^N,\ a(x)\xi\cdot\xi\geq\alpha|\xi|^2$$

When solving a partial differential equation by the method variational, as a general rule we do it by changing the notion of solution somewhat. While sometimes in applications we need to find classic solutions (for example of class C^2), the variational method only allows, a priori, to find weak solutions in the sense that the equation is only verified in $\mathcal{D}'(\Omega)$. For example if we consider the Dirichlet problem (1) in a bounded and regular open Ω, even if the data of the second member $f \in C^\infty(\bar{\Omega})$, the application of the Lax-Milgram theorem only gives us a weak solution in $H_0^1(\Omega)$ while the use of other techniques, certainly more complicated but finer, would make it possible to obtain a solution $u \in C^\infty(\bar{\Omega})$. We therefore see that it is essential to ask the question of the regularity of the solution u of the problem, more precisely we will ask the question of knowing under which hypotheses the weak solution thus found is, for example a classical solution. This will not always be true and we will see that it depends in particular on the open Ω on which we work.

Let us note that there are several notions of regularity (which partly overlap). First there is regularity in the classical sense of spaces C^k or Holder spaces $C^{k,\alpha}$. But there is also regularity in the sense of Sobolev spaces (the more the function u is in a Sobolev space with a high exponent, the more regular it is). We can move from this last notion to the previous one by using Sobolev injections. In general, regularity results are proven by distinguishing the regularity inside the open set from that on the boundary of the open set. Here we will simply give the general results without proofs. The proofs are indeed very technical and long. The principle is: we start by studying the case $\Omega = \mathbb{R}^N_+$ then we deduce the general case using local maps.

Theorem 1.0.23. (Regularity for the Dirichlet problem) *Let Ω be a bounded open set of class C^2 and f be a function of $L^2(\Omega)$. Then the solution to the problem*

$$\begin{cases} Lu = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

is in the Sobolev space $H^2(\Omega)$ and moreover, for a constant C that depends only on Ω we have

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

Remark: If f is more regular, and we want to deduce more regularity for u, we must assume that the open Ω is also more regular. For example, if we suppose $f \in H^m(\Omega)$ and Ω of class C^{m+2} then $u \in H^{m+2}(\Omega)$ with an estimate of the norm of u by the norm of f.

Lemma 1.0.2. (Gronwall lemma) *Let f, g and y be three continuous functions on the interval $[a,b]$ with positive values, and verifying the following inequality: for all $t \in [a,b]$,*

$$y(t) \leq f(t) + \int_a^t g(s)y(s)\ ds$$

Then , for all t of $[a,b]$, we have

$$y(t) \leq f(t) + \int_a^t f(s)g(s)\exp\Big(\int_s^t g(\tau)\ d\tau\Big)\ ds.$$

1.0.9 Derivation of Kirchhof Equation

We are now in a convenient space to derive the Gustav Kirchhof equation. This is done to give the reader some basic concept of the Kirchhof equation. Let τ_0 be the initial tension of the string at the rest position $[\alpha_0,\beta_0]$, $\dot{\tau}(t)$ be the tension of the string in the position $[\alpha(t),\beta(t)]$ which is the deformation of $[\alpha_0,\beta_0]$ and $\tau(t)$ the tension of the curve deformation $u(x,t)$ of $[\alpha(t),\beta_(t)]$. The tension at each point of the curve $u(x,t)$ is a vector that has the direction of the tangent vector of this curve at this point and has modulus $\tau(t)$. Thus its vertical component is

$$\tau(t)\sin\vartheta$$

where ϑ is the angle of the direction 0x with the tangent vector. From the hypothesis of small deformations we don't consider the horizontal component of the tension. We have $\sin\vartheta \approx \text{tg}\ \vartheta = \frac{\partial u}{\partial x}$. Thus the vertical component $\overrightarrow{\tau(t)}$ is

$$\tau(t)\sin\vartheta = \tau(t)\text{tg}\ \vartheta = \tau(t)\frac{\partial u}{\partial x}.$$

The variation of the tension generates a force and by Newton's second Law, we obtain

$$\frac{\partial}{\partial x}(\tau(t)\sin\vartheta) = m\frac{\partial^2 u}{\partial t^2}$$

where $\frac{\partial^2 u}{\partial t^2}$ is the acceleration of the deformation $u(x,t)$ and m is the mass of the string. We have

$$m\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{1.1}$$

Calculation of the Tension $\tau(t)$: We calculate the tension $\tau(t)$ in function of $\alpha(t)$, $\beta(t)$ and $u(xt)$. In fact, by Hooke's Law, we have

$$\dot{\tau}(t) = \tau_0 + k\frac{\gamma(t)-\gamma_0}{\gamma_0} \tag{1.2}$$

where $\gamma(t) = \beta(t) - \alpha(t)$ is the length of $[\alpha(t), \beta(t)]$ and $\gamma_0 = \gamma(0)$.

If we represent by $S(t)$ the length of the arc of the curve $u(x,t)$, deformation of $[\alpha(t), \beta(t)]$, we obtain:

$$S(t) =: \int_{\alpha(t)}^{\beta(t)} \sqrt{1 + \Big(\frac{\partial u}{\partial x}\Big)^2}\; dx \approx \int_{\alpha(t)}^{\beta(t)} \Big(\Big(1 + \frac{1}{2}\Big(\frac{\partial u}{\partial x}\Big)^2\Big)\; dx.$$

By hypothesis of small deformation $|\frac{\partial u}{\partial x}| \ll 1$. Thus

$$S(t) - \gamma(t) = \frac{1}{2} \int_{\alpha(t)}^{\beta(t)} \Big|\frac{\partial u}{\partial x}\Big|^2\; dx.$$

By Hooke's Law we obtain

$$\tau(t) - \hat{\tau}(t) = k \frac{S(t) - \gamma(t)}{\gamma(t)}$$

and therefore

$$\tau(t) - \hat{\tau}(t) = \frac{k}{2\gamma(t)} \int_{\alpha(t)}^{\beta(t)} \Big|\frac{\partial u}{\partial x}\Big|^2\; dx \tag{1.3}$$

From (1.2) and (1.3) we obtain the tension $\tau(t)$

$$\tau(t) = \tau_0 + k\frac{\gamma(t) - \gamma_0}{\gamma_0} + \frac{k}{2\gamma(t)} \int_{\alpha(t)}^{\beta(t)} \Big|\frac{\partial u}{\partial x}\Big|^2\; dx. \tag{1.4}$$

Substituting $\tau(t)$ given by (1.4) in (1.1) and dividing by m, we obtain

$$\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\tau_0}{m} + \frac{k}{m}\frac{\gamma(t) - \gamma_0}{\gamma_0} + \frac{k}{2m\gamma(t)} \int_{\alpha(t)}^{\beta(t)} \Big|\frac{\partial u}{\partial x}\Big|^2\; dx\Big) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.5}$$

This is a model for small vertical deformations $u(x,t)$ when the ends of the strings are not fixed, that is it has small displacements $\alpha(t) < \alpha_0$ and $\beta(t) > \beta_0$.

Remark If we suppose the extremes of the string fixed, that is $\alpha(t) = a$ and $\beta(t) = b$ for all $t > 0$, the model (1.5) reduces to

$$\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\tau_0}{m} + \frac{k}{2m(b-a)} \int_a^b \Big|\frac{\partial u}{\partial x}\Big|^2\; dx\Big) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.6}$$

This model was proposed by Kirchhof in 1883.

Chapter 2

Local solution

Our main focus in this thesis is on the existence of global solutions and their asymptotic behavior. Global existence is obtained using a continuation argument based on a local existence theorem and adequate a priori estimates of the local solution. This chapter is devoted to the existence and uniqueness of the local solution following the standard approach that combines the Galerkin approximation method with the a priori estimates and compactness arguments. We follow here the change of variable method described above. We will first study our problem in the cylinder $Q = \Omega\times]0,\infty[$. The domains Q and $\hat{Q}$ are related by the diffeomorphism $\tau : \hat{Q} \longrightarrow Q$ defined by

$$\tau(x,t) := (y,t) = (\frac{x}{\sigma(t)}, t) \quad \text{for} \quad (x,t) \in \hat{Q}. \tag{2.1}$$

Whose inverse $\tau^{-1} : Q \longrightarrow \hat{Q}$ is given by

$$\tau^{-1}(y,t) := (x,t) = (y\sigma(t), t). \tag{2.2}$$

If we set

$$v(y,t) := u \circ \tau^{-1}(y,t) = u(y\sigma(t), t), \tag{2.3}$$

then the initial boundary value problem (0.1)-(0.3) becomes

$$v_{tt} + \mu v_t - \frac{1}{\sigma^2}\tilde{M}\Big(\int_\Omega |\sigma^{\frac{n-2}{2}}\nabla v|^2 dy\Big)\Delta v = \tilde{F}(t,v)\,, \tag{2.4}$$

$$v|_{\partial\Omega} = 0, \qquad v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1, \tag{2.5}$$

where

$$\tilde{F}(t,v) := -\Big(\frac{\sigma'}{\sigma}\Big)^2 \sum_{i,j=1}^{n} \partial_{y_i}(y_i y_j \partial_{y_j} v) + a_1(t,y)\cdot\nabla v_t + a_2(t)\cdot\nabla v, \tag{2.6}$$

$$a_1(t,y) := 2\frac{\sigma'}{\sigma}y, \quad a_2(t,y) := \sigma^{-2}y(\sigma\sigma'' + \mu\sigma\sigma' + (n-1)\sigma'^2). \tag{2.7}$$

Remark 2.0.4. *Note that the initial data (v_0, v_1) is determined by the given couple (0.3) (u_0, u_1) and depends of course on the initial position $\sigma(0)$ and the initial velocity $\sigma'(0)$, thus (see (2.21)) on σ_0 and σ_1. But considering subsequent assumption (see (3.1)) on σ_0 and σ_1, the only dependency of (v_0, v_1) in terms of σ_0 is meaningful. To emphasize this dependency, when required it will be noted $(v^0_{\sigma_0}, v^1_{\sigma_0})$ instead of (v_0, v_1).*

Indeed, given (u_0, u_1), the couple of initial data (v_0, v_1) is determined using equations

$$x \in \Omega_0 = \sigma(0)\Omega, \quad u_0(x) = u(\sigma(0)y, 0) = v_0(y), \quad y \in \Omega \tag{2.8}$$

and (see (2.3) and (2.21))

$$u_1(x) = v_1(y) - \alpha\frac{\sigma_1}{\sigma_0}y\cdot\nabla v_0(y), \quad v_1(y) = v_t(y,t)|_{t=0} \tag{2.9}$$

We set

$$M(s) := \tilde{M}(s) - \frac{m_0}{2}, \tag{2.10}$$

$$a_{ij}(t,y) := \frac{m_0}{2\sigma^2}\delta_{ij} - \Big(\frac{\sigma'}{\sigma}\Big)^2 y_i y_j \quad \big(i,j = 1,n\big)\,. \tag{2.11}$$

According to (2.10) and (0.4), it follows that

$$M(\lambda) \geq \frac{m_0}{2}, \quad M \in C^2([0,\infty[), \quad M'(\lambda) \geq 0, \qquad \forall\lambda \in [0,\infty[\,. \tag{2.12}$$

Given (2.10)-(2.11), the problem (2.4) and (2.5) is rewritten as

$$v_{tt} + \mu v_t - \frac{1}{\sigma^2}M\Big(\int_\Omega |\sigma^{\frac{n-2}{2}}\nabla v|^2 dy\Big)\Delta v = F(t,v)\,, \tag{2.13}$$

$$v|_{\partial\Omega} = 0\,, \tag{2.14}$$

$$v|_{t=0} = v_0, \quad v_t|_{t=0} = v_1, \tag{2.15}$$

with

$$F(t,v) = A(t)v + a_1(t,y)\cdot\nabla v_t + a_2(t,y)\cdot\nabla v, \tag{2.16}$$

where

$$A(t) = \sum_{i,j=1}^{n} \partial_{y_i}(a_{ij}(t,y)\partial_{y_j} v).$$

We set

$$a(t,u,v) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(t,y)(\partial_{y_i} u)(\partial_{y_j} v) dy \tag{2.17}$$

$$a'(t,u,v) = \sum_{i,j=1}^{n} \int_{\Omega} a'_{ij}(t,y)(\partial_{y_i} u)(\partial_{y_j} v) dy. \tag{2.18}$$

To study (2.13)-(2.15) we need some hypotheses on the function σ. Let us first recall that the function σ is positive, increasing and unbounded from above. Moreover, we assume that

$$\sigma \in C^3([0,\infty[), \quad \sigma(0) > 0, \quad 0 \leq \sigma'(t) \leq \frac{1}{d}\sqrt{\frac{m_0}{2}} \quad \forall t > 0 \tag{2.19}$$

where $d = \mathrm{diam}(\Omega)$. The second condition (2.19) implies that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq 0 \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}. \tag{2.20}$$

In order to avoid tedious abstract computations, we work throughout the paper with a typical family of functions σ which satisfy (2.19), that is

$$\sigma(t) = (\sigma_0 + \sigma_1 t)^{\alpha}, \quad 0 < \alpha < \frac{1}{2}. \tag{2.21}$$

where σ_0 and σ_1 are positive constants chosen so that (2.19) is satisfied. Note that this assumption means that $\hat{Q}$ is increasing in the sense that, if $t > t'$ then Ω_t contains $\Omega_{t'}$.

2.1 Faedo-Galerkin approximations

The existence of a solution will be obtained by the constructive Faedo-Galerkin approximation process. Let us denote by A the operator

$$Aw = -\Delta w, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

It is well know that A is a positive self adjoint operator in the Hilbert space $L^2(\Omega)$ for which there exist sequences $\{w_m\}_{m\geq 1}$ and $\{\lambda_m\}_{n\geq 1}$ of eigenfunctions and eigenvalues. The sequence $\{w_m\}_{m\geq 1}$ is dense in $H_0^1(\Omega)$ and we suppose it is orthonormalized in $L^2(\Omega)$.

Let us denote by

$$v_{0,n} = \sum_{k=1}^{n}(v_0, w_k)w_k, \quad v_{1,n} = \sum_{k=1}^{n}(v_1, w_k)w_k.$$

If $(v_0, v_1) \in D(A) \times H_0^1(\Omega)$, we have

$$(v_{0,m}, v_{1,m}) \longrightarrow (v_0, v_1) \text{ strongly in } D(A) \times H_0^1(\Omega). \tag{2.22}$$

Finally by V_m we denote the space generated by $w_1, w_2, ..., w_m$. In this condition there exists a local solution on some interval $[0, T]$ of the form

$$v_m(t) = \sum_{j=1}^{m} g_{j,m}(t)w_j. \tag{2.23}$$

of the ordinary differential equation given by

$$\begin{aligned}(v''_m, w_j) + \mu(v'_m, w_j) + \frac{1}{\sigma^2}M\Big(\|\sigma^{\frac{n-2}{2}}\nabla v_n\|^2_{L^2}\Big)(\nabla v_m, \nabla w_j) + a(t, v_m, w_j)\\ = (a_1 \cdot \nabla v'_m, w_j) + (a_2 \cdot \nabla v_m, w_j)\end{aligned} \tag{2.24}$$

with the initial conditions

$$v_m(0) = v_{0,m}, \qquad v'_m(0) = v_{1,m} \tag{2.25}$$

for any $w_m \in V_m$ and

$$v_m(t) = \sum_{k=1}^{m} g_{k,m}(t)w_k. \tag{2.26}$$

2.2 Estimates of approximate solutions

The estimates corresponding to the approximate solutions v_m will be established in the following lemmas.

Lemma 2.2.1. *Let* $(v_0, v_1) \in H^2(\Omega) \times H^1(\Omega)$*. Then we have*

$$\|\sigma^{\frac{n-2}{2}} \nabla v_m\|_{L^\infty(0,T,L^2)} \leq C_\Omega(\|v_0\|_{H^1} + \|v_1\|_{L^2}) \tag{2.27}$$

$$\|\sigma(\sigma^{\frac{n-2}{2}} v_m)'\|_{L^\infty(0,T,L^2)} \leq C_\Omega(\|v_0\|_{H^1} + \|v_1\|_{L^2}) \tag{2.28}$$

$$\sup_{0\leq t\leq T} \left|M^{(i)}\big(\|\sigma^{\frac{n-2}{2}} \nabla v_m(t)\|_{L^2}^2\big)\right| \leq M_0, \quad i = 0, 1, 2 \tag{2.29}$$

where $M^{(i)}$ *is the ith derivative of* M *and* M_0 *is some constant which depend only on* v_0 *and* v_1*.*

Proof. We first multiply equation (2.24) with

$$\sigma^2 \sigma^{\frac{n-2}{2}} (\sigma^{\frac{n-2}{2}} v_m)'. \tag{2.30}$$

By integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt} E_{1,m}(t) + \mu\sigma^2 \|(\sigma^{\frac{n-2}{2}} v_m)'\|_{L^2}^2 = \sum_{k=1}^{5} I_k \tag{2.31}$$

where

$$E_{1,m}(t) := \sigma^2\|(\sigma^{\frac{n-2}{2}} v_m)'\|_{L^2}^2 + \sigma^2 a(t, \sigma^{\frac{n-2}{2}} v_m, \sigma^{\frac{n-2}{2}} v_m) + \hat{M}(\|\sigma^{\frac{n-2}{2}} \nabla v_m\|_{L^2}^2) \tag{2.32}$$

$$\hat{M}(\lambda) := \int_0^\lambda M(s)ds. \tag{2.33}$$

and

$$\begin{aligned}
I_{1,m} &:= (n-1)\frac{\sigma'}{\sigma}\sigma^2\|(\sigma^{\frac{n-2}{2}} v_m)'\|_{L^2}^2 - \frac{1}{2}\sigma^2 \int_\Omega (\nabla \cdot a_1)|(\sigma^{\frac{n-2}{2}} v_m)'|^2 dy\,,\\
I_{2,m} &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^2 \int_\Omega (a_1.\nabla(\sigma^{\frac{n-2}{2}} v_m)(\sigma^{\frac{n-2}{2}} v_m)' dy\,,\\
I_{3,m} &:= \sigma^2 \int_\Omega (a_2.\nabla(\sigma^{\frac{n-2}{2}} v_m)(\sigma^{\frac{n-2}{2}} v_m)' dy\,,\\
I_{4,m} &:= \frac{n-2}{2}\Big[\frac{\sigma''}{\sigma} - \frac{n}{2}\Big|\frac{\sigma'}{\sigma}\Big|^2 + \mu\frac{\sigma'}{\sigma}\Big]\sigma^2 \int_\Omega (\sigma^{\frac{n-2}{2}} v_m)(\sigma^{\frac{n-2}{2}} v_m)' dy\,,\\
I_{5,m} &:= \frac{1}{2}\sigma^2 a'(t, \sigma^{\frac{n-2}{2}} v_m, \sigma^{\frac{n-2}{2}} v_m) + \frac{\sigma'}{\sigma}\sigma^2 a(t, \sigma^{\frac{n-2}{2}} v_m, \sigma^{\frac{n-2}{2}} v_m)\,.
\end{aligned}$$

Recalling the expressions (2.7) of a_1 and a_2, it is easy to see that

$$I_{1,m} = -\frac{\sigma'}{\sigma}\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)_t\|_{L^2}^2 \le 0, \tag{2.34}$$

$$\begin{aligned}|I_{2,m}+I_{3,m}+I_{4,m}| \le {} & \frac{\mu}{4}\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2 \\ & + C_\Omega\big(|\sigma'|^2+|\sigma''|^2+|\sigma'|^2\big|\frac{\sigma'}{\sigma}\big|^2\big)\|\sigma^{\frac{n-2}{2}}\nabla v_n\|_{L^2}^2\,.\end{aligned} \tag{2.35}$$

Furthermore, by recalling (2.17), (2.18) and (2.11)) we get

$$I_{5,m} = -\sigma''\sigma'\sum_{i,j=1}^{n}\int_\Omega y_iy_j(\sigma^{\frac{n-2}{2}}\partial_{y_i}v_m)(\sigma^{\frac{n-2}{2}}\partial_{y_j}v_m)dy \le \mathcal{C}_\Omega(|\sigma''|^2+|\sigma'|^2)\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2\,.$$

Therefore, on account of (3.16) and (3.15)

$$\sum_{k=1}^{5} I_k \le \frac{\mu}{2}\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2 + C_\Omega\big(|\sigma'|^2+|\sigma''|^2+|\sigma'|^2\big|\frac{\sigma'}{\sigma}\big|^2\big)\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2 \tag{2.36}$$

and by adding (3.17) to (2.31) it follows

$$\frac{d}{dt}E_{1,m}(t) + \mu\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2 \le C_\Omega\varphi_1(t)\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2. \tag{2.37}$$

where

$$\varphi = |\sigma''|^2 + |\sigma'|^2 + \left|\frac{\sigma'}{\sigma}\right|^2|\sigma'|^2. \tag{2.38}$$

We consider now the scalar product in $L^2(\Omega)$ of equation (2.24) with

$$\sigma^2\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v_m), \tag{2.39}$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_{2,m}(t) + M(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2 + {} & \\ +\sigma^2 a(t,\sigma^{\frac{n-2}{2}}v_m,\sigma^{\frac{n-2}{2}}v_m) = \sum_{k=1}^{4} I_k\,, &\end{aligned} \tag{2.40}$$

where

$$E_{2,m}(t) = \mu\sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|_{L^2}^2 + 2\sigma^2\int_\Omega(\sigma^{\frac{n-2}{2}}v_m)(\sigma^{\frac{n-2}{2}}v_m')dy \tag{2.41}$$

$$
\begin{aligned}
I_{1,m} &:= \frac{1}{2}\Big[\mu n\frac{\sigma'}{\sigma} + (n-2)\frac{\sigma''}{\sigma} + \frac{3}{2}n(n-2)\Big|\frac{\sigma'}{\sigma}\Big|^2\Big]\sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|^2_{L^2},\\
I_{2,m} &:= \sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|^2_{L^2} - n\frac{\sigma'}{\sigma}\sigma^2\int_\Omega(\sigma^{\frac{n-2}{2}}v_m)(\sigma^{\frac{n-2}{2}}v_m)'dy,\\
I_{3,m} &:= -\frac{1}{2}\sigma^2\int_\Omega(\nabla\cdot a_2)|\sigma^{\frac{n-2}{2}}v_m|^2dy - \sigma^2\int_\Omega a_1\cdot(\sigma^{\frac{n-2}{2}}\nabla v_m)(\sigma^{\frac{n-2}{2}}v_m)'\,dy\,.
\end{aligned}
$$

Given the expressions (2.7) of a_1 and a_2, we can estimate the terms $I_{1,m}$, $I_{2,m}$ and $I_{3,m}$ so that

$$
I_{1,m}+I_{2,m}+I_{3,m} \le \frac{m_0}{4}\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2}+2\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|^2_{L^2}+C_\Omega\varphi(t)\sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|^2_{L^2}
$$

(see (2.38) for φ_1). By adding these estimates to (2.40), we obtain

$$
\begin{aligned}
\frac{d}{dt}E_{2,m}(t) + \frac{m_0}{4}\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2} \le 2\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'|^2_{L^2}&\\
+ C_\Omega\varphi(t)(\sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|^2_{L^2}).&
\end{aligned}
\tag{2.42}
$$

This being so, by posing

$$
L_{1,m}(t) = \frac{4}{\mu}E_{1,m}(t) + E_{2,m}(t) \tag{2.43}
$$

and by adding (2.37) multiplied by $4/\mu$ to (2.42), we obtain

$$
\begin{aligned}
\frac{d}{dt}L_{1,m}(t) + 2\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|^2_{L^2} + \frac{m_0}{4}\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2}&\\
\le C_\Omega\varphi(t)(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2} + \sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|^2_{L^2}).&
\end{aligned}
\tag{2.44}
$$

Furthermore, by recalling (2.32), (2.41) and (2.12)) , we easily see that

$$
L_{1,m}(t) \ge \frac{2}{\mu}\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|^2_{L^2} + \frac{\mu}{2}\sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|^2_{L^2} + \frac{m_0}{2}\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2} \tag{2.45}
$$

so that (2.44) gives us

$$
\frac{d}{dt}L_{1,m}(t) \le C_\Omega\varphi(t)L_{1,m}(t)
$$

and, since (see (2.21) and (2.38)) $\varphi \in L^1([0,+\infty[)$, from the previous inequality it follows

$$
L_{1,m}(t) \le L_{1,m}(0)\exp(\mathcal{C}_\Omega\|\varphi\|_{L^1(0,+\infty)}).
$$

Finally, note that given (2.22), it is easy to see that

$$L_{1,m}(0) \le C_\Omega(\|v_0\|^2_{H^1} + \|v_1\|^2_{L^2})$$

Furthermore, since (see (2.12)) $M \in C^2([0,\infty[)$, we set

$$\sup_{0\le\lambda\le\Lambda_0} |M^{(i)}(\lambda)| = M_i, \quad M_0 = \max(M_1, M_1, M_2). \tag{2.46}$$

Given (2.45),

$$\left|M^{(i)}\big(\|\sigma^{\frac{n-2}{2}}\nabla v_m(t)\|^2_{L^2}\big)\right| \le \sup_{0\le\lambda\le\Lambda_0} |M^{(i)}(\lambda)| \le M_0$$

which completes the proof of the lemma 2.2.1 □

Lemma 2.2.2. *Let*

$$\begin{aligned} L_{2,m}(t) &:= \sigma^2\|(\sigma^{\frac{n-2}{2}}v'_m)'\|^2_{L^2} + \sigma^2 a(t,\sigma^{\frac{n-2}{2}}v'_m,\sigma^{\frac{n-2}{2}}v'_m) \\ &+M(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v'_m\|^2_{L^2} + \frac{1}{2}M'(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2})[\frac{d}{dt}\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2}]^2 \\ &+2\sigma^2 a'(t,\sigma^{\frac{n-2}{2}}v_m,\sigma^{\frac{n-2}{2}}v'_m) - 4\frac{\sigma'}{\sigma}M(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2})\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_{mt})(\sigma^{\frac{n-2}{2}}\nabla v_m)dy. \end{aligned} \tag{2.47}$$

We have

$$\begin{aligned} &\frac{d}{dt}L_{2,m}(t) + \mu\sigma^2\|(\sigma^{\frac{n-2}{2}}v'_m)'|^2_{L^2} \\ &\qquad \le C_0(\varphi(t) + \|\sigma^{\frac{n-2}{2}}\Delta v_m\|_{L^2}(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}}\nabla v'_m\|^2_{L^2}). \end{aligned} \tag{2.48}$$

Proof. If we differentiate (2.24) with respect to t and we take the scalar product in $L^2(\Omega)$ with

$$\sigma^2\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v'_m)',$$

we obtain

$$\frac{1}{2}\frac{d}{dt}L_{2,m}(t) + \mu\sigma^2\|(\sigma^{\frac{n-2}{2}}v'_m)'\|^2_{L^2} + 2\frac{\sigma'}{\sigma}M(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v'_m\|^2_{L^2} \tag{2.49}$$

$$= \sum_{k=1}^{7} I_{k,m} + \sum_{k=1}^{6} J_{k,m}$$

where

$$
\begin{aligned}
I_{1,m} &:= (n-1)\frac{\sigma'}{\sigma}\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2 - \frac{1}{2}\sigma^2\int_\Omega |(\sigma^{\frac{n-2}{2}}v_m')'|^2\nabla\cdot a_1 dy\,,\\
I_{2,m} &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^2\int_\Omega (a_1.\nabla(\sigma^{\frac{n-2}{2}}v_m')(\sigma^{\frac{n-2}{2}}v_m')'dy\,,\\
I_{3,m} &:= \frac{n-2}{2}\big[\frac{\sigma''}{\sigma}-(n-1)\big(\frac{\sigma'}{\sigma}\big)^2+\mu\frac{\sigma'}{\sigma}\big]\sigma^2\int_\Omega(\sigma^{\frac{n-2}{2}}v_m')(\sigma^{\frac{n-2}{2}}v_m')'dy\,,\\
I_{4,m} &:= \sigma^2\int_\Omega a_2\cdot\nabla(\sigma^{\frac{n-2}{2}}v_m')(\sigma^{\frac{n-2}{2}}v_m')'dy\,,\\
I_{5,m} &:= \sigma^2\int_\Omega a_2'.\nabla(\sigma^{\frac{n-2}{2}}v_m))(\sigma^{\frac{n-2}{2}}v_m')'dy\,,\\
I_{6,m} &:= \sigma^2\int_\Omega a_1'\cdot\nabla(\sigma^{\frac{n-2}{2}}v_m')(\sigma^{\frac{n-2}{2}}v_m')'dy\,,\\
I_{7,m} &:= \frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^2 a'(t,\sigma^{\frac{n-2}{2}}v_m',\sigma^{\frac{n-2}{2}}v_m)+\sigma^2 a''(t,\sigma^{\frac{n-2}{2}}v_m',\sigma^{\frac{n-2}{2}}v_m)\\
&\qquad + \frac{3}{2}\sigma^2 a'(t,\sigma^{\frac{n-2}{2}}v_m',\sigma^{\frac{n-2}{2}}v_m')+\frac{2}{2}\frac{\sigma'}{\sigma}\sigma^2 a(t,\sigma^{\frac{n-2}{2}}v_m',\sigma^{\frac{n-2}{2}}v_m')\,,
\end{aligned}
$$

and the nonlinear terms J_k they are given by

$$
\begin{aligned}
J_{1,m} &:= -\big[2\frac{\sigma''}{\sigma}+n\big(\frac{\sigma'}{\sigma}\big)^2\big]M(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_m')(\sigma^{\frac{n-2}{2}}\nabla v_m)dy\,,\\
J_{2,m} &:= \frac{n-6}{2}\frac{\sigma'}{\sigma}M'(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_m')(\sigma^{\frac{n-2}{2}}\nabla v_m)dy\,,\\
J_{3m} &:= \big((\frac{n-2}{2}\big)\frac{\sigma'}{\sigma}M'(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\big[\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\big]^2\,,\\
J_{4,m} &:= \frac{n-2}{2}\big(\frac{\sigma'}{\sigma}\big)'M'(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2\,,\\
J_{5,m} &:= \frac{1}{4}M''(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\big[\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\big]^3\,,\\
J_{6,m} &:= \frac{3}{2}M'(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|^2.
\end{aligned}
$$

The terms I_k, $k=1,\cdots,7$, are similar to those in the identity (2.31) and therefore, can be (in particular the first and last term) estimated in the same way. We then obtain taking into account (2.21)

$$\sum_{k=1}^{7} I_{k,m} \le \frac{\mu}{2}\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2 + C_\Omega\varphi(t)(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2+\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2), \quad (2.50)$$

where, let us recall here

$$\varphi(t) := |\sigma''|^2 + |\sigma'|^2 + \left|\frac{\sigma'}{\sigma}\right|^2 |\sigma'|^2.$$

As for the remaining nonlinear terms, considering (2.29), we have

$$J_{1,m} \leq C_\Omega \left|\frac{\sigma'}{\sigma}\right|^2 (\|\sigma^{\frac{n-2}{2}} \nabla v_m\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}} \nabla v'_m\|^2_{L^2}) \tag{2.51}$$

Regarding the terms $J_2, \cdots, J_6$, let us first note that, given (2.27)-(2.29), we have

$$\begin{aligned} &M^{(i)}(\|\sigma^{\frac{n-2}{2}} \nabla v_m\|^2_{L^2}) \frac{d}{dt} (\|\sigma^{\frac{n-2}{2}} \nabla v_m\|^2_{L^2}) \\ &\qquad = -M^{(i)}(\|\sigma^{\frac{n-2}{2}} \nabla v_m\|^2_{L^2}) \int_\Omega (\sigma^{\frac{n-2}{2}} \Delta v)(\sigma^{\frac{n-2}{2}} v_m)' dy \\ &\qquad\qquad \leq C_\Omega M_0 (\|v_0\|^2_{H^1} + \|v_1\|^2_{L^2}) [\sigma^{-1} \|\sigma^{\frac{n-2}{2}} \Delta v_m\|_{L^2}] \end{aligned}$$

so that, we have

$$\sum_{k=2}^{6} J_k \leq C_0 [\sigma^{-1} \|\sigma^{\frac{n-2}{2}} \Delta v_m\|_{L^2}] (\|\sigma^{\frac{n-2}{2}} \nabla v'_m\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}} \nabla v_m\|^2_{L^2}). \tag{2.52}$$

where C_0 is a constant which depends on the size of the initial data v_0 and v_1. Finally, putting together (3.46), (2.51) and (3.44), from (2.49) it follows the inequality (2.48) and thus this achieves the proof of lemma 2.2.2. □

Lemma 2.2.3. *Let*

$$L_{3,m}(t) := 2\sigma^2 \int_\Omega (\sigma^{\frac{n-2}{2}} v_m)' (\sigma^{\frac{n-2}{2}} v'_m)' dy. \tag{2.53}$$

Then the following inequality holds

$$\begin{aligned} \frac{d}{dt} L_{3,m}(t) + \frac{m_0}{2} \|(\sigma^{\frac{n-2}{2}} \nabla v'_m)\|^2_{L^2} &\leq C_1 (\sigma^2 \|(\sigma^{\frac{n-2}{2}} v_m)'\|^2_{L^2} + \sigma^2 \|(\sigma^{\frac{n-2}{2}} v'_m)'\|^2_{L^2}) \\ &+ C_4 \varphi(t) (\|\sigma^{\frac{n-2}{2}} \nabla v_m\|^2_{L^2} + \sigma^2 \|(\sigma^{\frac{n-2}{2}} v_m)'\|^2_{L^2} + \sigma^2 \|(\sigma^{\frac{n-2}{2}} v'_m)'\|^2_{L^2}). \end{aligned} \tag{2.54}$$

Proof. As in the proof of lemma 2.2.2, if we differentiate (2.24) with respect to t and we take the scalar product in $L^2(\Omega)$ of the new equation with

$$\sigma^2 \sigma^{\frac{n-2}{2}} (\sigma^{\frac{n-2}{2}} v_m)',$$

we obtain

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}L_{3,m}(t)+M(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2+ \\ +\sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v_m',\sigma^{\frac{n-2}{2}}v_m')+\ M'(\sigma^{\frac{n-2}{2}}\|\nabla v_m\|_{L^2}^2)\Big[\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\Big]^2=\sum_{i=1}^{8}I_i\end{gathered} \tag{2.55}$$

where

$$\begin{aligned}
I_{1,m} &:= \frac{3n-2}{2}\frac{\sigma'}{\sigma}\sigma^2\int_\Omega(\sigma^{\frac{n-2}{2}}v_m)'(\sigma^{\frac{n-2}{2}}v_m')_t dy+\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2\,,\\
I_{2,m} &:= [\frac{n-2}{2}(\frac{\sigma''}{\sigma}+\frac{n}{2}|\frac{\sigma'}{\sigma}|^2)+\mu(n-1)\frac{\sigma'}{\sigma}]\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2\,,\\
I_{3,m} &:= \frac{n-2}{2}(\frac{\sigma''}{\sigma}-\frac{n}{2}|\frac{\sigma'}{\sigma}|^2)[\frac{n-2}{2}\frac{\sigma'}{\sigma}+\mu]\sigma^2\int_\Omega(\sigma^{\frac{n-2}{2}}v_m)(\sigma^{\frac{n-2}{2}}v_m)'dy\,,\\
I_{4,m} &:= \sigma^{3-r}\int_\Omega[(a_2+a_1')\cdot(\sigma^{\frac{n-2}{2}}\nabla v_m'+a_2'\cdot(\sigma^{\frac{n-2}{2}}\nabla v_m)](\sigma^{\frac{n-2}{2}}v_m)'dy\,,\\
I_{5,m} &:= -\sigma^2a'(t,\sigma^{\frac{n-2}{2}}v_m,(\sigma^{\frac{n-2}{2}}v_m)')\,,\\
I_{6,m} &:= \sigma^2\int_\Omega(a_1\cdot(\sigma^{\frac{n-2}{2}}\nabla v_m''))(\sigma^{\frac{n-2}{2}}v_m)'dy\,,\\
I_{7,m} &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}M(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v_m)dy\,,\\
I_{8,m} &:= (n-2)\Big|\frac{\sigma'}{\sigma}\Big|^2M(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2\,.
\end{aligned}$$

Recalling the expression (2.7) of a_1 and a_2 (see also (2.18), (2.17) and (2.11)), we can estimate the first six terms so that

$$\begin{gathered}\sum_{i=1}^{6}I_k\le\frac{m_0}{12}\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2\\ +\,C_\Omega(\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2+\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2)\\ +\,C_\Omega\varphi(t)(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2+\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2+\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2).\end{gathered}$$

Furthermore, by integrating by parts, the term I_6 can be estimated ensure that

$$\begin{gathered}I_6\le\frac{m_0}{12}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2\\ +\,C_\Omega\varphi(t)(\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2+\sigma^2\|(\sigma^{\frac{n-2}{2}}v)_m'\|_{L^2}^2+\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2).\end{gathered}$$

As for the latter terms, we have

$$I_{7,m}+I_{8,m}\le \frac{m_0}{12}\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2+C_\Omega\varphi(t)\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2.$$

The proof of the lemma is completed by adding the above estimates of the terms I_i to (2.55). □

Lemma 2.2.4. *We set*

$$\begin{aligned}\mathcal{D}_m(t):=\sigma^2\big[\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2+\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2\big]\\ +\big[\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2+\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2\big]\,,\end{aligned}\tag{2.56}$$

$$L_m(t):=\sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|_{L^2}^2+\mathcal{D}_m(t)\tag{2.57}$$

$$\mathcal{L}_m(t):=k_1L_{1,m}(t)+k_2L_{2,m}(t)+L_{3,m}(t)\,,\tag{2.58}$$

where $L_{i,m}(t)$ $(i=1,\dots 3)$ are given by (2.45), (2.47) and (2.53) as for k_1 and k_2 are positive constants. Then the following inequality hold

$$\frac{d}{dt}\mathcal{L}_m(t)+\alpha\mathcal{D}_m(t)\le N_0(\varphi(t)+\sigma^{-1}\|\sigma^{\frac{n-2}{2}}\Delta v_m\|_{L^2})\mathcal{L}_m(t)\tag{2.59}$$

$$\mathcal{L}_m(t)\ge\beta L_m(t)\tag{2.60}$$

with α and N_0 positive constants (N_0 depend on the size of initial data v_0 and v_1).

Proof. Given (2.45), (2.47) and (2.53), it is easy to see that

$$\begin{aligned}L_{1,m}(t)&\ge\frac{2}{\mu}\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2+\frac{\mu}{2}\sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|_{L^2}^2+\frac{m_0}{2}\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2,\\ L_{2,m}(t)&\ge\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2+\frac{m_0}{4}\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2-C_\Omega\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2,\\ L_{3,m}(t)&\ge-\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2-\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2\,,\end{aligned}$$

From which (see also (2.58)) we get

$$\begin{aligned}\mathcal{L}_m(t)\ge\lambda_1\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2+\lambda_2\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2+\frac{k_2\mu}{2}\sigma^2\|\sigma^{\frac{n-2}{2}}v_m\|_{L^2}^2\\ +\lambda_3\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2+\frac{m_0}{4}k_3\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2,\end{aligned}\tag{2.61}$$

where

$$\lambda_1:=\frac{2k_1}{\mu}-1,\quad \lambda_2:=\frac{k_1m_0}{2}-C_\Omega k_2,\quad \lambda_3=k_2-1\,.\tag{2.62}$$

On the order hand, if we multiply inequality (2.44), (2.48) and (2.54) by k_1, k_2 and $k_3 = 1$ respectively and summing, we obtain

$$\begin{aligned} &\frac{d}{dt}\mathcal{L}_m(t) + \lambda_4\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2 + \lambda_5\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2 \\ &\quad + \lambda_6\sigma^2\|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2 + \lambda_7\|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2 \\ &\le C_\Omega(1+C_0)(\varphi(t)+\|\Delta v_m\|_{L^2})\Big[\sigma^2\big[\|\sigma^{\frac{n-2}{2}}v_m\|_{L^2}^2 + \|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2}^2 + \|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}^2\big] \\ &\quad + \|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2}^2 + \|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}^2\Big]. \end{aligned} \tag{2.63}$$

where

$$\lambda_4 := 2k_1 - C_1, \quad \lambda_5 := \mu k_2 - C_1, \quad \lambda_6 := k_1\frac{m_0}{2}, \quad \lambda_7 := \frac{m_0}{2} \tag{2.64}$$

By choosing k_1 and k_2 so that $\lambda_i > 0$ $(i = 1, \dots, 7)$ (see (2.62) and (2.64), the inequalities (2.59) and (2.60) follows easily from (3.58) and (2.61). $\square$

Lemma 2.2.5. *Assume that*

$$\int_0^{T_0} \sigma^{-1}(t)\|\sigma^{\frac{n-2}{2}}\Delta v_m(t)\|_{L^2}\,dt \le \bar{N}_0 \tag{2.65}$$

where the positive constant $\bar{N}_0$ *and* $T_0 \in [0, T[$ *depend only on the size of initial data. Then, for all* $t \in [0, T_0]$, *we have*

$$L_m(t) + \alpha\int_0^t \mathcal{D}_m(s)ds \le C_\Omega(\|v_0\|_{H^2}^2 + \|v_1\|_{H^1}^2)\exp(N_0(\|\varphi\|_{L^1(0,+\infty} + \bar{N}_0)). \tag{2.66}$$

Proof. By the Gronwall lemma and on the basis of (2.65), we obtain from (2.59),

$$\mathcal{L}_m(t) + \alpha\int_0^t \mathcal{D}(s)ds \le \mathcal{L}_m(0)\exp(N_0(\|\varphi\|_{L^1(0,+\infty} + \bar{N}_0)). \tag{2.67}$$

Moreover, by recalling the expressions (2.47) and (2.53) of $L_{2,m}(t)$ and $L_{3,m}(t)$, we can see that $\mathcal{L}_m(0)$ contains the L^2-norm of the term $v_m''|_{t=0}$. This term is defined (see (2.24) by

$$\begin{aligned} &(v_m''(0), w_j) + \mu(v_m'(0), w_j) + \frac{1}{\sigma^2(0)}M\Big(\|\sigma^{\frac{n-2}{2}}\nabla v_m(0)\|_{L^2}^2\Big)(\nabla v_m(0), \nabla w_j) \\ &\qquad + a(0, v_m, w) = (a_1(0)\cdot\nabla v_m'(0), w_j) + (a_2(0)\cdot\nabla v_m(0), w_j). \end{aligned}$$

Recalling (see (2.17) the expression of the bilinear form $a(t,\cdot,\cdot)$ and (2.25) , we have

$$\begin{aligned}(v_m''(0),w_j) &= -\mu(v_{1,m},w_j) - \frac{1}{\sigma^2(0)}M\Big(\|\sigma^{\frac{n-2}{2}}\nabla v_{0,m}\|^2_{L^2}\Big)(\Delta v_{0,m},w_j) \\ &\quad + (A_0 v_{0,m},w_j) = (a_1(0)\cdot\nabla v_{1,m},w_j) + (a_2(0)\cdot\nabla v_{1,m},w_j)\end{aligned} \tag{2.68}$$

where

$$A_0 v = \sum_{i,j=1}^{n} \partial_{y_i}(a_{i,j}(0,y)\partial_{y_i} v).$$

Since w_j is an orthonormal system in L^2 and $(v_{0,m},v_{1,m}) \in H^2(\Omega)\times H^1(\Omega)$, also considering(2.22), from (2.68) it follows

$$\|v_m''(0)\|_{L^2} \le C_\Omega(\|(v_{0,m}\|^2_{H^2} + \|(v_{0,m}\|^2_{H^1}) \le C_\Omega(\|v_0\|^2_{H^2} + \|v_1\|^2_{H^1}).$$

This being, now recalling the expression (2.58) of $\mathcal{L}_m(t)$ (see also (2.43), (2.47) and (2.53)), taking into account the inequality above, we can easily convince ourselves that

$$\mathcal{L}_m(0) \le C_\Omega(\|v_0\|^2_{H^2} + \|v_1\|^2_{H^1})$$

so that from (2.67) (see also (2.60)), we have (2.66). □

Finally, we still have to verify the hypothesis (2.65). To do this, let's first rewrite the equation (2.24) in the following form

$$(\tilde{A}(t)v_m,w_j) = (\tilde{F}_m,w_j) \tag{2.69}$$

where

$$\begin{aligned}\tilde{F}_m &= \sigma^2(-v_m'' - \mu v_m' + a_1\cdot\nabla v_m' + a_2\cdot\nabla v_m) \\ \tilde{A}(t)v_m &= -\sum_{i,j=1}^{n}\partial_{y_i}(\tilde{a}_{ij}(t)\partial_{y_j}v_m), \quad \tilde{a}_{ij} = \tilde{M}(\sigma^{\frac{n-2}{2}}\|\nabla v\|^2_{L^2})\delta_{ij} - |\sigma'|^2 y_i y_j.\end{aligned}$$

with the same arguments as above, we have

$$\|\tilde{A}(t)v_m\|_{L^2} \le C_\Omega\|\tilde{F}_m\|_{L^2}$$

and, since (see the second condition of (2.19)) we have

$$\sum_{i,j=1}^{n}\tilde{a}_{ij}\xi_j\xi_i \ge \frac{m_0}{2}|\xi|^2,$$

by standard regularity arguments of elliptic equations we have

$$\|\Delta v_m\|_{L^2} \leq C_\Omega \|\tilde{F}_m\|_{L^2}. \tag{2.70}$$

Furthermore, by recalling the expressions (see (2.7)) of a_1 and a_2 (see also (2.21)), we can easily see that

$$\begin{aligned} \|\sigma^{\frac{n-2}{2}}\tilde{F}_m\|_{L^2} \leq \sigma^2\big[\|(\sigma^{\frac{n-2}{2}}v_m)'\|_{L^2} + \|(\sigma^{\frac{n-2}{2}}v_m')'\|_{L^2}\big] \qquad (2.71)\\ + \sigma\big[\|\sigma^{\frac{n-2}{2}}\nabla v_m\|_{L^2} + \|\sigma^{\frac{n-2}{2}}\nabla v_m'\|_{L^2}\big], \end{aligned}$$

Now by combining (2.70) and (2.71), considering (2.56) and (2.66), we have

$$\begin{aligned} \int_0^{T_0} \sigma^{-1}(t)\|\Delta v_m(t)\|_{L^2} dt &\leq C_\Omega \int_0^{T_0} \mathcal{D}_m^{\frac{1}{2}}(s)ds \qquad (2.72)\\ &\leq \sqrt{T^*}\Big(\int_0^{T_0} \mathcal{D}_m(t)dt\Big)^{\frac{1}{2}} \leq \tilde{N}_0\sqrt{T_0} \qquad (2.73) \end{aligned}$$

where

$$\tilde{N}_0 = C_\Omega(\|v_0\|_{H^2} + \|v_1\|_{H^1})\exp(\frac{1}{2}N_0(\|\varphi\|_{L^1(0,+\infty} + \frac{1}{2}\bar{N}_0))$$

By choosing

$$T_0 \leq \Big(\frac{\tilde{N}_0}{\bar{N}_0}\Big)^2$$

we can easily see that the hypothesis (2.65) which we used to obtain the estimate (2.66) is verified. Now, we can say that the latter remains true without the hypothesis (2.65).

2.3 Existence and uniqueness of local solutions

Having established the estimates of approximate solutions (see lemmas 2.2.1-2.2.5), one is now in a position to prove the following local existence theorem.

Theorem 2.3.1. *Let*

$$v_0 \in H_0^2(\Omega), \quad v_1 \in H_0^1(\Omega)$$

If the assumptions (0.4), (2.10) and (2.19) hold, then there exists T_0 which depends on the size of the initial data (v_0, v_1) such that the boundary value problem (2.13)-(2.15) admits a unique local solution v such that

$$v \in L^\infty\big(0, T_0, H_0^1(\Omega)\cap H^2(\Omega)\big), \ \ v_t \in L^\infty\big(0, T_0, H_0^1(\Omega)\big), \ \ v_{tt} \in L^\infty\big(0, T_0, L^2(\Omega)\big)$$

Proof From (2.66) (see also (2.56) and (2.57)), we can easily deduce that

$$\begin{array}{lll} v''_m & \text{is bounded in} & L^\infty(0,T_0,L^2(\Omega)) \cap L^2(0,T_0,L^2(\Omega)) \\ v'_m & \text{is bounded in} & L^\infty(0,T_0,H_0^1(\Omega)) \cap L^2(0,T_0,H_0^1(\Omega)) \\ v_m & \text{is bounded in} & L^\infty(0,T_0,H_0^1(\Omega)) \cap L^2(0,T_0,H_0^1(\Omega)). \end{array}$$

thus, we can now pass to the limit in (2.24). The argument for the linear terms is classical, we only have to justify the limit in the non linear term. Indeed, from the above estimates, it follows that ∇v_m is bounded in $C^0(0,T_0,L^2(\Omega))$. Moreover, since

$$\begin{aligned} &\big|\|\nabla v_m(t)\|_{L^2}^2 - \|\nabla v_m(s)\|_{L^2}^2\big| \\ &\qquad\qquad \leq 2|t-s|^{\frac{1}{2}}\|\nabla v_m\|_{C^0(0,T_0,L^2)}\|\nabla v'_m\|_{L^2(0,T_0,L^2)}, \end{aligned}$$

we deduce from the Ascoli-Arzelà theorem that we can extract a subsequence v_{k_m} such that the sequence $\|\nabla v_{k_m}(\cdot)\|_{L^2}^2$ converges uniformly in $[0,T_0]$ to $|\|\nabla v(\cdot)\|_{L^2}^2$.

Since the function M is continuous and $\|\nabla v_{k_m}\|_{L^2}^2$ is bounded in $L^2(0,T_0,L^2)$ (maybe passing to the new subsequence), we have for m large enough so that $k_m > j$

$$M(\|\nabla v_{k_m}(t)\|_{L^2}^2)(\nabla v_{k_m},\nabla w_j)_{L^2} \longrightarrow M(\|\nabla v(t)\|_{L^2}^2)(\nabla v,\nabla w_j)_{L^2}. \tag{2.74}$$

Now if we replace in (2.24) m by k_m and we pass to the limit when m tends to infinity, we obtain, taking (3.70) into account and the usual classical arguments for the limit in the linear terms,

$$\begin{aligned} (v'',w_j) + \mu(v',w_j) + \frac{1}{\sigma^2}M\Big(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2\Big)(\nabla v,\nabla w_j) + a(t,v,w_j) \\ = (a_1\cdot\nabla v',w_j) + (a_2\cdot\nabla v,w_j) \quad \text{in} \;\; L^2(0,T_0). \end{aligned} \tag{2.75}$$

This equality is true for all $j \geq 1$. It implies by finite linear combinations that (2.75) is verified for all $w \in \cup_{m\geq1}V_m$ dense in $H_0^1(\Omega)$ so (2.75) is verified for all $w \in H_0^1(\Omega)$. Thus, we obtain a function v satisfying equation (2.4) in the sense $L^2(0,T_0,L^2)$. About the uniqueness of solution it's easy to show by classical argument that the solution is unique. Consequently, the theorem is proved . □

Chapter 3

Global Solution

The global solution will be obtained by the standard continuation argument based on a local existence and the a priori estimates. These estimates which will be obtained in the following lemmas require a more elaborate treatment unlike the case σ bounded, because the assumption $\sigma(t) \to \infty$ for $t \to \infty$ makes the equation (2.13) degenerate at infinity. However, these estimates will be established under the assumptions (see (2.21))

$$0 < \varepsilon_1 \leq \sigma_0 \leq \varepsilon_0 < 1, \quad 0 \leq \sigma_1 \leq K\varepsilon_1^{1+\alpha}, \tag{3.1}$$

where ε_0, ε_1, K are positive constants. It should be noted (see (2.21)) that for all $j \geq 2$

$$\begin{gathered} |\sigma^{(j)}(t)| \leq K^{j-1}|(\alpha-1)(\alpha-2)\cdots(\alpha-(j-1))|\varepsilon_0^{\alpha(j-1)}\sigma'(t) \text{ for all } t \geq 0, \\ 0 < \sigma'(t) \leq \alpha K\varepsilon_0^{2\alpha}, \quad \frac{\sigma'}{\sigma} \leq \alpha K\varepsilon_0^{\alpha} \text{ for all } t \geq 0, \end{gathered} \tag{3.2}$$

which follows immediately from (3.1), (2.21) and the inequality

$$|\sigma^{(j)}(t)| \leq K\varepsilon_0^{\alpha}|\alpha - j + 1||\sigma^{(j-1)}(t)| \text{ for all } t \geq 0.$$

Let us bear in mind that, given the initial expansion $\Omega_0 = \sigma_0\Omega$ which will be assumed (see (3.1)) small enough and initial data $(u_0, u_1) \in H^2(\Omega_0) \times H^1(\Omega_0)$, our goal here is to show that the problem (0.1)-(0.3) admits a global solution u if the size

$$R(\Omega_0) = \|u_0\|^2_{H^2(\Omega_0)} + \|u_1\|^2_{H^1(\Omega_0)} \tag{3.3}$$

of initial data is large enough. Since $u = vo\tau$ (see (1.6)), it suffices to show that the problem (2.13) and (2.15) has a global solution v if the size of initial data

$(v^0_{\sigma_0}, v^1_{\sigma_0}) \in H^2(\Omega) \times H^1(\Omega)$ is large enough. For that purpose, we set

$$\begin{cases} R(\sigma_0) = \|v^0_{\sigma_0}\|^2_{H^2(\Omega)} + \|v^1_{\sigma_0}\|^2_{H^1(\Omega)}, \\ \lambda(\sigma_0) = \sigma_0^{\alpha(n-3+r)} R(\sigma_0), \quad n \geq 3, \quad 0 < r < 1. \end{cases} \tag{3.4}$$

and we suppose

$$\lim_{\sigma_0 \to 0} \lambda(\sigma_0) = 0. \tag{3.5}$$

The assumption (3.5) specifies in what sense the size of our initial data $(v^0_{\sigma_0}, v^1_{\sigma_0})$ can be considered rather large if σ_0 is small enough. Moreover, if (3.5) is satisfied, then the size $R(\Omega_0)$ can be considered large enough. More precisely, we have

$$R(\Omega_0) \leq \frac{C_\Omega}{|\Omega_0|^{\frac{1+r}{n}}}. \tag{3.6}$$

In fact, recalling (2.8), (2.9) and (3.4), by easy computations, we can verify that

$$\begin{aligned} \|u_0\|^2_{H^2(\Omega_0)} + \|u_1\|^2_{H^1(\Omega_0)} &\leq C_\Omega \sigma_0^{\alpha(n-4)} (\|v^0_{\sigma_0}\|^2_{H^2(\Omega)} + \|v^1_{\sigma_0}\|^2_{H^1(\Omega)}) \\ &\leq C_\Omega \sigma_0^{-\alpha(1+r)} \lambda(\sigma_0) \end{aligned} \tag{3.7}$$

and considering (3.5), necessarily we have

$$\sigma_0^{\alpha(1+r)} \|u_0\|^2_{H^2(\Omega_0)} + \|u_1\|^2_{H^1(\Omega_0)} \leq C_\Omega \lambda(\sigma_0) \leq C_\Omega \tag{3.8}$$

whence (3.8) because $|\Omega_0| = \sigma_0^{\alpha n} |\Omega|$.

In statements following lemmas, we denote by C_i $(i = 0, \ldots, 4)$ the constants which depend on Ω, n , μ, m_0 and (see (3.10)) M_0 but (see (3.1)) neither on σ_0 nor σ_1. In addition, in the proof of each lemma, we will denote by $\tilde{C}_i$ $(i = 1, \cdots 8)$ the constants that depend only on Ω, n, μ and possibly on m_0 and M_0. As for the constants that depend only on Ω and n, they will be designated by C_Ω. Moreover, we denote $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^m}$ for the usual norms in the spaces $L^2 = L^2(\Omega)$ and $H^m = H^m(\Omega)$ respectively. In the sequel, fixed σ_0 small enough we consider the family initial data $(v^0_{\sigma_0}, v^1_{\sigma_0}) \in H^2(\Omega) \cap H^1(\Omega)$ verifying (3.5) and we will derive estimates of the local solution of problem (2.13)-(2.15) allow us to extend this to a global solution.

3.1 A priori estimates

We begin by showing a crucial estimates established in lemma 3.1.1and 3.1.2 which will be used essentially in the proof of lemmas 3.1.3-3.1.6.

Lemma 3.1.1. *Let σ_0 small enough and $(v^0_{\sigma_0}, v^1_{\sigma_0}) \in H^2(\Omega) \times H^1(\Omega)$. Given (3.5), (3.1) and (2.21), we have*

$$\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2} \le C_0 \sigma_0^{\alpha(n-2)} (\|v^1_{\sigma_0}\|^2_{L^2} + \|v^0_{\sigma_0}\|^2_{H^1}), \quad n \ge 3 \tag{3.9}$$

$$\big| M^{(i)} \big(\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2} \big) \big| \le M_0, \quad i = 0, 1, 2 \tag{3.10}$$

where $M^{(i)}$ is the ith derivative of M and M_0 a positive constant independing on σ_0 and σ_1.

Proof. To prove (3.9), we first multiply equation (2.13) with

$$\sigma^2 \sigma^{\frac{n-2}{2}} (\sigma^{\frac{n-2}{2}} v)_t. \tag{3.11}$$

By integrating over Ω, we obtain

$$\frac{1}{2} \frac{d}{dt} E(t) + \mu \sigma^2 \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} = \sum_{k=1}^{5} I_k \tag{3.12}$$

where

$$E(t) := \sigma^2 \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} + \sigma^2 a(t, \sigma^{\frac{n-2}{2}} v, \sigma^{\frac{n-2}{2}} v) + \hat{M}(\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2}), \tag{3.13}$$

$$\hat{M}(\lambda) := \int_0^\lambda M(s) ds. \tag{3.14}$$

and

$$\begin{aligned}
I_1 &:= (n-1) \frac{\sigma'}{\sigma} \sigma^2 \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} - \frac{1}{2} \sigma^2 \int_\Omega (\nabla \cdot a_1) |(\sigma^{\frac{n-2}{2}} v)_t|^2 dy, \\
I_2 &:= -\frac{n-2}{2} \frac{\sigma'}{\sigma} \sigma^2 \int_\Omega (a_1 . \nabla (\sigma^{\frac{n-2}{2}} v)(\sigma^{\frac{n-2}{2}} v)_t dy, \\
I_3 &:= \sigma^2 \int_\Omega (a_2 . \nabla (\sigma^{\frac{n-2}{2}} v)(\sigma^{\frac{n-2}{2}} v)_t dy, \\
I_4 &:= \frac{n-2}{2} \Big[\frac{\sigma''}{\sigma} - \frac{n}{2} \Big| \frac{\sigma'}{\sigma} \Big|^2 + \mu \frac{\sigma'}{\sigma} \Big] \sigma^2 \int_\Omega (\sigma^{\frac{n-2}{2}} v)(\sigma^{\frac{n-2}{2}} v)_t dy, \\
I_5 &:= \frac{1}{2} \sigma^2 a'(t, \sigma^{\frac{n-2}{2}} v, \sigma^{\frac{n-2}{2}} v) + \frac{\sigma'}{\sigma} \sigma^2 a(t, \sigma^{\frac{n-2}{2}} v, \sigma^{\frac{n-2}{2}} v).
\end{aligned}$$

Recalling the expressions (2.7) of a_1 and a_2, it is easy to see that

$$I_1 = -\frac{\sigma'}{\sigma} \sigma^2 \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} \le 0, \tag{3.15}$$

$$\begin{aligned}
|I_2 + I_3 + I_4| \le{}& \frac{\mu}{4} \sigma^2 \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} \\
&+ C_\Omega \big(|\sigma'|^2 + |\sigma''|^2 + |\sigma'|^2 \Big| \frac{\sigma'}{\sigma} \Big|^2 \big) \|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2}.
\end{aligned} \tag{3.16}$$

Furthermore, by recalling (2.17), (2.18) and (2.11)) we get

$$I_5 = -\sigma''\sigma' \sum_{i,j=1}^{n} \int_\Omega y_i y_j (\sigma^{\frac{n-2}{2}}\partial_{y_i} v)(\sigma^{\frac{n-2}{2}}\partial_{y_j} v)dy \le \mathcal{C}_\Omega(|\sigma''|^2 + |\sigma'|^2)\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\,.$$

Therefore, on account of (3.16) and (3.15)

$$\sum_{k=1}^{5} I_k \le \frac{\mu}{2}\sigma^2\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + C_\Omega\big(|\sigma'|^2 + |\sigma''|^2 + |\sigma'|^2\big|\frac{\sigma'}{\sigma}\big|^2\big)\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \quad (3.17)$$

and by adding (3.17) to (3.12) it follows

$$\frac{d}{dt}E(t) \le C_\Omega\big(|\sigma'|^2 + |\sigma''|^2 + |\sigma'|^2\big|\frac{\sigma'}{\sigma}\big|^2\big)\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}. \quad (3.18)$$

Since (see (3.13), (3.14) and (2.12))

$$E(t) \ge \hat{M}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})) \ge \frac{m_0}{2}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}, \quad (3.19)$$

and (see (3.2)) if ε_0 is small enough

$$|\sigma''|^2 + |\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2 \le 2\big(|\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2\big)$$

from (3.18) it follows

$$\frac{d}{dt}E(t) \le \frac{C_\Omega}{m_0}\varphi(t)E(t), \quad \varphi(t) = |\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2. \quad (3.20)$$

Note that, given (3.1) and (2.21) it is easy to see that φ is, relative to σ_0 and σ_1 , uniformly bounded in $L^1(0,\infty)$. By applying the Gronwall lemma, we get

$$E(t) \le E(0)\exp(\frac{C_\Omega}{m_0}\|\varphi\|_{L^1}) \quad (3.21)$$

and thanks to (3.19), we have

$$\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \le \frac{2}{m_0}E(0)\exp(\frac{C_\Omega}{m_0}\|\varphi\|_{L^1}). \quad (3.22)$$

From (3.13) (see also (2.21) and (2.17)), we have

$$\begin{aligned} E(0) := \sigma_0^{\alpha n}\Big\|v^1_{\sigma_0} + \alpha\frac{n-2}{2}v^0_{\sigma_0}\Big\|^2_{L^2} \\ + \sigma_0^{\alpha(n-2)}\sum_{i,j=1}^{n}\int_\Omega \big(\frac{m_0}{2}\delta_{ij} - \alpha^2\sigma_0^{2\alpha}\frac{\sigma_1^2}{\sigma_0^2}y_i y_j\big)(\partial_{y_i}v^0_{\sigma_0})(\partial_{y_j}v^0_{\sigma_0})dy \\ + \hat{M}(\|\sigma_0^{\alpha(n-2)}\nabla v^0_{\sigma_0}\|^2_{L^2}). \end{aligned}$$

According to (3.1), we have

$$E(0) \leq \tilde{C}_0 \sigma_0^{\alpha(n-2)}(\|v^1_{\sigma_0}\|^2_{L^2} + \|v^0_{\sigma_0}\|^2_{H^1}) + \hat{M}(\sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2}).$$

Furthermore, since (see (3.5) and (3.4)) $\sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2} \leq \lambda(\sigma_0) \leq 1$, thanks to (2.12), we have

$$M\big(\sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2}\big) \leq M(\lambda(\sigma_0)) \leq \sup_{0\leq\lambda\leq 1} M(\lambda) \tag{3.23}$$

given (3.14), we can see that

$$\hat{M}(\|\sigma_0^{\alpha(n-2)}\nabla v^0_{\sigma_0}\|^2_{L^2}) \leq \sigma_0^{\alpha(n-2)}\|\nabla v^0_{\sigma_0}\|^2_{L^2} \sup_{0\leq\lambda\leq 1} M(\lambda) \tag{3.24}$$

and so

$$E(0) \leq \tilde{C}_0 \sigma_0^{\alpha(n-2)}(\|v^1_{\sigma_0}\|^2_{L^2} + \|v^0_{\sigma_0}\|^2_{H^1}) \tag{3.25}$$

which, together with (3.22) gives us (3.9).

As to (3.10), since (see (2.12)) $M \in C^2([0,\infty[)$, we set

$$\sup_{0\leq\lambda\leq 1} |M^{(i)}(\lambda)| = N_i, \quad M_0 = \max(N_0, N_1, N_2). \tag{3.26}$$

Given (3.9) and (3.5), we have $\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \leq C_0\lambda(\sigma_0) \leq 1$ for σ_0 small enough and so

$$\big|M^{(i)}\big(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big)\big| \leq \sup_{0\leq\lambda\leq 1}|M^{(i)}(\lambda)| \leq M_0$$

from which follows (3.10). □

Lemma 3.1.2. *Let $0 < r < 1$. Under the same assumptions as the lemma 3.1.1, we have*

$$\Big[\frac{1}{\sigma^{\frac{1-r}{2}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\Big]^2 \leq C_0\sigma_0^{\alpha(1-r)}\lambda_0^2 \leq 1, \quad \frac{1}{\sigma^{\frac{1-r}{2}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2} \leq C_0\lambda_0^{\frac{1}{2}} \leq 1 \tag{3.27}$$

$$\|\sigma^{\frac{n-2}{2}}\nabla v\|^4_{L^2} \leq C_0\sigma_0^{2\alpha(1-r)}\lambda_0^2 \leq 1, \quad \Big[\frac{1}{\sigma^{\frac{1-r}{3}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\Big]^{\frac{3}{2}} \leq C_0\sigma_0^{\alpha(1-r)}\lambda_0^{\frac{3}{2}} \leq 1 \tag{3.28}$$

if (see (3.1)) σ_0 is small enough.

Proof. The proof follows immediately from (3.9) and (3.5) (see also (3.4) and (2.21)). □

Lemma 3.1.3. *Let* $0 < r < 1$ *and*

$$L_1(t) := \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + \sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v) + \sigma^{1-r}\hat{M}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2). \tag{3.29}$$

Given (3.1) and (2.21), we have the following inequality

$$\begin{aligned}\frac{d}{dt}L_1(t) \,+\, \mu\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2& \\ \le C_1\varepsilon_0^{\alpha}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 \,&+\, C_1\varphi(t)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2,\end{aligned} \tag{3.30}$$

where φ *is given by (3.20).*

Proof. An easy computation of the scalar product in $L^2(\Omega)$ of the equation (2.13) with

$$\sigma^{3-r}\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v)_t \tag{3.31}$$

gives

$$\frac{1}{2}\frac{d}{dt}L_1(t) + \mu\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 = \sum_{k=1}^{5} I_k \tag{3.32}$$

with

$$\begin{aligned}
I_1 &:= \big(n-2+\frac{3-r}{2}\big)\frac{\sigma'}{\sigma}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 - \frac{1}{2}\sigma^{3-r}\int_\Omega(\nabla\cdot a_1)|(\sigma^{\frac{n-2}{2}}v)_t|^2dy,\\
I_2 &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}\int_\Omega(a_1.\nabla(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t dy\,,\\
I_3 &:= \sigma^{3-r}\int_\Omega(a_2\cdot\nabla(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t dy
\end{aligned}$$

$$\begin{aligned}
I_4 &:= \frac{n-2}{2}\Big[\frac{\sigma''}{\sigma} - \frac{n}{2}\Big|\frac{\sigma'}{\sigma}\Big|^2 + \mu\frac{\sigma'}{\sigma}\Big]\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t dy\,,\\
I_5 &:= \frac{1}{2}\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v) + \frac{3-r}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v)\\
&\quad + \frac{1-r}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}\hat{M}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\,.
\end{aligned}$$

Recalling the expression (2.7) of a_1 and of a_2, it is easy to see that

$$\begin{aligned}
I_1 &= -\frac{1}{2}(1+r)\frac{\sigma'}{\sigma}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 \le 0\,,\\
I_2 &\le \frac{\mu}{4}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + C_\Omega\Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2\,,\\
I_3 + I_4 &\le \frac{\mu}{4}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + C_\Omega\big(|\sigma''|^2 + |\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2.
\end{aligned}$$

Furthermore, on account of (2.17), (2.18) and (2.11)) we get

$$\begin{aligned} I_5 = {} & \frac{1-r}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}\big[\frac{m_0}{2}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + \hat{M}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\big] \\ & - \big(\sigma''\sigma' - \frac{\sigma'}{\sigma}|\sigma'|^2\big)\sigma^{1-r}\sum_{i,j=1}^{n}\int_\Omega y_i y_j(\sigma^{\frac{n-2}{2}}\partial_{y_i}v)(\sigma^{\frac{n-2}{2}}\partial_{y_i}v)dy. \end{aligned}$$

Therefore, considering (2.21), (3.1) and (3.10), we have

$$\begin{aligned} I_5 \leq {} & \frac{1-r}{2}\alpha K\varepsilon_0^\alpha\big(\frac{m_0}{2} + M_0\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 \\ & + C_\Omega(|\sigma''|^2 + |\sigma'|^2 + \frac{\sigma'}{\sigma}|\sigma'|^2\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 \end{aligned}$$

and so

$$I_5 \leq \tilde{C}_1\varepsilon_0^\alpha\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + C_\Omega(|\sigma''|^2 + |\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2.$$

Given estimates of terms I_i $(i = 1, \dots, 5)$, we obtain

$$\sum_{k=1}^{5} I_k \leq \frac{\mu}{2}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + \tilde{C}_1\varepsilon_0^\alpha\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 \tag{3.33}$$

$$+ \tilde{C}_1\big(|\sigma''|^2 + |\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2\big)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 \tag{3.34}$$

Recalling the expression (3.20) of φ and taking account of (3.2), we have if ε_0 is small enough

$$|\sigma''|^2 + |\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2|\sigma'|^2 + \Big|\frac{\sigma'}{\sigma}\Big|^2 \leq 2\varphi(t), \tag{3.35}$$

and given (3.33) and (3.32) we have (3.30) and therefore Lemma 3.1.3. □

Lemma 3.1.4. *Let* $0 < r < 1$ *and*

$$L_2(t) := \mu\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2 + 2\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t dy\,. \tag{3.36}$$

Given (3.1) and (2.21), the following inequality holds

$$\begin{aligned} \frac{d}{dt}L_2(t) + \frac{m_0}{2}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 \leq {} & \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 \\ & C_2\varphi(t)(\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2 + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2). \end{aligned} \tag{3.37}$$

Proof. Taking the scalar product in $L^2(\Omega)$ of equation (2.13) with

$$\sigma^{3-r}\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v), \tag{3.38}$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}L_2(t) + \sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \\ + \sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v) = \sum_{k=1}^{4} I_k\,,\end{aligned} \tag{3.39}$$

where

$$\begin{aligned}I_1 &:= \frac{1}{2}\Big[\mu(n+1-r)\frac{\sigma'}{\sigma} + (n-2)\frac{\sigma''}{\sigma} + \frac{1}{2}n(n-2)\Big|\frac{\sigma'}{\sigma}\Big|^2\Big]\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|^2_{L^2},\\ I_2 &:= \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + (n+1-r)\frac{\sigma'}{\sigma}\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t dy,\\ I_3 &:= -\frac{1}{2}\sigma^{3-r}\int_\Omega(\nabla\cdot a_2)|\sigma^{\frac{n-2}{2}}v|^2 dy,\\ I_4 &:= -\sigma^{3-r}\int_\Omega(\nabla\cdot a_1)(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v_t)dy - \sigma^{3-r}\int_\Omega(a_1\cdot\sigma^{\frac{n-2}{2}}\nabla v)(\sigma^{\frac{n-2}{2}}v_t)dy\,.\end{aligned}$$

We have

$$\begin{aligned}I_1 &\le \frac{m_0}{16}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \tilde{C}_2\big[|\sigma''|^2 + |\sigma'|^2 + \big|\frac{\sigma'}{\sigma}\big|^2|\sigma'|^2\big]\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|^2_{L^2},\\ I_2 &\le \frac{m_0}{16}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + (1+\tilde{C}_2|\sigma'|^2)\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2}.\end{aligned}$$

Moreover, given the expressions (2.7) of a_1 and a_2, we can estimate the last terms so that

$$\begin{aligned}I_3 + I_4 \le \frac{m_0}{8}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \\ + \tilde{C}_2(|\sigma''|^2 + |\sigma'|^2)(\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|^2_{L^2} + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2}).\end{aligned}$$

By adding these estimates to (3.39) and taking into account of (3.35), we obtain (3.37) and so the lemma 3.1.4. □

Lemma 3.1.5. *Let* $0 < r < 1$ *and*

$$\begin{aligned}L_3(t) := \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2} + \sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v_t) \\ + \sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2} + \frac{1}{2}\sigma^{1-r}M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})[\frac{d}{dt}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}]^2 \\ +2\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v,\sigma^{\frac{n-2}{2}}v_t) - 4\frac{\sigma'}{\sigma}\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v)dy.\end{aligned} \tag{3.40}$$

Given (3.5), (3.1) and (2.21), the following inequality holds

$$\begin{aligned}
&\frac{d}{dt}L_3(t)+\mu\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2} \qquad (3.41)\\
&\le C_3\varepsilon_0^{\alpha}(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}+\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\\
&+C_3\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}+\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})+C_3\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}}\\
&+C_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{2}}.
\end{aligned}$$

Proof. If we differentiate (2.13) with respect to t and we take the scalar product in $L^2(\Omega)$ with

$$\sigma^{3-r}\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v_t)_t, \tag{3.42}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}L_3(t)+\frac{3+r}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\,, \qquad (3.43)\\
&\qquad+\mu\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}=\sum_{k=1}^{7}I_k+\sum_{k=1}^{6}J_k
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= (n-2+\frac{3-r}{2})\frac{\sigma'}{\sigma}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}-\frac{1}{2}\sigma^{3-r}\int_\Omega|(\sigma^{\frac{n-2}{2}}v_t)_t|^2\nabla\cdot a_1dy\,,\\
I_2 &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}\int_\Omega(a_1.\nabla(\sigma^{\frac{n-2}{2}}v_t)(\sigma^{\frac{n-2}{2}}v_t)_tdy\,,\\
I_3 &:= \frac{n-2}{2}\Big[\frac{\sigma''}{\sigma}-(n-1)\Big(\frac{\sigma'}{\sigma}\Big)^2+\mu\frac{\sigma'}{\sigma}\Big]\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v_t)(\sigma^{\frac{n-2}{2}}v_t)_tdy\,,\\
I_4 &:= \sigma^{3-r}\int_\Omega a_2\cdot\nabla(\sigma^{\frac{n-2}{2}}v_t)(\sigma^{\frac{n-2}{2}}v_t)_tdy\,,\\
I_5 &:= \sigma^{3-r}\int_\Omega a_2'.\nabla(\sigma^{\frac{n-2}{2}}v))(\sigma^{\frac{n-2}{2}}v_t)_tdy\,,\\
I_6 &:= \sigma^{3-r}\int_\Omega a_1'\cdot\nabla(\sigma^{\frac{n-2}{2}}v_t)(\sigma^{\frac{n-2}{2}}v_t)_tdy\,,\\
I_7 &:= \frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v)+\sigma^{3-r}a''(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v)\\
&\qquad+\frac{3}{2}\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v_t)+\frac{3-r}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v_t)\,,
\end{aligned}$$

and the nonlinear terms J_k they are given by

$$\begin{aligned}
J_1 &:= -\big[2\frac{\sigma''}{\sigma} + (n+2-2r)\big(\frac{\sigma'}{\sigma}\big)^2\big]\sigma^{1-r} M(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\int_\Omega (\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v)dy\,,\\
J_2 &:= \frac{n-6}{2}\frac{\sigma'}{\sigma}\sigma^{1-r} M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\int_\Omega (\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v)dy\,,\\
J_3 &:= \big((\frac{n-2}{2} + \frac{1-r}{4}\big)\frac{\sigma'}{\sigma}\sigma^{1-r} M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\big[\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\big]^2\,,\\
J_4 &:= \frac{n-2}{2}\big(\frac{\sigma'}{\sigma}\big)'\sigma^{1-r} M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2\,,\\
J_5 &:= \frac{1}{4}\sigma^{1-r} M''(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\big[\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\big]^3\,,\\
J_6 &:= \frac{3}{2}\sigma^{1-r} M'(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2.
\end{aligned}$$

The terms I_k are similar to those in the identity (3.32) and therefore, can be (in particular the first and last term) estimated in the same way. We then obtain taking into account (3.2), (3.1) and (2.21)

$$\sum_{k=1}^{7} I_k \le \frac{\mu}{2}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2 + C_\Omega\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + \sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2, \tag{3.44}$$

where, let us recall the here

$$\varphi(t) := |\sigma'|^2 + \big|\frac{\sigma'}{\sigma}\big|^2. \tag{3.45}$$

As for the non-linear terms, given (3.10), (3.9), (3.5), (3.2), (3.1) and (2.21), one has

$$\sum_{k=1}^{4} J_k \le \tilde{C}_3\varepsilon_0^{\alpha}(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2 + \sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2) \tag{3.46}$$

Regarding the last terms J_5 and J_6, given (3.10), we get

$$\begin{aligned}
J_5 \le \;& \tilde{C}_3\big|\frac{\sigma'}{\sigma}\big|^3(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2)\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^4\\
&+ \tilde{C}_3\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2\big)^{\frac{3}{2}}\big(\frac{1}{\sigma^{\frac{1-r}{3}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2\big)^{\frac{3}{2}}
\end{aligned}$$

and in view of (3.28)

$$J_5 \le \tilde{C}_3 \Big|\frac{\sigma'}{\sigma}\Big|^3 (\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}) + \tilde{C}_3 \big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{2}}.$$

As for the last term J_6, we have (see (3.10))

$$\begin{aligned} J_6 \ &\le \ \tilde{C}_3 \Big[\frac{1}{\sigma^{\frac{1-r}{2}}}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}\Big]\,\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}} \\ &\qquad + \tilde{C}_3 \frac{\sigma'}{\sigma}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}. \end{aligned}$$

Since

$$\begin{aligned} &\frac{\sigma'}{\sigma}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2} \\ &= \frac{\sigma'}{\sigma}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{4}}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{4}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \\ &\le \frac{|\sigma'|^2}{2\sigma^2}\big(\frac{1}{\sigma^{1-r}}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{2}} \\ &\qquad + \frac{1}{2}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{2}}, \end{aligned}$$

then thanks to (3.27) (see also (3.2))

$$J_6 \le \tilde{C}_3\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}} + \tilde{C}_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{2}}$$

and consequently, we have

$$\begin{aligned} &J_5 + J_6 \\ &\le \tilde{C}_3\Big|\frac{\sigma'}{\sigma}\Big|^3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \tilde{C}_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{1}{2}} \\ &\qquad + \tilde{C}_3\big(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big)^{\frac{3}{2}} \end{aligned} \tag{3.47}$$

Finally, putting together (3.47), (3.46) and (3.44), from (3.43) it follows the inequality (3.41) and thus this achieves the proof of lemma 3.1.5. □

Lemma 3.1.6. *Let* $0 < r < 1$ *and*

$$L_4(t) := 2\sigma^{3-r}\int_\Omega (\sigma^{\frac{n-2}{2}}v)_t(\sigma^{\frac{n-2}{2}}v_t)_t dy. \tag{3.48}$$

Given (3.5), (3.1) and (2.21), the following inequality holds

$$\begin{aligned} &\frac{d}{dt}L_4(t) + \frac{m_0}{2}\sigma^{1-r}\|(\sigma^{\frac{n-2}{2}}\nabla v_t)\|^2_{L^2} \\ \leq\ & C_4(\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}) \\ +\ & C_4\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}). \end{aligned} \tag{3.49}$$

Proof. As in the proof of lemma 3.1.5, if we differentiate (2.13) with respect to t and we take the scalar product in $L^2(\Omega)$ of the new equation with

$$\sigma^{3-r}\sigma^{\frac{n-2}{2}}(\sigma^{\frac{n-2}{2}}v)_t\,,$$

we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}L_4(t) + \sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2} + \sigma^{3-r}a(t,\sigma^{\frac{n-2}{2}}v_t,\sigma^{\frac{n-2}{2}}v_t) \\ &+ \sigma^{1-r}M'(\sigma^{\frac{n-2}{2}}\|\nabla v\|^2_{L^2})\Big[\frac{d}{dt}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\Big]^2 = \sum_{i=1}^{8} I_i \end{aligned} \tag{3.50}$$

where

$$\begin{aligned} I_1 &:= \frac{3n-2r}{2}\frac{\sigma'}{\sigma}\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)_t(\sigma^{\frac{n-2}{2}}v_t)_t dy + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2}\,, \\ I_2 &:= [\frac{n-2}{2}(\frac{\sigma''}{\sigma} + \frac{n}{2}|\frac{\sigma'}{\sigma}|^2) + \mu(\frac{3-r}{2} + n - 2)\frac{\sigma'}{\sigma}]\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2}\,, \\ I_3 &:= \frac{n-2}{2}(\frac{\sigma''}{\sigma} - \frac{n}{2}|\frac{\sigma'}{\sigma}|^2)[\frac{n-2}{2}\frac{\sigma'}{\sigma} + \mu]\sigma^{3-r}\int_\Omega(\sigma^{\frac{n-2}{2}}v)(\sigma^{\frac{n-2}{2}}v)_t dy\,, \\ I_4 &:= \sigma^{3-r}\int_\Omega[(a_2 + a_1')\cdot(\sigma^{\frac{n-2}{2}}\nabla v_t + a_2'\cdot(\sigma^{\frac{n-2}{2}}\nabla v)](\sigma^{\frac{n-2}{2}}v)_t dy\,, \\ I_5 &:= -\sigma^{3-r}a'(t,\sigma^{\frac{n-2}{2}}v,(\sigma^{\frac{n-2}{2}}v)_t)\,, \\ I_6 &:= \sigma^{3-r}\int_\Omega(a_1\cdot(\sigma^{\frac{n-2}{2}}\nabla v_{tt}))(\sigma^{\frac{n-2}{2}}v)_t dy\,, \\ I_7 &:= -\frac{n-2}{2}\frac{\sigma'}{\sigma}\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\int_\Omega(\sigma^{\frac{n-2}{2}}\nabla v_t)(\sigma^{\frac{n-2}{2}}\nabla v)dy\,, \\ I_8 &:= (n-2)\Big|\frac{\sigma'}{\sigma}\Big|^2\sigma^{1-r}M(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2})\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\,. \end{aligned}$$

Recalling the expression (2.7) of a_1 and a_2 (see also (2.18), (2.17) and (2.11)) and taking account of (3.2) and (3.1), we can estimate the first six terms so that

$$\begin{aligned}\sum_{i=1}^{5} I_k \leq \frac{m_0}{12}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2 \\ + C_\Omega(\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2) \\ + \tilde{C}_4\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2 + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2).\end{aligned}$$

Furthermore, by integrating by parts, the term I_6 can be estimated ensure that

$$\begin{aligned}I_6 \leq \frac{m_0}{12}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2 \\ + \tilde{C}_4\varphi(t)(\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2).\end{aligned}$$

As for the latter terms, considering (3.10), (3.9), (3.2) and (3.1), we have

$$I_7 + I_8 \leq \frac{m_0}{12}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2 + \tilde{C}_4\varphi(t)\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2.$$

The proof of the lemma is completed by adding the above estimates of the terms I_i to (3.50). □

Lemma 3.1.7. *Let $0 < r < 1$. We set*

$$D(t) := \sigma^{3-r}\left[\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + \|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2\right] + \sigma^{1-r}\left[\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2 + \|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2\right], \tag{3.51}$$

$$L(t) = \sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2 + D(t) \tag{3.52}$$

$$\mathcal{L}(t) := k_1L_1(t) + k_2L_2(t) + k_3L_3(t) + L_4(t), \tag{3.53}$$

where $L_i(t)$ $(i = 1, \ldots 4)$ are given by (3.29), (3.36), (3.40) and (3.48) as for k_1, k_2 and k_3 are positive constants. Then the following inequalities hold

$$\mathcal{L}(t) \geq b_0L(t), \tag{3.54}$$

$$\frac{d}{dt}\mathcal{L}(t) + \frac{b_1}{4}D(t) \leq b_2D(t)\mathcal{L}(t), \tag{3.55}$$

with positive constants $b_0, \ldots, b_2$ independent of σ_0.

Proof. Given (3.29), (3.36), (3.40) and (3.48), it is easy to see that

$$\begin{aligned}L_1(t) &\geq \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 + \frac{m_0}{2}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2, \\ L_2(t) &\geq \frac{\mu}{2}\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|_{L^2}^2 - \frac{2}{\mu}\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2 \\ L_3(t) &\geq \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2 + \frac{m_0}{4}\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|_{L^2}^2 - C_\Omega\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|_{L^2}^2, \\ L_4(t) &\geq -\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|_{L^2}^2 - \sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|_{L^2}^2,\end{aligned}$$

From which (see also (3.52)) we get

$$\mathcal{L}(t) \geq \lambda_1\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + \lambda_2\sigma^{1+r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \frac{k_2\mu}{2}\sigma^{3-r}\|\sigma^{\frac{n-2}{2}}v\|^2_{L^2} \tag{3.56}$$
$$+ \lambda_3\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2} + \frac{m_0}{4}k_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2},$$

where

$$\lambda_1 := k_1 - \frac{2k_2}{\mu} - 1, \quad \lambda_2 := \frac{k_1 m_0}{2} - C_\Omega k_3, \quad \lambda_3 := k_3 - 1\,. \tag{3.57}$$

On the order hand, if we multiply inequality (3.30), (3.37), (3.41) and (3.49) by k_1, k_2, k_3 and $k_4 = 1$ respectively and summing, we obtain

$$\frac{d}{dt}\mathcal{L}(t) + \lambda_4\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + \lambda_5\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} \tag{3.58}$$
$$+\lambda_6\sigma^{3-r}\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2} + \lambda_7\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}$$
$$\leq (C_1k_1 + C_2k_2 + C_3k_3 + C_4)\varphi(t)\big[\sigma^{3-r}(\|\sigma^{\frac{n-2}{2}}v\|^2_{L^2} + \|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2} + \|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2})$$
$$+ \sigma^{1-r}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\big] + k_3C_3\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}}$$
$$+ k_3C_3\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2}\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2}\big]^{\frac{1}{2}}.$$

where

$$\lambda_4 := k_1\mu - k_2 - C_4, \quad \lambda_5 := \frac{k_2m_0}{2} - k_1C_1\varepsilon_0^\alpha - k_3C_3\varepsilon_0^\alpha, \tag{3.59}$$
$$\lambda_6 := k_3\mu - C_4, \quad \lambda_7 := \frac{m_0}{2} - k_3C_3\varepsilon_0^\alpha.$$

If ε_0 is small enough, it is easy to see that we can choose k_1, k_2 and k_3 such as (see (3.57) and (3.59))

$$\lambda_i > 0 \quad i = 1, \dots, 7. \tag{3.60}$$

Indeed, we first choose

$$k_3 = 2\max\big(1, \frac{C_4}{\mu}, \frac{2C_4m_0}{3C_\Omega\mu}, \frac{2m_0}{3C_\Omega\mu}\big), \quad k_1 = \frac{3C_\Omega k_3}{m_0}, \quad 0 < \varepsilon_0^\alpha < \min\big(1, \frac{m_0}{4C_3k_3}\big) \tag{3.61}$$

so that

$$\lambda_2 = \frac{1}{2}C_\Omega k_3, \quad \lambda_3 \geq 1, \quad \lambda_6 \geq C_4, \quad \lambda_7 \geq \frac{m_0}{4}. \tag{3.62}$$

On the other hand, by choosing (see (3.61))

$$k_2 = \frac{3C_\Omega k_3}{m_0}\mu - \max(2C_4, \mu) > 0, \quad 0 < \varepsilon_0^\alpha \leq \min\big(1, \frac{k_2m_0}{4(k_1C_1 + C_3k_3)}, \frac{m_0}{4C_3k_3}\big) \tag{3.63}$$

we have

$$\lambda_4 = k_1\mu - k_2 - C_4 \geq C_4, \quad \lambda_1 := k_1 - \frac{2k_2}{\mu} - 1 \geq 1, \qquad (3.64)$$
$$\lambda_5 = \frac{k_2 m_0}{2} - k_1 C_1 \varepsilon_0^\alpha - C_3 \varepsilon_0^\alpha k_3 \geq \frac{k_2 m_0}{4}.$$

So, considering (3.63)-(3.64), from (3.52) it follows

$$\mathcal{L}(t) \geq b_0 L(t), \quad b_0 = \min\big(1, \frac{k_3 C_\Omega}{2}, \frac{k_2\mu}{2}, \frac{k_3 m_0}{4}\big) \qquad (3.65)$$

i.e (3.54). Furthermore from (3.58) (see also (3.51) it follows

$$\frac{d}{dt}\mathcal{L}(t) + b_1 D(t) \qquad (3.66)$$
$$\leq \tilde{C}_5 \varphi(t)\big[\sigma^{3-r}(\|\sigma^{\frac{n-2}{2}} v\|^2_{L^2} + \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} + \|(\sigma^{\frac{n-2}{2}} v_t)_t\|^2_{L^2})$$
$$+ \sigma^{1-r}(\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}} \nabla v_t\|^2_{L^2})\big] + k_3 C_3 \big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}} \nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}}$$
$$+ \ + \ k_3 C_3 \sigma^{1-r}\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2}\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}} \nabla v_t\|^2_{L^2}\big]^{\frac{1}{2}},$$

where (see (3.63)-(3.64)

$$b_1 = \min(\lambda_j, j = 4, \cdots 7) \geq \min(C_4, \frac{m_0}{4}, \frac{m_0 k_2}{4}), \quad \tilde{C}_5 = C_1 k_1 + C_2 k_2 + C_3 k_3 + C_4. \qquad (3.67)$$

Given (3.51), by recalling the expression (3.45) of φ (see also (3.2) and (3.1)), we have

$$\tilde{C}_5 \varphi(t)\big[\sigma^{3-r}(\|\sigma^{\frac{n-2}{2}} v\|^2_{L^2} + \|(\sigma^{\frac{n-2}{2}} v)_t\|^2_{L^2} + \|(\sigma^{\frac{n-2}{2}} v_t)_t\|^2_{L^2}) \qquad (3.68)$$
$$+ \sigma^{1-r}(\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}} \nabla v_t\|^2_{L^2})\big] = \tilde{C}_5 \varphi(t)(\sigma^{3-r}\|\sigma^{\frac{n-2}{2}} v\|^2_{L^2} + D(t))$$
$$\leq \tilde{C}_5 C_\Omega \varphi(t)\sigma^2 \sigma^{1-r}\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2} + \tilde{C}_5 \varphi(t) D(t)) \leq \tilde{C}_6 \varepsilon_0^\alpha D(t)$$

and (see (3.65), (3.51) and (3.52))

$$k_3 C_3 \big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}} \nabla v_t\|^2_{L^2}\big]^{\frac{3}{2}} + k_3 C_3 \sigma^{1-r}\|\sigma^{\frac{n-2}{2}} \nabla v\|^2_{L^2}\big[\sigma^{1-r}\|\sigma^{\frac{n-2}{2}} \nabla v_t\|^2_{L^2}\big]^{\frac{1}{2}} \qquad (3.69)$$
$$\leq \frac{k_3 C_3}{\sqrt{b_0}} \mathcal{L}^{\frac{1}{2}}(t)(t) D(t) \leq \frac{b_1}{2} D(t) + b_2 D(t)\mathcal{L}(t)$$

where

$$b_2 = \frac{k_3^2 C_3^2}{2 b_1 b_0}. \qquad (3.70)$$

By adding (3.69) and (3.68) to (3.58) and choosing (see (3.63)) ε_0 so that

$$\varepsilon_0^\alpha \leq \frac{b_1}{4\tilde{C}_6}$$

we obtain (3.55). □

Lemma 3.1.8. *Let be* $(v_{\sigma_0}^0, v_{\sigma_0}^1) \in H^2(\Omega) \cap H^1(\Omega)$*. We set*

$$\tilde{R}(\sigma_0) = \sigma_0^{-2\alpha r} \|v_{\sigma_0}^0\|_{H^2}^2 + \|v_{\sigma_0}^1\|_{H^1}^2, \quad \tilde{\lambda}(\sigma_0) = \sigma_0^{\alpha(n-3+r)} \tilde{R}(\sigma_0), \tag{3.71}$$

and we suppose

$$\lim_{\sigma_0 \to 0} \tilde{\lambda}(\sigma_0) = 0. \tag{3.72}$$

Then, if (see (3.1)) σ_0 *is small enough, we have*

$$0 \leq \mathcal{L}(0) < \frac{b_1}{8b_2} \tag{3.73}$$

(see (3.70) (3.67) and (3.65)) for b_1 *and* b_2*.*

Proof. Let us first note that (3.72) implies the hypothesis (3.5) under which lemmas 3.1.2-3.1.6 and therefore lemma 3.1.7 are established. That said, by recalling the expressions (see (3.40), (3.48) and (3.53)) of $L_3(t)$ and $L_4(t)$, we can see that $\mathcal{L}(0)$ contains the L^2-norm of the term $v_{tt}|_{t=0}$. This term is defined (see (2.7), (2.11) and (2.13)) by

$$v_{tt}|_{t=0} = -\mu v_{\sigma_0}^1 + \frac{1}{\sigma_0^{2\alpha}} \tilde{A} v_{\sigma_0}^0 + a_1(0,y).\nabla v_{\sigma_0}^1 + a_2(0,y).\nabla v_{\sigma_0}^0\,,$$

where

$$\tilde{A} v_{\sigma_0}^0 = \sum_{i,j}^{n} \partial_{y_i}\Big(\big(\tilde{M}(\sigma_0^{\alpha(n-2)} \|\nabla v_{\sigma_0}^0\|_{L^2}^2)\delta_{ij} - \alpha^2 \Big|\frac{\sigma_1}{\sigma_0}\Big|^2 \sigma_0^{2\alpha} y_i y_j\big) \partial_{y_j} v_{\sigma_0}^0\Big).$$

Therefore, considering (3.26), (2.10) and (3.1), we get

$$\|v_{tt}|_{t=0}\|_{L^2} \leq C_\Omega (\|v_{\sigma_0}^1\|_{H^1} + \|v_{\sigma_0}^0\|_{H^1}) + C_\Omega \sigma_0^{-2\alpha} \|v_{\sigma_0}^0\|_{H^2}. \tag{3.74}$$

Recalling (3.48), (3.40) and (3.74), given (3.1) and (3.26) the easy computations give us

$$L_3(0) + L_4(0) \leq \tilde{C}_7 \sigma_0^{\alpha(n-3+r)} (\sigma_0^{-2\alpha r} \|v_{\sigma_0}^0\|_{H^2}^2 + \|v_{\sigma_0}^1\|_{H^1}^2). \tag{3.75}$$

Moreover, one can easily see that

$$L_1(0) + L_2(0) \leq \tilde{C}_7 \sigma_0^{\alpha(n-3+r)} (\|v_{\sigma_0}^0\|_{H^1}^2 + \|v_{\sigma_0}^1\|_{L^2}^2). \tag{3.76}$$

So, from (3.76), (3.75) and (3.53) (see also (3.71)) it follows $\mathcal{L}(0) \leq \tilde{C}_8 \tilde{\lambda}(\sigma_0)$ and from (3.72) it follows (3.73) □

3.2 Global solution and its asymptotic behaviour

Lemmas 3.1.1-3.1.7 being established, now we are in position to prove our main result on the existence and asymptotic behaviour of global solution of the initial boundary value problem (0.1)-(0.3). More precisely, fixed the initial expansion Ω_0, we give initial data $(u_0, u_1) \in H^2(\Omega_0) \times H^1(\Omega_0)$ verifying (3.6), we suppose Ω_0 small enough and we ask the question of the existence of global solution u of the initial boundary value problem (1.1)-(1.3). Here, we insist on the fact the initial data (u_0, u_1) can be large enough. In fact, recalling (3.6) (see also (3.3)) and fixed R_0 large enough, it can be seen that if

$$0 < |\Omega_0| \le \frac{C_\Omega}{R_0^{\frac{n}{1+r}}} \quad n \ge 3, \quad 0 < r < 1$$

then

$$R(\Omega_0) = \|u_0\|^2_{H^2(\Omega_0)} + \|u_1\|^2_{H^1(\Omega_0)} \le R_0.$$

Our main result enunciated above is a non trivial generalization in higher dimension of our previous papers [3] and [4], where the results are obtained in dimension one and two with an unbounded expansion of the domain and with initial data sufficiently small.

Theorem 3.2.1. *Let σ_0 small enough, $\Omega_0 = \sigma_0^\alpha \Omega$ and $(u_0, u_1) \in H^2(\Omega_0) \times H^1(\Omega_0)$ such that (3.6) is satisfied then the initial boundary value problem (0.1)-(0.3) has a unique global solution*

$$u \in L^\infty(0, \infty; H_0^1(\Omega_t) \cap H^2(\Omega_t)) \tag{3.77}$$

$$u_t \in L^\infty(0, \infty; H^1(\Omega_t)), \quad u_{tt} \in L^\infty(0, \infty; L^2(\Omega_t)).$$

Moreover,

$$\|u_{tt}\|^2_{L^2(\Omega_t)} + \|u_t\|^2_{L^2(\Omega_t)} + \|u\|^2_{H^1(\Omega_t)} \le \frac{C_\Omega}{|\Omega_t|^{\frac{1-r}{n}}}, \quad 0 < r < 1\,. \tag{3.78}$$

Remark 3.2.1. *Theorem 3.2.1 results from the existence of a global solution v of problem (2.13) and (2.15) under the assumption (3.72) on the initial data $(v^0_{\sigma_0}, v^1_{\sigma_0})$.*

Indeed, if under hypothesis (3.72) (see also (3.71)) such a solution v exists, one can easily verify that $u = v o \tau$ (see (1.6)) is a global solution of the initial boundary value problem (0.1)-(0.3) with the initial data (u_0, u_1) large enough. Thus the proof of theorem 4.1 is reduced to that of the existence of a global solution v of problem (2.13) and (2.15) under the assumption (3.72) on the initial data $(v^0_{\sigma_0}, v^1_{\sigma_0})$. The latter follows from combination of its local solution and some of these a priori estimates allowing to get the uniform boundedness with respect to $t \in [0,\infty[$ of the weighted norm $L(t)$ (see (3.52)). In fact, if this norm is bounded for all t by the same constant, as will be seen, we can then step by step extend the local solution v to the whole interval $[0,\infty[$.

Proof of Theorem 3.2.1. From (3.55) it follows that

$$\frac{d}{dt}\mathcal{L}(t) + \frac{b_1}{8}D(t) + b_2 D(t)\big[\frac{b_1}{8b_2} - \mathcal{L}(t)\big] \leq 0\,, \tag{3.79}$$

Considering (3.73), there exists $\tau_0 > 0$ small enough (which depend only on the size of initial data) such that

$$\mathcal{L}(t) \leq \mathcal{L}(0) \quad \forall t \in [0,\tau_0]. \tag{3.80}$$

Indeed, suppose that for all $\tau_0 > 0$ there exists $t_0 \in]0,\tau_0[$ such that $\mathcal{L}(t_0) > \mathcal{L}(0)$, and pose

$$t^* := \inf\{t \in]0,\tau_0[: \mathcal{L}(t) > \mathcal{L}(0)\}. \tag{3.81}$$

Whence $\mathcal{L}(t) \leq \mathcal{L}(0)$ for all $t \in]0,t^*[$ and, by recalling (3.73), we have

$$\frac{b_1}{8b_2} - \mathcal{L}(t) > 0 \quad \text{for all} \;\; t \in]0,t^*[.$$

Therefore from (3.79) it follows,

$$\frac{d}{dt}\mathcal{L}(t) < 0 \quad \forall t \in]0,t^*[$$

which implies $\mathcal{L}(t^*) < \mathcal{L}(0)$ which is contradictory with (3.81).

From (3.80), it follows $\mathcal{L}(\tau_0) \leq \mathcal{L}(0)$ so that by reiterating the arguments which leads us (3.80), we easily see that $\mathcal{L}(t) \leq \mathcal{L}(0)$ for all $t \in [0,\bar{t}[$ with some positive $\bar{t}$. Let then $\bar{T} := \sup\{T \geq 0 : \;\mathcal{L}(t) \leq \mathcal{L}(0) \;\; \forall t \in [0,T[\} \in [0,+\infty]$. If $\bar{T}$ is finite, we have $\mathcal{L}(\bar{T}-\varepsilon) \leq \mathcal{L}(0)$ for all $\varepsilon > 0$ and with the same reasoning as above, we show that there exists a some $\delta > 0$ such that $\mathcal{L}(t) \leq \mathcal{L}(\bar{T}-\varepsilon) \leq \mathcal{L}(0)$ for all $t \in [\bar{T}-\varepsilon, \bar{T}-\varepsilon+\delta[$. By choosing $\varepsilon = \delta/2$, we obtain $\mathcal{L}(t) \leq \mathcal{L}(0)$ for all $t \in [\bar{T}-\delta/2, \bar{T}+\delta/2[$ which contradicts the definition of $\bar{T}$ so we have $\mathcal{L}(t) \leq \mathcal{L}(0)$ for all $t \in [0,\infty[$. This last inequality gives us (see (3.54)) $L \in L^\infty(0,\infty)$ and from (3.79) is follows that $D \in L^1(0,\infty)$. By reminding (see (3.51) and (3.52)) expressions of L and D, we obtain

$$\begin{gathered}\sigma^{\frac{3-r}{2}}(\sigma^{\frac{n-2}{2}}v) \in L^\infty(0,\infty;L^2), \quad \sigma^{\frac{1-r}{2}}(\sigma^{\frac{n-2}{2}}v) \in L^\infty \cap L^2(0,\infty;H^1), \quad (3.82)\\ \sigma^{\frac{3-r}{2}}(\sigma^{\frac{n-2}{2}}v_t) \in L^\infty \cap L^2(0,\infty;L^2), \quad \sigma^{\frac{1-r}{2}}(\sigma^{\frac{n-2}{2}}v_t) \in L^\infty \cap L^2(0,\infty;H^1),\\ \sigma^{\frac{3-r}{2}}(\sigma^{\frac{n-2}{2}}v_{tt}) \in L^\infty \cap L^2(0,\infty;L^2).\end{gathered}$$

Now, we rewrite the equation (2.13) in the following form

$$-\sum_{i,j=1}^{n} \partial_{y_i}(\tilde{a}_{ij}\partial_{y_j}v) = \tilde{F},$$

where

$$\tilde{a}_{ij} = \tilde{M}(\sigma^{\frac{n-2}{2}}\|\nabla v\|^2_{L^2})\delta_{ij} - |\sigma'|^2 y_i y_j, \quad \tilde{F} = \sigma^2(-v_{tt} - \mu v_t + a_1.\nabla v_t + a_2.\nabla v).$$

From (3.2), (0.4) and (2.10) it is easy to see that

$$\sum_{i,j=1}^{n} \tilde{a}_{ij}\xi_j\xi_i \geq \frac{m_0}{2}|\xi|^2.$$

So, by standard regularity arguments of elliptic equations we have

$$\begin{gathered}\|\sigma^{\frac{n-2}{2}}v\|_{H^2} \leq C_\Omega\|\sigma^{\frac{n-2}{2}}\tilde{F}\|_{L^2} \leq \qquad (3.83)\\ C_\Omega\frac{\sigma^4}{\sigma^{3-r}}\big[\sigma^{3-r}(\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}}v_t\|^2_{L^2}) + \sigma^{1-r}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\big].\end{gathered}$$

The last inequality follows from the above expression of $\tilde{F}$ (see also (3.2), (2.21) and (2.7),). From (3.83) and (3.82) it follows that

$$\sigma^{\frac{1+r}{2}}(\sigma^{\frac{n-2}{2}}v) \in L^\infty \cap L^2(0,\infty,H^2) \quad 0 < r < 1\,. \qquad (3.84)$$

Now, if we use $u = v \circ \tau$ (see (1.6) for definition of τ), given (3.84) and (3.82), by easy computations we can see that

$$u \in L^\infty(0,\infty;H^1_0(\Omega_t) \cap H^2(\Omega_t)), \;\; u_t \in L^\infty(0,\infty;H^1(\Omega_t)), \;\; u_{tt} \in L^2(0,\infty;L^2(\Omega_t)).$$

In order to complete the proof of Theorem 3.2.1, it remains to prove the asymptotic behaviour of global solution. Indeed, given (3.1) and (2.21), easy computations gives us

$$\|u\|^2_{H^1(\Omega_t)} + \|u_t\|^2_{H^1(\Omega_t)} + \|u_{tt}\|^2_{L^2(\Omega_t)} \leq$$
$$\frac{C_\Omega}{\sigma^{1-r}}\big[\sigma^{3-r}(\|(\sigma^{\frac{n-2}{2}}v_t)_t\|^2_{L^2} + \|(\sigma^{\frac{n-2}{2}}v)_t\|^2_{L^2}) + \sigma^{1-r}(\|\sigma^{\frac{n-2}{2}}\nabla v\|^2_{L^2} + \|\sigma^{\frac{n-2}{2}}\nabla v_t\|^2_{L^2})\big]$$

and from (3.82) follows easily

$$\|u\|^2_{H^1(\Omega_t)} + \|u_t\|^2_{H^1(\Omega_t)} + \|u_{tt}\|^2_{L^2(\Omega_t)} \leq \frac{C_\Omega}{\sigma^{1-r}(t)},$$

that is to say (3.78) because $|\Omega_t| = \sigma^n(t)|\Omega|$. This conclude the proof of Theorem 3.2.1. $\square$

Conclusion

This problem has already been studied in [2, 3] in the the two-dimensional space case. Our goal here is to extend the results in the articles [2, 3] in higher dimensional space and for opportunely large initial data. The standard results in the literature include local existence and global in time well-posedness for small enough initial data. Here we consider a case of a domain slowly increasing in time, obtaining a condition for global existence that involves the size of the initial domain and the initial data. Under this condition, the initial data are bounded above by the reciprocal of the domain size (see (4.6)), allowing for large (but not arbitrarily large) initial data at the expense of the domain size. The domain is allowed to increase to infinity as $t \to \infty$, but that is compensated by time decay of the solution. The estimate on the rate of decay is obtained in Theorem 4.1. The main idea for the proof is transforming to an equation in a fixed domain in order to obtain a "big initial conditions"-type (see (4.4) and (4.5)) result for the transformed problem using local in time existence and a priori estimates. As such, the main result is not surprising, although delicate a priori bounds had to be established in order to accommodate growing domain. let us point out that the general case of arbitrarily large initial data, independent on the domain size is not yet resolved, although the idea of the evolving domain, compensating its increase by the decay of the solutions seems interesting.

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